



L^p regularity of homogeneous elliptic differential operators with constant coefficients on \mathbb{R}^N

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Abstract. Let A be a homogeneous elliptic differential operator of order m on \mathbb{R}^N with constant complex coefficients. A special case of the main result is as follows: suppose that $u \in L^1_{loc}$ and that $Au \in L^p$ for some $1 < p < \infty$. Then, all the partial derivatives of order m of u are in L^p if and only if $|u|$ grows slower than $|x|^m$ at infinity, provided that growth is measured in an L^1 -averaged sense over balls with increasing radii. The necessity provides an alternative answer to the pointwise growth question investigated with mixed success in the literature. Only very few special cases of the sufficiency are already known, even when $A = \Delta$.

The full result gives a similar necessary and sufficient growth condition for the derivatives of u of any order $k \geq 0$ to be in L^p when Au satisfies a suitable (necessary) condition. This is generalized to exterior domains, which sometimes introduces mandatory restrictions on N and p , and to Douglis–Nirenberg elliptic systems whose entries are homogeneous operators with constant coefficients but possibly different orders, as the Stokes system.

1. Introduction

It is understood that \mathbb{R}^N is the domain of all function spaces. The vast PDE literature offers only surprisingly few answers to the basic question: if $u \in \mathcal{D}'$ (distributions) and $\Delta u \in L^p$ for some $1 < p < \infty$, what extra condition should be required of u to ensure that all the second order derivatives of u are in L^p ? If so, $u \in \mathcal{S}'$ (tempered distributions), but this necessary condition is obviously not sufficient.

The same question with Δ replaced with, say, $\Delta - 1$, is answered by the classical L^p regularity theory of elliptic PDEs: $u \in \mathcal{S}'$ is now necessary and sufficient since every distribution with second order derivatives in L^p is in \mathcal{S}' and since $u \in \mathcal{S}'$ and $\Delta u - u \in L^p$ imply $u \in W^{2,p}$ (classical Sobolev space).

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For the Laplace operator, the known sufficient conditions, such as $u \in L^p$ (for then $\Delta u - u \in L^p$) or the weaker $(1 + |x|^2)^{-1}u \in L^p$ (an implicit by-product of a result of Nirenberg and Walker in weighted spaces, Theorem 3.1 in [37]) or $\nabla u \in (L^q)^N$ for some $1 < q < \infty$ (by duality and bootstrapping; see Remark V.5.3, p. 349 in Galdi [15]), do not point to any recognizable common feature, especially since the proof of their sufficiency is each time completely different.

In this paper, we show, among other things, that if A is any *homogeneous* elliptic operator of order m with constant complex coefficients and $Au \in L^p$, all the partial derivatives of order m of u are in L^p if and only if u satisfies a very simple *necessary and sufficient* side condition. We shall actually prove significantly more general results in the same spirit. Here and everywhere in the paper, “homogeneous” is synonymous with “pure order”, that is, A is of the form

$$(1.1) \quad Au = i^m \sum_{|\alpha|_1=m} a_\alpha \partial^\alpha u,$$

where $m \in \mathbb{N}$ (to avoid trivialities, we rule out $m = 0$) and $a_\alpha \in \mathbb{C}$, and where $|\alpha|_1 := \alpha_1 + \dots + \alpha_N$. Recall that the ellipticity assumption means

$$(1.2) \quad A(\xi) := \sum_{|\alpha|_1=m} a_\alpha \xi^\alpha \neq 0 \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\}.$$

If $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and $1 \leq p \leq \infty$, we define the homogeneous Sobolev space (also known as Beppo Levi space, after Deny and Lions [11]; various other notations, e.g., $L^{k,p}, \hat{W}^{k,p}, BL^{k,p}$, etc., are used in the literature) as

$$(1.3) \quad D^{k,p} := \{u \in \mathcal{D}' : \partial^\alpha u \in L^p, |\alpha|_1 = k\} = \{u \in L^1_{\text{loc}} : \partial^\alpha u \in L^p, |\alpha|_1 = k\},$$

where the second equality follows from the well-known fact that a distribution with first-order partial derivatives in L^p_{loc} is itself in L^p_{loc} (Theorem XV, p. 181 in Schwartz [43]).

When $1 < p < \infty$, the necessary and sufficient condition for $Au \in L^p$ to imply $u \in D^{m,p}$ given in this paper is just a *growth limitation* on $|u|$ at infinity, but the correct concept of growth is not a pointwise one. This is made precise through the introduction of spaces $M^{s,q}$ and subspaces $M_0^{s,q}$ for $s \in \mathbb{R}$ and $1 \leq q \leq \infty$ (Section 2). In essence, $u \in M^{s,q}$ ($M_0^{s,q}$) if and only if $|u|$ does not grow faster (grows slower) than $|x|^s$ after both are L^q -averaged over balls with increasing radii.

On the other hand, since growth slower than $|x|^0 = 1$ must be viewed as decay, the functions of $M_0^{s,q}$ with $s \leq 0$, all contained in $M_0^{0,1}$, tend to 0 at infinity in an L^q -average sense. These functions are related to, but have more structure than, the “functions vanishing at infinity” of Lieb and Loss [26].

The simplest special case of the main result reads:

Theorem 1.1. *If A in (1.1) is elliptic and $u \in \mathcal{D}'$, then $u \in D^{m,p}$ for some $1 < p < \infty$ if and only if $Au \in L^p$ and $u \in M_0^{m,1}$. In other words,*

$$(1.4) \quad D^{m,p} = \{u \in M_0^{m,1} : Au \in L^p\}.$$

It is rather remarkable that $u \in D^{m,p}$ – a matter of integrability of $\nabla^m u$ at infinity since $Au \in L^p$ – depends only upon the growth of u itself and, in addition,

that this growth can be evaluated in a p -independent L^1 sense. Although the spaces $M^{s,q}$ with $q > 1$ are not involved in this criterion, they are still important for various technical reasons and in the applications.

In particular, the characterization (1.4) yields an estimate of the growth at infinity of the functions of the space $D^{m,p}$. Their *pointwise* growth was investigated by Mizuta [33] and, earlier, by Fefferman [14], Portnov [39], Uspenskiĭ [48], etc. When $mp > N$, Mizuta’s estimates are uniform, but when $mp \leq N$ (so that functions of $D^{m,p}$ need not be continuous), they are only valid outside some set thinning out at infinity, which makes them much harder to use in practice. Uniform pointwise estimates when $m = 1$ can also be found in Galdi’s book [15], but only for functions of $D^{1,p} \cap D^{1,q}$ for some $q > N$. They coincide with Mizuta’s when $p = q > N$. It has not been proved, or even suggested, that such pointwise estimates, plus $Au \in L^p$, imply $u \in D^{m,p}$. In other words, there is no prior variant of Theorem 1.1.

With a suitable (standard) definition of $D^{k,p}$ when $k < 0$, we actually prove that, more generally, $u \in D^{m+\kappa,p}$ for some integer $\kappa \geq -m$ if and only if $Au \in D^{\kappa,p}$ and $u \in M_0^{m+\kappa,1}$ (Theorem 4.5). This does not follow inductively from Theorem 1.1. When $k < 0$, not only the distributions of $D^{k,p}$ are generally not functions, but there is no limitation, pointwise or averaged, to the growth at infinity of the functions of $D^{k,p}$ (Remark 4.3). For that reason, there is no predictable generalization of Theorem 1.1 when $\kappa < -m$. Such a generalization could perhaps rely on a suitable refinement of the inherent “slow growth” of the distributions of \mathcal{S}' , but this is only speculative at this time.

Also, Theorem 1.1 breaks down when A is not homogeneous. Its validity when A is homogeneous with variable coefficients is a delicate issue that we shall not address here. It may be false even for uniformly elliptic operators with smooth, bounded and Lipschitz continuous coefficients.

The characterization (1.4) calls for a closer look at the functions that can be found in $M_0^{m,1}$. For the sake of argument, assume $m = 2$. It can be shown that $u \in M_0^{2,1}$ in a variety of special cases, including:

- (i) $u \in L^{q,\sigma}$ or $\nabla u \in (L^{q,\sigma})^N$ (Lorentz spaces, $q > 1, \sigma \leq \infty$ or $q = \sigma = 1$; see Example 2.1 and Theorem 3.2).
- (ii) $u \in L_\Phi$ or $\nabla u \in (L_\Phi)^N$ where L_Φ is any Orlicz space; see Example 2.2 and Theorem 3.2.
- (iii) $u \in L^{\mathbf{q}}$ or $\nabla u \in (L^{\mathbf{q}})^N$ (mixed norm spaces, $\mathbf{q} = (q_1, \dots, q_N)$ with $1 \leq q_1, \dots, q_N \leq \infty$; see Example 2.3 and Theorem 3.2).
- (iv) $u \in W_{\text{loc}}^{k,\infty}$ and $\lim_{|x| \rightarrow \infty} |x|^{-s} |\nabla^k u(x)| = 0$ with $0 \leq k \leq 2$ and $s \leq 2 - k$ (see Theorem 2.7 (ii) and Theorem 3.2).
- (v) $(1 + |x|)^{-s-N/q} |\nabla^k u| \in L^q$ with $1 \leq q \leq \infty, 0 \leq k \leq 2$ and $s < 2 - k$ (see Theorem 2.7, Remark 2.8 and Theorem 3.2).

Thus, any of the conditions (i) to (v) together with $\Delta u \in L^p$, or more generally $Au \in L^p$ where A is homogeneous second order elliptic with constant coefficients, implies $u \in D^{2,p}$. Note that if the coefficients are not real, A is not reducible to Δ by a linear change of variables. The known cases mentioned earlier when $A = \Delta$

are covered by one or more of these conditions. Evidently, $u \in L^p$ fits within (i), (ii), (iii) and (v), whereas $(1 + |x|^2)^{-1}u \in L^p$ is (v) with $s = 2 - N/p, k = 0$ and $q = p$. On the other hand, $\nabla u \in (L^q)^N$ with $1 < q < \infty$ is also accounted for by (i), (ii), (iii) and (v). Any of these conditions shows that the values $q = 1$ and $q = \infty$ can be included, even though the argument used in [15] breaks down. In fact, by (v) with $k = 1$, it suffices that $(1 + |x|)^{-t}|\nabla u| \in L^q$ with $t < 1 + N/q$ and $1 \leq q \leq \infty$.

By (v) with $k = 2$ and Theorem 1.1, $(1 + |x|)^{-t}|\nabla^2 u| \in L^q$ with $1 \leq q \leq \infty$ and $t < N/q$ and $\Delta u \in L^p$ with $1 < p < \infty$ imply $u \in D^{2,p}$. In particular, if $u \in D^{2,q}$ for some $1 \leq q < \infty$ and $\Delta u \in L^p$, then $u \in D^{2,p}$ (let $t = 0$; if $q = \infty$, quadratic harmonic polynomials are counter examples). Also, if $(1 + |x|)^{-t}|\nabla^2 u| \in L^p$ with $t < N/p$ (weaker than $u \in D^{2,p}$ if $t > 0$) and $\Delta u \in L^p$, then $u \in D^{2,p}$.

We now come to the organization of the paper. Section 2 is devoted to the definition and basic properties of the spaces $M^{s,q}$. With occasional minor modifications, the growth limitations embodied in these spaces have already been used extensively when $q = \infty$ or $q = 2$ and, in some instances, when $q = 1$, in connection with Liouville-type theorems ([3], [4], [25], [27], [35]) but not in regularity issues. There seems to have been no prior incentive to incorporate these growth limitations into a family of function spaces and the other values of q have apparently been ignored.

The most important feature of the spaces $M^{s,q}$ is that “integration”, i.e., passing from ∇u to u , takes $(M^{s,q})^N$ into $M^{s+1,q}$ when $s > -1$. This is shown in Section 3 (Theorem 3.2), where we also obtain the embedding of $D^{k,p}$ into $M^{s,p}$ for suitable s as a straightforward by-product (Theorem 3.3).

This embedding is one of the main ingredients for the proof of Theorem 4.5 (the general form of Theorem 1.1), given in Section 4. This proof also depends on properties of homogeneous elliptic operators acting on homogeneous Sobolev spaces (Theorem 4.4). In addition to new $W^{k,p}$ regularity results which, in particular, do not require strong ellipticity (Corollary 4.6), four examples show how Theorem 4.5 and the various properties of the spaces $M^{s,q}$ can be used in practice.

In Section 5, we generalize Theorem 4.5 to exterior domains (Theorem 5.4). When homogeneous Sobolev spaces of negative order are involved, this is not a routine variant because passing to an exterior domain introduces *necessary* restrictions on N and p , not needed in the whole space.

We also take advantage of the exterior domain setting to show how the Kelvin transform method yields solutions of boundary value problems in $M_0^{0,1}$ (Theorem 5.6). Thus, as noted earlier, these solutions vanish at infinity in a generalized sense. The physical relevance of solutions vanishing at infinity has been discussed at length in the literature, notably in Dautray and Lions [10]. For obvious reasons, $M_0^{0,1}$ (larger than any L^p space with $p < \infty$) has not previously been part of this discussion.

In Section 6, Theorem 4.5 is extended to Douglis–Nirenberg elliptic systems with constant coefficients when the entries are homogeneous operators with possibly different orders (Theorem 6.2), as is the case with the Stokes system. A variant of a trick used long ago by Malgrange [30] for other purposes allows for a convenient reduction to the scalar case.

Notation. The general notation is standard. Everywhere, B_R is the Euclidean open ball with center 0 and radius $R > 0$ in \mathbb{R}^N and, depending on context, $|\cdot|$ is either the Euclidean norm or the Lebesgue measure. On the other hand, $|\cdot|_1$ is the ℓ^1 norm (used only with multi-indices). The notation $\|\cdot\|_{p,E}$, abbreviated $\|\cdot\|_p$ when $E = \mathbb{R}^N$, is used for the norm of $L^p(E)$. Also, p' denotes the Hölder conjugate of $p \in [1, \infty]$.

Differentiation is always understood in the weak (distribution) sense and $\nabla^k u$ is the symmetric tensor of partial derivatives of order $k \in \mathbb{N}$ of the distribution u . Of course, ∇u is used instead of $\nabla^1 u$. As is customary, \mathcal{S} and \mathcal{S}' refer to the Schwartz space and its topological dual (tempered distributions), respectively. Fourier transform on those spaces is denoted by \mathcal{F} , with inverse \mathcal{F}^{-1} . We shall also use the convenient “hat” notation $\hat{u} := \mathcal{F}u$.

If $d \in \mathbb{N}_0$, we let \mathcal{P}_d denote the space of polynomials on \mathbb{R}^N of degree at most d with complex coefficients and $[u]_d$ is the equivalence class of the function u modulo \mathcal{P}_d . It will be convenient to set $\mathcal{P}_d := \{0\}$ if $d < 0$ and $\mathcal{P} := \cup_d \mathcal{P}_d$. Lastly, if X and Y are topological spaces, $X \hookrightarrow Y$ means that X is continuously embedded into Y .

In inequalities, $C > 0$ denotes a constant independent of the functions involved, whose value may change from place to place.

2. The spaces $M^{s,q}$

Unless stated otherwise, $s \in \mathbb{R}$ and $1 \leq q \leq \infty$. We define

$$(2.1) \quad M^{s,q} := \left\{ u \in L^q_{\text{loc}} : \sup_{R \geq 1} R^{-s} |B_R|^{-1/q} \|u\|_{q,B_R} < \infty \right\}$$

and

$$(2.2) \quad M_0^{s,q} := \left\{ u \in L^q_{\text{loc}} : \lim_{R \rightarrow \infty} R^{-s} |B_R|^{-1/q} \|u\|_{q,B_R} = 0 \right\},$$

where $|B_R|^{-1/q} := 1$ if $q = \infty$. Obviously, $R^{-s-N/q}$ may -and often will- be substituted for $R^{-s} |B_R|^{-1/q}$ in (2.1) and (2.2). To reconcile these definitions with the comments in the Introduction, observe that $R^{s+N/q}$ is proportional to $\| |x|^s \|_{q,B_R}$ when $s > -N/q$.

The possible resemblance with Morrey spaces, maximal functions, etc., is formal at best. In (2.1), the center 0 of the balls B_R is fixed and the supremum is not taken over all radii $R > 0$. However, there is no difficulty in showing that as long as $R_0 > 0$ and $x_0 \in \mathbb{R}^N$ are fixed, the definition of $M^{s,q}$ is unchanged if the condition $R \geq 1$ is replaced with $R \geq R_0$ and if all the balls are centered at x_0 .

Notice that $M^{s,q} = \{0\}$ if $s < -N/q$ and $M_0^{s,q} = \{0\}$ if $s \leq -N/q$. Accordingly, all the results quoted without limitation about s are trivial in these cases. It is equally obvious that $M^{-N/q,q} = L^q$ and that $M^{s_1,q} \subset M^{s_2,q}$ and $M_0^{s_1,q} \subset M_0^{s_2,q}$ if $s_1 \leq s_2$, whereas $M^{s_1,q} \subset M_0^{s_2,q}$ if $s_1 < s_2$. Also, by Hölder’s inequality, $M^{s,q_2} \subset M^{s,q_1}$ and $M_0^{s,q_2} \subset M_0^{s,q_1}$ if $q_1 \leq q_2$.

To summarize, given q , the nontrivial spaces $M^{s,q}$ start with L^q when $s = -N/q$ and get larger as s is increased. On the other hand, given s , they get smaller as q is increased. The practical value of this double linear ordering cannot be overemphasized. The spaces $M_0^{s,q}$ have similar properties, except that, given q , there is no smallest nontrivial space $M_0^{s,q}$.

Many classical function spaces are subspaces of some $M^{s,q}$ space. Examples follow.

Example 2.1. The Lorentz space $L^{q,\sigma}$, $1 < q < \infty, 1 \leq \sigma \leq \infty$, is contained in $M^{-N/q,1} \subset M_0^{s,1}$ if $s > -N/q$. Since $L^{q,\sigma} \subset L^{q,\infty}$, it suffices to prove that $L^{q,\infty} \subset M^{-N/q,1}$. The norm of $u \in L^{q,\infty}$ is $\|u\|_{(q,\infty)} := \sup_{t>0} t^{-1+1/q} \int_0^t u^*(\tau) d\tau < \infty$, where u^* is the decreasing rearrangement of u . On the other hand, if E is a measurable subset of finite measure, then $|E|^{-1} \int_E |u| \leq |E|^{-1} \int_0^{|E|} u^*(\tau) d\tau$ (see Theorem 7.3.1, p. 82 in [17] or Lemma 3.17, p. 201 in [47]). Thus, $|E|^{-1} \int_E |u| \leq |E|^{-1/q} \|u\|_{(q,\infty)}$. By using this with $E = B_R$, it follows that $u \in M^{-N/q,1}$. On the other hand, $L^{1,\sigma} \not\subset L_{loc}^1$ if $\sigma > 1$ is not a subspace of any $M^{s,q}$. \square

Example 2.2. Let L_Φ be the Orlicz space corresponding to the Young function Φ ([6], [38]). Given $1 \leq q < \infty$, assume that $t^q \leq \Phi(\lambda t)$ if $t \geq t_0$ for some $\lambda > 0$ and $t_0 \geq 0$ (λ and t_0 always exist if $q = 1$; just choose $t_0 > 0$, pick λ large enough that $\Phi(\lambda t_0) \geq t_0$ and use the convexity of Φ). If v is Lebesgue measurable, then $\int_{B_R} |v|^q \leq t_0^q |B_R| + \int_{\mathbb{R}^N} \Phi(\lambda |v|)$ for every $R > 0$. In particular, if $u \in L_\Phi \setminus \{0\}$, the choice $v := u/\lambda \|u\|_\Phi$ (Luxemburg norm) yields $|B_R|^{-1} \int_{B_R} |u|^q \leq \lambda^q \|u\|_\Phi^q (t_0^q + |B_R|^{-1})$. This shows that (i) $L_\Phi \subset M^{0,q} \subset M_0^{s,q}$ for every $s > 0$ (true for every Φ if $q = 1$), (ii) $L_\Phi \subset M^{-N/q,q} = L^q$ if $t_0 = 0$ and (iii) $L_\Phi \subset M_0^{0,q}$ if λt_0 can be chosen arbitrarily small. In particular: (iv) $L_\Phi \subset M_0^{0,1}$ if and only if $1 \notin L_\Phi$. The necessity is obvious. Conversely, $1 \notin L_\Phi$ if and only if $\Phi > 0$ on $(0, \infty)$. If so, Φ has a continuous inverse Φ^{-1} defined on some interval $[0, b)$ with $0 < b \leq \infty$. For $t_0 < b$, let $\lambda := \Phi^{-1}(t_0)/t_0$. Then, $t \leq \Phi(\lambda t)$ if $t \geq t_0$ by the monotonicity of $\Phi(\lambda t)/t$ and $\Phi(\lambda t_0)/t_0 = 1$. Since $\lambda t_0 = \Phi^{-1}(t_0) \rightarrow 0$ as $t_0 \rightarrow 0$, the result follows from (iii). \square

In Example 2.2, $L_\Phi \subset M^{0,1}$ for every Orlicz space L_Φ can be quickly seen from $L_\Phi \subset L^1 + L^\infty$ and $L^1 = M^{-N,1} \subset M^{0,1}, L^\infty = M^{0,\infty} \subset M^{0,1}$.

Example 2.3. If $\mathbf{q} := (q_1, \dots, q_N)$ with $1 \leq q_1, \dots, q_N \leq \infty$, the space $L^{\mathbf{q}}$ (see [8]) is contained in $M^{s,q}$ with $s := -\sum_{i=1}^N q_i^{-1}$ and $q := \min\{q_1, \dots, q_N\}$. This follows from the remark that the definition of $M^{s,q}$ is unchanged if balls are replaced with cubes. \square

It is readily checked that

$$(2.3) \quad \|u\|_{M^{s,q}} := \sup_{R \geq 1} R^{-s-N/q} \|u\|_{q, B_R},$$

is a well defined norm on $M^{s,q}$. The proofs of the first two theorems are routine and left to the reader (parts (ii) and (iv) of Theorem 2.4 were noticed earlier).

Theorem 2.4. (i) $M^{s,q}$ is a Banach space for the norm (2.3).

(ii) $M^{s_1,q} \hookrightarrow M^{s_2,q}$ if $s_1 \leq s_2$ and $M^{s,q_2} \hookrightarrow M^{s,q_1}$, $M_0^{s,q_2} \hookrightarrow M_0^{s,q_1}$ if $q_1 \leq q_2$.

(iii) $M_0^{s,q}$ is a closed subspace of $M^{s,q}$.

(iv) $M^{s_1,q} \hookrightarrow M_0^{s_2,q}$ if $s_1 < s_2$.

Theorem 2.5. If $s_1, s_2 \in \mathbb{R}$ and if $1 \leq q_1, q_2 \leq \infty$ satisfy $1/q_1 + 1/q_2 \leq 1$, define $s_3 \in \mathbb{R}$ and $q_3 \geq 1$ by $s_3 := s_1 + s_2$ and $1/q_3 := 1/q_1 + 1/q_2$. Then, the multiplication $(u, v) \mapsto uv$ is defined and continuous from $M^{s_1,q_1} \times M^{s_2,q_2}$ to M^{s_3,q_3} and from $M_0^{s_1,q_1} \times M^{s_2,q_2}$ (or $M^{s_1,q_1} \times M_0^{s_2,q_2}$) to $M_0^{s_3,q_3}$. More precisely,

$$\|uv\|_{M^{s_3,q_3}} \leq \|u\|_{M^{s_1,q_1}} \|v\|_{M^{s_2,q_2}}.$$

The above inequality generalizes Hölder’s inequality, which is recovered when $q_1 = q$, $q_2 = q'$, $s_1 = -N/q$ and $s_2 = -N/q'$.

Example 2.6. If $a \geq 0$, the function $(1 + |x|)^a$ is in $M^{a,\infty}$. Thus, if $(1 + |x|)^{-a}u \in L^q = M^{-N/q,q}$, then $u \in M^{a-N/q,q}$ and $\|u\|_{M^{a-N/q,q}} \leq \|((1 + |x|)^a)\|_{M^{a,\infty}} \|(1 + |x|)^{-a}u\|_q = 2^a \|(1 + |x|)^{-a}u\|_q$ by Theorem 2.5. \square

The definition of the spaces $M^{s,q}$ hints that they should be related to weighted Lebesgue spaces with weights behaving like $|x|^{-sq-N}$ for large $|x|$. To make this connection precise, we introduce the spaces

$$L_s^q := \{u : (1 + |x|)^{-s-N/q}u \in L^q\},$$

equipped with the obvious norm. This definition makes sense if $s \in \mathbb{R}$ and $1 \leq q \leq \infty$ and $L_s^q = L^q(\mathbb{R}^N; (1 + |x|)^{-sq-N} dx)$ when $q < \infty$. The only motivation for introducing the spaces L_s^q is the proof of Theorem 2.9 (i) later. They will not be used beyond that point, but they play a key role in other issues ([42]).

Theorem 2.7. (i) $M^{s,\infty} = L_s^\infty$ for every $s \geq 0$, with equivalent norms.

(ii) If $s > 0$, then $u \in M_0^{s,\infty}$ if and only if $u \in L_{\text{loc}}^\infty$ and, for every $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that $|u(x)| < \varepsilon|x|^s$ for a.e. x with $|x| > R_\varepsilon$ (i.e., $|u(x)| = o(|x|^s)$ at infinity after modifying u on a null set).

(iii) If $1 \leq q < \infty$, then $L_s^q \hookrightarrow M^{s,q} \hookrightarrow L_t^q$ for every $t > s \geq -N/q$ and $L_s^q \hookrightarrow M_0^{s,q}$ if $s > -N/q$.

Proof. (i) $L_s^\infty \hookrightarrow M^{s,\infty}$ by Example 2.6 with $a = s \geq 0$ and $q = \infty$. To prove $M^{s,\infty} = L_s^\infty$ with equivalent norms, it suffices to show that $M^{s,\infty} \subset L_s^\infty$, for then the equivalence of norms follows from the inverse mapping theorem since both $M^{s,\infty}$ and L_s^∞ are Banach spaces.

Let then $u \in M^{s,\infty}$ be given and let $n \in \mathbb{N}$. For a.e. $x \in B_{n+1} \setminus B_n$, we have $|u(x)| \leq \|u\|_{\infty, B_{n+1}} \leq C(n+1)^s$ where $C > 0$ is independent of x and n and so $(1+|x|)^{-s}|u(x)| \leq |x|^{-s}|u(x)| \leq Cn^{-s}(n+1)^s \leq 2^s C$. Thus, $(1+|x|)^{-s}|u(x)| \leq 2^s C$ for a.e. $x \in \mathbb{R}^N \setminus B_1$. Since $u \in L_{\text{loc}}^\infty$, it follows that $(1+|x|)^{-s}u \in L^\infty$, i.e., $u \in L_s^\infty$.

(ii) The sufficiency is straightforward: given $\varepsilon > 0$, let $R_\varepsilon > 0$ be such that $|u(x)| < \varepsilon|x|^s$ for a.e. x with $|x| > R_\varepsilon$. If $R > R_\varepsilon$, then $\|u\|_{\infty, B_R} < \|u\|_{\infty, B_{R_\varepsilon}} + \varepsilon R^s$, so that $R^{-s}\|u\|_{\infty, B_R} < 2\varepsilon$ if R is large enough since $s > 0$.

Conversely, if $u \in M_0^{s,\infty}$ and $\varepsilon > 0$, then $\|u\|_{\infty, B_R} \leq 2^{-s}\varepsilon R^s$ for R large enough. In particular, $|u(x)| \leq 2^{-s}\varepsilon(n+1)^s$ for a.e. $x \in B_{n+1} \setminus B_n$ and $n \in \mathbb{N}$ large enough, say $n \geq n_\varepsilon$, and then $|x|^{-s}|u(x)| \leq 2^{-s}\varepsilon n^{-s}(n+1)^s \leq \varepsilon$. Thus, $|u(x)| \leq \varepsilon|x|^s$ for a.e. x with $|x| > R_\varepsilon := n_\varepsilon$.

(iii) $L_s^q \hookrightarrow M^{s,q}$ by Example 2.6 with $a = s + N/q \geq 0$. The proof that $M^{s,q} \hookrightarrow L_t^q$ when $t > s \geq -N/q$ is more delicate. Let $u \in M^{s,q}$ be given. By using $B_n = \cup_{j=1}^n (B_j \setminus B_{j-1})$ (with $B_0 = \emptyset$) and since $(1 + |x|)^{-tq-N} \leq j^{-tq-N}$ for $x \in B_j \setminus B_{j-1}$, we get

$$\begin{aligned} \int_{B_n} (1 + |x|)^{-tq-N} |u|^q &\leq \sum_{j=1}^n j^{-tq-N} \int_{B_j \setminus B_{j-1}} |u|^q \\ &= n^{-tq-N} \int_{B_n} |u|^q + \sum_{j=2}^n ((j-1)^{-tq-N} - j^{-tq-N}) \int_{B_{j-1}} |u|^q. \end{aligned}$$

In the right-hand side, $n^{-tq-N} \int_{B_n} |u|^q \leq n^{-(t-s)q} \|u\|_{M^{s,q}}^q$ tends to 0 as $n \rightarrow \infty$ since $t > s$. Thus, to prove that $u \in L_t^q$, it suffices to show that the sum

$$\sum_{j=2}^n ((j-1)^{-tq-N} - j^{-tq-N}) \int_{B_{j-1}} |u|^q$$

is uniformly bounded.

Note that $(j-1)^{-tq-N} - j^{-tq-N} = (j-1)^{-tq-N} (1 - (1-j^{-1})^{tq+N})$ and that $1 - (1-j^{-1})^{tq+N} = O(j^{-1}) = O((j-1)^{-1})$ for $j \geq 2$. Thus, $(j-1)^{-tq-N} - j^{-tq-N} \leq C(j-1)^{-tq-N-1}$, where $C > 0$ is independent of n (and of u), and so,

$$\begin{aligned} 0 &\leq \sum_{j=2}^n ((j-1)^{-tq-N} - j^{-tq-N}) \int_{B_{j-1}} |u|^q \leq C \sum_{j=2}^n (j-1)^{-tq-N-1} \int_{B_{j-1}} |u|^q \\ &= C \sum_{j=1}^{n-1} j^{-(t-s)q-1} j^{-sq-N} \int_{B_j} |u|^q \leq C \left(\sum_{j=1}^{n-1} j^{-(t-s)q-1} \right) \|u\|_{M^{s,q}}^q \end{aligned}$$

and the right-hand side is bounded since $\sum_{j=1}^\infty j^{-(t-s)q-1} < \infty$. It follows that $u \in L_t^q$ and, in fact, that

$$\|u\|_{L_t^q}^q \leq C \left(\sum_{j=1}^\infty j^{-(t-s)q-1} \right) \|u\|_{M^{s,q}}^q,$$

so that the embedding $M^{s,q} \subset L_t^q$ is continuous.

To complete the proof, we now assume $s > -N/q$ and show that $L_s^q \hookrightarrow M_0^{s,q}$. Since $L_s^q \hookrightarrow M^{s,q}$ was proved above and $M_0^{s,q} \subset M^{s,q}$, it suffices to show that $L_s^q \subset M_0^{s,q}$. Let $u \in L_s^q$ and $\varepsilon > 0$ be given. Write $u = (1 + |x|)^{s+N/q} v$ with $v := (1 + |x|)^{-s-N/q} u \in L^q$ and choose $\varphi \in \mathcal{C}_0^\infty$ such that $\|v - \varphi\|_q < \varepsilon$. This yields

$$R^{-s-N/q} \|u\|_{q, B_R} \leq R^{-s-N/q} (1 + R)^{s+N/q} \varepsilon + R^{-s-N/q} \|(1 + |x|)^{s+N/q} \varphi\|_{q, B_R}.$$

Now,

$$\|(1 + |x|)^{s+N/q}\varphi\|_{q, B_R} = \|(1 + |x|)^{s+N/q}\varphi\|_{q, B_{R_0}}$$

if $R \geq R_0$ and R_0 is chosen so that $\text{Supp } \varphi \subset B_{R_0}$. Since $s > -N/q$, it follows that $\limsup_{R \rightarrow \infty} R^{-s-N/q}\|u\|_{q, B_R} \leq \varepsilon$ and so $u \in M_0^{s,q}$ since $\varepsilon > 0$ is arbitrary. \square

Remark 2.8. If $s < -N/q \leq \tilde{s}$, then $L_s^q \hookrightarrow L_{\tilde{s}}^q$ and Theorem 2.7 is applicable with s replaced with \tilde{s} .

Although Theorem 2.7 suggests that the gap between the spaces $M^{s,q}$ and L_s^q is negligible when $s \geq -N/q$, this gap allows for major differences. Most notably, the spaces L_s^q are *not* ordered by inclusion when s is fixed and q is varied. For example, $(1 + |x|)^{-N/q'} \notin L^{q'}$ if $q > 1$, so that there is $v \in L^q$ such that $(1 + |x|)^{-N/q'}v \notin L^1$ (indeed, if f is a function on \mathbb{R}^N and $fv \in L^1$ for every $v \in L^q$, then $f \in L^{q'}$; use truncation and the uniform boundedness principle.) As a result, $u := (1 + |x|)^{s+N/q} \in L_s^q$ but $u \notin L_s^1$. An elaboration on this example shows that $L_s^{q_2} \not\subset L_s^{q_1}$ if $q_1 < q_2$ (or $q_1 > q_2$, which is trivial). For that reason, it will be technically essential to use the $M^{s,q}$ and $M_0^{s,q}$ scales rather than the L_s^q scale, even though, by Theorem 2.7, they often end up being interchangeable.

Theorem 2.9. (i) $M^{s,q} \hookrightarrow \mathcal{S}'$.

(ii) If u is a polynomial, then $u \in M^{s,q}$ (respectively, $M_0^{s,q}$) if and only if $\deg u \leq s$ (respectively, $\deg u < s$).

Proof. (i) With no loss of generality, assume $s \geq -N/q$. Since $M^{s,q} \hookrightarrow M^{s,1}$ by Theorem 2.4 (ii), it is not restrictive to assume $q = 1$ and then, by Theorem 2.7 (iii), it suffices to check that $L_t^1 \hookrightarrow \mathcal{S}'$ for every $t \in \mathbb{R}$. Since $\sup_{x \in \mathbb{R}^N} (1 + |x|)^{t+N} |\varphi(x)|$ is a continuous seminorm on \mathcal{S} , this follows from

$$\begin{aligned} \left| \int_{\mathbb{R}^N} u \varphi \right| &\leq \|(1 + |x|)^{-t-N} u\|_1 \sup_{x \in \mathbb{R}^N} (1 + |x|)^{t+N} |\varphi(x)| \\ &= \|u\|_{L_t^1} \sup_{x \in \mathbb{R}^N} (1 + |x|)^{t+N} |\varphi(x)|, \end{aligned}$$

for every $u \in L_t^1$ and every $\varphi \in \mathcal{S}$.

(ii) Let $d := \deg u$. It is plain that $u \in M^{d,\infty} \subset M^{d,q} \subset M^{s,q}$ for every $1 \leq q \leq \infty$ and every $s \geq d$. In particular, $u \in M_0^{s,q}$ if $s > d$.

Conversely, assume by contradiction that $u \in M^{s,q}$ for some $1 \leq q \leq \infty$ and some $s < d$. Choose a system of coordinates such that $x = (x_1, x')$ with $x_1 \in \mathbb{R}^N, x' \in \mathbb{R}^{N-1}$ and that $u(x) = a_d x_1^d + \sum_{j=0}^{d-1} a_j(x') x_1^j$, where $a_j \in \mathcal{P}_{d-j}(\mathbb{R}^{N-1})$ for $0 \leq j \leq d$ and $a_d \in \mathbb{C} \setminus \{0\}$.

Given $\varepsilon > 0$, denote by $\Sigma_\varepsilon \subset \mathbb{R}^N$ the sector $|x'| < \varepsilon x_1$ around the positive x_1 -axis. For $0 \leq j \leq d-1$, there are constants $C_j \geq 0$ independent of ε and of $x \in \Sigma_\varepsilon$ such that $|a_j(x')| \leq C_j(1 + \varepsilon^{d-j} x_1^{d-j})$ for every $x \in \Sigma_\varepsilon$. Thus, if ε is small enough and $R_0 > 0$ is large enough, $|u(x)| \geq (|a_d|/2)x_1^d$ for $x \in \Sigma_\varepsilon$ with $|x| \geq R_0$. Since

$|x| \leq x_1(1 + \varepsilon^2)^{1/2}$ when $x \in \Sigma_\varepsilon$, it follows that $|u(x)| \geq (|a_d|/2)(1 + \varepsilon^2)^{-d/2}|x|^d$ for every $x \in \Sigma_\varepsilon$ with $|x| \geq R_0$. As a result, if $R > R_0$,

$$\begin{aligned} \int_{\Sigma_\varepsilon \cap B_R} |u| &\geq \frac{|a_d|}{2} (1 + \varepsilon^2)^{-d/2} \int_{\Sigma_\varepsilon \cap (B_R \setminus B_{R_0})} |x|^d \\ &= \frac{\omega_\varepsilon |a_d|}{2(d + N)} (1 + \varepsilon^2)^{-d/2} (R^{d+N} - R_0^{d+N}), \end{aligned}$$

where $\omega_\varepsilon := |\Sigma_\varepsilon \cap B_1|/|B_1| > 0$ is also the ratio of the $N - 1$ dimensional measures of $\Sigma_\varepsilon \cap \partial B_1$ and ∂B_1 . Thus,

$$R^{-s-N} \|u\|_{1, B_R} \geq R^{-s-N} \left(\int_{\Sigma_\varepsilon \cap B_R} |u| \right) \geq c R^{d-s} (1 - R^{-d-N} R_0^{d+N}),$$

where $c = \omega_\varepsilon (|a_d|/2)(d + N)^{-1}(1 + \varepsilon^2)^{-d/2} > 0$ is independent of R . Since $d > s$, this contradicts the assumption $u \in M^{s,q} \subset M^{s,1}$.

If it is assumed that $u \in M_0^{s,q} \subset M_0^{s,1}$, a contradiction still arises when $s = d$. This completes the proof of (ii). \square

As a corollary, we obtain an elementary Liouville-type property that will be instrumental in the proof of Theorem 4.5 (and convenient, but not essential, in that of Lemma 4.1). Although it will only be used here with the elliptic operator A in (1.1), we give a more general statement since the proof is the same. Recall that \mathcal{P} is the space of polynomials on \mathbb{R}^N with complex coefficients.

Corollary 2.10. *Let $B := \sum_{|\alpha|_1 \leq m} i^{|\alpha|_1} b_\alpha \partial^\alpha$ be a differential operator with constant coefficients such that $B(\xi) := \sum_{|\alpha|_1 \leq m} b_\alpha \xi^\alpha \neq 0$ on $\mathbb{R}^N \setminus \{0\}$. If $u \in M^{s,q}$ (respectively, $M_0^{s,q}$) for some $1 \leq q \leq \infty$ and $Bu \in \mathcal{P}$, then $u \in \mathcal{P}$ and $\deg u \leq s$ (respectively, $\deg u < s$).*

Proof. By Theorem 2.9 (i), $u \in \mathcal{S}'$, so that $Bu = \pi$ with $\pi \in \mathcal{P} \subset \mathcal{S}'$ implies $B(\xi)\widehat{u} = \widehat{\pi}$. Now, $\text{Supp } \widehat{\pi} \subset \{0\}$ since $\widehat{\pi}$ is a linear combination of partial derivatives of δ (Dirac delta). Since $B(\xi) \neq 0$ when $\xi \neq 0$, it follows that $\text{Supp } \widehat{u} \subset \{0\}$. Hence, \widehat{u} is a linear combination of partial derivatives of δ , which amounts to saying that u is a polynomial. The bound on $\deg u$ follows from Theorem 2.9 (ii). \square

Corollary 2.10 is related in various ways to a number of results in the literature. Among many others, we mention Agmon, Douglis and Nirenberg [2], p. 662 (when B is homogeneous elliptic and $q = \infty$), Weck [49] (when $Bu = 0$ and $q = \infty$), Hörmander [21], Murata [36] (when $Bu = 0, s = 0$ and $q = 2$). The last two papers deal with the solutions of $Bu = 0$ when B is a general operator with constant coefficients and $u \in L^2_{\text{loc}}$. They are much deeper but only cover the special case of Corollary 2.10 when $Bu = 0, s \leq 0$ and $q \geq 2$. For polyharmonic functions, Liouville theorems stronger than Corollary 2.10 with $q = 1$, but in the same spirit, can be found in Armitage [3] and the references therein.

3. Embedding of $D^{k,p}$ into $M^{s,p}$

Arguably, the most important feature of the spaces $M^{s,q}$ is that a function with first order derivatives in $M^{s,q}$ is in $M^{s+1,q}$ if $s > -1$. This will be proved in this section (Theorem 3.2). The examples given in the Introduction made repeated use of this property. In addition, the embedding $D^{k,p} \subset M^{s,p}$ for suitable values of s is a straightforward by-product (Theorem 3.3). For brevity, we do not discuss the embedding $D^{k,p} \subset M^{s,q}$ when $q \neq p$, which will not be needed.

We begin with a lemma on real-valued functions of one variable.

Lemma 3.1. *Let $H: (0, \infty) \rightarrow \mathbb{R}$ be a function bounded above on every compact subset of $(0, \infty)$. Suppose that there are $\lambda, \mu \in (0, 1)$ and $c \in \mathbb{R}, R_0 > 0$ such that*

$$(3.1) \quad H(R) \leq \mu H(\lambda R) + c, \quad \forall R \geq R_0.$$

Then, H is bounded above on $[1, \infty)$ and $\limsup_{R \rightarrow \infty} H(R) \leq c(1 - \mu)^{-1}$.

Proof. Since H is bounded above on the compact subsets of $(0, \infty)$, it suffices to show that if $\varepsilon > 0$, then $H(R) \leq \varepsilon + c(1 - \mu)^{-1}$ for $R > 0$ large enough. By contradiction, if this is false, there is a sequence $R_n \rightarrow \infty$ such that $\varepsilon + c(1 - \mu)^{-1} < H(R_n)$ for every $n \in \mathbb{N}$. With no loss of generality, assume $R_n \geq R_0$ and let $j_n \in \mathbb{N}$ denote the largest integer j such that $\lambda^j R_n \geq R_0$, so that $R_0 \leq \lambda^{j_n} R_n < \lambda^{-1} R_0$. Evidently, $j_n \rightarrow \infty$. On the other hand, since $R_n \geq R_0$, it follows from (3.1) that $H(R_n) \leq \mu H(\lambda R_n) + c$ and so $\varepsilon + c(1 - \mu)^{-1} < \mu H(\lambda R_n) + c$. As a result, $\mu^{-1}\varepsilon + c(1 - \mu)^{-1} < H(\lambda R_n)$. If $\lambda R_n \geq R_0$, then (3.1) can be used again with R replaced with λR_n , which yields $\mu^{-2}\varepsilon + c(1 - \mu)^{-1} < H(\lambda^2 R_n)$ and so, by induction, $\mu^{-j_n}\varepsilon + c(1 - \mu)^{-1} < H(\lambda^{j_n} R_n)$ for every n . Since $\lim \mu^{-j_n} = \infty$ and $\lambda^{j_n} R_n \in [R_0, \lambda^{-1} R_0) \subset [R_0, \lambda^{-1} R_0]$, this contradicts the assumption that H is bounded above on the compact subsets of $(0, \infty)$. \square

Theorem 3.2. *Let $k \in \mathbb{N}_0$ and let $s \in (-1, \infty)$ and $1 \leq q \leq \infty$ be given. If $u \in \mathcal{D}'$ and $\partial^\beta u \in M^{s,q}$ (respectively, $M_0^{s,q}$) when $|\beta|_1 = k$, then $u \in M^{s+k,q}$ (respectively, $M_0^{s+k,q}$). Furthermore, if $k \geq 1, s = 0, q = \infty$ and $\lim_{|x| \rightarrow \infty} |\nabla^k u(x)| = 0$, then $u \in M_0^{k,\infty}$ (since $M_0^{0,\infty} = \{0\}$, this is false if $k = 0$).*

Proof. If $k = 0$, there is nothing to prove (and $s > -1$ is not needed). By induction, it suffices to consider the case when $k = 1$. The same thing is true for the “furthermore” part, for if the result is true when $k = 1$, it implies that $\partial^\beta u \in M_0^{1,\infty}$ when $|\beta|_1 = k - 1$ and $k > 1$ and it suffices to use the first part.

From now on, $k = 1$. We first settle the case $q = \infty$. Assume that $\nabla u \in (M^{s,\infty})^N$. There is a constant $C_u > 0$ (for instance, $C_u = \|\|\nabla u\|\|_{M^{s,\infty}}$) such that $\|\|\nabla u\|\|_{\infty, B_R} \leq C_u R^s$ for every $R \geq 1$. Thus, u is Lipschitz continuous with constant $C_u R^s$ on B_R , so that $|u(x)| \leq |u(0)| + C_u R^s |x| \leq |u(0)| + C_u R^{s+1}$ for $x \in B_R$. As a result, $R^{-s-1} \|u\|_{\infty, B_R} \leq R^{-s-1} |u(0)| + C_u \leq |u(0)| + C_u$ since $s > -1$. This shows that $u \in M^{s+1,\infty}$.

If now $\nabla u \in (M_0^{s,\infty})^N$, just note that the constant C_u above may be chosen arbitrarily small provided that R is large enough. The result then follows from $\limsup_{R \rightarrow \infty} R^{-s-1} \|u\|_{\infty, B_R} \leq C_u$.

If $\nabla u \in (M^{0,\infty})^N = (L^\infty)^N$ and it is only assumed that $\lim_{|x| \rightarrow \infty} |\nabla u(x)| = 0$ (rather than the trivial $\nabla u \in (M_0^{0,\infty})^N = \{0\}$), the proof must be modified to show that $u \in M_0^{1,\infty}$. Note that u is continuous on \mathbb{R}^N .

Given $\varepsilon > 0$, there is $R_0 > 0$ such that $\|\nabla u\|_{\infty, \mathbb{R}^N \setminus B_{R_0}} \leq \varepsilon$. In particular, u is Lipschitz continuous with constant ε in every ball not intersecting B_{R_0} . If $x \notin B_{R_0}$, set $x_0 := R_0 x / |x| \in \partial B_{R_0}$, so that x_0 is in the closed ball with center x and radius $|x| - R_0$ (not intersecting B_{R_0}). As a result, $|u(x) - u(x_0)| \leq \varepsilon|x - x_0|$ and $|x - x_0| \leq |x|$ since x_0 lies on the line segment between 0 and x . Hence, $|u(x)| \leq |u(x_0)| + \varepsilon|x| \leq \|u\|_{\infty, B_{R_0}} + \varepsilon|x|$ and so $|u(x)| \leq \|u\|_{\infty, B_{R_0}} + \varepsilon|x|$ for every $x \in \mathbb{R}^N$ since the inequality is trivial when $x \in B_{R_0}$. This yields $\|u\|_{\infty, B_R} \leq \|u\|_{\infty, B_{R_0}} + \varepsilon R$ for every $R > 0$, whence $\limsup_{R \rightarrow \infty} R^{-1} \|u\|_{\infty, B_R} \leq \varepsilon$. Thus, $\lim_{R \rightarrow \infty} R^{-1} \|u\|_{\infty, B_R} = 0$, i.e., $u \in M_0^{1,\infty}$.

In the remainder of the proof, $q < \infty$. Let $R > 0$ be given. As a first step, we prove the inequality

$$(3.2) \quad \|u\|_{q, B_R} \leq 2\lambda^{-N/q} \|u\|_{q, B_{\lambda R}} + 2\lambda^{(1-N)/q} R \|\nabla u\|_{q, B_R},$$

for every $0 < \lambda < 1$ and every $u \in W^{1,q}(B_R)$. Since $q < \infty$, it suffices to prove (3.2) when $u \in C^\infty(\overline{B_R})$. In what follows, $\partial_\rho u$ denotes the radial derivative of u .

For $0 \leq t < R$ and $\theta \in \mathbb{S}^{N-1}$, write

$$u(t, \theta) = u(\lambda t, \theta) + \int_{\lambda t}^t \partial_\rho u(\tau, \theta) d\tau.$$

By Hölder's inequality and since $0 < 1 - \lambda < 1$,

$$\begin{aligned} |u(t, \theta)|^q &\leq 2^{q-1} \left(|u(\lambda t, \theta)|^q + \left(\int_{\lambda t}^t |\partial_\rho u(\tau, \theta)| d\tau \right)^q \right) \\ &\leq 2^{q-1} \left(|u(\lambda t, \theta)|^q + t^{q-1} \int_{\lambda t}^t |\partial_\rho u(\tau, \theta)|^q d\tau \right). \end{aligned}$$

Multiply by t^{N-1} and use $t \leq \tau/\lambda$ for $\tau \geq \lambda t$ to get

$$\begin{aligned} t^{N-1} |u(t, \theta)|^q &\leq 2^{q-1} \lambda^{1-N} (\lambda t)^{N-1} |u(\lambda t, \theta)|^q \\ &\quad + 2^{q-1} \lambda^{1-N} t^{q-1} \int_{\lambda t}^t \tau^{N-1} |\partial_\rho u(\tau, \theta)|^q d\tau. \end{aligned}$$

Since

$$\int_{\lambda t}^t \tau^{N-1} |\partial_\rho u(\tau, \theta)|^q d\tau \leq \int_0^R \tau^{N-1} |\partial_\rho u(\tau, \theta)|^q d\tau,$$

integration over \mathbb{S}^{N-1} yields

$$\begin{aligned} &\int_{\mathbb{S}^{N-1}} t^{N-1} |u(t, \theta)|^q d\theta \\ &\leq 2^{q-1} \lambda^{1-N} \int_{\mathbb{S}^{N-1}} (\lambda t)^{N-1} |u(\lambda t, \theta)|^q d\theta + 2^{q-1} \lambda^{1-N} t^{q-1} \|\nabla u\|_{q, B_R}^q, \end{aligned}$$

so that (3.2) follows by t -integration over $(0, R)$, by using $(a + b)^{1/q} \leq a^{1/q} + b^{1/q}$ for $a, b \geq 0$, $2^{(q-1)/q} < 2$ and $q^{-1/q} \leq 1$.

Assume now that $\nabla u \in (M^{s,q})^N$, so that $u \in W_{\text{loc}}^{1,q}$. In particular, (3.2) holds for every $R > 0$, and so,

$$R^{-s-1-N/q} \|u\|_{q, B_R} \leq 2\lambda^{s+1} (\lambda R)^{-s-1-N/q} \|u\|_{q, B_{\lambda R}} + 2\lambda^{(1-N)/q} R^{-s-N/q} \|\nabla u\|_{q, B_R},$$

that is,

$$(3.3) \quad H(R) \leq 2\lambda^{s+1} H(\lambda R) + 2\lambda^{(1-N)/q} R^{-s-N/q} \|\nabla u\|_{q, B_R},$$

where $H(R) := R^{-s-1-N/q} \|u\|_{q, B_R}$. Note that H is continuous on $(0, \infty)$ since $q < \infty$. Choose $\lambda \in (0, 1)$ small enough that $2\lambda^{s+1} < 1$, which is possible since $s > -1$. The assumption $\nabla u \in (M^{s,q})^N$ ensures that $2\lambda^{(1-N)/q} R^{-s-N/q} \|\nabla u\|_{q, B_R}$ is bounded irrespective of $R \geq 1$. By Lemma 3.1 with $\mu = 2\lambda^{s+1}$, it follows that H is bounded on $[1, \infty)$ and so $u \in M^{s+1,q}$.

If $\nabla u \in (M_0^{s,q})^N$ and $\varepsilon > 0$, there is $R_\varepsilon > 0$ such that $R^{-s-N/q} \|\nabla u\|_{q, B_R} < \varepsilon$ for every $R \geq R_\varepsilon$. Thus, by (3.3), $H(R) \leq 2\lambda^{s+1} H(\lambda R) + 2\lambda^{(1-N)/q} \varepsilon$ if $R \geq R_\varepsilon$. By Lemma 3.1 (still with $\mu = 2\lambda^{s+1} < 1$), $\limsup_{R \rightarrow \infty} H(R) \leq 2\lambda^{(1-N)/q} \varepsilon (1 - 2\lambda^{s+1})^{-1}$, so that $\lim_{R \rightarrow \infty} H(R) = 0$ since $\varepsilon > 0$ is arbitrary. This shows that $u \in M_0^{s+1,q}$. \square

If $k \geq 1$ and $s < -1$, Theorem 3.2 is always false: if u is a polynomial of degree exactly $k - 1 \geq 0$, then $\nabla^k u = 0 \in M_0^{s,q}$ but $u \notin M^{s+k,q}$ since $s + k < k - 1$ (Theorem 2.9(ii)). If $s = -1$, it takes a bit more work to show that it is true from $M^{-1,q}$ to $M^{k-1,q}$ (but never from $M_0^{-1,q}$ to $M_0^{k-1,q}$) in only two cases with little to no interest: if $q > N$ (trivial since $M^{-1,q} = \{0\}$ and $\nabla^k u = 0$ if and only if $u \in \mathcal{P}_{k-1} \subset M^{k-1,q}$), or if $q = N = 1$ (if $u^{(k)} \in M^{-1,1} = L^1$, then $u^{(k-1)} \in L^\infty = M^{0,\infty}$, so that $u \in M^{k-1,\infty} \subset M^{k-1,1}$ by Theorem 3.2 with $s = k - 1 \geq 0$).

Theorem 3.3. (i) If $k \in \mathbb{N}_0$ and $N < p \leq \infty$, then $D^{k,p} \subset M^{s,p}$ for every $s \geq k - N/p$ and $D^{k,p} \subset M_0^{s,p}$ for every $s > k - N/p$.

(ii) If $k \in \mathbb{N}_0$ and $1 \leq p \leq N$, then $D^{k,p} \subset M_0^{s,p}$ for every $s > k - 1$.

Proof. If $u \in D^{k,p}$, then $\partial^\beta u \in L^p = M^{-N/p,p}$ when $|\beta|_1 = k$. In case (i), Theorem 3.2 with $q = p$ and $s - k$ instead of s directly yields $u \in M^{s,p}$ for $s \geq k - N/p$ because $s - k \geq -N/p > -1$ and $M^{-N/p,p} \subset M^{s-k,p}$. That $D^{k,p} \subset M_0^{s,p}$ for every $s > k - N/p$ follows from Theorem 2.4(iv).

In case (ii), $-N/p \leq -1$, so that $M^{-N/p,p} \subset M^{-1,p} \subset M_0^{-1+\varepsilon,p}$ for every $\varepsilon > 0$ (Theorem 2.4(iv)) and then $u \in M_0^{k-1+\varepsilon,p}$ by Theorem 3.2 with $s = -1 + \varepsilon$. \square

Since $\|\nabla^k u\|_p$ is only a seminorm on $D^{k,p}$ when $k \geq 1$, the embeddings of Theorem 3.3 are not continuous if $D^{k,p}$ is equipped with this seminorm, but it is easy to get around this problem. For $k \in \mathbb{N}_0$, we define

$$(3.4) \quad \dot{D}^{k,p} := D^{k,p} / \mathcal{P}_{k-1},$$

a (reflexive if $1 < p < \infty$) Banach space for the norm $\|[u]_{k-1}\|_{\dot{D}^{k,p}} := \|\nabla^k u\|_p$ with $u \in D^{k,p}$, where $[u]_{k-1}$ denotes the equivalence class modulo \mathcal{P}_{k-1} (see [15], and [31], p. 22). Recall that $\mathcal{P}_{-1} = \{0\}$, so that $\dot{D}^{0,p} = D^{0,p} = L^p$.

Remark 3.4. By a theorem of Sobolev [45], \mathcal{C}_0^∞ is dense in $\dot{D}^{k,p}$ if $1 < p < \infty$. See Hajlasz and Kalamajska [18] for a simple proof and for the case $p = 1, N > 1$.

The next theorem asserts that the set-theoretic embedding $D^{k,p} \subset M^{s,q}$ (or $\dot{D}^{k,p} \subset M_0^{s,q}$) is equivalent to the topological embedding $\dot{D}^{k,p} \hookrightarrow M^{s,q}/\mathcal{P}_{k-1}$ (or $\dot{D}^{k,p} \hookrightarrow M_0^{s,q}/\mathcal{P}_{k-1}$). In particular, the embedding of quotient spaces corresponding to the embeddings of Theorem 3.3 are continuous.

Theorem 3.5. *If $k \in \mathbb{N}, s \geq k - 1$ (respectively, $s > k - 1$) and $1 \leq p, q \leq \infty$, then $\dot{D}^{k,p} \hookrightarrow M^{s,q}/\mathcal{P}_{k-1}$ (respectively, $\dot{D}^{k,p} \hookrightarrow M_0^{s,q}/\mathcal{P}_{k-1}$) if and only if $D^{k,p} \subset M^{s,q}$ (respectively, $D^{k,p} \subset M_0^{s,q}$).*

Proof. The necessity is obvious. For the sufficiency, we only prove that if $s \geq k - 1$ and $D^{k,p} \subset M^{s,q}$, then $\dot{D}^{k,p} \hookrightarrow M^{s,q}/\mathcal{P}_{k-1}$. The exact same argument works in the other case.

Both $\dot{D}^{k,p}$ and $M^{s,q}/\mathcal{P}_{k-1}$ are Banach spaces, the latter since $M^{s,q}$ is complete (Theorem 2.4(i)) and \mathcal{P}_{k-1} is finite dimensional. Therefore, by the closed graph theorem, it suffices to show that if $[u_n]_{k-1} \in \dot{D}^{k,p}$ is a sequence such that $[u_n]_{k-1} \rightarrow [u]_{k-1}$ in $\dot{D}^{k,q}$ and $[u_n]_{k-1} \rightarrow [v]_{k-1}$ in $M^{s,q}/\mathcal{P}_{k-1}$, then $[u]_{k-1} = [v]_{k-1}$.

That $[u_n]_{k-1} \rightarrow [u]_{k-1}$ in $\dot{D}^{k,p}$ means that $u_n, u \in D^{k,p}$ and $\partial^\alpha u_n \rightarrow \partial^\alpha u$ in L^p when $|\alpha|_1 = k$. On the other hand, since $M^{s,q}$ is a normed space, $[u_n]_{k-1} \rightarrow [v]_{k-1}$ in $M^{s,q}/\mathcal{P}_{k-1}$ means that $v \in M^{s,q}$ and that there is a sequence $\pi_n \in \mathcal{P}_{k-1}$ such that $u_n - \pi_n \rightarrow v$ in $M^{s,q}$.

By Theorem 2.9(i), $u_n - \pi_n \rightarrow v$ in \mathcal{S}' . Since $\deg \pi_n \leq k - 1$ and differentiation is continuous on \mathcal{S}' , $\partial^\alpha u_n \rightarrow \partial^\alpha v$ in \mathcal{S}' when $|\alpha|_1 = k$. As a result, $\partial^\alpha u = \partial^\alpha v$ since $\partial^\alpha u_n \rightarrow \partial^\alpha u$ in $L^p \hookrightarrow \mathcal{S}'$. Thus, $u - v \in \mathcal{P}_{k-1}$, i.e., $[u]_{k-1} = [v]_{k-1}$. \square

4. Regularity

In this section, we prove a more general form of Theorem 1.1. The proof will follow from Theorem 3.3 combined with another preliminary result with somewhat of a folklore status (Theorem 4.4 below). For instance, related inequalities are quickly mentioned in Bergh and Löfström [7], p. 167, with a reference to Hörmander [20], where apparently nothing relevant is to be found. When $A = \Delta$, special cases have been proved “as needed” ([13] when $\kappa = -1$, [16] when $\kappa = 0$, [44] when $\kappa \geq 0$). For completeness, we give a full self-contained proof. The following lemma is the first step.

Lemma 4.1. *If $\kappa \in \mathbb{N}_0$ and $1 < p < \infty$, the homogeneous elliptic operator A in (1.1) is a linear isomorphism from $\dot{D}^{m+\kappa,p}$ onto $\dot{D}^{\kappa,p}$.*

Proof. If $v, w \in \dot{D}^{m+\kappa,p}$ and $[v]_{m+\kappa-1} = [w]_{m+\kappa-1}$, then $w = v + \pi$ where $\pi \in \mathcal{P}_{m+\kappa-1}$, whence $Aw = Av + A\pi$ with $A\pi \in \mathcal{P}_{\kappa-1}$ (since A is homogeneous of order m). Thus, $[Av]_{\kappa-1}$ is independent of the representative of $[v]_{m+\kappa-1}$, so that $A : \dot{D}^{m+\kappa,p} \rightarrow \dot{D}^{\kappa,p}$ is well defined by $A[v]_{m+\kappa-1} := [Av]_{\kappa-1}$. From the definitions of the norms of $\dot{D}^{m+\kappa,p}$ and $\dot{D}^{\kappa,p}$, it follows at once that $A : \dot{D}^{m+\kappa,p} \rightarrow \dot{D}^{\kappa,p}$ is continuous.

If $[v]_{m+\kappa-1} \in \dot{D}^{m+\kappa,p}$ and $A[v]_{m+\kappa-1} = [0]_{\kappa-1}$, then $[Av]_{\kappa-1} = [0]_{\kappa-1}$ for every representative v of $[v]_{m+\kappa-1}$, i.e., $Av \in \mathcal{P}_{\kappa-1}$. By Theorem 3.3, $D^{m+\kappa,p} \subset M_0^{m+\kappa,p}$ and so, by Corollary 2.10, $v \in \mathcal{P}_{m+\kappa-1}$. Thus, $[v]_{m+\kappa-1} = [0]_{m+\kappa-1}$ and so A is one-to-one on $\dot{D}^{m+\kappa,p}$.

It remains to prove the surjectivity of A . A distribution $E \in \mathcal{S}'$ is a fundamental solution of A if and only if $E = \mathcal{F}^{-1}\widehat{E}$, where $\widehat{E} \in \mathcal{S}'$ is any distribution such that $A(\cdot)\widehat{E} = 1$. Although the existence of a tempered fundamental solution has long been known for all nonzero differential operators with constant coefficients (Hörmander [19], Lojasiewicz [29]), we need a more precise result in the simpler case of this lemma. The construction below, needed to clarify an important point, is implicit in Hörmander [22]; see also Camus [9].

If $m < N$, it follows from the ellipticity and homogeneity of A that the function $\widehat{E} := A(\cdot)^{-1} \in L^1_{\text{loc}} \cap \mathcal{S}'$ solves the division problem. If $m \geq N$, we may proceed as follows. Given $\phi(= \phi(\xi)) \in \mathcal{C}_0^\infty$ and $\rho \geq 0$, set

$$\psi_\phi(\rho) := \int_{\mathbb{S}^{N-1}} A(\sigma)^{-1} \phi(\rho\sigma) d\sigma.$$

This makes sense since $A(\sigma)$ is bounded away from 0 on \mathbb{S}^{N-1} . Clearly, $\psi_\phi \in \mathcal{C}_0^\infty([0, \infty))$ with

$$(4.1) \quad \psi_\phi^{(j)}(\rho) = \int_{\mathbb{S}^{N-1}} A(\sigma)^{-1} D^j \phi(\rho\sigma) \sigma^j d\sigma, \quad \forall j \in \mathbb{N}_0.$$

Formally, $\widehat{E} \text{ “} = \text{” } A(\cdot)^{-1}$ should be given by $\langle \widehat{E}, \phi \rangle = \int_0^\infty \rho^{N-1-m} \psi_\phi(\rho) d\rho$ but, since $\rho^{N-1-m} \notin L^1_{\text{loc}}([0, \infty))$ when $m \geq N$, the integral is not defined. We replace it with its finite part (see Schwartz [43], p. 42, and Hörmander [22], p. 69 ff), thereby defining \widehat{E} by

$$(4.2) \quad \begin{aligned} \langle \widehat{E}, \phi \rangle &= \lim_{\varepsilon \rightarrow 0^+} \left[\int_\varepsilon^\infty \rho^{N-1-m} \psi_\phi(\rho) d\rho + \sum_{j=0}^{m-N-1} \frac{\psi_\phi^{(j)}(0)}{j!} \frac{\varepsilon^{N-m+j}}{N-m+j} + \frac{\psi_\phi^{(m-N)}(0)}{(m-N)!} \log \varepsilon \right] \\ &= \frac{1}{(m-N)!} \left[- \int_0^\infty (\log \rho) \psi_\phi^{(m-N+1)}(\rho) d\rho + \left(\sum_{j=1}^{m-N} j^{-1} \right) \psi_\phi^{(m-N)}(0) \right]. \end{aligned}$$

It is readily checked that \widehat{E} is a distribution on \mathbb{R}^N and that $\widehat{E} \in \mathcal{S}'$. Furthermore, $\psi_{A(\cdot)\phi}(\rho) = \rho^m \int_{\mathbb{S}^{N-1}} \phi(\rho\sigma) d\sigma$ (in particular, $\psi_{A(\cdot)\phi}^{(j)}(0) = 0$ if $j \leq m-1$), which shows that $\langle \widehat{E}, A(\cdot)\phi \rangle = \int_{\mathbb{R}^N} \phi$, i.e., that $A(\cdot)\widehat{E} = 1$, as desired.

Thus, $E := \mathcal{F}^{-1}\widehat{E} \in \mathcal{S}'$ is a fundamental solution. For our purposes, the key property of \widehat{E} is that $\xi^\alpha \widehat{E}$ is the bounded function $\xi^\alpha A(\xi)^{-1}$ when $|\alpha|_1 = m$.

This is obvious if $m < N$, for then \widehat{E} is already a function. If $m \geq N$, it follows from (4.1) that $\psi_{\xi^\alpha \phi}^{(j)}(0) = 0$ for every $\phi \in C_0^\infty$ when $j \leq m - 1$, so that, by (4.2),

$$\begin{aligned} \langle \widehat{E}, \xi^\alpha \phi \rangle &= \lim_{\varepsilon \rightarrow 0^+} \int_\varepsilon^\infty \rho^{N-1-m} \psi_{\xi^\alpha \phi}(\rho) d\rho \\ &= \lim_{\varepsilon \rightarrow 0^+} \int_{|\xi| \geq \varepsilon} \xi^\alpha A(\xi)^{-1} \phi(\xi) d\xi = \int_{\mathbb{R}^N} \xi^\alpha A(\xi)^{-1} \phi(\xi) d\xi, \end{aligned}$$

which proves the claim.

In the remainder of the proof, α, β and γ denote multi-indices and $|\alpha|_1 = m$. If $\varphi (= \varphi(x)) \in C_0^\infty$, then $E * \varphi \in \mathcal{S}'$ (Theorem XI, p. 247 in [43]) solves $A(E * \varphi) = \varphi$. Also, $\partial^{\alpha+\beta}(E * \varphi) = \partial^\alpha E * \partial^\beta \varphi \in \mathcal{S}'$ and so $\mathcal{F}(\partial^{\alpha+\beta}(E * \varphi)) = \widehat{\partial^\alpha E} \widehat{\partial^\beta \varphi}$ (e.g., because $\partial^\beta \varphi \in C_0^\infty$ and $\partial^\alpha E \in \mathcal{S}'$; see [43], p. 268). Since $\widehat{\partial^\alpha E} = (-i)^m \xi^\alpha \widehat{E}$ and since $\xi^\alpha \widehat{E}$ is the function $\xi^\alpha A(\xi)^{-1}$, it follows that

$$\partial^{\alpha+\beta}(E * \varphi) = (-i)^m \mathcal{F}^{-1}(\xi^\alpha A(\xi)^{-1} \widehat{\partial^\beta \varphi}).$$

By the Mikhlin multiplier theorem ([46], p. 96), $\mathcal{F}^{-1}(\xi^\alpha A(\xi)^{-1} \mathcal{F})$ is a bounded operator on L^p . As a result, $\partial^{\alpha+\beta}(E * \varphi) \in L^p$ and there is a constant $C_\alpha > 0$ independent of $\varphi \in C_0^\infty$ such that $\|\partial^{\alpha+\beta}(E * \varphi)\|_p \leq C_\alpha \|\partial^\beta \varphi\|_p$. Since every γ with $|\gamma|_1 = m + \kappa$ can be split in the form $\gamma = \alpha + \beta$ with $|\alpha|_1 = m$ and $|\beta|_1 = \kappa$, this shows that $E * \varphi \in D^{m+\kappa, p}$ and that $\|[E * \varphi]_{m+\kappa-1}\|_{\dot{D}^{m+\kappa, p}} \leq C \|\varphi\|_{\kappa-1, \dot{D}^{\kappa, p}}$ where $C > 0$ is independent of φ .

As noted in Remark 3.4, C_0^∞ is dense in $\dot{D}^{\kappa, p}$. Given $f \in D^{\kappa, p}$, let then $\varphi_n \in C_0^\infty$ tend to $[f]_{\kappa-1}$ in $\dot{D}^{\kappa, p}$, i.e., $\partial^\beta \varphi_n \rightarrow \partial^\beta f$ in L^p when $|\beta|_1 = \kappa$. From the above, the sequence $[E * \varphi_n]_{m+\kappa-1}$ is a Cauchy sequence in $\dot{D}^{m+\kappa, p}$, so that it has a limit $[v]_{m+\kappa-1} \in \dot{D}^{m+\kappa, p}$. By the continuity of A , the convergence of $[E * \varphi_n]_{m+\kappa-1}$ to $[v]_{m+\kappa-1}$ in $\dot{D}^{m+\kappa, p}$ implies

$$A[v]_{m+\kappa-1} = \lim A[E * \varphi_n]_{m+\kappa-1} = \lim [A(E * \varphi_n)]_{\kappa-1}.$$

Since $A(E * \varphi_n) = \varphi_n$, this yields $A[v]_{m+\kappa-1} = \lim [\varphi_n]_{\kappa-1} = [f]_{\kappa-1}$. Thus, A is onto $\dot{D}^{\kappa, p}$ and the proof is complete. \square

When $\kappa > 0$, neither Lemma 4.1 nor its proof implies that A maps $D^{m+\kappa, p}$ onto $D^{\kappa, p}$. This issue will be resolved in Theorem 4.4.

If $\ell \in \mathbb{N}_0$ and $1 < p < \infty$, we now set

$$(4.3) \quad D^{-\ell, p} := (\dot{D}^{\ell, p'})^*,$$

a Banach space for the dual norm. This gives again $D^{0, p} = L^p$, consistent with (1.3). Denote by

$$(4.4) \quad \nu(\ell, N) := \binom{N+\ell-1}{\ell},$$

the number of multi-indices α such that $|\alpha|_1 = \ell$. Since the mapping $[u]_{\ell-1} \mapsto \nabla^\ell u$ is an isometric isomorphism of $\dot{D}^{\ell,p'}$ onto a closed subspace of $(L^{p'})^{\nu(\ell,N)}$, it follows from the Hahn–Banach theorem that every $f \in D^{-\ell,p}$ has the form

$$\langle f, [u]_{\ell-1} \rangle = \sum_{|\alpha|_1=\ell} \int_{\mathbb{R}^N} f_\alpha \partial^\alpha u,$$

where $f_\alpha \in L^p$ and $\|f\|_{D^{-\ell,p}} = \|(\sum_{|\alpha|_1=\ell} |f_\alpha|^2)^{1/2}\|_p$.

Conversely, if $f \in D^{-\ell,p}$ is defined by

$$\langle f, [u]_{\ell-1} \rangle := \sum_{|\alpha|_1=\ell} \int_{\mathbb{R}^N} f_\alpha \partial^\alpha u$$

for some $f_\alpha \in L^p$, then $\|f\|_{D^{-\ell,p}} \leq \|(\sum_{|\alpha|_1=\ell} |f_\alpha|^2)^{1/2}\|_p$ and, by the denseness of \mathcal{C}_0^∞ in $\dot{D}^{\ell,p'}$ (Remark 3.4), $f \in D^{-\ell,p}$ is uniquely determined by the distribution $\sum_{|\alpha|_1=1} (-1)^\ell \partial^\alpha f_\alpha$. By changing f_α into $(-1)^\ell f_\alpha$, it follows that

$$(4.5) \quad D^{-\ell,p} = \{f = \sum_{|\alpha|_1=\ell} \partial^\alpha f_\alpha : f_\alpha \in L^p\},$$

equipped with the norm $\inf \|(\sum_{|\alpha|_1=\ell} |f_\alpha|^2)^{1/2}\|_p$ (always a minimum). In particular, this shows that ∂^β maps $D^{\kappa,p}$ into $D^{\kappa-|\beta|_1,p}$ for every $\kappa \in \mathbb{Z}$.

We shall now extend Lemma 4.1 when $\kappa \in \mathbb{Z}$. To do this, we need another lemma, in the spirit of Corollary 2.10.

Lemma 4.2. *If $1 < p < \infty$ and $k \in \mathbb{Z}$, then $D^{k,p} \cap \mathcal{P} = \mathcal{P}_{k-1}$.*

Proof. If $k \geq 0$, the result is trivial since L^p contains no nonzero polynomial. If $k < 0$, it must be shown that if $u \in D^{k,p}$ is a polynomial, then $u = 0$.

Set $k = -\ell$ with $\ell \in \mathbb{N}$. We first prove that u cannot be a nonzero constant. By contradiction, if $1 \in D^{-\ell,p}$, it follows from (4.5) that $1 = \sum_{|\alpha|_1=\ell} \partial^\alpha f_\alpha$ for some $f_\alpha \in L^p$ and so $|\int_{\mathbb{R}^N} \varphi| \leq C \|\nabla^\ell \varphi\|_{p'}$ for every $\varphi \in \mathcal{C}_0^\infty$, where $C > 0$ depends only upon the norms $\|f_\alpha\|_p$. Let $\psi \in \mathcal{C}_0^\infty$ be such that $\int_{\mathbb{R}^N} \psi = 1$. With $\varphi(x) := \psi(\lambda x)$ and $\lambda > 0$, we get $1 = |\int_{\mathbb{R}^N} \psi| \leq C \lambda^{\ell+N/p} \|\nabla^\ell \psi\|_{p'}$ with the same constant C independent of λ , and a contradiction arises by letting $\lambda \rightarrow 0$.

If now $u \in D^{-\ell,p}$ is a nonzero polynomial, some derivative of u is a nonzero constant and this derivative is in $D^{-\tilde{\ell},p}$ with $\tilde{\ell} \geq \ell > 0$, which contradicts $1 \notin D^{-\tilde{\ell},p}$. □

Remark 4.3. Lemma 4.2 may suggest that when $k < 0$, the functions of $D^{k,p}$ continue to be subject to growth limitations at infinity. This is false. For example, $g_n(x) := (1 + |x|^2)^{-N/2} e^{i|x|^{2n}}$ is in L^p for every $1 < p < \infty$ and every $n \in \mathbb{N}$, whence $f_n := \partial_1 g_n \in D^{-1,p}$ by (4.5), but $f_n \notin M^{s,q}$ for any prescribed s and q if n is large enough. Nonetheless, depending on N, p and k , suitable integrability conditions suffice for membership to $D^{k,p}$ when $k < 0$ (Lemma 4.9), but this is not always true (Example 4.11).

To give uniform statements for all $k \in \mathbb{Z}$, we henceforth drop the “dot” notation $\dot{D}^{k,p}$ when $k \geq 0$ and return to the usual quotient space notation. Of course, $D^{k,p}/\mathcal{P}_{k-1} = D^{k,p}$ when $k \leq 0$.

Theorem 4.4. *If $\kappa \in \mathbb{Z}$ and $1 < p < \infty$, the homogeneous elliptic operator A in (1.1) is a linear isomorphism from $D^{m+\kappa,p}/\mathcal{P}_{m+\kappa-1}$ onto $D^{\kappa,p}/\mathcal{P}_{\kappa-1}$ and a homomorphism of $D^{m+\kappa,p}$ onto $D^{\kappa,p}$.*

Proof. We begin with the isomorphism property. Since it was proved in Lemma 4.1 when $\kappa \geq 0$, we assume $\kappa < 0$. Note first that $D^{k,p} \subset \mathcal{S}'$ for every $k \in \mathbb{Z}$. This follows for instance from Theorem 2.9 (i) and Theorem 3.3 if $k \geq 0$ (alternatively, [43], pp. 244–245, shows that $u \in \mathcal{S}'$ if and only if all the derivatives of u of some order $k \geq 0$ are in \mathcal{S}') and from (4.5) if $k < 0$.

By the homogeneity and ellipticity of A , the only solutions $u \in \mathcal{S}'$ of $Au = 0$ are polynomials. This is a simple exercise on Fourier transform (see the proof of Corollary 2.10). Consequently, if $u \in D^{m+\kappa,p}$ and $Au = 0$, then $u \in \mathcal{P}$ and so $u \in \mathcal{P}_{m+\kappa-1}$ by Lemma 4.2. Thus, A is one-to-one on $D^{m+\kappa,p}/\mathcal{P}_{m+\kappa-1}$. Since $\kappa < 0$ (hence $\mathcal{P}_{\kappa-1} = \{0\}$), it remains to show that A maps $D^{m+\kappa,p}$ onto $D^{\kappa,p}$.

Set $\kappa = -\ell$ with $\ell \in \mathbb{N}$, so that, by (4.5), every $f \in D^{\kappa,p} = D^{-\ell,p}$ has the form $f = \sum_{|\alpha|_1=\ell} \partial^\alpha f_\alpha$ for some $f_\alpha \in L^p$. By Lemma 4.1, there is $v_\alpha \in D^{m,p}$ such that $Av_\alpha = f_\alpha$. Thus, if $u := \sum_{|\alpha|_1=\ell} \partial^\alpha v_\alpha$, then $u \in D^{m-\ell,p} = D^{m+\kappa,p}$ and $Au = f$. This completes the proof that A is an isomorphism of $D^{m+\kappa,p}/\mathcal{P}_{m+\kappa-1}$ onto $D^{\kappa,p}/\mathcal{P}_{\kappa-1}$ for every $\kappa \in \mathbb{Z}$.

We now prove that A maps $D^{m+\kappa,p}$ onto $D^{\kappa,p}$. This was just done above when $\kappa \leq 0$. If $\kappa > 0$ and $f \in D^{\kappa,p}$, the first part of the proof (or Lemma 4.1) ensures that there are $\pi \in \mathcal{P}_{\kappa-1}$ and $u \in D^{m+\kappa-1,p}$ such that $Au = f + \pi$. Thus, it suffices to show that there is $\varpi \in \mathcal{P}_{m+\kappa-1}$ such that $A\varpi = \pi$, for then $u - \varpi \in D^{m+\kappa,p}$ and $A(u - \varpi) = f$.

The dimension of the space of homogeneous A -harmonic polynomials of degree ℓ , as calculated by Horváth [23], is $\nu(\ell, N) - \nu(\ell - m, N)$ with ν from (4.4), where $\nu(\ell - m, N) := 0$ if $\ell < m$. Thus, the subspace of A -harmonic polynomials in $\mathcal{P}_{m+\kappa-1}$ has dimension

$$\sum_{\ell=0}^{m+\kappa-1} \nu(\ell, N) - \sum_{\ell=0}^{\kappa-1} \nu(\ell, N).$$

Since $\nu(\ell, N)$ is also the dimension of the space of homogeneous polynomials of degree ℓ , this is just $\dim \mathcal{P}_{m+\kappa-1} - \dim \mathcal{P}_{\kappa-1}$. As a result, the rank of $A : \mathcal{P}_{m+\kappa-1} \rightarrow \mathcal{P}_{\kappa-1}$ is $\dim \mathcal{P}_{\kappa-1}$. Thus, A maps $\mathcal{P}_{m+\kappa-1}$ onto $\mathcal{P}_{\kappa-1}$ and the proof is complete. \square

In Theorem 4.4, the isomorphism property is equivalent to the generalized Calderón–Zygmund inequality (the reverse inequality is trivial)

$$\| [u]_{m+\kappa-1} \|_{D^{m+\kappa,p}/\mathcal{P}_{m+\kappa-1}} \leq C \| [Au]_{\kappa-1} \|_{D^{\kappa,p}/\mathcal{P}_{\kappa-1}},$$

for $u \in D^{m+\kappa,p}$.

We can now prove a sharper and more general form of Theorem 1.1.

Theorem 4.5. *Let A denote the homogeneous elliptic operator (1.1). If $u \in \mathcal{D}'$ and $Au \in D^{\kappa,p}$ for some integer $\kappa \geq -m$ and $1 < p < \infty$, the following properties are equivalent:*

- (i) $u \in D^{m+\kappa,p}$.
- (ii) $u \in M_0^{s,p}$ for every $s > m+\kappa-N/p$ if $p > N$ and every $s > m+\kappa-1$ if $p \leq N$.
- (iii) $u \in M_0^{m+\kappa,1}$.

Proof. (i) \Rightarrow (ii) by Theorem 3.3.

(ii) \Rightarrow (iii) by letting $s = m + \kappa$ in (ii) and by using $M_0^{m+\kappa,p} \subset M_0^{m+\kappa,1}$.

(iii) \Rightarrow (i). Assume $u \in M_0^{m+\kappa,1}$ and $Au \in D^{\kappa,p}$. By Theorem 4.4, there is $v \in D^{m+\kappa,p}$ such that $Av = Au$ and, by (i) \Rightarrow (iii) already proved above, $v \in M_0^{m+\kappa,1}$. Hence, $u - v \in M_0^{m+\kappa,1}$. Since $A(u - v) = 0$, Corollary 2.10 yields $u - v \in \mathcal{P}_{m+\kappa-1}$, so that $u = v + (u - v) \in D^{m+\kappa,p}$. \square

A straightforward corollary of Theorem 4.5 addresses the same issue when $D^{m+\kappa,p}$ is replaced with $W^{m+\kappa,p}$.

Corollary 4.6. *Let A denote the homogeneous elliptic operator (1.1), and let $\kappa \geq -m$ be an integer. Then, $u \in W^{m+\kappa,p}$ if and only if*

- (i) $Au \in D^{\kappa,p}$ and $u \in L^p$

or

- (ii) $Au \in D^{\kappa,p} \cap D^{-m,p}$ and $u \in M_0^{0,1}$.

In particular, if $m < N$, $Au \in D^{\kappa,p} \cap L^{Np/(N+m)}$ with $N/(N - m) < p < \infty$ and $u \in M_0^{0,1}$, then $u \in W^{m+\kappa,p}$.

Proof. In both (i) and (ii), the necessity is trivial and the “in particular” part follows from (ii) and Lemma 4.9 below (with an independent proof). To prove the sufficiency of (i), just use $L^p = M^{-N/p,p} \subset M_0^{m+\kappa,1}$ since $m + \kappa \geq 0 > -N/p$ to get $u \in D^{m+\kappa,p}$ by (iii) \Rightarrow (i) of Theorem 4.5. Thus, $u \in L^p \cap D^{m+\kappa,p} = W^{m+\kappa,p}$ (Theorem 4.10, p. 337 in [6]).

In (ii), use (iii) \Rightarrow (i) of Theorem 4.5 with $\kappa = -m$ to get $u \in L^p$, so that the result follows from (i). \square

Part (i) is trivial if $\kappa = -m$, or if $-m < \kappa \leq 0$ and $A - z$ is an isomorphism of $W^{m+\kappa,p}$ to $W^{\kappa,p}$ for some $z \in \mathbb{C}$ (e.g., $A = \Delta$ or, more generally, A strongly elliptic), but the latter can only happen if $A(\xi) - z \neq 0$ for every $\xi \in \mathbb{R}^N$, which need not hold for any z if A is merely elliptic. The simplest counter-examples are given by the powers $\bar{\partial}^m$ of the Cauchy–Riemann operator $\bar{\partial}$ when $N = 2$. On the other hand, if A is strongly elliptic, part (i) remains true with $D^{\kappa,p}$ replaced with $W^{\kappa,p}$, which is more general when $\kappa < 0$.

When $\kappa > 0$, part (i) does not follow right away from the classical elliptic theory even if $A = \Delta$. Indeed, if $\Delta u = f \in D^{\kappa,p}$ and $u \in L^p$, then $\Delta u - u = f - u$ but since (a priori) $f - u$ need not be in $W^{\kappa,p}$, the regularity properties of $\Delta - 1$ do not yield $u \in W^{2+\kappa,p}$. In fact, they do, but only with extra work (differentiation and bootstrapping; details are left to the reader) and the result cannot be called well known.

Since part (ii) involves the space $M_0^{0,1}$, it is new irrespective of A . Recall that $u \in M_0^{0,1}$ is much more general than the necessary $u \in L^p$; see the examples of Section 2, but more information than just $Au \in D^{\kappa,p}$ is needed. A nonstandard example of (i) and (ii) with $N = 2$ and $m = 1$ is that $u \in W^{1,p}$ if either $u \in L^p$ and $\bar{\partial}u \in L^p$ with $1 < p < \infty$, or $u \in M_0^{0,1}$ and $\bar{\partial}u \in L^p \cap L^{2p/(p+2)}$ with $2 < p < \infty$. Of course, $u = 0$ if $\bar{\partial}u = 0$ in both cases, consistent with Corollary 2.10.

We complete this section with four very different applications of Theorem 4.5. We begin with a “consistency” property. These properties are very useful, but not trivial in scales of spaces which are not ordered by inclusion.

Example 4.7. Suppose that $u \in D^{m+\kappa_1,p_1}$ for some $\kappa_1 \in \mathbb{Z}$ and some $1 < p_1 < \infty$. If $Au \in D^{\kappa_2,p_2}$ for some $\kappa_2 \geq \kappa_1$ and some $1 < p_2 < \infty$, then $u \in D^{m+\kappa_2,p_2}$. If $\kappa_1 \geq -m$, this follows at once from (i) \Leftrightarrow (iii) in Theorem 4.5 and from $M_0^{m+\kappa_1,1} \subset M_0^{m+\kappa_2,1}$. If $\kappa_1 < -m$, choose $\ell \in \mathbb{N}$ such that $\kappa_1 \geq -\ell m$. By Theorem 4.4 for $A^{\ell-1}$, there is $v \in D^{\ell m+\kappa_1,p_1}$ such that $A^{\ell-1}v = u$. Hence, $A^\ell v = Au \in D^{\kappa_2,p_2}$. Since $\kappa_2 \geq \kappa_1 \geq -\ell m$, the first step yields $v \in D^{\ell m+\kappa_2,p_2}$, whence $u = A^{\ell-1}v \in D^{m+\kappa_2,p_2}$. This argument shows that a general result may be useful even if a single operator is of interest. \square

The use of Theorem 4.5 (Corollary 4.6(ii)) may be simplified in problems $Au = G(u)$ when G has at least linear growth and $\kappa \geq 0$ ($\kappa = 0$):

Example 4.8. Let $G : \mathbb{C} \rightarrow \mathbb{C}$ be continuous. If $u \in L^p$ and $Au = G(u) \in D^{\kappa,p}$ for some integer $\kappa \geq -m$ and $1 < p < \infty$, then $u \in W^{m+\kappa,p}$ by Corollary 4.6 (i).

Suppose now that $\liminf_{|z| \rightarrow \infty} |G(z)|/|z| > 0$ and that $\kappa \geq 0$, but only that $u \in L^1_{\text{loc}}$. Then, $u \in D^{m+\kappa,p}$. By Theorem 4.5, it suffices to prove that $u \in M_0^{m+\kappa,1}$. In fact, $u \in M^{\kappa,1}$, which is stronger since $m > 0$. To see this, set $g(t) := \min_{\theta \in [0,2\pi]} |G(te^{i\theta})|$ for $t \geq 0$, so that $g \geq 0$ and $\liminf_{t \rightarrow \infty} g(t)/t > 0$. If $h(0) = 0$ and $h(t) = g(t)$ if $t > 0$, the convex hull Φ of h is a Young function. By Theorem 3.3, $G(u) \in D^{\kappa,p} \subset M_0^{\kappa,p} \subset M_0^{\kappa,1}$. Since $0 \leq \Phi(|u|) \leq |G(u)|$, we infer that $\Phi(|u|) \in M_0^{\kappa,1}$. Let $t_0 \geq 0$ and $\lambda > 0$ be chosen as in Example 2.2 with $q = 1$, so that $t \leq \Phi(\lambda t)$ if $t \geq t_0$. Equivalently, $t \leq \lambda \Phi(t)$ if $t \geq \lambda t_0$. Thus, $|u| \leq \lambda t_0 + \lambda \Phi(|u|) \in M^{\kappa,1}$, whence $u \in M^{\kappa,1}$, as claimed, and even $u \in M_0^{\kappa,1}$ if $\kappa > 0$ (because $\lambda t_0 \in M^{0,1} \subset M_0^{\kappa,1}$) or if $\kappa = 0$ and $1 \notin L_\Phi$ (because $\lambda t_0 > 0$ can be chosen arbitrarily small; see part (iv) of Example 2.2). In particular, by Corollary 4.6 (ii), $u \in W^{m,p}$ if $G(z) \neq 0$ when $z \neq 0$ (for then $1 \notin L_\Phi$), $u \in L^1_{\text{loc}}$ and $G(u) \in L^p \cap D^{-m,p}$ (e.g., $m < N$ and $G(u) \in L^p \cap L^{Np/(N+mp)}$ with $N/(N - m) < p < \infty$). \square

It is easy to generalize Example 4.8 when $G = G(x, u, \nabla u, \dots, \nabla^k u)$ (possibly $k > m$) and there is a Young function Φ such that $|G(x, u, \nabla u, \dots, \nabla^k u)| \geq \Phi(|\nabla^j u|)$ for some $0 \leq j \leq m - 1$. Indeed, by the same argument, $\Phi(|\nabla^j u|) \in M_0^{\kappa,1}$ implies $|\nabla^j u| \in M^{\kappa,1}$ and then $u \in M^{j+\kappa,1} \subset M_0^{m+\kappa,1}$ by Theorem 3.2. This also works if $j = m$ and either $\kappa > 0$ or $1 \notin L_\Phi$, for then $|\nabla^m u| \in M_0^{\kappa,1}$ and so $u \in M_0^{m+\kappa,1}$ by Theorem 3.2.

The next example shows how the properties of the spaces $M^{s,q}$, notably Theorems 2.5, 3.2 and 3.3, can be combined with Theorem 4.5 to convert growth

assumptions on the coefficients into regularity results for the solutions. Similar issues have been discussed by various authors; see [32], [40] and the references therein, but it is safe to say that the results given in Example 4.10 below cannot be proved by previously known arguments.

The following equivalent dual form of Sobolev’s inequality will be useful.

Lemma 4.9. *If $k < N$ is a positive integer and if $N/(N - k) < p < \infty$, then $Np/(N + kp) > 1$ and $L^{Np/(N+kp)} \hookrightarrow D^{-k,p}$ (dense embedding).*

Proof. Let $q > 1$ be such that $kq < N$. By Sobolev’s inequality, there is a constant $C > 0$ independent of $\varphi \in \mathcal{C}_0^\infty$ such that $\|\varphi\|_{q^{*k}} \leq C\|\nabla^k \varphi\|_q$, where $q^{*k} := Nq/(N - kq)$. By the denseness of \mathcal{C}_0^∞ in $\dot{D}^{k,q}$ (Remark 3.4), this yields the embedding¹ $\dot{D}^{k,q} \hookrightarrow L^{q^{*k}}$. Since $\mathcal{C}_0^\infty \subset \dot{D}^{k,q}$ is dense in $L^{q^{*k}}$, this embedding is dense and so, by duality, $L^{(q^{*k})'} \hookrightarrow (\dot{D}^{k,q})^* = D^{-k,q'}$. The embedding is dense since $\dot{D}^{k,q}$ is reflexive. Since $(q^{*k})' = Nq/((N + k)q - N)$, the result follows by letting $q = p'$. \square

From the proof, the embedding of Lemma 4.9 identifies $f \in L^{Np/(N+kp)}$ with the linear form $[v]_{k-1} \mapsto \int_{\mathbb{R}^N} f v_0$, where v_0 is the unique representative of $[v]_{k-1}$ in $L^{Np'/(N-kp')}$.

Example 4.10. In this example, all the functions are real-valued. Consider the problem $-\nabla \cdot (a \nabla u) + cu = f$ on \mathbb{R}^N , where $a, c > 0$ are \mathcal{C}^∞ (for simplicity) and satisfy the conditions (i) $a^{-1/2} \in D^{1,\infty}$ and (ii) $a^{-1}c^{1/2} \in L^\infty$.

Neither a nor c needs to be bounded or bounded below by a positive constant but, since $a^{-1/2} \in M^{1,\infty}$ by (i) and Theorem 3.3 (i), $a(x)$ cannot decay (pointwise) faster than $|x|^{-2}$ at infinity. The function c can decay arbitrarily fast but, by (ii), it cannot grow faster than a^2 .

It is readily checked that the space $V := \{u \in \mathcal{D}' : a^{1/2} \nabla u \in (L^2)^N, c^{1/2} u \in L^2\}$ is a Hilbert space for the inner product $\int_{\mathbb{R}^N} a \nabla u \cdot \nabla v + \int_{\mathbb{R}^N} cuv$. Hence, if $f \in L^2(\mathbb{R}^N; c^{-1} dx)$, which is henceforth assumed, there is a unique $u \in V$ such that $-\nabla \cdot (a \nabla u) + cu = f$. Equivalently,

$$(4.6) \quad -\Delta u = (a^{-1} \nabla a) \cdot \nabla u - a^{-1} cu + a^{-1} f.$$

The right-hand side is in L^2 . Indeed, $(a^{-1} \nabla a) \cdot \nabla u = (a^{-3/2} \nabla a) \cdot (a^{1/2} \nabla u) \in L^2$ by (i) since $u \in V$ and $a^{-1} cu = a^{-1} c^{1/2} (c^{1/2} u) \in L^2$ and $a^{-1} f = a^{-1} c^{1/2} (c^{-1/2} f) \in L^2$ by (ii) since $u \in V$ and since $c^{-1/2} f \in L^2$. We claim that $u \in D^{2,2}$. By Theorem 4.5, it suffices to show that $u \in M_0^{2,1}$. As noted above, $a^{-1/2} \in M^{1,\infty}$. Since $a^{1/2} \nabla u \in (L^2)^N$ and $L^2 = M^{-N/2,2}$, it follows from Theorem 2.5 that $\nabla u = (a^{-1/2}) a^{1/2} \nabla u \in (M^{1-N/2,2})^N \subset (M^{1/2,1})^N$. By Theorem 3.2, $u \in M^{3/2,1} \subset M_0^{2,1}$.

Assume now $N > 2$ and replace (i) and (ii) with (i') $a^{-1/2} \in D^{1,N}$ and (ii') $a^{-1} c^{1/2} \in L^N$. By using $u \in V$ and (i'), $(a^{-1} \nabla a) \cdot \nabla u = (a^{-3/2} \nabla a) \cdot (a^{1/2} \nabla u) \in L^{2N/(N+2)}$. By Lemma 4.9 (with $k = 1, p = 2$ and since $N/(N - 1) < 2$), it follows

¹Explicitly, this embedding is given by $[u]_{k-1} \mapsto u - \pi_u$, where $\pi_u \in \mathcal{P}_{k-1}$ is the only polynomial such that $u - \pi_u \in L^{q^{*k}}$; clearly, $u - \pi_u$ is independent of the representative u .

that $(a^{-1}\nabla a) \cdot \nabla u \in D^{-1,2}$. Likewise, by using $u \in V$ and (ii'), $a^{-1}cu \in D^{-1,2}$ and $a^{-1}f \in D^{-1,2}$. Thus, the right-hand side of (4.6) is in $D^{-1,2}$. We claim that $u \in M_0^{1,1}$, so that $u \in D^{1,2}$ by Theorem 4.5. First, $a^{-1/2} \in M_0^{s,N}$ for every $s > 0$ by Theorem 3.3 (ii). In particular, $a^{-1/2} \in M_0^{N/2,N}$. Next, $a^{1/2}\nabla u \in (L^2)^N = (M^{-N/2,2})^N$. Hence, by Theorem 2.5, $\nabla u = a^{-1/2}(a^{1/2}\nabla u) \in (M_0^{0,2N/(N+2)})^N \subset (M_0^{0,1})^N$ and so $u \in M_0^{1,1}$ by Theorem 3.2.

Let $2^* := 2N/(N - 2)$ (recall $N > 2$). By the Sobolev inequality, $\nabla u - U_\infty \in (L^{2^*})^N$ with $U_\infty \in \mathbb{R}^N$ if (i) and (ii) hold and $u - u_\infty \in L^{2^*}$ with $u_\infty \in \mathbb{R}$ if (i') and (ii') hold. In particular, $U_\infty = 0$ if (i), (ii), (i') and (ii') hold (because $u \in D^{1,2}$) and so $u \in D^{1,2^*}$. With this, it is not hard to find further conditions on a, c and f (compatible with previous assumptions) ensuring that the right-hand side of (4.6) is in L^{2^*} . Then, $u \in D^{2,2^*}$ by Theorem 4.5 since $u \in M_0^{2,1}$ is already known. In turn, this implies $u - u_\infty \in D^{2,2^*} \cap L^{2^*} = W^{2,2^*}$. \square

By a scaling argument, $\mathcal{C}_0^\infty \not\subset D^{-k,p}$ if $k \geq N$ and $1 < p < \infty$ or $0 < k < N$ and $1 < p < N/(N - k)$. Example 4.11 below shows that the latter result, extended to the optimal² range $1 < p \leq N/(N - k)$, can be derived from Theorem 4.5.

Example 4.11. Let u be a smooth function equal to $|x|^{2-N}$ for $|x|$ large enough if $N > 2$, or equal to $\log|x|$ for $|x|$ large enough if $N = 2$. Then, $u \in D^{1,p}$ with $p > N/(N - 1)$ and so $u \in M_0^{1,p} \subset M_0^{1,1}$ by Theorem 3.3. Therefore, $\Delta u \notin D^{-1,p}$ if $1 < p \leq N/(N - 1)$, for otherwise $u \in D^{1,p}$ by Theorem 4.5, which is obviously false. Since $\Delta u \in \mathcal{C}_0^\infty$, this shows that $\mathcal{C}_0^\infty \not\subset D^{-1,p}$ if $N > 1$ and $1 < p \leq N/(N - 1)$. Likewise, $\mathcal{C}_0^\infty \not\subset D^{-2,p}$ if $N > 2$ and $1 < p \leq N/(N - 2)$ because $u \in L^p$ if and only if $p > N/(N - 2)$. More generally, $\mathcal{C}_0^\infty \not\subset D^{-k,p}$ when $N > k > 0$ and $1 < p \leq N/(N - k)$ can be seen by using the function $|x|^{2\ell-N}$ and the operator Δ^ℓ with $\ell = k/2$ when k is even or $\ell = (k + 1)/2$ when k is odd. By Lemma 4.9, these non-embeddings are sharp. \square

5. Exterior domains

In this section, $\Omega \subset \mathbb{R}^N$ is an exterior domain (i.e., $\mathbb{R}^N \setminus \Omega$ is compact). To fix ideas, we shall also assume that $0 \notin \bar{\Omega}$. In particular, $\Omega \neq \mathbb{R}^N$ but also $\Omega \neq \mathbb{R}^N \setminus \{0\}$. We shall extend Theorem 4.5 to this setting, but unexpected necessary restrictions on N and p arise when $\kappa < 0$, which are not needed when $\Omega = \mathbb{R}^N$.

If $k \in \mathbb{N}_0$ and $1 < p < \infty$, the homogeneous Sobolev space $D^{k,p}(\Omega)$ is defined by (1.3) after merely replacing \mathbb{R}^N with Ω . If $\ell \in \mathbb{N}$, the space $D^{-\ell,p}(\Omega)$ is the dual of the completion $D_0^{\ell,p'}(\Omega)$ of $\mathcal{C}_0^\infty(\Omega)$ for the norm $\|\cdot\|_{p',\Omega}$. (If $\Omega = \mathbb{R}^N$, the definition (4.3) is recovered since \mathcal{C}_0^∞ is dense in $\dot{D}^{\ell,p'}$.) With this definition, $D^{-\ell,p}(\Omega)$ is a space of distributions, ∂^α maps $D^{k,p}(\Omega)$ into $D^{k-|\alpha|,p}(\Omega)$ for every $k \in \mathbb{Z}$ and ∂^α is continuous from $D^{k,p}(\Omega)/\mathcal{P}_{k-1}$ to $D^{k-|\alpha|,p}(\Omega)/\mathcal{P}_{k-|\alpha|-1}$. For more details, see e.g. [15].

² Since we did not define $D^{-k,p}$ when $p = 1$.

The first task will be to adjust Theorem 4.5 to the new setting (Theorem 5.4). To begin with, spaces $M^{s,p}(\Omega)$ and $M_0^{s,p}(\Omega)$ can also be defined on Ω in the obvious way, by merely replacing B_R with

$$(5.1) \quad \Omega_R := B_R \cap \Omega,$$

in (2.1) and (2.2) and by choosing $R > 0$ large enough that $|\Omega_R| > 0$. However, to ensure the L^p -integrability on Ω_R , the definition of $M^{s,p}(\Omega)$ must incorporate $u \in L^p_{\text{loc}}(\bar{\Omega})$ rather than just $u \in L^p_{\text{loc}}(\Omega)$. This is of course immaterial when $\Omega = \mathbb{R}^N$.

Remark 5.1. The extension by 0 outside Ω maps $M^{s,p}(\Omega)$ ($M_0^{s,p}(\Omega)$) into $M^{s,p}$ ($M_0^{s,p}$). As a result, Theorems 2.4, 2.5 and 2.7 have obvious generalizations that we shall not spell out explicitly.

The aforementioned possible restrictions about N and p originate in part (i) of the following lemma, related to Lemma 4.9.

Lemma 5.2. *Let $\omega \subset \mathbb{R}^N$ be an open subset.*

(i) *Let $\varphi \in C^\infty$ be such that $\text{Supp } \varphi \subset \omega$ and that $\text{Supp } \nabla \varphi$ is compact. Then $\varphi v \in D^{k,p}$ for every $v \in D^{k,p}(\omega)$ if either $k \geq 0$ and $1 \leq p \leq \infty$ or $-N < k < 0$ and $N/(N+k) < p < \infty$.*

(ii) *If ω is bounded and $\partial\omega$ has the cone property, then $D^{k,p}(\omega) = W^{k,p}(\omega)$ if either $k \geq 0$ and $1 \leq p \leq \infty$ or $k < 0$ and $1 < p < \infty$.*

Proof. (i) If $k \geq 0$, this follows from Leibnitz' rule and from $D^{k,p}(\omega) \subset W^{k,p}_{\text{loc}}(\omega)$ (use $\text{Supp } \nabla \varphi$ compact; in particular, φ is locally constant outside a ball and therefore bounded). Below, we give a proof when $k = -1$ (hence $N > 1$ and $N/(N-1) < p < \infty$). The modifications are obvious when $k < 0$ is arbitrary.

First, $p > N/(N-1)$ amounts to $p' < N$, so that $p'^* := Np'/(N-p') < \infty$. Let $\psi \in C^\infty_0$ be given. Since $v \in D^{-1,p}(\omega)$, it follows that $|\langle \varphi v, \psi \rangle| \leq C \|\nabla(\varphi\psi)\|_{p'}$ with $C > 0$ independent of ψ . Now, use $\|\nabla(\varphi\psi)\|_{p'} \leq C_\varphi (\|\nabla\psi\|_{p'} + \|\psi\|_{p',S_\varphi})$, where $S_\varphi := \text{Supp } \nabla\varphi$ and $C_\varphi > 0$ is independent of ψ . Next, by Hölder's inequality, $\|\psi\|_{p',S_\varphi} \leq |S_\varphi|^{1/N} \|\psi\|_{p'^*,S_\varphi}$, whereas $\|\psi\|_{p'^*,S_\varphi} \leq \|\psi\|_{p'^*} \leq C \|\nabla\psi\|_{p'}$ by Sobolev's inequality. Altogether, $|\langle \varphi v, \psi \rangle| \leq C \|\nabla\psi\|_{p'}$ for every $\psi \in C^\infty_0$, whence $\varphi v \in D^{-1,p}$ by Remark 3.4.

(ii) is trivial if $k = 0$ and proved in [31], p. 21, if $k > 0$. If so and if $1 < p < \infty$, then $D_0^{k,p}(\omega) = W_0^{k,p}(\omega)$ with equivalent norms (Poincaré's inequality), so that $D^{-k,p'}(\omega) = W^{-k,p'}(\omega)$ by duality. Exchange the roles of p and p' and of k and $-k$ to get $D^{k,p}(\omega) = W^{k,p}(\omega)$ when $k < 0$. \square

In part (i), the restrictions on N and p when $k < 0$ are needed even if ω is bounded. In particular, if $v \in D^{k,p}(\omega)$ has compact support, the extension of v by 0 need *not* be in $D^{k,p}$ without these restrictions; see Example 4.11 and preceding comments (indeed, $C^\infty_0(\omega) \subset D^{k,p}(\omega)$ for every $k \in \mathbb{Z}$ and $1 < p < \infty$ when ω is bounded). On the other hand, no restriction on N and p is necessary if ω is bounded and \mathbb{R}^N is replaced with a *bounded* open subset $\tilde{\omega} \supset \omega$, because Poincaré's inequality can be substituted for Sobolev's inequality in the proof.

The following generalization of Theorem 3.3 is straightforward.

Lemma 5.3. *If $\partial\Omega$ has the cone property, Theorem 3.3 remains true upon replacing $D^{k,p}$ and $M^{s,p}$ with $D^{k,p}(\Omega)$ and $M^{s,p}(\Omega)$, respectively.*

Proof. Let $R_0 > 0$ be large enough that $\mathbb{R}^N \setminus \Omega \subset B_{R_0}$. If $u \in D^{k,p}(\Omega)$, then $u \in D^{k,p}(\Omega_{R_0}) = W^{k,p}(\Omega_{R_0})$ (Lemma 5.2 (ii)) and so $u \in L^p_{\text{loc}}(\overline{\Omega})$. Let $\varphi \in C^\infty(\Omega)$ be such that $\varphi = 1$ outside B_{R_0} and $\varphi = 0$ on some neighborhood of $\mathbb{R}^N \setminus \Omega$. By Lemma 5.2 (i) (with no restriction on N or p since $k \geq 0$), $\varphi u \in D^{k,p}$ and so, by Theorem 3.3 for φu , it follows that $\varphi u \in M^{s,p}$ or $\varphi u \in M_0^{s,p}$ for the specified values of s . This trivially implies $u \in M^{s,p}(\Omega)$ or $u \in M_0^{s,p}(\Omega)$, as the case may be. \square

It is now easy to prove a variant of Theorem 4.5.

Theorem 5.4. *Suppose that $\partial\Omega \in C^{0,1}$ and let $R_0 > 0$ be such that $\mathbb{R}^N \setminus \Omega \subset B_{R_0}$. If $u \in \mathcal{D}'(\Omega)$ and $Au \in D^{\kappa,p}(\Omega)$ for some integer $\kappa \geq \max\{1 - N, -m\}$ and $\max\{1, N/(N + \kappa)\} < p < \infty$, the following properties are equivalent:*

- (i) $u \in D^{m+\kappa,p}(\Omega)$.
- (ii) $u \in D^{m+\kappa,p}(\Omega_{R_0}) \cap M_0^{s,p}(\Omega)$ for every $s > m + \kappa - N/p$ if $p > N$ and every $s > m + \kappa - 1$ if $p \leq N$.
- (iii) $u \in D^{m+\kappa,p}(\Omega_{R_0}) \cap M_0^{m+\kappa,1}(\Omega)$.

Furthermore, (i) \Rightarrow (ii) if it is only assumed that $\partial\Omega$ has the cone property, $\kappa \geq -m$ and $1 \leq p \leq \infty$ and (ii) \Rightarrow (iii) is always true.

Proof. (i) \Rightarrow (ii) \Rightarrow (iii) If $u \in D^{m+\kappa,p}(\Omega)$, it is obvious that $u \in D^{m+\kappa,p}(\Omega_{R_0})$, whereas $u \in M_0^{s,1}(\Omega)$ for every $s > m + \kappa - N/p$ if $p > N$ and every $s > m + \kappa - 1$ if $p \leq -N$ by Lemma 5.3. In particular, $u \in M_0^{m+\kappa,1}(\Omega)$ by Remark 5.1 and Theorem 2.4 (ii). This also proves the “furthermore” part.

(iii) \Rightarrow (i) Suppose that $u \in D^{m+\kappa,p}(\Omega_{R_0})$, that $Au \in D^{\kappa,p}(\Omega)$ and that $u \in M_0^{m+\kappa,1}(\Omega)$. By Lemma 5.2 (ii), $D^{m+\kappa,p}(\Omega_{R_0}) = W^{m+\kappa,p}(\Omega_{R_0})$ and so, by the Stein extension theorem (this uses $\partial\Omega \in C^{0,1}$; see [1], p. 154, or Chapter 6 of [46]), u can be extended to all of \mathbb{R}^N as a function $\tilde{u} \in W^{m+\kappa,p}(B_{R_0})$. In particular, $A\tilde{u} \in D^{\kappa,p}(B_{R_0})$ and $A\tilde{u} \in D^{\kappa,p}(\Omega)$ since $\tilde{u} = u$ on Ω .

Choose $\varphi \in C^\infty(B_{R_0})$ with $\varphi = 1$ on a neighborhood of $\mathbb{R}^N \setminus \Omega$ and write $A\tilde{u} = \varphi A\tilde{u} + (1 - \varphi)A\tilde{u}$. By Lemma 5.2 (i) with $\omega = B_{R_0}$ and, next, $\omega = \Omega$, we get $\varphi A\tilde{u} \in D^{\kappa,p}$ and $(1 - \varphi)A\tilde{u} \in D^{\kappa,p}$. This shows that $A\tilde{u} \in D^{\kappa,p}$. Since $u \in M_0^{m+\kappa,1}(\Omega)$ implies $\tilde{u} \in M_0^{m+\kappa,1}$, Theorem 4.5 yields $\tilde{u} \in D^{m+\kappa,p}$, whence $u \in D^{m+\kappa,p}(\Omega)$. \square

If $N > 2$ and $u(x) := |x|^{2-N}$, then $\Delta u = 0$ in Ω , $u \in L^p(\Omega_{R_0})$ for every $1 < p < \infty$ and $u \in L^p(\Omega)$ if and only if $p > N/(N - 2)$. In particular, $u \in M_0^{0,1}(\Omega)$. Thus, the hypotheses of Theorem 5.4 are satisfied with $m = 2$, $\kappa = -2$ and $N/(N - 2) < p < \infty$. Since $u \notin L^p(\Omega)$ when $p \leq N/(N - 2)$, this shows that the condition $p > N/(N - 2)$ cannot be dropped. Of course, the similarities with Example 4.11 are no coincidence and the functions and operators of that example also show that, more generally, $p > N/(N + \kappa)$ cannot be dropped in Theorem 5.4.

We shall not spell out the obvious analog of Corollary 4.6 (just note that $L^p(\Omega) \cap D^{m+\kappa,p}(\Omega) = W^{m+\kappa,p}(\Omega)$ when $\kappa \geq -m$ follows, by extension, from the same property when $\Omega = \mathbb{R}^N$ and from Lemma 5.2 (ii)). The consistency question, similar to Example 4.7 when $\Omega = \mathbb{R}^N$, is settled in the following corollary, but in a (necessarily) more limited setting. It has not been addressed in works discussing existence, even when $A = \Delta$ (for instance, [44]).

Corollary 5.5. *Suppose that $u \in D^{m+\kappa_1,p_1}(\Omega)$ for some integer $\kappa_1 \geq \max\{1 - N, -m\}$ and some $\max\{1, N/(N + \kappa_1)\} < p_1 < \infty$ and that $Au \in D^{\kappa_2,p_2}(\Omega)$ for some $\kappa_2 \geq \kappa_1$ and some $\max\{1, N/(N + \kappa_2)\} < p_2 < \infty$.*

(i) *If $\partial\Omega \in C^{0,1}$ and $u \in D^{m+\kappa_2,p_2}(\Omega_{R_0})$ with $R_0 > 0$ such that $\mathbb{R}^N \setminus \Omega \subset B_{R_0}$ (whence $u \in W^{m+\kappa_2,p_2}(\Omega_{R_0})$ by Lemma 5.2 (ii)), then $u \in D^{m+\kappa_2,p_2}(\Omega)$.*

(ii) *If Ω' is an open subset such that $\overline{\Omega'} \subset \Omega$, then $u \in D^{m+\kappa_2,p_2}(\Omega')$.*

Proof. (i) First, $u \in M_0^{m+\kappa_1,1}(\Omega) = M_0^{m+\kappa_2,1}(\Omega)$ by Theorem 5.4 with $\kappa = \kappa_1$ and $p = p_1$ and then $u \in D^{m+\kappa_2,p_2}(\Omega)$ by Theorem 5.4 with $\kappa = \kappa_2$ and $p = p_2$.

(ii) After enlarging Ω' , it is not restrictive to assume $\partial\Omega' \in C^{0,1}$. Let $\varphi \in C_0^\infty(\Omega)$ be such that $\varphi = 1$ on some open ball $B \subset \Omega$. By Lemma 5.2 (i), $\varphi Au \in D^{\kappa_2,p_2}$, so that, by Theorem 4.4, there is $v \in D^{m+\kappa_2,p_2}$ such that $Av = \varphi Au$. In particular, $A(v - u) = 0$ on B . By hypoellipticity, $v - u \in C^\infty(B)$, whence $u \in D^{m+\kappa_2,p_2}(B') = W^{m+\kappa_2,p_2}(B')$ for every ball $B' \Subset B$. This shows that $u \in W_{\text{loc}}^{m+\kappa_2,p_2}(\Omega)$ and, hence, that $u \in W^{m+\kappa_2,p_2}(\Omega'_{R_0}) = D^{m+\kappa_2,p_2}(\Omega'_{R_0})$ for every $R_0 > 0$ such that $\mathbb{R}^N \setminus \Omega' \subset B_{R_0}$. Thus, $u \in D^{m+\kappa_2,p_2}(\Omega')$ by (i) with Ω replaced with Ω' . \square

In part (i) of Corollary 5.5, the condition $u \in D^{m+\kappa_2,p_2}(\Omega_{R_0})$ depends only upon the behavior of u near $\partial\Omega$. This may be provable by elliptic regularity arguments. For instance, if A is properly elliptic (hence $m = 2\ell$ is even), $\kappa_1 \geq 0$ (hence $\kappa_2 \geq 0$), $\partial\Omega \in C^{m+\kappa_2}$ and $\partial^j u / \partial\nu^j \in W^{m+\kappa_2-j-1/p_2,p_2}(\partial\Omega)$ for $0 \leq j \leq \ell - 1$, classical elliptic regularity yields $u \in W^{m+\kappa_2,p_2}(\Omega_{R_0})$ (note that $u \in W_{\text{loc}}^{m+\kappa_2,p_2}(\Omega)$ since $Au \in D^{\kappa_2,p_2}(\Omega) \subset W_{\text{loc}}^{\kappa_2,p_2}(\Omega)$, whence $\partial^j u / \partial\nu^j \in W^{m+\kappa_2-j-1/p_2,p_2}(\partial\Omega_{R_0})$ for $0 \leq j \leq \ell - 1$). This remains true under much more general boundary conditions on $\partial\Omega$ under suitable smoothness requirements; see Corollary 2.1 in [41] and its “obvious” generalizations when $\kappa > 0$ in that paper. If $\kappa_1 < 0$, there may or may not be an elliptic regularity result to answer the question.

We complete this section with an example showing how solutions of boundary value problems on Ω can be found in the space $M_0^{0,1}(\Omega)$. As pointed out in the Introduction, the functions of $M_0^{0,1}$ vanish at infinity in a generalized (averaged) sense. On the exterior domain Ω , this property is obviously shared by the functions of $M_0^{0,1}(\Omega)$. In spite of having no direct connection with regularity, this short digression is included since it involves the $M^{s,q}$ scale introduced earlier, which has not been used elsewhere to discuss the asymptotic behavior of solutions of PDEs. In the next theorem, $W_{\text{loc}}^{1,q}(\overline{\Omega})$ refers to the space of distributions $u \in \mathcal{D}'(\Omega)$ such that $\varphi u \in W^{1,q}(\Omega)$ for every $\varphi \in C_0^\infty(\overline{\Omega})$.

Theorem 5.6. *Suppose that $N > 2$ and that $\partial\Omega \in C^1$. If $|x|^{(N+2)-2N/q} f \in L^q(\Omega)$ and $g \in W^{1-1/q,q}(\partial\Omega)$ for some $1 < q < \infty$, the Dirichlet boundary value problem*

$$(5.2) \quad \begin{cases} \Delta u = f \text{ in } \Omega, \\ u = g \text{ on } \partial\Omega, \end{cases}$$

has a solution $u \in W_{\text{loc}}^{1,q}(\overline{\Omega}) \cap M_0^{0,1}(\Omega)$. If $\partial\Omega \in C^{0,1}$, this remains true when $3/2 \leq q \leq 3$ (and, more generally, when $3(\varepsilon + 2)^{-1} < q < 3(1 - \varepsilon)^{-1}$ for some $0 < \varepsilon \leq 1$ depending only upon Ω).

Proof. We reformulate the problem (5.2) with the help of the Kelvin transform method ([5], Vol. 1 of [10]). Denote by Ω^K the bounded open subset of \mathbb{R}^N obtained by the inversion $x \mapsto y := |x|^{-2}x$ of $\Omega \cup \{\infty\}$. The boundary $\partial\Omega^K$ is the inverse of $\partial\Omega$, so that $\partial\Omega^K \in C^1$. If h is a function defined on a subset of $\mathbb{R}^N \setminus \{0\}$, set

$$h^K(y) := |y|^{2-N} h(|y|^{-2}y),$$

(Kelvin transform of f). Note that $(h^K)^K = h$.

Therefore, u is a solution of (5.2) if and only if u^K solves $\Delta u^K = |y|^{-4} f^K$ on $\Omega^K \setminus \{0\}$ and $u^K = g^K$ on $\partial\Omega^K$. In particular, if $|y|^{-4} f^K$ can be extended as a distribution on Ω^K , solutions $u := (u^K)^K$ of (5.2) can be found by solving the Dirichlet problem

$$(5.3) \quad \begin{cases} \Delta u^K = |y|^{-4} f^K \text{ in } \Omega^K, \\ u^K = g^K \text{ on } \partial\Omega^K. \end{cases}$$

The assumption $|x|^{(N+2)-2N/q} f \in L^q(\Omega)$ is equivalent to $|y|^{-4} f^K \in L^q(\Omega^K)$. Since $L^q(\Omega^K) \subset W^{-1,q}(\Omega^K)$ and since $\partial\Omega^K \in C^1$ and $g^K \in W^{1-1/q,q}(\partial\Omega^K)$, there is a unique solution $u^K \in W^{1,q}(\Omega^K)$ of (5.3). By standard trace theorems, this follows by reduction to the case $g^K = 0$. When $g^K = 0$, see Theorem 1.2, p. 45 in Simader and Sohr [44], the remarks on p. 157 of Morrey [34] (when $q \geq 2$) or Dautray and Lions, p. 409 of [10] (with proof in [28]), when $\partial\Omega^K \in C^\infty$.

By Theorem 1.1 in Jerison and Kenig [24], the case when $\partial\Omega^K \in C^{0,1}$ (i.e., $\partial\Omega \in C^{0,1}$) introduces the necessary restrictions $3(\varepsilon + 2)^{-1} < q < 3(1 - \varepsilon)^{-1}$ for some $0 < \varepsilon \leq 1$ depending only upon Ω^K (i.e., upon Ω). This range includes $3/2 \leq q \leq 3$.

Clearly, $u^K \in W^{1,q}(\Omega^K)$ implies $u \in W_{\text{loc}}^{1,q}(\overline{\Omega})$, but this alone does not give any information about the behavior of u at infinity. Choose $R_0 > 0$ such that $\mathbb{R}^N \setminus \Omega \subset B_{R_0}$. Then,

$$I_0 := \int_{\Omega_{R_0}} |u| < \infty$$

since $u \in W^{1,q}(\Omega_{R_0})$ and, if $R > R_0$ (observe that $-|y|^{-2N}$ is the Jacobian of $|y|^{-2}y$),

$$\int_{\Omega_R} |u| = I_0 + \int_{B_R \setminus B_{R_0}} |u| = I_0 + \int_{B_{R_0^{-1}} \setminus B_{R^{-1}}} |y|^{-(N+2)} |u^K(y)| dy.$$

Now, by interior elliptic regularity, $u^K \in W_{\text{loc}}^{2,q}(\Omega^K) \subset W^{2,q}(B_{R_0^{-1}})$ and so, by Hölder’s inequality,

$$\int_{B_{R_0^{-1}} \setminus B_{R^{-1}}} |y|^{-(N+2)} |u^K(y)| \, dy \leq C_r R^{N+2-N/r'} \|u^K\|_{r, B_{R_0^{-1}}}$$

if $W^{2,q}(B_{R_0^{-1}}) \hookrightarrow L^r(B_{R_0^{-1}})$, where $C_r > 0$ is independent of R . Thus, $u \in M^{2-N/r',1}(\Omega)$ for every such r . By the Sobolev embedding theorem, we can choose $r = Nq/(N - 2q)$ if $q < N/2$ or r arbitrarily large if $q \geq N/2$. In summary, $u \in M^{s,1}(\Omega)$ with $s = -N/q'$ if $q < N/2$ or with $s > 2 - N$ if $q \geq N/2$. Since $N > 2$ and $q > 1$, it follows that $u \in M^{s,1}(\Omega)$ for some $s < 0$, whence $u \in M_0^{0,1}(\Omega)$. \square

If $q > 2N/(N + 2)$, it is a bit tedious but not difficult to check that the solution u of Theorem 5.6 is even in $L^p(\Omega)$ for some $1 < p < \infty$ (hence in $M_0^{0,1}(\Omega)$), but $u \in M_0^{0,1}(\Omega)$ does not follow from membership to any $L^p(\Omega)$ if $1 < q \leq 2N/(N + 2)$. Also, $|u(x)| = O(|x|^{2-N}) = o(1)$ for large $|x|$ if u^K is continuous at the origin. This requires $|x|^{(N+2)-2N/q} f \in L^q(\Omega)$ with $q > N/2$ (so that $u^K \in W_{\text{loc}}^{2,q}(\Omega^K)$), which incidentally may not be compatible with $q < 3(1 - \varepsilon)^{-1}$ when $N \geq 7$ and $\partial\Omega \in \mathcal{C}^{0,1}$. At any rate, this is stronger³ than $|x|^{N-2} f \in L^{N/2}(\Omega)$. By comparison, Theorem 5.6 shows that when $\partial\Omega \in \mathcal{C}^1$, solutions vanishing at infinity still exist under the (much) more general condition $|x|^{(N+2)-2N/q} f \in L^q(\Omega)$ for some $q > 1$, only slightly stronger than $|x|^{2-N} f \in L^1(\Omega)$. For example, this amounts to $\alpha < -2$ versus $\alpha < -N$ if $f(x) = |x|^\alpha$ for large $|x|$.

The method of Theorem 5.6 can readily be used with other boundary conditions and other operators. Neither the homogeneity nor the constancy of the coefficients is important, as long as the problem on Ω^K fits within the elliptic theory.

6. Systems

In what follows, $n \in \mathbb{N}$, $\mathbf{m} := (m_1, \dots, m_n) \in (\mathbb{N}_0)^n$ and $\boldsymbol{\kappa} := (\kappa_1, \dots, \kappa_n) \in \mathbb{Z}^n$ are given and $\mathbf{A} := (A_{jk})_{1 \leq j, k \leq n}$ is a matrix differential operator where

$$A_{jk} := i^{m_{jk}} \sum_{|\alpha|_1 = m_{jk}} a_{jk\alpha} \partial^\alpha,$$

is homogeneous of order $m_{jk} := m_k + \kappa_k - \kappa_j$, with the understanding that $A_{jk} = 0$ if $m_{jk} < 0$. With these assumptions, the n -tuples $\mathbf{m} + \boldsymbol{\kappa}$ and $-\boldsymbol{\kappa}$ are a system of DN numbers for the operator \mathbf{A} (Douglis and Nirenberg [12], Wloka *et al.* [50]).

Let $\mathbf{A}(\xi)$ denote the $n \times n$ matrix with entries

$$A_{jk}(\xi) := \sum_{|\alpha|_1 = m_{jk}} a_{jk\alpha} \xi^\alpha.$$

³ Since the condition $|x|^{(N+2)-2N/q} f \in L^q(\Omega)$ is equivalent to $|y|^{-4} f^K \in L^q(\Omega^K)$ and Ω^K is bounded, it becomes more restrictive as q is increased.

Since $A_{jk}(\xi)$ is homogeneous of degree m_{jk} , it follows that $\det(\mathbf{A}(\xi))$ is homogeneous of degree

$$M := \sum_{j=1}^n m_j.$$

We shall assume that \mathbf{A} is DN elliptic. This means that

$$\det(\mathbf{A}(\xi)) \neq 0 \quad \text{for every } \xi \in \mathbb{R}^N \setminus \{0\}.$$

The above assumptions are unaffected by changing κ into $\kappa + \iota \mathbf{1}$, where $\iota \in \mathbb{Z}$ and $\mathbf{1} := (1, \dots, 1) \in \mathbb{N}^n$.

Homogeneous Petrovsky-elliptic systems ($m_1 = \dots = m_n$ and $\kappa_1 = \dots = \kappa_n$), such as the linear elasticity system and diagonal systems of homogeneous elliptic operators (arbitrary m_j and κ_j) are the simplest examples satisfying the above conditions. The Stokes system, with $n = N + 1$ and $m_1 = \dots = m_N = 2, \kappa_1 = \dots = \kappa_N = \kappa \in \mathbb{Z}$ and $m_{N+1} = 0, \kappa_{N+1} = \kappa + 1$ is a less obvious example.

Since the space of formal scalar differential operators with constant coefficients is a commutative ring, $\det \mathbf{A}$ is defined as a scalar differential operator with constant coefficients. This remark was first used long ago by Malgrange [30] to prove the existence of a fundamental solution for systems with constant coefficients. We shall use it in a technically different way, but in a similar spirit, to generalize Theorem 4.4. In practice, $\det \mathbf{A}$ is obtained by replacing ξ^α with $i^{|\alpha|_1} \partial^\alpha$ in $\det(\mathbf{A}(\xi))$, so that it is homogeneous of order M and elliptic.

For simplicity of notation, we set

$$\begin{aligned} D^{\mathbf{m}+\kappa,p} &:= \prod_{j=1}^n D^{m_j+\kappa_j,p}, & \mathcal{P}_{\mathbf{m}+\kappa-1} &:= \prod_{j=1}^n \mathcal{P}_{m_j+\kappa_j-1} \\ D^{\kappa,p} &:= \prod_{j=1}^n D^{\kappa_j,p}, & \mathcal{P}_{\kappa-1} &:= \prod_{j=1}^n \mathcal{P}_{\kappa_j-1}. \end{aligned}$$

Thus, if $\mathbf{u} = (u_j)_{1 \leq j \leq n} \in D^{\mathbf{m}+\kappa,p}$ ($\mathbf{f} = (f_j)_{1 \leq j \leq n} \in D^{\kappa,p}$), the equivalence class of \mathbf{u} in $D^{\mathbf{m}+\kappa,p}/\mathcal{P}_{\mathbf{m}+\kappa-1}$ (of \mathbf{f} in $D^{\kappa,p}/\mathcal{P}_{\kappa-1}$) is $[\mathbf{u}]_{\mathbf{m}+\kappa-1} = ([u]_{m_j+\kappa_j-1})_{1 \leq j \leq n}$ ($[\mathbf{f}]_{\kappa-1} = ([f_j]_{\kappa_j-1})_{1 \leq j \leq n}$).

Theorem 6.1. *If $1 < p < \infty$, the operator \mathbf{A} is a linear isomorphism from $D^{\mathbf{m}+\kappa,p}/\mathcal{P}_{\mathbf{m}+\kappa-1}$ onto $D^{\kappa,p}/\mathcal{P}_{\kappa-1}$ and a homomorphism of $D^{\mathbf{m}+\kappa,p}$ onto $D^{\kappa,p}$.*

Proof. A routine verification shows that \mathbf{A} maps $D^{\mathbf{m}+\kappa,p}$ into $D^{\kappa,p}$ and $\mathcal{P}_{\mathbf{m}+\kappa-1}$ into $\mathcal{P}_{\kappa-1}$, so that \mathbf{A} is well defined from $D^{\mathbf{m}+\kappa,p}/\mathcal{P}_{\mathbf{m}+\kappa-1}$ to $D^{\kappa,p}/\mathcal{P}_{\kappa-1}$. Furthermore, in that setting, \mathbf{A} is one-to-one, for if $\mathbf{u} \in D^{\mathbf{m}+\kappa,p}$ and $\mathbf{A}\mathbf{u} \in \mathcal{P}_{\kappa-1}$, the usual Fourier transform argument shows that $\text{Supp } \hat{\mathbf{u}} = \{0\}$. Hence, the components $u_j \in D^{m_j+\kappa_j,p}$ of \mathbf{u} are polynomials and so $u_j \in \mathcal{P}_{m_j+\kappa_j-1}$ by Lemma 4.2, which in turn yields $[\mathbf{u}]_{\mathbf{m}+\kappa-1} = [\mathbf{0}]_{\mathbf{m}+\kappa-1}$.

We now prove that \mathbf{A} is onto $D^{\kappa,p}/\mathcal{P}_{\kappa-1}$ by exhibiting a right inverse. For every $1 \leq j, k \leq n$, denote by C_{jk} the (j, k) cofactor of \mathbf{A} . This is the scalar differential operator obtained by replacing ξ^α with $i^{|\alpha|_1} \partial^\alpha$ in the corresponding cofactor

$C_{jk}(\xi)$ of $\mathbf{A}(\xi)$. As a result, C_{jk} is homogeneous of order $M - m_k - \kappa_k + \kappa_j$. In particular, $C_{k\ell}$ (homogeneous of order $M - m_\ell - \kappa_\ell + \kappa_k$) maps $D^{M+\kappa_k,p}$ into $D^{m_\ell+\kappa_\ell,p}$ and $\mathcal{P}_{M+\kappa_k-1}$ into $\mathcal{P}_{m_\ell+\kappa_\ell-1}$ and so it is a well defined operator from $D^{M+\kappa_k,p}/\mathcal{P}_{M+\kappa_k-1}$ to $D^{m_\ell+\kappa_\ell,p}/\mathcal{P}_{m_\ell+\kappa_\ell-1}$. On the other hand, by the very definition of C_{jk} ,

$$(6.1) \quad \sum_{\ell=1}^n A_{j\ell} C_{k\ell} = \delta_{jk} \det \mathbf{A} \quad (\text{Kronecker delta}).$$

It follows from Theorem 4.4 that $\det \mathbf{A}$ is an isomorphism of $D^{M+\kappa_k,p}/\mathcal{P}_{M+\kappa_k-1}$ onto $D^{\kappa_k,p}/\mathcal{P}_{\kappa_k-1}$ for $1 \leq k \leq n$. Denote by B_k the inverse isomorphism and let $[f]_{\kappa_k-1} \in D^{\kappa_k,p}/\mathcal{P}_{\kappa_k-1}$, so that $B_k[f]_{\kappa_k-1} \in D^{M+\kappa_k,p}/\mathcal{P}_{M+\kappa_k-1}$. From the above, $C_{k\ell} B_k[f]_{\kappa_k-1} \in D^{m_\ell+\kappa_\ell,p}/\mathcal{P}_{m_\ell+\kappa_\ell-1}$ and so $A_{j\ell} C_{k\ell} B_k[f]_{\kappa_k-1} \in D^{\kappa_j,p}/\mathcal{P}_{\kappa_j-1}$ since $A_{j\ell}$ (homogeneous of order $m_\ell + \kappa_\ell - \kappa_j$) maps $D^{m_\ell+\kappa_\ell,p}$ into $D^{\kappa_j,p}$ and $\mathcal{P}_{m_\ell+\kappa_\ell-1}$ into \mathcal{P}_{κ_j-1} . Consequently, by (6.1),

$$\sum_{\ell=1}^n A_{j\ell} C_{k\ell} B_k[f]_{\kappa_k-1} = [0]_{\kappa_j-1}$$

if $j \neq k$, and

$$\sum_{\ell=1}^n A_{k\ell} C_{k\ell} B_k[f]_{\kappa_k-1} = (\det \mathbf{A}) B_k[f]_{\kappa_k-1} = [f]_{\kappa_k-1}.$$

Therefore, the operator $\mathbf{B} := (B_{jk})_{1 \leq j,k \leq n}$ with $B_{jk} := C_{kj} B_k$ acts from $D^{\kappa,p}/\mathcal{P}_{\mathbf{k}-1}$ to $D^{\mathbf{m}+\kappa,p}/\mathcal{P}_{\mathbf{m}+\kappa-1}$ and is a right inverse of \mathbf{A} .

To show that \mathbf{A} maps $D^{\mathbf{m}+\kappa,p}$ onto $D^{\kappa,p}$, recall that, by Theorem 4.4, $\det \mathbf{A}$ maps $D^{M+\kappa_k,p}$ onto $D^{\kappa_k,p}$ for $1 \leq k \leq n$. Given $\mathbf{f} = (f_k)_{1 \leq k \leq n} \in D^{\kappa,p}$, choose $v_k \in D^{M+\kappa_k,p}$ such that $(\det \mathbf{A})v_k = f_k$ and, for $1 \leq \ell \leq n$, set $u_\ell := \sum_{k=1}^n C_{k\ell} v_k$. Then, $u_\ell \in D^{m_\ell+\kappa_\ell,p}$ and, with $\mathbf{u} := (u_\ell)_{1 \leq \ell \leq n} \in D^{\mathbf{m}+\kappa,p}$, we have

$$(\mathbf{A}\mathbf{u})_j = \sum_{\ell=1}^n A_{j\ell} u_\ell = \sum_{k=1}^n \sum_{\ell=1}^n A_{j\ell} C_{k\ell} v_k = \sum_{k=1}^n \delta_{jk} (\det \mathbf{A}) v_k = f_j.$$

Thus, $\mathbf{A}\mathbf{u} = \mathbf{f}$ and the proof is complete. □

When \mathbf{A} is the Stokes system, partial results related to Theorem 6.1 have been proved, with the help of fundamental solutions, under more restrictive assumptions about \mathbf{f} ([13], [15]). In that regard, we point out that there are technical difficulties in proving Theorem 6.1 in full generality based on the construction of a suitable fundamental solution, as was done in Lemma 4.1 in the scalar case. Note that if \mathbf{A} is the Stokes system, then $\det \mathbf{A} = (-1)^N \Delta^N$ has order $2N$.

Upon using Theorem 6.1 instead of Theorem 4.4 in the proof, it is now obvious how Theorem 4.5 can be generalized. It suffices to introduce a convenient notation.

If $\mathbf{a} = (a_j)_{1 \leq j \leq n}$ and $\mathbf{b} = (b_j)_{1 \leq j \leq n}$, the inequality $\mathbf{a} \geq \mathbf{b}$ ($\mathbf{a} > \mathbf{b}$) means $a_j \geq b_j$ ($a_j > b_j$) for $1 \leq j \leq n$. Also, if $\mathbf{s} = (s_j)_{1 \leq j \leq n}$, we set

$$M_0^{\mathbf{s},p} := \prod_{j=1}^n M_0^{s_j,p}.$$

With this, we can state:

Theorem 6.2. *If $\mathbf{u} \in (\mathcal{D}')^n$ and $\mathbf{A}\mathbf{u} \in D^{\boldsymbol{\kappa},p}$ with $\boldsymbol{\kappa} \geq -\mathbf{m}$ and $1 < p < \infty$, the following properties are equivalent:*

- (i) $\mathbf{u} \in D^{\mathbf{m}+\boldsymbol{\kappa},p}$.
- (ii) $\mathbf{u} \in M_0^{\mathbf{s},p}$ for every $\mathbf{s} > \mathbf{m} + \boldsymbol{\kappa} - (N/p)\mathbf{1}$ if $p > N$ and every $\mathbf{s} > \mathbf{m} + \boldsymbol{\kappa} - \mathbf{1}$ if $p \leq N$.
- (iii) $\mathbf{u} \in M_0^{\mathbf{m}+\boldsymbol{\kappa},1}$.

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