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The approximation property for spaces of Lipschitz functions with the bounded weak* topology

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Abstract. Let X be a pointed metric space and let $\mathrm{Lip}_0(X)$ be the space of all scalar-valued Lipschitz functions on X which vanish at the base point. We prove that $\mathrm{Lip}_0(X)$ with the bounded weak* topology τ_{bw^*} has the approximation property if and only if the Lipschitz-free Banach space $\mathcal{F}(X)$ has the approximation property if and only if, for each Banach space F, each Lipschitz operator from X into F can be approximated by Lipschitz finite-rank operators within the unique locally convex topology $\gamma\tau_\gamma$ on $\mathrm{Lip}_0(X,F)$ such that the Lipschitz transpose mapping $f \mapsto f^t$ is a topological isomorphism from $(\mathrm{Lip}_0(X,F),\gamma\tau_\gamma)$ to $(\mathrm{Lip}_0(X),\tau_{bw^*})\epsilon F$.

Introduction

Let (X,d) be a pointed metric space with a base point which we will always denote by 0 and let F be a Banach space. The space $\mathrm{Lip}_0(X,F)$ is the Banach space of all Lipschitz mappings f from X to F that vanish at 0, with the Lipschitz norm defined by

$$\mathrm{Lip}(f) = \sup\left\{ \frac{\|f(x)-f(y)\|}{d(x,y)} : x,y \in X,\ x \neq y \right\}.$$

The elements of $\mathrm{Lip}_0(X,F)$ are frequently called Lipschitz operators. If $\mathbb{K}$ is the field of real or complex numbers, $\mathrm{Lip}_0(X,\mathbb{K})$ is denoted by $\mathrm{Lip}_0(X)$. The closed linear subspace of the dual of $\mathrm{Lip}_0(X)$ spanned by the functionals $\delta(x)$ on $\mathrm{Lip}_0(X)$ with $x \in X$, given by $\delta(x)(f) = f(x)$, is a predual of $\mathrm{Lip}_0(X)$. This predual is called the Lipschitz-free space over X and denoted by $\mathcal{F}(X)$ in [11]. We refer the reader to Weaver's book [26] for the basic theory of $\mathrm{Lip}_0(X)$ and its predual $\mathcal{F}(X)$, which is called the Arens–Eells space of X and denoted by Æ(X) there.

The study of the approximation property is a topic of interest for many researchers. Let us recall that a Banach space E has the approximation property

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(in short, (AP)) if for each compact set $K \subset E$ and each $\varepsilon > 0$, there exists a bounded finite-rank linear operator $T\colon E \to E$ such that $\sup_{x\in K} \|T(x) - x\| < \varepsilon$. If $\|T\| \leq \lambda$ for some $\lambda \geq 1$, it is said that E has the λ-bounded approximation property (in short, λ-(BAP)).

Several authors have tackled the (AP) for $\mathrm{Lip}_0(X)$. Johnson [17] observed that if X is the closed unit ball of Enflo's space [10], then $\mathrm{Lip}_0(X)$ fails the (AP). Godefroy and Ozawa [12] showed that there exists a compact pointed metric space X such that $\mathcal{F}(X)$ fails the (AP) and hence so does $\mathrm{Lip}_0(X)$. For positive results, $\mathrm{Lip}_0([0,1])$ is isomorphic to $L^\infty[0,1]$ (see page 224 in [14]) and thus $\mathrm{Lip}_0([0,1])$ has the (AP). If (X,d) is a doubling compact pointed metric space, in particular a compact subset of a finite dimensional Banach space, and $X^{(\alpha)}$ with $\alpha \in (0,1)$ denotes the metric space (X,d^α), then the space $\mathrm{Lip}_0(X^{(\alpha)})$ is isomorphic to ℓ_∞ by Theorem 6.5 in [18], and therefore $\mathrm{Lip}_0(X^{(\alpha)})$ has the (AP). In [16] (see also [15]), Johnson proved that $\mathrm{Lip}_0(X)$ has the (AP) if and only if, for each Banach space F, every Lipschitz compact operator from X to F can be approximated in the Lipschitz norm by Lipschitz finite-rank operators. Godefroy and Kalton [11] proved that a Banach space E has the λ-(BAP) if and only if $\mathcal{F}(E)$ has the same property.

The most recent research on the (AP) has been directed toward $\mathcal{F}(X)$ rather than to $\mathrm{Lip}_0(X)$ (see [21] and the references therein). The results in those works provide limited information about the (AP) for $\mathrm{Lip}_0(X)$ since the (AP) of a Banach space follows from the (AP) of its dual space but the converse does not always hold.

In this paper we follow an entirely different approach to the study of the (AP) for the space $\mathrm{Lip}_0(X)$ equipped with the bounded weak* topology. In the seminal paper [2], Aron and Schottenloher initiated the investigation about the (AP) for spaces of holomorphic mappings on Banach spaces. Mujica [20] extended this study to the preduals of such spaces. Their techniques, based on the tensor product, the ϵ-product and the linearization of holomorphic mappings, work just as well for spaces of Lipschitz mappings. Our analysis is similar to that carried out by Aron, Maestre and Rueda [1], Caliskan [5] and Dineen and Mujica [8] on the approximation property for spaces of holomorphic functions on infinite dimensional spaces.

We now describe the contents of this paper. In Section 1, we briefly recall some topologies on spaces $\mathrm{Lip}_0(X,F)$, the approximation property and the ϵ-product, and the linearization of Lipschitz mappings.

In Section 2, we address the study of the topology of bounded compact convergence τ_γ on $\mathrm{Lip}_0(X)$. It is the largest topology on $\mathrm{Lip}_0(X)$ which coincides with the compact-open topology τ_0 on each norm bounded subset of $\mathrm{Lip}_0(X)$. Its study is motivated by the fact that $\mathcal{F}(X)$ agrees with the space of all linear functionals on $\mathrm{Lip}_0(X)$ whose restrictions to the closed unit ball of $\mathrm{Lip}_0(X)$ are continuous with respect to the topology τ_0. In the terminology of Cooper [6], [7], we prove that τ_γ is the mixed topology $\gamma[\mathrm{Lip},\tau_0]$, and $(\mathrm{Lip}_0(X),\tau_\gamma)$ is a Saks space. Furthermore, it is shown that τ_γ agrees with the bounded weak* topology τ_{bw^*}. This description of τ_{bw^*} as a mixed topology provides a useful tool for obtaining their properties easily. This approach was used by Prieto [22] to study strict and mixed topologies on spaces of continuous and holomorphic functions on a Banach space.

In Section 3, we give a pair of descriptions of τ_γ by means of seminorms. Assuming X is compact, we first identify τ_γ with the classical strict topology β introduced by Buck [4]. A second seminorm description for τ_γ motivates the introduction of a very useful locally convex topology $\gamma\tau_\gamma$ on $\mathrm{Lip}_0(X,F)$.

Section 4 deals with the (AP) for $(\mathrm{Lip}_0(X),\tau_\gamma)$. For every $f\in\mathrm{Lip}_0(X,F)$, the Lipschitz transpose $f^t\colon F'\to\mathrm{Lip}_0(X)$, $\psi\mapsto\psi\circ f$ for $\psi\in F'$, is a bounded linear mapping. The Lipschitz transpose mapping $f\mapsto f^t$ identifies the space $(\mathrm{Lip}_0(X,F),\gamma\tau_\gamma)$ with the ϵ-product of $(\mathrm{Lip}_0(X),\tau_\gamma)$ and F, and this permits us to prove that $(\mathrm{Lip}_0(X),\tau_\gamma)$ has the (AP) if and only if $\mathcal{F}(X)$ has the (AP) if and only if every Lipschitz operator from X into F can be approximated by Lipschitz finite-rank operators within the topology $\gamma\tau_\gamma$ for all Banach spaces F.

In Section 5, we establish a representation of the dual space of $(\mathrm{Lip}_0(X,F),\gamma\tau_\gamma)$.

1. Preliminaries

Topologies on spaces of Lipschitz functions

Let X be a pointed metric space and let F be a Banach space. The compact-open topology or topology of compact convergence on $\mathrm{Lip}_0(X,F)$ is the locally convex topology generated by the seminorms of the form

$$|f|_K=\sup_{x\in K}\|f(x)\|,\quad f\in\mathrm{Lip}_0(X,F),$$

where K varies over the family of all compact subsets of X. We denote by τ_0 the compact-open topology on $\mathrm{Lip}_0(X,F)$, or on any vector subspace of $\mathrm{Lip}_0(X,F)$.

The topology of pointwise convergence on $\mathrm{Lip}_0(X,F)$ is the locally convex topology τ_p generated by the seminorms of the form

$$|f|_G=\sup_{x\in G}\|f(x)\|,\quad f\in\mathrm{Lip}_0(X,F),$$

where G ranges over the family of all finite subsets of X.

Finally, we denote by τ_{Lip} the topology on $\mathrm{Lip}_0(X,F)$ generated by the Lipschitz norm Lip. It is clear that $\tau_p\subset\tau_0$, and the inclusion $\tau_0\subset\tau_{\mathrm{Lip}}$ follows easily since $|f|_K\le\mathrm{Lip}(f)\,\mathrm{diam}(K\cup\{0\})$ for all $f\in\mathrm{Lip}_0(X,F)$ and each compact set $K\subset X$.

Approximation property and epsilon-product

Let E and F be locally convex Hausdorff spaces. Let $\mathcal{L}(E;F)$ denote the vector space of all continuous linear mappings from E into F, let $\mathcal{L}_b(E;F)$ denote the vector space $\mathcal{L}(E;F)$ with the topology of uniform convergence on the bounded subsets of E and let $\mathcal{L}_c(E;F)$ denote the vector space $\mathcal{L}(E;F)$ with the topology of uniform convergence on the convex balanced compact subsets of E. That last topology coincides with the compact-open topology if the closed convex hull of each compact subset of E is compact (for example, if E is quasi-complete).

When $F=\mathbb{K}$, we write E' instead of $\mathcal{L}(E;\mathbb{K})$, E'_b in place of $\mathcal{L}_b(E;\mathbb{K})$, and E'_c instead of $\mathcal{L}_c(E;\mathbb{K})$. Unless stated otherwise, if E and F are normed spaces, $\mathcal{L}(E;F)$ is endowed with its natural norm topology. Let $E\otimes F$ denote the tensor product of E and F, then $E'\otimes F$ can be identified with the subspace of all finite-rank mappings in $\mathcal{L}(E;F)$.

A locally convex space E is said to have the approximation property (in short, (AP)) if the identity mapping on E lies in the closure of $E'\otimes E$ in $\mathcal{L}_c(E;E)$. This is Schwartz's definition of the (AP) in [24], which is slightly different from Grothendieck's definition in [13], though both definitions coincide for quasi complete locally convex spaces.

The ϵ-product of E and F, denoted by $E\epsilon F$ and introduced by Schwartz ([24], [25]), is the space $\mathcal{L}_\epsilon(F'_c;E)$, that is, the vector space $\mathcal{L}(F'_c;E)$, with the topology of uniform convergence on the equicontinuous subsets of F'. Notice that if F is a normed space, then equicontinuous sets and norm bounded sets in F' coincide. The topology on $\mathcal{L}_\epsilon(F'_c;E)$ is generated by the seminorms

$$\alpha\epsilon\beta(T)=\sup\left\{|\langle T(\mu),\nu\rangle| : \mu\in F',\ |\mu|\le\alpha,\ \nu\in E',\ |\nu|\le\beta\right\},\quad T\in\mathcal{L}_\epsilon(F'_c;E),$$

where α ranges over the continuous seminorms on F and β over the continuous seminorms on E.

We will use the subsequent results which follow from results of Grothendieck [13], Schwartz [24] and Bierstedt and Meise [3].

Proposition 1.1 ([24]). *Let E and F be locally convex spaces. Then the transpose mapping $T\mapsto T^t$ from $E\epsilon F$ to $F\epsilon E$ is a topological isomorphism.*

Theorem 1.2 ([3], [13], [24]). *A locally convex space E has the* (AP) *if and only if $E\otimes F$ is dense in $E\epsilon F$ for every Banach space F.*

Proposition 1.3 ([3], [13], [24]). *A locally convex space E has the* (AP) *if E'_c has the* (AP).

Detailed proofs of the preceding results can be found in the paper [8] by Dineen and Mujica.

Linearization of Lipschitz mappings

The study of the preduals of $\mathrm{Lip}_0(X)$ was approached by Weaver [26] by using a procedure to linearize Lipschitz mappings. A similar process of linearization was presented by Mujica for bounded holomorphic mappings on Banach spaces in [20].

Theorem 1.4 ([26]). *Let X be a pointed metric space.*

(i) *The Dirac mapping $\delta=\delta_X\colon x\mapsto\delta(x)$ is an isometric embedding from X into $\mathcal{F}(X)$.*

(ii) *For each Banach space F and each $f\in\mathrm{Lip}_0(X,F)$, there is a unique operator $T_f\in\mathcal{L}(\mathcal{F}(X);F)$ such that $T_f\circ\delta=f$. Furthermore, the evaluation mapping $f\mapsto T_f$ from $\mathrm{Lip}_0(X,F)$ to $\mathcal{L}(\mathcal{F}(X);F)$, defined by $T_f(\varphi)=\varphi(f)$, is an isometric isomorphism.*

(iii) *$\mathcal{F}(X)$ coincides with the space of all linear functionals φ on $\mathrm{Lip}_0(X)$ such that the restriction of φ to the closed unit ball $B_{\mathrm{Lip}_0(X)}$ of $\mathrm{Lip}_0(X)$ is continuous when $B_{\mathrm{Lip}_0(X)}$ is equipped with the compact-open topology τ_0.*

The statements (i)–(ii) follow from Theorem 2.2.4 in [26]. The affirmation (iii) was stated in Lemma 1.1 of [15] when $B_{\mathrm{Lip}_0(X)}$ is endowed with the topology τ_p, and let us recall that τ_p agrees with τ_0 on the equicontinuous subsets of $\mathrm{Lip}_0(X)$ and, in particular, on $B_{\mathrm{Lip}_0(X)}$.

Viewing $\mathrm{Lip}_0(X)$ as the dual of $\mathcal{F}(X)$, we can consider its weak* topology. We recall that the weak* topology on $\mathrm{Lip}_0(X)$ is the locally convex topology τ_{w^*} generated by the seminorms of the form

$$p_G(f) = \sup_{\varphi\in G} |\varphi(f)|, \quad f \in \mathrm{Lip}_0(X),$$

where G ranges over the family of all finite subsets of $\mathcal{F}(X)$. Let us recall that τ_{w^*} is the smallest topology for $\mathrm{Lip}_0(X)$ such that, for each $\varphi \in \mathcal{F}(X)$, the linear functional $f \mapsto \varphi(f)$ on $\mathrm{Lip}_0(X)$ is continuous with respect to that topology. It is easy to check that $\tau_p \subset \tau_{w^*} \subset \tau_{\mathrm{Lip}}$.

Let E be a locally convex space, $M \subset E$ and $N \subset E'$. The polar of M and the prepolar of N are denoted by

$$M^\circ = \Big\{f \in E' \colon \sup_{x\in M} |f(x)| \leq 1\Big\}, \quad N_\circ = \Big\{x \in E \colon \sup_{f\in N} |f(x)| \leq 1\Big\},$$

respectively. $\overline{\Gamma} M$ stands for the closed, convex, balanced hull of M in E.

2. Topology of bounded compact convergence for $\mathrm{Lip}_0(X)$

According Definition I.3.2 in [7], we recall that a Saks space is a triple $(E, \|\cdot\|, \tau)$, where E is a vector space, τ is a locally convex topology on E and $\|\cdot\|$ is a norm on E so that the closed unit ball B_E of $(E, \|\cdot\|)$ is τ-bounded and τ-closed.

Given a pointed metric space X, we consider on $\mathrm{Lip}_0(X)$ the topologies:

- τ_p : the topology of pointwise convergence.
- τ_0 : the topology of compact convergence.
- τ_{w^*} : the weak* topology.
- τ_{Lip} : the norm topology.

The triple $(\mathrm{Lip}_0(X), \mathrm{Lip}, \tau_0)$ is a Saks space since $B_{\mathrm{Lip}_0(X)}$ is τ_0-compact by the Ascoli theorem. Then we can form the mixed topology $\gamma[\mathrm{Lip}, \tau_0]$ on $\mathrm{Lip}_0(X)$ (see I.3.4 in [7]). Following Definition I.1.4 of [7], $\gamma[\mathrm{Lip}, \tau_0]$ is the locally convex topology on $\mathrm{Lip}_0(X)$ generated by the base of neighborhoods of zero $\{\gamma(U)\}$, where $U = \{U_n\}$ is a sequence of convex balanced τ_0-neighborhoods of zero and $\gamma(U)$ denotes

$$\bigcup_{n=1}^{\infty} \left(U_1 \cap B_{\mathrm{Lip}_0(X)} + U_2 \cap 2B_{\mathrm{Lip}_0(X)} + U_3 \cap 3B_{\mathrm{Lip}_0(X)} + \cdots + U_n \cap nB_{\mathrm{Lip}_0(X)}\right).$$

Since $\tau_p \subset \tau_{w^*} \subset \tau_0$ on $B_{\mathrm{Lip}_0(X)}$ (the second inclusion follows from Theorem 1.4 (iii)) and $B_{\mathrm{Lip}_0(X)}$ is τ_0-compact, we have $\tau_p = \tau_{w^*} = \tau_0$ on $B_{\mathrm{Lip}_0(X)}$. Then Corollary I.1.6 of [7] yields

$$\gamma[\mathrm{Lip}, \tau_p] = \gamma[\mathrm{Lip}, \tau_{w^*}] = \gamma[\mathrm{Lip}, \tau_0],$$

and we denote this topology by τ_γ. We gather next some properties of τ_γ.

Theorem 2.1. *Let X be a pointed metric space.*

(1) *τ_0 is smaller than τ_γ, and τ_γ is smaller than τ_{Lip}.*

(2) *τ_γ coincides with τ_0 on each norm bounded subset of* $\mathrm{Lip}_0(X)$.

(3) *If F is a locally convex space and $T\colon \mathrm{Lip}_0(X) \to F$ is linear, then T is τ_γ-continuous if and only if $T|_B$ is τ_0-continuous for all norm bounded subsets B of* $\mathrm{Lip}_0(X)$.

(4) *A sequence in $\mathrm{Lip}_0(X)$ is τ_γ-convergent to zero if and only if it is norm bounded and τ_0-convergent to zero.*

(5) *A subset of $\mathrm{Lip}_0(X)$ is τ_γ-bounded if and only if it is norm bounded.*

(6) *A subset of $\mathrm{Lip}_0(X)$ is τ_γ-compact if and only if it is norm bounded and τ_0-compact.*

(7) *$(\mathrm{Lip}_0(X), \tau_\gamma)$ is a complete semi-Montel space.*

(8) *τ_γ is the largest topology on $\mathrm{Lip}_0(X)$ which agrees with τ_0 on each norm bounded subset of* $\mathrm{Lip}_0(X)$.

(9) *A subset U of $\mathrm{Lip}_0(X)$ is open (closed) in $(\mathrm{Lip}_0(X), \tau_\gamma)$ if and only if the set $U \cap nB_{\mathrm{Lip}_0(X)}$ is open (closed) in $(nB_{\mathrm{Lip}_0(X)}, \tau_0)$ for each $n \in \mathbb{N}$.*

Proof. It follows from Proposition 1 in [6], which gives the main properties of mixed topologies. For (8) and (9), see Corollary I.4.2 in [7]. □

Remark 2.2. (1) All assertions of Theorem 2.1 are valid if the topology τ_0 is replaced by τ_p or τ_{w^*}.

(2) The property (2) justifies the name of topology of bounded compact convergence for τ_γ.

Let us recall that if E is a Banach space, then the bounded weak* topology on its dual E', denoted by τ_{bw^*}, is the largest topology on E' agreeing with the topology τ_{w^*} on norm bounded sets (see Definition V.5.3 in [9]). According to the Banach–Dieudonné theorem (see Lemma V.5.4 in [9]), τ_{bw^*} is just the topology of uniform convergence on sequences in E which tend in norm to zero.

Since $\tau_{w^*} = \tau_0$ on $B_{\mathrm{Lip}_0(X)}$, the assertion (8) of Theorem 2.1 gives the following.

Corollary 2.3. *Let X be a pointed metric space. On the space $\mathrm{Lip}_0(X)$, the bounded weak* topology τ_{bw^*} is the topology τ_γ.*

The next theorem follows from Theorem 2.1 and some facts on (DFC)-spaces.

Theorem 2.4. *Let X be a pointed metric space.*

(i) $(\mathrm{Lip}_0(X), \tau_\gamma)$ *is a* (DFC)*-space. More precisely,* $(\mathrm{Lip}_0(X), \tau_\gamma)'_b$ *is a Fréchet space and the evaluation mapping from* $(\mathrm{Lip}_0(X), \tau_\gamma)$ *to* $((\mathrm{Lip}_0(X), \tau_\gamma)'_b)'_c$ *is a topological isomorphism.*

(ii) $\mathcal{F}(X) = (\mathrm{Lip}_0(X), \tau_\gamma)'_b = (\mathrm{Lip}_0(X), \tau_\gamma)'_c$.

(iii) *The evaluation mapping* $f \mapsto T_f$ *from* $(\mathrm{Lip}_0(X), \tau_\gamma)$ *to* $\mathcal{F}(X)'_c$ *is a topological isomorphism.*

Proof. (i) Using Theorem 2.1, we can check that $\{nB_{\mathrm{Lip}_0(X)}\}$ is an increasing sequence of convex, balanced and τ_γ-compact subsets of $\mathrm{Lip}_0(X)$ with the property that a set $U \subset \mathrm{Lip}_0(X)$ is τ_γ-open whenever $U \cap nB_{\mathrm{Lip}_0(X)}$ is open in $(nB_{\mathrm{Lip}_0(X)}, \tau_\gamma)$ for every $n \in \mathbb{N}$. Then (i) follows by applying Theorem 4.1 of [19].

(ii) From Theorem 1.4 (iii) and Theorem 2.1 (3), we deduce that $\mathcal{F}(X) = (\mathrm{Lip}_0(X), \tau_\gamma)'$ algebraically. Since $\mathcal{F}(X)$ is a linear subspace of $(\mathrm{Lip}_0(X), \tau_{\mathrm{Lip}})'$ by its very definition, and both spaces $(\mathrm{Lip}_0(X), \tau_{\mathrm{Lip}})$ and $(\mathrm{Lip}_0(X), \tau_\gamma)$ have the same bounded sets by Theorem 2.1 (5), we infer that $\mathcal{F}(X) = (\mathrm{Lip}_0(X), \tau_\gamma)'_b$. The identification $(\mathrm{Lip}_0(X), \tau_\gamma)'_b = (\mathrm{Lip}_0(X), \tau_\gamma)'_c$ follows from Theorem 2.1 (7).

(iii) follows from (i) and (ii). □

3. Seminorm descriptions of τ_γ on $\mathrm{Lip}_0(X)$

Our aim in this section is to give a pair of descriptions for τ_γ by means of seminorms. For our purposes, we will need the next lemma. Given a pointed metric space X, we denote

$$\widetilde{X} = \left\{(x, y) \in X^2 \colon x \neq y\right\}.$$

For $f \in \mathrm{Lip}_0(X)$ and $A \subset \widetilde{X}$, define

$$\mathrm{Lip}_A(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} \colon (x, y) \in A\right\}.$$

Notice that if $F \subset \widetilde{X}$ is finite, then $\mathrm{Lip}_F(f) = p_G(f)$ (see Section 1), where G is the finite subset of $\mathcal{F}(X)$ given by

$$G = \left\{\frac{\delta(x) - \delta(y)}{d(x, y)} \colon (x, y) \in F\right\},$$

and hence, for $\varepsilon > 0$, the set $\{f \in \mathrm{Lip}_0(X) \colon \mathrm{Lip}_F(f) \leq \varepsilon\}$ is a neighborhood of zero in $(\mathrm{Lip}_0(X), \tau_{w^*})$.

Lemma 3.1. *Let X be a pointed metric space. Then the sets of the form*

$$U = \bigcap_{n=1}^{\infty} \left\{f \in \mathrm{Lip}_0(X) \colon \mathrm{Lip}_{F_n}(f) \leq \lambda_n\right\},$$

where $\{F_n\}$ is a sequence of finite subsets of $\widetilde{X}$ and $\{\lambda_n\}$ is a sequence of positive numbers tending to ∞, form a base of neighborhoods of zero in $(\mathrm{Lip}_0(X), \tau_\gamma)$.

Proof. We first claim that if $\{F_k\}$ and $\{\lambda_k\}$ are sequences as above, then the set

$$\bigcap_{k=1}^{\infty}\left\{f\in \mathrm{Lip}_0(X)\colon\ \mathrm{Lip}_{F_k}(f)\leq\lambda_k\right\}$$

is a neighborhood of zero in $(\mathrm{Lip}_0(X),\tau_\gamma)$. Indeed, given $n\in\mathbb{N}$, if $m\in\mathbb{N}$ is chosen so that $\lambda_k\geq n$ for $k>m$, then

$$\bigcap_{k=1}^{\infty}\left\{f\in \mathrm{Lip}_0(X)\colon\ \mathrm{Lip}_{F_k}(f)\leq\lambda_k\right\}\cap nB_{\mathrm{Lip}_0(X)}$$
$$=\bigcap_{k=1}^{m}\left\{f\in \mathrm{Lip}_0(X)\colon\ \mathrm{Lip}_{F_k}(f)\leq\lambda_k\right\}\cap nB_{\mathrm{Lip}_0(X)}.$$

The latter is a neighborhood of zero in $(nB_{\mathrm{Lip}_0(X)},\tau_{w^*})$, and our claim follows by Theorem 2.1 (9).

We now must prove that if U is a neighborhood of zero in $(\mathrm{Lip}_0(X),\tau_\gamma)$, then there are sequences $\{F_k\}$ and $\{\lambda_k\}$ as above for which

$$\bigcap_{k=1}^{\infty}\left\{f\in \mathrm{Lip}_0(X)\colon\ \mathrm{Lip}_{F_k}(f)\leq\lambda_k\right\}\subset U.$$

Indeed, by Theorem 2.1 (9), we can take a set $U\subset\mathrm{Lip}_0(X)$ such that $U\cap nB_{\mathrm{Lip}_0(X)}$ is an open neighborhood of zero in $(nB_{\mathrm{Lip}_0(X)},\tau_{w^*})$ for every $n\in\mathbb{N}$. In particular, $U\cap B_{\mathrm{Lip}_0(X)}$ is a neighborhood of zero in $(B_{\mathrm{Lip}_0(X)},\tau_{\mathrm{Lip}})$ and then there exists $\varepsilon>0$ such that $\varepsilon B_{\mathrm{Lip}_0(X)}\subset U$. In order to prove that there exists a finite set $F_1\subset\widetilde{X}$ such that

$$\left\{f\in \mathrm{Lip}_0(X)\colon\ \mathrm{Lip}_{F_1}(f)\leq\varepsilon\right\}\cap B_{\mathrm{Lip}_0(X)}\subset U,$$

assume on the contrary that the set

$$\{f\in \mathrm{Lip}_0(X)\colon\ \mathrm{Lip}_{F}(f)\leq\varepsilon\}\cap(B_{\mathrm{Lip}_0(X)}\setminus U)$$

is nonempty for every finite set $F\subset\widetilde{X}$. These sets are closed in $(B_{\mathrm{Lip}_0(X)}\setminus U,\tau_{w^*})$ and have the finite intersection property. Since the set $B_{\mathrm{Lip}_0(X)}\setminus U$ is a closed, and therefore compact, subset of $(B_{\mathrm{Lip}_0(X)},\tau_{w^*})$, we infer that there exists some $f\in B_{\mathrm{Lip}_0(X)}\setminus U$ such that $\mathrm{Lip}_F(f)\leq\varepsilon$ for each finite set $F\subset\widetilde{X}$. This implies that $f\in\varepsilon B_{\mathrm{Lip}_0(X)}\setminus U$ which is impossible, and thus proving our assertion.

Proceeding by induction, suppose that we can find finite subsets $F_1,\dots,F_n$ of $\widetilde{X}$ such that

$$\bigcap_{k=1}^{n}\left\{f\in \mathrm{Lip}_0(X)\colon\ \mathrm{Lip}_{F_k}(f)\leq\lambda_k\right\}\cap nB_{\mathrm{Lip}_0(X)}\subset U\cap nB_{\mathrm{Lip}_0(X)},$$

where $\lambda_1=\varepsilon$ and $\lambda_k=k-1$ for $k>1$. We will prove that there exists a finite set $F_{n+1}\subset\widetilde{X}$ such that

$$\bigcap_{k=1}^{n+1}\left\{f\in \mathrm{Lip}_0(X)\colon\ \mathrm{Lip}_{F_k}(f)\leq\lambda_k\right\}\cap(n+1)B_{\mathrm{Lip}_0(X)}\subset U\cap(n+1)B_{\mathrm{Lip}_0(X)}.$$

We argue by contradiction. If no such finite set F_{n+1} exists, then the set

$$C_F := \bigcap_{k=1}^{n} \left\{f \in \mathrm{Lip}_0(X)\colon \ \mathrm{Lip}_{F_k}(f) \leq \lambda_k\right\} \cap \{f \in \mathrm{Lip}_0(X)\colon \ \mathrm{Lip}_F(f) \leq n\}$$

has nonempty intersection with the τ_{w^*}-compact set $(n+1)B_{\mathrm{Lip}_0(X)} \setminus U$ for each finite set $F \subset \widetilde{X}$. So, by the finite intersection property, there exists an element $f_0 \in \left((n+1)B_{\mathrm{Lip}_0(X)} \setminus U\right) \cap (\cap_F C_F)$. Therefore $\mathrm{Lip}_F(f_0) \leq n$ for each F and so $\mathrm{Lip}(f_0) \leq n$. Then $f_0 \in U \cap nB_{\mathrm{Lip}_0(X)} \subset U \cap (n+1)B_{\mathrm{Lip}_0(X)}$ which is a contradiction.

Then we can construct, by induction, a sequence $\{F_k\}$ of finite subsets of $\widetilde{X}$ so that

$$\bigcap_{k=1}^{n} \left\{f \in \mathrm{Lip}_0(X)\colon \ \mathrm{Lip}_{F_k}(f) \leq \lambda_k\right\} \cap nB_{\mathrm{Lip}_0(X)} \subset U$$

for every $n \in \mathbb{N}$. Since $\mathrm{Lip}_0(X) = \cup_{n=1}^{\infty} nB_{\mathrm{Lip}_0(X)}$, we conclude that

$$\bigcap_{k=1}^{\infty} \left\{f \in \mathrm{Lip}_0(X)\colon \ \mathrm{Lip}_{F_k}(f) \leq \lambda_k\right\} \subset U. \qquad \square$$

A first characterization of τ_γ by means of seminorms uses the concept of strict topology, introduced by Buck in [4], for spaces of continuous functions on locally compact spaces. Our result is essentially an adaptation of Proposition 3 in [6].

Let X be a pointed metric space. Let $C_b(\widetilde{X})$ be the space of bounded continuous scalar-valued functions on $\widetilde{X}$ with the supremum norm, and let Φ be de Leeuw's mapping from $\mathrm{Lip}_0(X)$ into $C_b(\widetilde{X})$ defined by

$$\Phi(f)(x,y) = \frac{f(x)-f(y)}{d(x,y)}.$$

Clearly, Φ is an isometric isomorphism from $\mathrm{Lip}_0(X)$ onto the closed subspace $\Phi(\mathrm{Lip}_0(X))$ of $C_b(\widetilde{X})$.

Definition 3.2. *Let X be a compact pointed metric space. The strict topology β on $\mathrm{Lip}_0(X)$ is the strict topology on $\Phi(\mathrm{Lip}_0(X))$, that is, the locally convex topology generated by the seminorms of the form*

$$\|f\|_\phi = \sup_{(x,y)\in\widetilde{X}} |\phi(x,y)| \frac{|f(x)-f(y)|}{d(x,y)}, \qquad f \in \mathrm{Lip}_0(X),$$

where ϕ runs through the space $C_0(\widetilde{X})$ of continuous functions from $\widetilde{X}$ into $\mathbb{K}$ which vanish at infinity.

Theorem 3.3. *Let X be a compact pointed metric space. On the space $\mathrm{Lip}_0(X)$, the strict topology β is the topology τ_γ.*

Proof. We first show that the identity is a continuous mapping from $(\mathrm{Lip}_0(X), \tau_\gamma)$ to $(\mathrm{Lip}_0(X), \beta)$. By Theorem 2.1 (3), it is enough to show that the identity on $nB_{\mathrm{Lip}_0(X)}$ is continuous from $(nB_{\mathrm{Lip}_0(X)}, \tau_0)$ to $(nB_{\mathrm{Lip}_0(X)}, \beta)$ for every $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and fix $\phi \in C_0(\widetilde{X})$ and $\varepsilon > 0$. Then there is a compact set $K \subset \widetilde{X}$ such that $|\phi(x,y)| < \varepsilon/2n$ if $(x,y) \in \widetilde{X} \setminus K$. Take

$$U = \left\{ f \in \mathrm{Lip}_0(X) \colon \sup_{(x,y)\in K} \frac{|f(x)-f(y)|}{d(x,y)} \le \frac{\varepsilon}{2(1+\|\phi\|_\infty)} \right\}.$$

We now prove that U is a neighborhood of zero in $(\mathrm{Lip}_0(X), \tau_\gamma)$. Indeed, define $\sigma \colon \widetilde{X} \to \mathcal{F}(X)$ by

$$\sigma(x,y) = \frac{\delta(x)-\delta(y)}{d(x,y)}.$$

Since the mappings $x \mapsto \delta(x)$ and $(x,y) \mapsto d(x,y)$ are continuous, so is σ. Then $\sigma(K)$ is a compact subset of $\mathcal{F}(X)$ and therefore the polar

$$\sigma(K)^\circ = \left\{ F \in \mathcal{F}(X)' \colon \sup_{(x,y)\in K} |F(\sigma(x,y))| \le 1 \right\}$$

is a neighborhood of zero in $\mathcal{F}(X)'_c$. Then, by Theorem 2.4 (iii), the set

$$\{ f \in \mathrm{Lip}_0(X) \colon T_f \in \sigma(K)^\circ \},$$

that is,

$$\left\{ f \in \mathrm{Lip}_0(X) \colon \sup_{(x,y)\in K} \frac{|f(x)-f(y)|}{d(x,y)} \le 1 \right\},$$

is a neighborhood of zero in $(\mathrm{Lip}_0(X), \tau_\gamma)$, and hence so is U as required. It follows that $U \cap nB_{\mathrm{Lip}_0(X)}$ is a neighborhood of zero in $(nB_{\mathrm{Lip}_0(X)}, \tau_0)$ by Theorem 2.1 (9). If $f \in U \cap nB_{\mathrm{Lip}_0(X)}$, we have

$$\begin{aligned}\|f\|_\phi &\le \sup_{(x,y)\in K} |\phi(x,y)| \frac{|f(x)-f(y)|}{d(x,y)} + \sup_{(x,y)\in \widetilde{X}\setminus K} |\phi(x,y)| \frac{|f(x)-f(y)|}{d(x,y)} \\ &\le \|\phi\|_\infty \frac{\varepsilon}{2(1+\|\phi\|_\infty)} + \frac{\varepsilon}{2n} n < \varepsilon.\end{aligned}$$

Conversely, let U be a neighborhood of zero in $(\mathrm{Lip}_0(X), \tau_\gamma)$. By Lemma 3.1, we can suppose that

$$U = \bigcap_{n=1}^{\infty} \{ f \in \mathrm{Lip}_0(X) \colon \mathrm{Lip}_{F_n}(f) \le \lambda_n \},$$

where $\{F_n\}$ is a sequence of finite subsets of $\widetilde{X}$ and $\{\lambda_n\}$ is a sequence of positive numbers tending to ∞. We can further suppose that $F_n \subset F_{n+1}$ and $\lambda_n < \lambda_{n+1}$ for all $n \in \mathbb{N}$. We can construct a function ϕ in $C_0(\widetilde{X})$ with

$$\{(x,y) \in \widetilde{X} \colon \phi(x,y) \ne 0\} \subset \bigcup_{n=1}^{\infty} F_n,$$

so that $\phi(x,y) = 1/\lambda_1$ if $(x,y) \in F_1$ and $1/\lambda_{n+1} \le \phi(x,y) \le 1/\lambda_n$ for all (x,y) in $F_{n+1} \setminus F_n$. Then $\{f \in \mathrm{Lip}_0(X) \colon \|f\|_\phi \le 1\} \subset U$ and this proves the theorem. □

The second description of τ_γ in terms of seminorms is the following.

Theorem 3.4. *Let X be a pointed metric space. The topology τ_γ is generated by the seminorms of the form*

$$p(f) = \sup_{n\in\mathbb{N}} \alpha_n \frac{|f(x_n) - f(y_n)|}{d(x_n, y_n)}, \quad f \in \mathrm{Lip}_0(X),$$

where $\{\alpha_n\}$ varies over all sequences in $\mathbb{R}^+$ tending to 0 and $\{(x_n, y_n)\}$ runs over all sequences in $\widetilde{X}$.

Proof. Let $\mathcal{V}$ be the base of neighborhoods of zero in $(\mathrm{Lip}_0(X), \tau_\gamma)$ formed by the sets of the form

$$U = \bigcap_{n=1}^{\infty} \left\{ f \in \mathrm{Lip}_0(X) \colon \mathrm{Lip}_{F_n}(f) \leq \lambda_n \right\},$$

where $\{F_n\}$ and $\{\lambda_n\}$ are sequences as in Lemma 3.1. If, for each $U \in \mathcal{V}$, p_U is the Minkowski functional of U, then the family of seminorms $\{p_U : U \in \mathcal{V}\}$ generates the topology τ_γ on $\mathrm{Lip}_0(X)$, but justly we have

$$p_U(f) = \sup_{n\in\mathbb{N}} \lambda_n^{-1} \mathrm{Lip}_{F_n}(f)$$

for all $f \in \mathrm{Lip}_0(X)$, and the result follows. □

4. The approximation property for $(\mathrm{Lip}_0(X), \tau_\gamma)$

We devote this section to the study of the (AP) for the space $(\mathrm{Lip}_0(X), \tau_\gamma)$. For it, we introduce the subsequent topology on $\mathrm{Lip}_0(X, F)$.

Definition 4.1. Let X be a pointed metric space and let F be a Banach space. The topology $\gamma\tau_\gamma$ on $\mathrm{Lip}_0(X, F)$ is the locally convex topology generated by the seminorms of the form

$$q(f) = \sup_{n\in\mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}, \quad f \in \mathrm{Lip}_0(X, F),$$

where $\{\alpha_n\}$ ranges over the sequences in $\mathbb{R}^+$ tending to 0 and $\{(x_n, y_n)\}$ over the sequences in $\widetilde{X}$.

We now study the relationships between the topologies $\gamma\tau_\gamma$, τ_γ and τ_0. We will first need the next lemma, which is a special case of a well-known result on compact sets in Banach spaces.

Lemma 4.2. *Let X be a pointed metric space. A closed subset L of $\mathcal{F}(X)$ is compact if and only if there exist sequences $\{\alpha_n\} \in c_0(\mathbb{R}^+)$ and $\{(x_n, y_n)\} \in \widetilde{X}^{\mathbb{N}}$ such that*

$$L \subset \overline{\Gamma}\left\{ \alpha_n \frac{\delta(x_n) - \delta(y_n)}{d(x_n, y_n)} : n \in \mathbb{N} \right\}.$$

Proposition 4.3. *Let X be a pointed metric space and let F be a Banach space.*

(i) *τ_γ agrees with $\gamma\tau_\gamma$ on $\mathrm{Lip}_0(X)$.*

(ii) *τ_0 is smaller than $\gamma\tau_\gamma$ on $\mathrm{Lip}_0(X, F)$.*

Proof. (i) is deduced immediately from Theorem 3.4 and Definition 4.1.

To prove (ii), let K be a compact subset of X. Then $\delta(K)$ is a compact subset of $\mathcal{F}(X)$ and, by Lemma 4.2, there are sequences $\{\alpha_n\} \in c_0(\mathbb{R}^+)$ and $\{(x_n, y_n)\} \in \widetilde{X}^{\mathbb{N}}$ such that

$$\delta(K) \subset \overline{\Gamma}\Big\{\alpha_n \frac{\delta(x_n) - \delta(y_n)}{d(x_n, y_n)} : n \in \mathbb{N}\Big\}.$$

Since that last set is justly

$$\Big\{\sum_{n=1}^{\infty} \beta_n \alpha_n \frac{\delta(x_n) - \delta(y_n)}{d(x_n, y_n)} : \{\beta_n\} \in B_{\ell_1}\Big\},$$

it follows that

$$|f|_K = \sup_{x \in K} \|f(x)\| \leq \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)} = q(f),$$

for all $f \in \mathrm{Lip}_0(X, F)$, and this proves (ii). □

Theorem 3.1 in [15] asserts that the Lipschitz transpose mapping $f \mapsto f^t$ is an isometric isomorphism from $\mathrm{Lip}_0(X, F)$ to $\mathcal{L}((F', \tau_{w^*}); (\mathrm{Lip}_0(X), \tau_{w^*}))$. Our next result shows that the mapping $f \mapsto f^t$ is a topological isomorphism from $(\mathrm{Lip}_0(X, F), \gamma\tau_\gamma)$ to $\mathcal{L}_\epsilon(F'_c; (\mathrm{Lip}_0(X), \tau_\gamma))$. By Section 1, the seminorms

$$\sup\Big\{\alpha_n \frac{|T(\psi)(x_n) - T(\psi)(y_n)|}{d(x_n, y_n)} : n \in \mathbb{N},\ \psi \in F',\ \|\psi\| \leq 1\Big\}$$

for all $T \in (\mathrm{Lip}_0(X), \tau_\gamma)\epsilon F$, with $\{\alpha_n\}$ and $\{(x_n, y_n)\}$ being as in Definition 4.1, determine the topology of $(\mathrm{Lip}_0(X), \tau_\gamma)\epsilon F$.

Theorem 4.4. *Let X be a pointed metric space and let F be a Banach space. Then each of the mappings*

$$f \in (\mathrm{Lip}_0(X, F), \gamma\tau_\gamma) \longmapsto f^t \in (\mathrm{Lip}_0(X), \tau_\gamma)\epsilon F$$

and

$$f \in (\mathrm{Lip}_0(X, F), \gamma\tau_\gamma) \longmapsto f^{tt} \in F\epsilon(\mathrm{Lip}_0(X), \tau_\gamma) = \mathcal{L}_\epsilon(\mathcal{F}(X); F)$$

is a topological isomorphism.

Proof. If $f \in \mathrm{Lip}_0(X, F)$, the mapping $f^t \colon F' \to \mathrm{Lip}_0(X)$ is continuous from F'_c into $(\mathrm{Lip}_0(X), \tau_\gamma)$. To prove this, let p be a continuous seminorm on $(\mathrm{Lip}_0(X), \tau_\gamma)$. By Theorem 3.4, we can suppose that

$$p(g) = \sup_{n \in \mathbb{N}} \alpha_n \frac{|g(x_n) - g(y_n)|}{d(x_n, y_n)}, \quad g \in \mathrm{Lip}_0(X),$$

where $\{\alpha_n\}$ is a sequence in $\mathbb{R}^+$ tending to 0 and $\{(x_n, y_n)\}$ is a sequence in $\widetilde{X}$. Since

$$\Big\|\alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)}\Big\| \leq \alpha_n \,\mathrm{Lip}(f)$$

for all $n \in \mathbb{N}$, the set

$$K = \left\{\alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)}\right\} \cup \{0\}$$

is compact in F. For each $\psi \in F'$, we have

$$\alpha_n \frac{|f^t(\psi)(x_n) - f^t(\psi)(y_n)|}{d(x_n, y_n)} = \left|\psi\left(\alpha_n \frac{f(x_n) - f(y_n)}{d(x_n, y_n)}\right)\right| \leq |\psi|_K$$

for all $n \in \mathbb{N}$, and consequently $p(f^t(\psi)) \leq |\psi|_K$ as required.

Clearly, the mapping $f \mapsto f^t$ from $\mathrm{Lip}_0(X, F)$ to $\mathcal{L}_\epsilon(F'_c; (\mathrm{Lip}_0(X), \tau_\gamma))$ is linear and injective since F' separates the points of F. To prove that it is surjective, let $T \in \mathcal{L}_\epsilon(F'_c; (\mathrm{Lip}_0(X), \tau_\gamma))$. Then its transpose T^t is in $\mathcal{L}_\epsilon((\mathrm{Lip}_0(X), \tau_\gamma)'_c; F) = \mathcal{L}_\epsilon(\mathcal{F}(X); F)$ by Proposition 1.1 and Theorem 2.4 (ii). Notice that T belongs to $\mathcal{L}(F'; \mathrm{Lip}_0(X))$ since the closed unit ball $B_{F'}$ of F' is a compact subset of (F', τ_0), so $T(B_{F'})$ is a bounded subset of $(\mathrm{Lip}_0(X), \tau_\gamma)$ and hence norm bounded by Theorem 2.1 (5). Take $f = T^t \circ \delta$. Clearly, f maps X into F, vanishes at 0 and is Lipschitz since

$$\|f(x) - f(y)\| \leq \|T^t\| \, \|\delta(x) - \delta(y)\| = \|T\| \, d(x, y)$$

for all $x, y \in X$. For every $\psi \in F'$ and $x \in X$, we have

$$f^t(\psi)(x) = \langle \psi, f(x) \rangle = \left\langle \psi, T^t\delta(x) \right\rangle = \langle T(\psi), \delta(x) \rangle = T(\psi)(x),$$

and thus $f^t = T$. Hence the mapping $f \mapsto f^t$ is a linear bijection from $\mathrm{Lip}_0(X, F)$ onto $(\mathrm{Lip}_0(X), \tau_\gamma)\epsilon F$ with inverse given by $T \mapsto T^t \circ \delta$.

It remains to show that it is continuous with continuous inverse. For it, let $\{\alpha_n\}$ and $\{(x_n, y_n)\}$ be sequences as above. By Definition 4.1,

$$q(f) = \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}, \quad f \in \mathrm{Lip}_0(X, F),$$

is a continuous seminorm on $(\mathrm{Lip}_0(X, F), \gamma\tau_\gamma)$. If $n \in \mathbb{N}$ and $\psi \in F'$ with $\|\psi\| \leq 1$, we have

$$\alpha_n \frac{|f^t(\psi)(x_n) - f^t(\psi)(y_n)|}{d(x_n, y_n)} = \alpha_n \frac{|\psi(f(x_n)) - \psi(f(y_n))|}{d(x_n, y_n)} \leq \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)},$$

therefore

$$\sup\left\{\alpha_n \frac{|f^t(\psi)(x_n) - f^t(\psi)(y_n)|}{d(x_n, y_n)} : n \in \mathbb{N},\ \psi \in F',\ \|\psi\| \leq 1\right\} \leq q(f),$$

and this proves that the mapping $f \mapsto f^t$ is continuous. To see that its inverse $T \mapsto T^t \circ \delta$ is continuous, let q be a continuous seminorm on $(\mathrm{Lip}_0(X, F), \gamma\tau_\gamma)$. By Definition 4.1, we can suppose that

$$q(f) = \sup_{n \in \mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}, \quad f \in \mathrm{Lip}_0(X, F),$$

where $\{\alpha_n\}$ and $\{(x_n, y_n)\}$ are sequences as above. For each $n \in \mathbb{N}$, take $\psi_n \in B_{F'}$ such that

$$\left\|T^t\delta(x_n) - T^t\delta(y_n)\right\| = \left|\left\langle \psi_n, T^t\delta(x_n) - T^t\delta(y_n)\right\rangle\right|$$

and then we have

$$\alpha_n \frac{\|T^t\delta(x_n) - T^t\delta(y_n)\|}{d(x_n, y_n)} = \alpha_n \frac{|\langle T\psi_n, \delta(x_n) - \delta(y_n)\rangle|}{d(x_n, y_n)} = \alpha_n \frac{|T(\psi_n)(x_n) - T(\psi_n)(y_n)|}{d(x_n, y_n)}.$$

It follows that

$$q(T^t \circ \delta) \le \sup\left\{\alpha_n \frac{|T(\psi)(x_n) - T(\psi)(y_n)|}{d(x_n, y_n)} : n \in \mathbb{N},\ \psi \in F',\ \|\psi\| \le 1\right\},$$

and this completes the proof that the first mapping is a topological isomorphism. By Proposition 1.1, the second mapping is a topological isomorphism too. □

Our next aim is to show that the topological isomorphism of Theorem 4.4 induces a linear isomorphism from the space of all Lipschitz finite-rank operators from X to F onto the tensor product $\mathrm{Lip}_0(X) \otimes F$. Let us recall that a mapping $f \in \mathrm{Lip}_0(X, F)$ is called a Lipschitz finite-rank operator if the linear hull of $f(X)$ in F has finite dimension in which case this dimension is called the rank of f and denoted by $\mathrm{rank}(f)$. We represent by $\mathrm{Lip}_{0F}(X, F)$ the vector space of all Lipschitz finite-rank operators from X to F and gather some of their properties in the next result.

Proposition 4.5. *Let X be a pointed metric space and F a Banach space.*

(i) *If $g \in \mathrm{Lip}_0(X)$ and $u \in F$, then the mapping $g \cdot u\colon X \to F$, given by $(g \cdot u)(x) = g(x)u$, belongs to $\mathrm{Lip}_{0F}(X, F)$ and $\mathrm{Lip}(g \cdot u) = \mathrm{Lip}(g)\,\|u\|$. Moreover, $\mathrm{rank}(g \cdot u) = 1$ if $g \ne 0$ and $u \ne 0$.*

(ii) *Every element $f \in \mathrm{Lip}_{0F}(X, F)$ has a representation as $f = \sum_{j=1}^m g_j \cdot u_j$, where $m = \mathrm{rank}(f)$, $g_1, \ldots, g_j \in \mathrm{Lip}_0(X)$ and $u_1, \ldots, u_m \in F$.*

(iii) *If $f = \sum_{j=1}^m g_j \cdot u_j \in \mathrm{Lip}_{0F}(X, F)$ as in (ii), then the bounded linear operator $f^t\colon F' \to \mathrm{Lip}_0(X)$ has finite rank and $f^t = \sum_{j=1}^m \kappa_F(u_j) \otimes g_j$, where κ_F denotes the canonical injection of F into its bidual F''.*

(iv) *If Y is a pointed metric space, E a Banach space, $h \in \mathrm{Lip}_0(Y, X)$, $f = \sum_{j=1}^m g_j \cdot u_j \in \mathrm{Lip}_{0F}(X, F)$ as in (ii) and $T \in \mathcal{L}(F; E)$, then Tfh belongs to $\mathrm{Lip}_{0F}(Y, E)$ and $Tfh = \sum_{j=1}^m (g_j \circ h) \cdot T(u_j)$.*

(v) *$\mathrm{Lip}_0(X) \otimes F$ is linearly isomorphic to $\mathrm{Lip}_{0F}(X, F)$ via the linear bijection $K\colon \mathrm{Lip}_0(X) \otimes F \to \mathrm{Lip}_{0F}(X, F)$ given by*

$$K\Big(\sum_{j=1}^m g_m \otimes u_j\Big) = \sum_{j=1}^m g_j \cdot u_j.$$

Moreover, its inverse map $K^{-1}\colon \mathrm{Lip}_{0F}(X, F) \to \mathrm{Lip}_0(X) \otimes F$ is given by

$$K^{-1}\Big(\sum_{j=1}^m g_j \cdot u_j\Big) = \Big(\sum_{j=1}^m g_j \cdot u_j\Big)^t.$$

Proof. (i) Clearly, $g \cdot u$ is well-defined. Let $x, y \in X$. We have

$$\|(g \cdot u)(x) - (g \cdot u)(y)\| = |g(x) - g(y)| \, \|u\| \leq \mathrm{Lip}(g) d(x, y) \, \|u\| \,,$$

and so $g \cdot u \in \mathrm{Lip}_0(X, F)$. By passing to the supremum over $x, y \in X$ on both sides of the equality, we obtain that $\mathrm{Lip}(g \cdot u) = \mathrm{Lip}(g) \, \|u\|$.

(ii) Suppose that the linear hull $\mathrm{lin}(f(X))$ of $f(X)$ in F is m-dimensional and let $\{u_1, \ldots, u_m\}$ be a base of $\mathrm{lin}(f(X))$. Then, for each $x \in X$, the element $f(x)$ is expressible in a unique form as $f(x) = \sum_{j=1}^m \lambda_j^{(x)} u_j$ with $\lambda_1^{(x)}, \ldots, \lambda_m^{(x)} \in \mathbb{K}$. For each $j \in \{1, \ldots, m\}$, define the linear mapping $y^j \colon \mathrm{lin}(f(X)) \to \mathbb{K}$ by $y^j(f(x)) = \lambda_j^{(x)}$ for all $x \in X$. Let $g_j = y^j \circ f$. Clearly, $g_j \in \mathrm{Lip}_0(X)$ and $f(x) = \sum_{j=1}^m \lambda_j^{(x)} u_j = \sum_{j=1}^m g_j(x) u_j$ for all $x \in X$. Hence $f = \sum_{j=1}^m g_j \cdot u_j$.

(iii) Fix $g \in \mathrm{Lip}_0(X)$ and $u \in F$. If $x \in X$ and $\phi \in F'$, we have

$$\begin{aligned}(g \cdot u)^t(\phi)(x) &= (\phi \circ (g \cdot u))(x) = \phi(g(x)u) \\ &= g(x)\phi(u) = g(x)\kappa_F(u)(\phi) = (\kappa_F(u) \otimes g)(\phi)(x),\end{aligned}$$

where we have used the identification of $F'' \otimes \mathrm{Lip}_0(X)$ with the space of bounded linear operators of finite rank from F' to $\mathrm{Lip}_0(X)$ (see page 8 in [23]). Hence $(g \cdot u)^t = \kappa_F(u) \otimes g$ and therefore $f^t = \sum_{j=1}^m \kappa_F(u_j) \otimes g_j$.

(iv) Clearly, $Tfh \in \mathrm{Lip}_0(Y, E)$. For any $y \in Y$, we have

$$Tfh(y) = T\Big(\sum_{j=1}^m g_j(h(y))u_j\Big) = \sum_{j=1}^m g_j(h(y))T(u_j).$$

Hence $Tfh = \sum_{j=1}^m (g_j \circ h) \cdot T(u_j)$ and thus $Tfh \in \mathrm{Lip}_{0F}(Y, E)$.

(v) Let $\sum_{j=1}^m g_j \otimes u_j \in \mathrm{Lip}_0(X) \otimes F$. The mapping K is well defined. Indeed, if $\sum_{j=1}^m g_j \otimes u_j = 0$, then $\sum_{j=1}^m \varphi(g_j) u_j = 0$ for every $\varphi \in \mathrm{Lip}_0(X)'$ by Proposition 1.2 of [23]. In particular, we have that $\sum_{j=1}^m \delta(x)(g_j) u_j = 0$ for every $x \in X$ and thus $\sum_{j=1}^m g_j \cdot u_j = 0$ as required. Clearly, K is linear and, by (ii), is onto. To see that it is one-to-one, assume that $K(\sum_{j=1}^m g_j \otimes u_j) = 0$. Then $\sum_{j=1}^m \delta(x)(g_j) u_j = 0$ for every $x \in X$, and since $\{\delta(x) \colon x \in X\}$ is a separating subset of $\mathrm{Lip}_0(X)'$, we infer that $\sum_{j=1}^m g_j \otimes u_j = 0$ (see pages 3–4 in [23]). Finally, we have

$$K^{-1}\Big(\sum_{j=1}^m g_j \cdot u_j\Big) = \sum_{j=1}^m g_j \otimes u_j = \sum_{j=1}^m \kappa_F(u_j) \otimes g_j = \Big(\sum_{j=1}^m g_j \cdot u_j\Big)^t$$

for any $\sum_{j=1}^m g_j \cdot u_j \in \mathrm{Lip}_{0F}(X, F)$. □

We now are ready to obtain the main result of this paper.

Theorem 4.6. *Let X be a pointed metric space. The following are equivalent.*

(i) $(\mathrm{Lip}_0(X), \tau_\gamma)$ *has the* (AP).

(ii) $\mathcal{F}(X)$ *has the* (AP).

(iii) $\mathrm{Lip}_0(X) \otimes F$ *is dense in* $(\mathrm{Lip}_0(X), \tau_\gamma)\epsilon F$ *for every Banach space* F.

(iv) $\mathcal{F}(X) \otimes F$ *is dense in* $\mathcal{F}(X)\epsilon F$ *for every Banach space* F.

(v) $\mathrm{Lip}_{0F}(X, F)$ *is dense in* $(\mathrm{Lip}_0(X, F), \gamma\tau_\gamma)$ *for every Banach space* F.

Proof. (i) $\Leftrightarrow$ (ii): Assume that $(\mathrm{Lip}_0(X), \tau_\gamma)$ has the (AP). Since $(\mathrm{Lip}_0(X), \tau_\gamma) = \mathcal{F}(X)'_c$ by Theorem 2.4 (iii), then $\mathcal{F}(X)$ has the (AP) by Proposition 1.3. Conversely, if $\mathcal{F}(X)$ has the (AP), we use that $\mathcal{F}(X) = (\mathrm{Lip}_0(X), \tau_\gamma)'_c$ by Theorem 2.4 (ii) to obtain that $(\mathrm{Lip}_0(X), \tau_\gamma)$ has the (AP) by Proposition 1.3.

(i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) are deduced from Theorem 1.2, and (iii) $\Leftrightarrow$ (v) follows from Theorem 4.4 and Proposition 4.5 (v). □

5. The dual space of $(\mathrm{Lip}_0(X, F), \gamma\tau_\gamma)$

The following theorem describes the dual of the space $(\mathrm{Lip}_0(X, F), \gamma\tau_\gamma)$. Recall that a linear functional T on a topological vector space Y is continuous if and only if there is a neighborhood U of zero in Y such that $T(U)$ is a bounded subset of $\mathbb{K}$. Hence $T \in (\mathrm{Lip}_0(X, F), \gamma\tau_\gamma)'$ if and only if there exist a constant $c > 0$ and sequences $\{\alpha_n\} \in c_0(\mathbb{R}^+)$ and $\{(x_n, y_n)\} \in \widetilde{X}^{\mathbb{N}}$ such that

$$|T(f)| \leq c \sup_{n\in\mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}$$

for every $f \in \mathrm{Lip}_0(X, F)$.

Theorem 5.1. *Let X be a pointed metric space and let F be a Banach space. Then a linear functional T on $\mathrm{Lip}_0(X, F)$ is in the dual of $(\mathrm{Lip}_0(X, F), \gamma\tau_\gamma)$ if and only if there exist sequences $\{\phi_n\}$ in F' and $\{(x_n, y_n)\}$ in $\widetilde{X}$ such that $\sum_{n=1}^{\infty} \|\phi_n\| < \infty$ and*

$$T(f) = \sum_{n=1}^{\infty} \phi_n\Big(\frac{f(x_n) - f(y_n)}{d(x_n, y_n)}\Big)$$

for all $f \in \mathrm{Lip}_0(X, F)$.

Proof. Assume that T is a linear functional on $\mathrm{Lip}_0(X, F)$ of the preceding form. Since $\sum_{n=1}^{\infty} \|\phi_n\| < \infty$, we can take a sequence $\{\lambda_n\}$ in $\mathbb{R}^+$ tending to ∞ so that $\sum_{n=1}^{\infty} \lambda_n \|\phi_n\| = c < \infty$. Then we have

$$|T(f)| \leq \sum_{n=1}^{\infty} \|\phi_n\| \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)} \leq c \sup_{n\in\mathbb{N}} \lambda_n^{-1} \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}$$

for all $f \in \mathrm{Lip}_0(X, F)$. This proves that T is continuous on $(\mathrm{Lip}_0(X, F), \gamma\tau_\gamma)$.

Conversely, if $T \in (\mathrm{Lip}_0(X, F), \gamma\tau_\gamma)'$, then there are sequences $\{\alpha_n\} \in c_0(\mathbb{R}^+)$ and $\{(x_n, y_n)\} \in \widetilde{X}^{\mathbb{N}}$ such that

$$|T(f)| \leq \sup_{n\in\mathbb{N}} \alpha_n \frac{\|f(x_n) - f(y_n)\|}{d(x_n, y_n)}$$

for every $f \in \mathrm{Lip}_0(X,F)$. Consider the linear subspace

$$Z = \Big\{\Big\{\alpha_n \frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\Big\}: f \in \mathrm{Lip}_0(X,F)\Big\}$$

of $c_0(F)$, and the functional S on Z given by

$$S\Big(\Big\{\alpha_n \frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\Big\}\Big) = T(f)$$

for every $f \in \mathrm{Lip}_0(X,F)$. It follows easily that S is well defined and linear. Since

$$\begin{aligned}\Big|S\Big(\Big\{\alpha_n \frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\Big\}\Big)\Big| &= |T(f)| \\ &\le \sup_{n\in\mathbb{N}} \alpha_n \frac{\|f(x_n)-f(y_n)\|}{d(x_n,y_n)} = \Big\|\Big\{\alpha_n \frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\Big\}\Big\|_\infty\end{aligned}$$

for all $f \in \mathrm{Lip}_0(X,F)$, S is continuous on Z. By the Hahn–Banach theorem, S has a norm-preserving continuous linear extension $\widehat{S}$ to all of $c_0(F)$. Since $c_0(F)'$ is just $\ell_1(F')$, there exists a sequence $\{\psi_n\}$ in F' such that $\sum_{n=1}^\infty \|\psi_n\| = ||\widehat{S}||$ and $\widehat{S}(\{u_n\}) = \sum_{n=1}^\infty \psi_n(u_n)$ for any $\{u_n\} \in c_0(F)$. Taking $\phi_n = \alpha_n\psi_n$ for each $n \in \mathbb{N}$, we conclude that $\sum_{n=1}^\infty \|\phi_n\| \le \|\{\alpha_n\}\|_\infty ||\widehat{S}|| < \infty$ and, for all $f \in \mathrm{Lip}_0(X,F)$,

$$T(f) = \widehat{S}\Big(\Big\{\alpha_n \frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\Big\}\Big) = \sum_{n=1}^\infty \phi_n\Big(\frac{f(x_n)-f(y_n)}{d(x_n,y_n)}\Big). \qquad \square$$

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