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Corrigendum to "On multilinear fractional strong maximal operator associated with rectangles and multiple weights" [Rev. Mat. Iberoam. 33 (2017), no. 2, 555–572]

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We wish to acknowledge and repair a gap in a proof in our paper "On multilinear fractional strong maximal operator associated with rectangles and multiple weights", which appeared in Revista Matemática Iberoamericana, volume 33 (2017), number 2, pages 555–572.

(We wish to thank Professor A. Meskhi and H. Tanaka for bringing this gap to our attention.)

In Lemma 3.3, we wish to show that the Carleson embedding theorem holds for all rectangles in $\mathbb{R}^n$ with sides parallel to the axes. However, the proof of Lemma 3.3 is valid only for all cubes. This corrigendum will be devoted to presenting a complete proof of Lemma 3.3. We notice that in the very special case that the rectangles are constructed by the product of cubes and the weights satisfy very strong conditions, some weighted estimates have been given in [3].

Throughout this note, the following notations $\mathcal{DR}$ and $\mathcal{DQ}$ will be used to denote the families of all dyadic rectangles and all dyadic cubes in $\mathbb{R}^n$, respectively:

$$\mathcal{DR} := \big\{2^{-k}(m+[0,1)) : \, k, m \in \mathbb{Z}\big\}^n,$$
$$\mathcal{DQ} := \big\{2^{-k}(m+[0,1)^n) : \, k \in \mathbb{Z}, \, m \in \mathbb{Z}^n\big\}.$$

The Carleson embedding theorem in Lemma 3.3 is stated as follows.

Theorem 1 (Carleson embedding theorem). *Let $1 < p < q < \infty$, and let ω be a nonnegative locally integrable function on $\mathbb{R}^n$. Assume that $\sigma := \omega^{1-p'}$ satisfies the dyadic reverse doubling condition with $\beta > 1$. Then, for all nonnegative $f \in L^p(\omega)$, the following inequality holds:*

$$\sum_{I \in \mathcal{DR}} \Big(\int_I \omega^{1-p'} \, dx \Big)^{-q/p'} \Big(\int_I f(x) \, dx \Big)^q \le C \Big(\int_{\mathbb{R}^n} f(x)^p \, \omega \, dx \Big)^{q/p}, \tag{1}$$

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where the constant C depends only on n, p, q and β.

The definition of dyadic reverse doubling condition in [1] will be slightly changed.

Definition 2. We say that a nonnegative measurable function ω satisfies the *dyadic reverse doubling condition*, or $\omega \in RD^{(\beta)}$, if ω is locally integrable on $\mathbb{R}^n$ and there is a constant $\beta > 1$ such that

$$\beta \int_I \omega(x)\, dx \leq \int_J \omega(x)\, dx$$

for any $I, J \in \mathcal{DR}$, where $I \subset J$ and $|I| = 2^{-1}|J|$.

Remark 3. The relationship between the above new $RD^{(\beta)}$ and the old class $RD^{(\beta)}$ in [1] is that: $\bigcup_{\beta>1} \text{new}\, RD^{(\beta)}(\mathbb{R}^n) \subsetneq \bigcup_{\beta>1} \text{old}\, RD^{(\beta)}(\mathbb{R}^n)$. Moreover, the new class $RD^{(\beta)}$ is still bigger than $A_{\infty,\mathcal{R}}$.

In order to prove Theorem 1, we first prepare some notions.

Let $x_i \in \mathbb{R}^{n_i}$, $R_i \in \mathcal{DR}(\mathbb{R}^{n_i})$ for $i = 1, 2$. Let $\sigma(x_1, x_2)$ be a weight defined on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$. Then, we define $\sigma(x_1, R_2)$ and $\sigma(R_1, R_2)$ by

$$\sigma(x_1, R_2) := \int_{R_2} \sigma(x_1, x_2)\, dx_2, \quad \text{and} \quad \sigma(R_1, R_2) := \int_{R_1} \int_{R_2} \sigma(x_1, x_2)\, dx_2\, dx_1.$$

The following two lemmas are needed in the proof of Theorem 1.

Lemma 4. *Let $n_1, n_2 \in \mathbb{N}$ and $n = n_1 + n_2$. Let σ be a nonnegative and locally integrable function defined on $\mathbb{R}^n$, $\sigma \in RD^{(\beta)}(\mathbb{R}^n)$ for some $\beta > 1$. Then for any $R_2 \in \mathcal{DR}(\mathbb{R}^{n_2})$, it holds that $\sigma(\cdot, R_2) \in RD^{(\beta)}(\mathbb{R}^{n_1})$. For a.e. $x_2 \in \mathbb{R}^{n_2}$, it holds that $\sigma(\cdot, x_2) \in RD^{(\beta)}(\mathbb{R}^{n_1})$.*

Proof. Let $I_1, J_1 \in \mathcal{DR}(\mathbb{R}^{n_1})$ with $I_1 \subset J_1$ and $|I_1| = 2^{-1}|J_1|$. Then it is easy to see that $I_1 \times R_2, J_1 \times R_2 \in \mathcal{DR}(\mathbb{R}^n)$ with $I_1 \times R_2 \subset J_1 \times R_2$ and $|I_1 \times R_2| = 2^{-1}|J_1 \times R_2|$. Hence, the assumption

$$\beta \int_{I_1 \times R_2} \sigma(x)\, dx \leq \int_{J_1 \times R_2} \sigma(x)\, dx,$$

implies that

$$\begin{aligned} \beta \int_{I_1} \sigma(x_1, R_2)\, dx_1 &= \beta \int_{I_1} \Big(\int_{R_2} \sigma(x_1, x_2)\, dx_2 \Big)\, dx_1 \\ &\leq \int_{J_1} \Big(\int_{R_2} \sigma(x_1, x_2)\, dx_2 \Big)\, dx_1 = \int_{J_1} \sigma(x_1, R_2)\, dx_1. \end{aligned}$$

Therefore, it follows that $\sigma(\cdot, R_2) \in RD^{(\beta)}(\mathbb{R}^{n_1})$.

Now, taking $R_2 \in \mathcal{DQ}(\mathbb{R}^{n_2})$, we have

$$\frac{1}{|R_2|} \int_{R_2} \Big(\beta \int_{I_1} \sigma(x_1, x_2)\, dx_1 \Big)\, dx_2 \leq \frac{1}{|R_2|} \int_{R_2} \Big(\int_{J_1} \sigma(x_1, x_2))\, dx_1 \Big)\, dx_2.$$

Moreover, by the Lebesgue differentiation theorem, it yields that

$$\beta \int_{I_1} \sigma(x_1, x_2)\, dx_1 \le \int_{J_1} \sigma(x_1, x_2)\, dx_1 \quad \text{a.e. } x_2 \in \mathbb{R}^{n_2}.$$

This shows that $\sigma(\cdot, x_2) \in RD^{(\beta)}(\mathbb{R}^{n_1})$ for a.e. $x_2 \in \mathbb{R}^{n_2}$. □

Lemma 5. *Let $1 < p \le q < \infty$. For any nonnegative measurable functions g and h, let σ be a weight defined on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ and satisfy*

$$\sum_{R_1 \in \mathcal{DR}(\mathbb{R}^{n_1})} \sigma(R_1, R_2)^{q/p} \Big(\frac{1}{\sigma_1(R_1, R_2)} \int_{R_1} g(x_1)\, \sigma(x_1, R_2)\, dx_1 \Big)^q \le C \Big(\int_{\mathbb{R}^{n_1}} g(x_1)^p \sigma(x_1, R_2)\, dx_1 \Big)^{q/p} \quad \text{for each } R_2 \in \mathcal{DR}(\mathbb{R}^{n_2}), \tag{2}$$

and

$$\sum_{R_2 \in \mathcal{DR}(\mathbb{R}^{n_2})} \sigma(x_1, R_2)^{q/p} \Big(\frac{1}{\sigma(x_1, R_2)} \int_{R_2} h(x_2)\, \sigma(x_1, x_2)\, dx_2 \Big)^q \le C \Big(\int_{\mathbb{R}^{n_2}} h(x_2)^p \sigma(x_1, x_2)\, dx_2 \Big)^{q/p} \quad \text{for a.e. } x_1 \in \mathbb{R}^{n_1}, \tag{3}$$

where the constants C in (2) and (3) are independent of R_2 and x_1.

Then, for any nonnegative measurable function f defined on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, it holds that

$$\sum_{R \in \mathcal{DR}(\mathbb{R}^{n_1+n_2})} \sigma(R)^{q/p} \Big(\frac{1}{\sigma(R)} \int_R f(x)\, d\sigma(x) \Big)^q \le C \Big(\int_{\mathbb{R}^{n_1+n_2}} f(x)^p\, d\sigma(x) \Big)^{q/p}.$$

Proof. First, we may rewrite

$$\begin{aligned}
&\sum_{R \in \mathcal{DR}(\mathbb{R}^{n_1+n_2})} \sigma(R)^{q/p} \Big(\frac{1}{\sigma(R)} \int_R f(x)\, d\sigma(x) \Big)^q \\
&= \sum_{R_2 \in \mathcal{DR}(\mathbb{R}^{n_2})} \Bigg[\sum_{R_1 \in \mathcal{DR}(\mathbb{R}^{n_1})} \sigma(R_1, R_2)^{q/p} \\
&\quad \times \Big\{ \frac{1}{\sigma(R_1, R_2)} \int_{R_1} \Big(\frac{1}{\sigma(x_1, R_2)} \int_{R_2} f(x_1, x_2)\, \sigma(x_1, x_2)\, dx_2 \Big) \sigma(x_1, R_2)\, dx_1 \Big\}^q \Bigg].
\end{aligned}$$

Using (2), the Minkowski inequality and (3), the left side of the above equation can be further controlled by

$$
\begin{aligned}
&\leq C \sum_{R_2 \in \mathcal{DR}(\mathbb{R}^{n_2})} \Big[\int_{\mathbb{R}^{n_1}} \Big(\frac{1}{\sigma(x_1, R_2)} \int_{R_2} f(x_1, x_2)\, \sigma(x_1, x_2)\, dx_2 \Big)^p \sigma(x_1, R_2)\, dx_1 \Big]^{q/p} \\
&= \Big(\Big\{ \sum_{R_2 \in \mathcal{DR}(\mathbb{R}^{n_2})} \Big[\int_{\mathbb{R}^{n_1}} \Big(\frac{1}{\sigma(x_1, R_2)} \int_{R_2} f(x_1, x_2)\, \sigma(x_1, x_2)\, dx_2 \Big)^p \\
&\qquad\qquad \times \sigma(x_1, R_2)\, dx_1 \Big]^{q/p} \Big\}^{p/q} \Big)^{q/p} \\
&\leq \Big(\int_{\mathbb{R}^{n_1}} \Big\{ \sum_{R_2 \in \mathcal{DR}(\mathbb{R}^{n_2})} \sigma(x_1, R_2)^{q/p} \\
&\qquad\qquad \times \Big(\frac{1}{\sigma(x_1, R_2)} \int_{R_2} f(x_1, x_2)\, \sigma(x_1, x_2)\, dx_2 \Big)^q \Big\}^{p/q} dx_1 \Big)^{q/p} \\
&\leq C \Big(\int_{\mathbb{R}^{n_1}} \Big\{ \Big(\int_{\mathbb{R}^{n_2}} f(x_1, x_2)^p\, \sigma(x_1, x_2)\, dx_2 \Big)^{q/p} \Big\}^{p/q} dx_1 \Big)^{q/p} \\
&= \Big(\int_{\mathbb{R}^{n_1}} \int_{\mathbb{R}^{n_2}} f(x_1, x_2)^p\, \sigma(x_1, x_2)\, dx_2\, dx_1 \Big)^{q/p} \\
&= \Big(\int_{\mathbb{R}^{n_1+n_2}} f(x)^p\, d\sigma(x) \Big)^{q/p}.
\end{aligned}
$$

□

Now, we are in the position to give the proof of Theorem 1.

Proof of Theorem 1. To prove inequality (1), by using the Sawyer's trick, it is equivalent to show that

$$
\sum_{R \in \mathcal{DR}} \sigma(R)^{q/p} \Big(\frac{1}{\sigma(R)} \int_R f\, d\sigma \Big)^q \leq C \Big(\int_{\mathbb{R}^n} f^p\, d\sigma \Big)^{q/p}. \tag{4}
$$

First, we note that (4) is valid in the one dimensional case because the former proof in [1] holds for dyadic cubes. Now, we assume that (4) holds for the case of dyadic rectangles in the $n-1$ dimension, and consider the n-dimensional case. Assume that $\sigma := \omega^{1-p'} \in RD^{(\beta)}(\mathbb{R}^n)$ for some $\beta > 1$. Then, for $R_2 \in \mathcal{DR}(\mathbb{R}^{n-1})$, by Lemma 4, we have $\sigma(\cdot, R_2) \in RD^{(\beta)}(\mathbb{R})$. Therefore, for any $R_2 \in \mathcal{DR}(\mathbb{R})$, as an application of the one dimensional case, there exists a constant $C > 0$, independent of $R_2 \in \mathcal{DR}(\mathbb{R}^{n-1})$, such that

$$
\begin{aligned}
\sum_{R_1 \in \mathcal{DR}(\mathbb{R})} \sigma(R_1, R_2)^{q/p} \Big(\frac{1}{\sigma_1(R_1, R_2)} \int_{R_1} g(x_1)\, \sigma(x_1, R_2)\, dx_1 \Big)^q& \\
\leq C \Big(\int_{\mathbb{R}} g(x_1)^p\, \sigma(x_1, R_2)\, dx_1 \Big)^{q/p}&.
\end{aligned}
$$

By Lemma 4, it holds immediately that $\sigma(x_1,\cdot) \in RD^{(\beta)}(\mathbb{R}^{n-1})$ uniformly in $x_1 \in \mathbb{R}$. Hence, for a.e. $x_1 \in \mathbb{R}$, the induction assumption yields that

$$\sum_{R_2\in\mathcal{DR}(\mathbb{R}^{n-1})} \sigma(x_1,R_2)^{q/p}\Big(\frac{1}{\sigma(x_1,R_2)}\int_{R_2} h(x_2)\,\sigma(x_1,x_2)\,dx_2\Big)^q \\ \leq C\Big(\int_{\mathbb{R}^{n-1}} h(x_2)^p\,\sigma(x_1,x_2)\,dx_2\Big)^{q/p},$$

where C is independent of x_1.

Therefore, by Lemma 5, we obtained the desired inequality (4). The proof of Theorem 1 is complete. □

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