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Parabolic Harnack inequality on fractal-type metric measure Dirichlet spaces

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Abstract. This paper proves the strong parabolic Harnack inequality for local weak solutions to the heat equation associated with time-dependent (nonsymmetric) bilinear forms. The underlying metric measure Dirichlet space is assumed to satisfy the volume doubling condition, the strong Poincaré inequality, and a cutoff Sobolev inequality. The metric is not required to be geodesic. Further results include a weighted Poincaré inequality, as well as upper and lower bounds for non-symmetric heat kernels.

1. Introduction

Parabolic Harnack inequalities are relevant in studying regularity of solutions to the heat equation, and to obtain heat kernel estimates. On some metric measure spaces, sharp two-sided bounds of (sub-)Gaussian type for the transition density of a diffusion process can be characterized by the parabolic Harnack inequality. Moreover, parabolic Harnack inequalities can be characterized by geometric conditions, namely the volume doubling property and the Poincaré inequality. This equivalence was first proved on complete Riemannian manifolds by Saloff-Coste [26], [27] and Grigor'yan [11]. We refer to [29], [30], and [19] for generalizations to symmetric and non-symmetric Dirichlet spaces. A proof of the elliptic Harnack inequality in Dirichlet spaces was given by Biroli and Mosco [5].

It is desirable to obtain similar results under minimal assumptions on the metric of the underlying Dirichlet space. Interesting and comprehensive results in this direction have been obtained in recent years. See, e.g., [16], [3], [15], [4], [14] and references therein for results in the context of fractal-type Dirichlet spaces. The main focus of these works is on bounds for symmetric heat kernels. Harnack inequalities are used to obtain or characterize these estimates. For this purpose, one may replace the parabolic Harnack inequality by the elliptic Harnack inequality together with some additional conditions, e.g., resistance estimate, or exit time estimate.

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In this paper, we present three main results. The first is the strong parabolic Harnack inequality on any metric measure Dirichlet space that satisfies volume doubling, strong Poincaré inequality, and the cutoff Sobolev inequality on annuli. We emphasize that we do not require the metric to be geodesic, though if the metric is geodesic then we also have the converse implication, namely that the parabolic Harnack inequality implies the strong Poincaré inequality. See Proposition 5.8.

More specifically, we show that the strong parabolic Harnack inequality

$$\sup_{Q^-} u \leq C \inf_{Q^+} u$$

holds for any non-negative local weak solution $u(t,x)$ of the heat equation on a time-space cylinder $Q(x,a,r) := (a, a+\Psi(r)) \times B(x,r)$, where $Q^- := (a + \tau_1 \Psi(r), a + \tau_2 \Psi(r)) \times B(x, \delta r)$ and $Q^+ := (a + \tau_3 \Psi(r), a + \tau_4 \Psi(r)) \times B(x, \delta r)$ are two smaller time-space cylinder of radius $\delta r < r$ that are separated by a time gap $(a + \tau_3 \Psi(r)) - (a + \tau_2 \Psi(r))$. Here a is any real number, $x \in X$ is any point in the underlying metric measure space, and C is a positive constant depending on the arbitrary choice of parameters $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 \leq 1$. The function Ψ describes the appropriate time-space scaling that is implicit in the assumed Poincaré inequality PI(Ψ) and the cutoff Sobolev inequality CSA(Ψ) whose definitions we recall in the main text. Our only condition on Ψ is that it satisfies a polynomial growth condition (2.4) given in Section 2.2.

In the absence of a geodesic metric, we must distinguish between the *strong* parabolic Harnack inequality as stated above, and the *weak* parabolic Harnack inequality (see [4]) in which the Harnack constant exists for *some* parameters $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 \leq 1$ but not necessarily for any arbitrary choice of parameters. See [4], [14] for equivalence results for the weak parabolic Harnack inequality on symmetric Dirichlet spaces.

The second main result concerns weak solutions of the heat equation associated with time-dependent and/or non-symmetric bilinear forms $(\mathcal{E}_t, \mathcal{F})$, $t \in \mathbb{R}$. These bilinear forms generalize Dirichlet forms: they may lack the Markovian property, non-negative definiteness, or symmetry. We think of these forms as perturbations of a symmetric strongly local regular reference Dirichlet form $(\mathcal{E}^*, \mathcal{F})$. Our hypothesis is that the bilinear forms $\mathcal{E}_t$ satisfy certain structural conditions (see Assumption 0) and quantitative conditions (Assumptions 1, 2). We establish the local boundedness of local weak solutions (Corollary 4.8) and the strong parabolic Harnack inequality for $\mathcal{E}_t$ (Theorem 5.3) under natural geometric conditions on the reference Dirichlet space. The local boundedness and the Hölder continuity (Corollary 5.5) of local weak solutions are well-known consequences of the parabolic Harnack inequality. A priori, however, the local boundedness of weak solutions is not obvious. We derive it from mean value estimates which we prove using a Steklov average technique similar to that in [19].

Third, we present upper and lower bounds for the nonsymmetric heat kernels or, in the time-dependent case, heat propagators associated with $\mathcal{E}_t$, $t \in \mathbb{R}$. As in [19], our assumptions on the non-symmetric perturbations cover plenty of examples on Euclidean space, Riemannian manifolds, or polytopal complexes. For instance, our results apply to uniformly elliptic second order differential operators with (time-dependend) bounded measurable coefficients. Examples of non-symmetric bilinear forms on an abstract Dirichlet space are not immediate. In Section 8, we construct a non-symmetric perturbation $\mathcal{E}$ of a symmetric strongly local regular Dirichlet form $(\mathcal{E}^*, \mathcal{F})$ so that $\mathcal{E}$ satisfies the strong parabolic Harnack inequality and heat kernel estimates.

Our setting includes fractal spaces like the Sierpiński carpet, though in this case the strong parabolic Harnack inequality is equivalent to the weak parabolic Harnack inequality because the metric is geodesic. Nevertheless, this case is interesting because we give a proof that does not rely on heat kernel estimates.

This work is in part motivated by applications to estimates for nonsymmetric Dirichlet heat kernels on inner uniform domains in fractal spaces [18]. A common hypothesis in the works [2], [3], [1], which treat fractal-type spaces, is the conservativeness of the Dirichlet form. Since the estimates in [18] are proved using Doob's transform and it is not clear a priori that the transformed Dirichlet space would be conservative, it was important to not assume conservativeness in the present work. We remark that the assumption of conservativeness was already dropped in, e.g., [14] in a similar context.

We prove our main results using the *parabolic* Moser iteration scheme [23], [24], and [25]. It was proved by Barlow and Bass in [2], [3] that the *elliptic* Moser iteration scheme can be applied to obtain the elliptic Harnack inequality on a fractal-type metric measure Dirichlet space which is symmetric strongly local regular and which satisfies the volume doubling property, the strong Poincaré inequality, and a cutoff Sobolev inequality. The parabolic Harnack inequality was then derived through an estimate for the resistance of balls in concentric larger balls. The approach in [2], [3] is to follow Moser's line of arguments with $d\mu$ replaced by a measure $d\gamma_{x,R} = \Psi(R)\, d\Gamma(\phi, \phi) + d\mu$, where $d\Gamma(\cdot, \cdot)$ is the energy measure of the Dirichlet form, and ϕ is a cutoff function for the ball $B(x, R/2)$ with compact support in the larger ball $B(x, R)$. This approach does not seem to generalize to the parabolic case: the estimates for sub- and supersolutions (cf. Lemmas 4.4 and 4.5), which are an important step in obtaining mean value estimates, are not available with $\gamma_{x,R}$ in place of μ. Therefore, the parabolic case requires that the energy measure $d\Gamma(\psi, \psi)$ of a suitable cutoff function ψ must be estimated through a cutoff Sobolev inequality very early in the line of arguments, that is, when proving sub- and supersolution estimates. This is possible thanks to the cutoff Sobolev inequality on annuli CSA(Ψ) which was introduced in [1]. The relevant property of this condition is that for *every* $\epsilon \in (0, 1)$ there exists a cutoff function ψ for $B(x, R)$ in $B(x, R + r)$ that satisfies the inequality

$$(1.1)\qquad \int f^2\, d\Gamma(\psi, \psi) \leq \epsilon \int \psi^2\, d\Gamma(f, f) + C\, \frac{\epsilon^{1-\beta_2/2}}{\Psi(r)} \int_{B(x,R+r)} f^2\, d\mu,$$

for all $f \in \mathcal{F}$, where C is a positive constant independent of $\psi, f, x, R, r, \epsilon$.

A slightly weaker condition is the *generalized capacity condition* introduced in [14]: it is inequality (1.1) for bounded functions $f \in \mathcal{F} \cap L^\infty(X)$ and the cutoff functions ψ are allowed to depend on f. The generalized capacity condition appears to be too weak to run the parabolic Moser iteration. Indeed, since the local boundedness of weak solutions is not known a priori, several approximation arguments are used in our proof. Because of this we need the cutoff functions to be independent of the functions that approximate the weak solution.

Once the mean value estimates for sub- and supersolutions are proved, we apply a weighted Poincaré inequality to complete the proof of the parabolic Harnack inequality. More specifically, we need the weight to be a cutoff function that satisfies CSA(Ψ). The weighted Poincaré inequality is obtained in Theorem 3.4.

It is worth pointing out that our arguments are local. Therefore, our hypotheses on the space (volume doubling and Poincaré inequality) are local. That is, they are stated for balls $B(x, R)$ that lie in some subset Y of the underlying space X, with radii R up to a fixed scale $R \leq R_0 \in (0, \infty]$.

Regarding the notion of (local) weak solutions to the heat equation, we adopt the definition that is natural from the viewpoint of existence and uniqueness theory (see, e.g., [20], [31], [9]). In order to clarify the relation of recent literature to our results, we verify that the space of local weak solutions to the heat equation associated with a symmetric strongly local regular Dirichlet form constitutes a space of caloric functions in the sense of [4]. Along the way, we obtain a proof of the parabolic maximum principle (Proposition 7.1) using the Steklov average technique. We remark that the axiomatic properties of caloric functions implicitly presume the strong locality of the Dirichlet form.

In part of this paper, we will work with the so-called very weak solutions introduced in [19]. Very weak solutions may lack continuity in the time-variable and are thus too general to satisfy the parabolic Harnack inequality unless we additionally assume continuity in the time-variable, which then leaves us with weak solutions.

Structure of the paper. In Section 2 we recall basic properties of the underlying metric measure Dirichlet space and introduce non-symmetric perturbations of the reference Dirichlet form $(\mathcal{E}^*, \mathcal{F})$. Since the assumptions we impose on the perturbations involve cutoff functions, we provide some background on cutoff Sobolev inequalities in the same section, and introduce a localized cutoff Sobolev condition.

In Section 3 we consider Sobolev and Poincaré inequalities for the reference form. The main result of this section is the weighted Poincaré inequality of Theorem 3.4.

In Section 4 we return to the setting of time-dependent non-symmetric local bilinear forms. We recall the definition of very weak solutions introduced in [19], Definition 3.1, in Section 4.1 and then follow Moser's reasoning: we first prove estimates for non-negative local weak sub- and supersolutions (Section 4.2) and then run the parabolic Moser iteration scheme to obtain mean value estimates (Section 4.3). A main result of the paper, the local boundedness of weak solutions, hides in Corollary 4.8.

Section 5 is devoted to parabolic Harnack inequalities. Section 5.2 contains main results, namely parabolic Harnack inequalities in the context of non-symmetric local bilinear forms. In Section 5.3 we take a closer look at the case of a symmetric strongly local regular Dirichlet form, relating the present paper to recent literature.

This subsection relies on a parabolic maximum principle and a super-mean value property for local weak solutions. We prove these in Section 7.

In Section 6 we present applications: estimates for symmetric and non-symmetric heat kernels and, in the time-dependent case, heat propagators. Some of these estimates are proved under the additional assumption that the metric is geodesic, and the bilinear forms satisfy a further quantitative condition (Assumption 4).

We conclude the paper by constructing an example of a non-symmetric local bilinear form on a fractal-type metric measure space, see Section 8.

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2. Cutoff Sobolev conditions and non-symmetric forms

2.1. The symmetric reference form

Let (X, d, μ) be a locally compact separable metric measure space, where μ is a Radon measure on X with full support. Throughout this paper we fix a symmetric strongly local regular Dirichlet form $(\mathcal{E}^*, \mathcal{F})$ on $L^2(X,\mu)$. The Dirichlet form $(\mathcal{E}^*, \mathcal{F})$ induces the norm

$$\|f\|_{\mathcal{F}}^2 := \mathcal{E}^*(f,f) + \int f^2 \, d\mu$$

on its domain $\mathcal{F}$. The energy measure Γ of $\mathcal{E}^*$ (in [10] denoted as $\frac{1}{2}\mu^c_{<\cdot,\cdot>}$) satisfies a Cauchy–Schwarz inequality, cf. Lemma 5.6.1 in [10],

$$\Big| \int fg \, d\Gamma(u,v) \Big| \leq \Big(\int f^2 \, d\Gamma(u,u) \Big)^{1/2} \Big(\int g^2 \, d\Gamma(v,v) \Big)^{1/2}, \tag{2.1}$$

for any $u, v \in \mathcal{F}$ and any bounded Borel measurable functions f, g on X. We have the following chain rule for Γ: for any $v, u_1, u_2, \ldots, u_m \in \mathcal{F} \cap L^\infty(X,\mu)$, $u = (u_1, \ldots, u_m)$, and $\Phi \in \mathcal{C}^1(\mathbb{R}^m)$ with $\Phi(0) = 0$, we have $\Phi(u) \in \mathcal{F} \cap L^\infty(X,\mu)$ and

$$d\Gamma(\Phi(u), v) = \sum_{i=1}^m \Phi_{x_i}(\tilde{u}) \, d\Gamma(u_i, v), \tag{2.2}$$

where $\Phi_{x_i} := \partial\Phi/\partial x_i$ and $\tilde{u}$ is a quasi-continuous version of u, see (3.2.27) and Theorem 3.2.2 in [10]. When Φ_{x_i} is bounded for every $i \in \{1, \ldots, m\}$ in addition, then $\Phi(u) \in \mathcal{F}$ and (2.2) hold for any $u_1, \ldots, u_m \in \mathcal{F}$ and any $v \in \mathcal{F} \cap L^\infty(X,\mu)$; see (3.2.28) in [10].

Inequality (2.1) together with a Leibniz rule ([10], Lemma 3.2.5) implies that

$$\int d\Gamma(fg, fg) \leq 2 \int f^2 \, d\Gamma(g,g) + 2 \int g^2 \, d\Gamma(f,f), \tag{2.3}$$

for any $f, g \in \mathcal{F} \cap L^\infty(X)$. Here, on the right-hand side, quasi-continuous versions of f and g must be used.

By definition, the (essential) support of $f \in L^2(X,\mu)$ is the support of the measure $|f|\,d\mu$. For an open set $U \subset X$, we set

$$\mathcal{F}_c(U) := \{f \in \mathcal{F} :\ \text{the support of } f \text{ is compact in } U\},$$
$$\mathcal{F}^0(U) := \ \text{closure of } \mathcal{F}_c(U) \text{ in } (\mathcal{F}, \|\cdot\|_{\mathcal{F}}),$$
$$\mathcal{F}_{\mathrm{loc}}(U) := \{f \in L^2_{\mathrm{loc}}(U) : \forall \text{ compact } K \subset U,\ \exists f^\sharp \in \mathcal{F}, f\big|_K = f^\sharp\big|_K\ \mu\text{-a.e.}\}.$$

For functions in $\mathcal{F}_{\mathrm{loc}}(U)$ we always take their quasi-continuous versions. Note that $\Gamma(f,g)$ can be defined locally on U for $f, g \in \mathcal{F}_{\mathrm{loc}}(U)$ by virtue of Corollary 3.2.1 in [10]. For any $v, u_1, \ldots, u_m \in \mathcal{F}_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U,\mu)$ and $\Phi \in \mathcal{C}^1(\mathbb{R}^m)$, we have $\Phi(u) \in \mathcal{F}_{\mathrm{loc}}(U) \cap L^\infty_{\mathrm{loc}}(U,\mu)$ and the chain rule (2.2) holds. For convenience, we set

$$\mathcal{F}_b := \mathcal{F} \cap L^\infty(X,\mu), \quad \mathcal{F}_c := \mathcal{F}_c(X) \quad \text{and} \quad \mathcal{F}_{\mathrm{loc}} := \mathcal{F}_{\mathrm{loc}}(X).$$

Throughout the paper we will use the notation $f \vee a := \max\{f, a\}$, $f \wedge a := \min\{f, a\}$, $f^+ := f \vee 0$ and $f^- := (-f)^+$, for a function f and a real number a.

2.2. Cutoff Sobolev inequalities

For the ease of readability, we suppose in this section that any metric ball $B(x, R+r) \subset X$ under consideration is relatively compact. Later, we will localize this assumption; see condition (A2-Y) in Subsection 3.1.

Let $\Psi\colon [0,\infty) \to [0,\infty)$ be a continuous strictly increasing bijection. Assume there exist $\beta_1, \beta_2 \in [2,\infty)$ and $C_\Psi \in [1,\infty)$ such that, for all $0 < s < R$,

$$C_\Psi^{-1}\Big(\frac{R}{s}\Big)^{\beta_1} \le \frac{\Psi(R)}{\Psi(s)} \le C_\Psi\Big(\frac{R}{s}\Big)^{\beta_2}. \tag{2.4}$$

Definition 2.1. A function $\psi \in \mathcal{F}$ is called a *cutoff function* for $B(x,R)$ in $B(x,R+r)$, where $x \in X$, $R > 0$, $r > 0$, if

(i) ψ is continuous,

(ii) $0 \le \psi \le 1$ μ-a.e.,

(iii) $\psi = 1$ on $B(x,R)$ μ-a.e.,

(iv) The compact support of ψ is contained in $B(x,R+r)$.

Definition 2.2. $(X, d, \mu, \mathcal{E}^*, \mathcal{F})$ satisfies the cutoff Sobolev condition on annuli, CSA(Ψ), if there exists a constant $C_0 \in (0,\infty)$ such that for any $\epsilon \in (0,1)$, $x \in X$, $R > 0$, $r > 0$, there exists a cutoff function ψ for $B(x,R)$ in $B(x,R+r)$ such that

$$\forall f \in \mathcal{F}, \quad \int_A f^2\, d\Gamma(\psi,\psi) \le \epsilon \int_A \psi^2\, d\Gamma(f,f) + \frac{C_0\epsilon^{1-\beta_2/2}}{\Psi(r)} \int_A \psi f^2\, d\mu, \tag{2.5}$$

where $A = B(x,R+r) \setminus B(x,R)$.

Abusing notation, we denote by CSA(Ψ) not only the cutoff Sobolev condition on annuli, but also the collection of all cutoff functions that satisfy (2.5) for some x, R, r. We will sometimes write $\psi \in \mathrm{CSA}(\Psi, \epsilon)$ or $\psi \in \mathrm{CSA}(\Psi, \epsilon, C_0)$ when ψ satisfies (2.5) for the specified ϵ and C_0. To keep notation simple, we will write $C_0(\epsilon)$ for $C_0 \epsilon^{1-\beta_2/2}$.

The cutoff Sobolev condition on annuli was introduced in [1] for fixed $\epsilon = 1/8$. From the proof of Lemma 5.1 in [1], it is clear that CSA(Ψ) holds with some fixed ϵ and for all $r > 0, R > 0$ if and only if it holds for all $\epsilon \in (0,1)$ and for all $r > 0, R > 0$ (with a different cutoff function for each ϵ). Thus, the two definitions are equivalent. More precisely, we have the following lemma which quantifies the scaling of the zero order term on the right-hand side of (2.5) as ϵ varies.

Lemma 2.3. *Let $B(x, R+r) \subset X$ be relatively compact. For every $\epsilon \in (0,1)$ there exists $\lambda \in (0,\infty)$ such that the following holds. For each non-negative integer n, let*

$$b_n = e^{-n\lambda} \quad \text{and} \quad s_n = c_\lambda r\, e^{-n\lambda/\beta_2},$$

where c_λ is chosen so that

$$\sum_{n=1}^{\infty} s_n =: r' < r.$$

Let $r_0 = 0$,

$$r_n = \sum_{k=1}^{n} s_k \tag{2.6}$$

and $B_n = B(x, R+r_n)$. Let ψ_n be a cutoff function for B_{n-1} in B_n which satisfies, for all $f \in \mathcal{F}$,

$$\int_{B_n \setminus B_{n-1}} f^2\, d\Gamma(\psi_n, \psi_n) \le c_1 \int_{B_n \setminus B_{n-1}} d\Gamma(f,f) + \frac{c_2}{\Psi(s_n)} \int_{B_n \setminus B_{n-1}} f^2\, d\mu,$$

for some fixed constants c_1 and c_2 that do not depend on f, n, x, r and R. Let

$$\psi := \sum_{n=1}^{\infty} (b_{n-1} - b_n)\, \psi_n.$$

Then ψ is a cutoff function for $B(x,R)$ in $B(x, R+r)$ and ψ satisfies (2.5) for the given ϵ with some constant $C_0 \in (0,\infty)$ that depends only on $\beta_2, C_\Psi, c_1, c_2$.

The cutoff function ψ constructed in Lemma 2.3 will serve as a weight function in the weighted Poincaré inequality of Theorem 3.4. We include the full proof of this lemma for the convenience of the reader, though it is essentially the same as the proof of Lemma 5.1 in [1].

Proof. Let $f \in \mathcal{F}$. Note that $\psi = 1$ on $B_0 = B(x,R)$, and $\psi - (b_{n-1} - b_n)\psi_n$ is constant on $B_n \setminus B_{n-1}$. Because of the strong locality and Theorem 4.3.8 in [7], we obtain

$$\begin{aligned}\int f^2\,d\Gamma(\psi,\psi) &= \int_{B_0} f^2\,d\Gamma(\psi,\psi) + \sum_{n=1}^{\infty}(b_{n-1}-b_n)^2\int_{B_n\setminus B_{n-1}} f^2\,d\Gamma(\psi_n,\psi_n)\\ &\quad + 2\sum_{n=1}^{\infty}(b_{n-1}-b_n)\int_{B_n\setminus B_{n-1}} f^2\,d\Gamma(\psi_n,\psi-(b_{n-1}-b_n)\,\psi_n)\\ &\quad + \sum_{n=1}^{\infty}\int_{B_n\setminus B_{n-1}} f^2\,d\Gamma(\psi-(b_{n-1}-b_n)\psi_n,\psi-(b_{n-1}-b_n)\psi_n)\\ &= \sum_{n=1}^{\infty}(b_{n-1}-b_n)^2\int_{B_n\setminus B_{n-1}} f^2\,d\Gamma(\psi_n,\psi_n)\\ &\le \sum_{n=1}^{\infty}(b_{n-1}-b_n)^2\Big(c_1\int_{B_n\setminus B_{n-1}} d\Gamma(f,f)+\frac{c_2}{\Psi(s_n)}\int_{B_n\setminus B_{n-1}} f^2\,d\mu\Big)\\ &\le (e^\lambda-1)^2\Big(\sum_{n=1}^{\infty}e^{-2n\lambda}c_1\int_{B_n\setminus B_{n-1}} d\Gamma(f,f)\Big)\\ &\quad + \sum_{n=1}^{\infty}(b_{n-1}-b_n)^2\frac{c_2}{\Psi(s_n)}\int_{B_n\setminus B_{n-1}} f^2\,d\mu\\ &\le (e^\lambda-1)^2c_1\int \psi^2\,d\Gamma(f,f)+\sum_{n=1}^{\infty}(b_{n-1}-b_n)^2\frac{c_2}{\Psi(s_n)}\int_{B_n\setminus B_{n-1}} f^2\,d\mu.\end{aligned}$$

The last inequality is where we needed the annuli (rather than balls). We also used the fact that $\psi_n \ge b_n = e^{-n\lambda}$ on $B_{n-1}\setminus B_n$. By (2.4), we have

$$\frac{\Psi(r)}{\Psi(s_n)} \le C_\Psi\Big(\frac{r}{c_\lambda r e^{-n\lambda/\beta_2}}\Big)^{\beta_2} \le C_\Psi\,\frac{e^\lambda-1}{c_\lambda^{\beta_2}(b_{n-1}-b_n)}. \tag{2.7}$$

Thus,

$$\sum_{n=1}^{\infty}(b_{n-1}-b_n)^2\frac{c_2}{\Psi(s_n)}\int_{B_n\setminus B_{n-1}} f^2\,d\mu \le \frac{(e^\lambda-1)^2}{c_\lambda^{\beta_2}}\,\frac{c_2\cdot C_\Psi}{\Psi(r)}\int \psi f^2\,d\mu.$$

Finally,

$$\int f^2\,d\Gamma(\psi,\psi)\le (e^\lambda-1)^2c_1\int\psi^2\,d\Gamma(f,f)+\frac{(e^\lambda-1)^2}{c_\lambda^{\beta_2}}\,\frac{c_2\cdot C_\Psi}{\Psi(r)}\int\psi f^2\,d\mu.$$

Choose $\lambda := \log(1+(\epsilon/c_1)^{1/2})$. Then $(e^\lambda-1)^2c_1=\epsilon$. By the choice of c_λ,

$$c_\lambda = e^{\lambda/\beta_2}(1-e^{-\lambda/\beta_2})\frac{r'}{r} = (e^{\lambda/\beta_2}-1)\frac{r'}{r}.$$

Hence,

$$\frac{(e^\lambda-1)^2}{c_\lambda^{\beta_2}} = \frac{(e^\lambda-1)^2}{(e^{\lambda/\beta_2}-1)^{\beta_2}}\Big(\frac{r'}{r}\Big)^{-\beta_2} = \frac{\epsilon}{c_1}(e^{\lambda/\beta_2}-1)^{-\beta_2}\Big(\frac{r'}{r}\Big)^{-\beta_2}$$
$$\le \mathrm{const}(\beta_2, c_1, r'/r)\cdot\Big(\frac{\epsilon}{c_1}\Big)^{1-\beta_2/2},$$

where we applied the trivial inequality $(e^x-1)^{-1} \le x^{-1}$ with $x = \log(1 + (\epsilon/c_1)^{1/2})/\beta_2$. This completes the proof. □

Let $Y \subset X$ be open and $R_0 > 0$.

Definition 2.4. The *cutoff Sobolev inequality on annuli,* CSA(Ψ), is satisfied on Y up to scale R_0 if there exists a constant $C_0 \in (0,\infty)$ such that, for any $\epsilon \in (0,1)$, $0 < r < R \le R_0$, $B(x,2R) \subset Y$, there exists a cutoff function ψ for $B(x,R)$ in $B(x,R+r)$ such that

$$\forall f \in \mathcal{F}, \quad \int_A f^2\, d\Gamma(\psi,\psi) \le \epsilon \int_A \psi^2\, d\Gamma(f,f) + \frac{C_0\epsilon^{1-\beta_2/2}}{\Psi(r)}\int_A \psi f^2\, d\mu, \tag{2.8}$$

where $A = B(x,R+r)\setminus B(x,R)$.

2.3. Structural assumptions on the bilinear forms

Let $(X,d,\mu,\mathcal{E}^*,\mathcal{F})$ be as in Section 2.1. We will refer to $(\mathcal{E}^*,\mathcal{F})$ as the *reference form* for the bilinear forms defined below. Let $(\mathcal{E}_t,\mathcal{F})$, $t\in\mathbb{R}$, be a family of (possibly non-symmetric) local bilinear forms that all have the same domain $\mathcal{F}$ as the reference form $(\mathcal{E}^*,\mathcal{F})$. We always assume that, for every $f,g\in\mathcal{F}$, the map $t\mapsto \mathcal{E}_t(f,g)$ is measurable.

For $f,g\in\mathcal{F}$, let $\mathcal{E}_t^{\mathrm{sym}}(f,g) := \frac{1}{2}\big[\mathcal{E}_t(f,g)+\mathcal{E}_t(g,f)\big]$ be the symmetric part of $\mathcal{E}_t(f,g)$ and let $\mathcal{E}_t^{\mathrm{skew}}(f,g) := \frac{1}{2}\big[\mathcal{E}_t(f,g)-\mathcal{E}_t(g,f)\big]$ be the skew-symmetric part. Notice that $1\in\mathcal{F}_{\mathrm{loc}}$, thus $\mathcal{E}_t(1,f)$ and $\mathcal{E}_t(f,1)$ are well-defined for any $f\in\mathcal{F}_{\mathrm{c}}$. We will use the decomposition

$$\mathcal{E}_t(f,g) = \mathcal{E}_t^{\mathrm{s}}(f,g) + \mathcal{E}_t^{\mathrm{sym}}(fg,1) + \mathcal{L}_t(f,g) + \mathcal{R}_t(f,g), \quad \forall f,g\in\mathcal{F} \text{ with } fg\in\mathcal{F}_{\mathrm{c}},$$

that we introduced in [19]. Here, the so-called *symmetric strongly local part* $\mathcal{E}_t^{\mathrm{s}}$ is defined by

$$\mathcal{E}_t^{\mathrm{s}}(f,g) := \mathcal{E}_t^{\mathrm{sym}}(f,g) - \mathcal{E}_t^{\mathrm{sym}}(fg,1), \quad f,g\in\mathcal{F} \text{ with } fg\in\mathcal{F}_{\mathrm{c}},$$

and the bilinear forms $\mathcal{L}_t$ and $\mathcal{R}_t$ are defined by

$$\mathcal{L}_t(f,g) := \frac{1}{4}\big[\mathcal{E}_t(fg,1)-\mathcal{E}_t(1,fg)+\mathcal{E}_t(f,g)-\mathcal{E}_t(g,f)\big],$$
$$\mathcal{R}_t(f,g) := \frac{1}{4}\big[\mathcal{E}_t(1,fg)-\mathcal{E}_t(fg,1)+\mathcal{E}_t(f,g)-\mathcal{E}_t(g,f)\big] = -\mathcal{L}_t(g,f),$$

for any $f, g \in \mathcal{F}$ with $fg \in \mathcal{F}_{\mathrm{c}}$. Due to the locality of $\mathcal{E}_t$, the bilinear forms $\mathcal{L}_t(f,g)$ and $\mathcal{R}_t(f,g)$ are well-defined whenever $f \in \mathcal{F}_{\mathrm{loc}} \cap L^\infty_{\mathrm{loc}}(X,\mu)$ and $g \in \mathcal{F}_{\mathrm{c}} \cap L^\infty_{\mathrm{loc}}(X,\mu)$, or vice versa.

Let $\mathcal{D}$ be a linear subspace of $\mathcal{F} \cap \mathcal{C}_{\mathrm{c}}(X)$ such that

(i) $\mathcal{D}$ is dense in $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$.

(ii) If $f \in \mathcal{D}$ then $(f \vee 0) \in \mathcal{D}$ and $(f \wedge 1) \in \mathcal{D}$.

(iii) If $f \in \mathcal{D}$ then $\Phi(f) \in \mathcal{D}$ for any function $\Phi \in \mathcal{C}^1(\mathbb{R}^m)$ with $\Phi(0) = 0$, where m is a positive integer.

By the regularity of the reference form $(\mathcal{E}^*, \mathcal{F})$, such a space $\mathcal{D}$ exists. We make the following assumption on the structure of the forms $\mathcal{E}_t$, $t \in \mathbb{R}$.

Assumption 0. For each $t \in \mathbb{R}$, $\mathcal{E}_t$ is a local bilinear form with domain $D(\mathcal{E}_t) = \mathcal{F}$. For every $f, g \in \mathcal{F}$, the map $t \mapsto \mathcal{E}_t(f,g)$ is measurable. Moreover,

(i) there exists a constant $C_* \in (0, \infty)$ such that

$$|\mathcal{E}_t(f,g)| \leq C_* \, \|f\|_{\mathcal{F}} \, \|g\|_{\mathcal{F}}, \quad \forall f, g \in \mathcal{F},$$

(ii) for all $f, g \in \mathcal{F}_{\mathrm{b}}$ with $fg \in \mathcal{F}_{\mathrm{c}}$,

$$|\mathcal{E}^{\mathrm{sym}}_t(fg, 1)| \leq C_* \, \|f\|_{\mathcal{F}} \, \|g\|_{\mathcal{F}},$$

(iii) there is a constant $C \in [1, \infty)$ such that

$$\frac{1}{C} \, \mathcal{E}^*(f,f) \leq \mathcal{E}^{\mathrm{s}}_t(f,f) \leq C \, \mathcal{E}^*(f,f), \quad \forall f \in \mathcal{F} \cap \mathcal{C}_{\mathrm{c}}(X).$$

(iv) (Product rule for $\mathcal{L}_t$) For any $u, v, f \in \mathcal{D}$,

$$\mathcal{L}_t(uf, v) = \mathcal{L}_t(u, fv) + \mathcal{L}_t(f, uv).$$

(v) (Chain rule for $\mathcal{L}_t$) For any $v, u_1, u_2, \ldots, u_m \in \mathcal{D}$ and $u = (u_1, \ldots, u_m)$, and for any $\Phi \in \mathcal{C}^2(\mathbb{R}^m)$,

$$\mathcal{L}_t(\Phi(u), v) = \sum_{i=1}^m \mathcal{L}_t(u_i, \Phi_{x_i}(u) v).$$

(vi) There exist constants $0 < c \leq \alpha < \infty$ such that, for all $f \in \mathcal{F}$,

$$\mathcal{E}_t(f,f) + \alpha \int f^2 \, d\mu \geq c \, \|f\|^2_{\mathcal{F}}.$$

Part (i) and (vi) of Assumption 0 ensure the existence of weak solutions to the heat equation. See, e.g., [20].

Under Assumption 0, the bilinear forms $\mathcal{E}_t$, $\mathcal{E}^{\mathrm{sym}}_t$, and $\mathcal{E}^{\mathrm{skew}}_t$ are continuous on $\mathcal{F} \times \mathcal{F}$. For results on extending the bilinear forms $\mathcal{L}_t$ and $\mathcal{R}_t$ and the maps $(f,g) \mapsto \mathcal{E}_t(fg, 1)$ and $(f,g) \mapsto \mathcal{E}_t(1, fg)$ to $\mathcal{F} \times \mathcal{F}$, see Section 7.2 of [19]. The elementary proof of the next lemma will be given elsewhere.

Lemma 2.5. *Under Assumption 0 (i)-(iii), the bilinear form $\mathcal{E}_t^{\mathrm{s}}$, defined for $f,g \in \mathcal{F}_{\mathrm{b}}$ with $fg \in \mathcal{F}_{\mathrm{c}}(X)$, extends continuously to $\mathcal{F} \times \mathcal{F}$, and the extension $(\mathcal{E}_t^{\mathrm{s}}, \mathcal{F})$ is a strongly local regular symmetric Dirichlet form.*

Under Assumption 0, the Dirichlet form $(\mathcal{E}_t^{\mathrm{s}}, \mathcal{F})$ admits an energy measure Γ_t which has all properties that are described in Section 2.1 for the energy measure Γ of $(\mathcal{E}^*, \mathcal{F})$. In particular, Γ_t satisfies the product rule, the chain rule, and a Cauchy–Schwarz type inequality.

Assumption 0 (ii) implies that there exists a constant $C_{10} \in [1,\infty)$ such that

$$\frac{1}{C_{10}} \int f^2\, d\Gamma(g,g) \le \int f^2\, d\Gamma_t(g,g) \le C_{10} \int f^2\, d\Gamma(g,g), \quad \forall f,g \in \mathcal{F} \cap \mathcal{C}_{\mathrm{c}}(X). \tag{2.9}$$

See [22]. Of course, this inequality extends to all bounded Borel measurable functions $f\colon X \to (-\infty,+\infty)$ and $g \in \mathcal{F}$. The inequality also holds when $f \in \mathcal{F}$ and $g \in \mathrm{CSA}(\Psi)$. If the reference form $(\mathcal{E}^*, \mathcal{F})$ satisfies $\mathrm{CSA}(\Psi,C_0)$ locally on Y up to scale R_0, and if $(\mathcal{E}_t, \mathcal{F})$ satisfies Assumption 0, then $(\mathcal{E}_t^{\mathrm{s}}, \mathcal{F})$ satisfies $\mathrm{CSA}(\Psi, \hat{C}_0)$ locally on Y up to scale R_0 (with $\hat{C}_0$ depending on C_0 and C_{10}).

We refer to Section 8 and to [19] for examples of forms $\mathcal{E}_t$ that satisfy Assumption 0.

2.4. Quantitative assumptions on the bilinear forms

Suppose Assumption 0 is satisfied. In this section we introduce quantitative assumptions on the zero-order part and on the skew-symmetric part of each of the forms $(\mathcal{E}_t, \mathcal{F})$, $t \in \mathbb{R}$. We will show in Section 4 below that our assumptions are sufficient to perform the Moser iteration technique to obtain L^2-mean value estimates. The statements of Assumption 1 and Assumption 2 are inspired by and weaker than Assumptions 1 and 2 in [19]. The new contribution here is that we state these quantitative conditions only for functions ψ that are cutoff functions and in $\mathrm{CSA}(\Psi)$.

As before, we fix an open connected set $Y \subset X$ and $R_0 > 0$. Let $C_0 \in (0,\infty)$ be given. Let

$$C_1(\epsilon) := \epsilon^{-1/2}\, C_0(\epsilon) = C_0 \cdot \epsilon^{(1-\beta_2)/2}, \quad \text{for } \epsilon \in (0,1]. \tag{2.10}$$

Assumption 1. There are constants $C_2, C_3, C_{11} \in [0,\infty)$ such that for all $t \in \mathbb{R}$, for any $\epsilon \in (0,1)$, any $0 < r < R \le R_0$, any ball $B(x,2R) \subset Y$, any cutoff function $\psi \in \mathrm{CSA}(\Psi,\epsilon,C_0)$ for $B(x,R)$ in $B(x,R+r)$, and any $0 \le f \in \mathcal{F}_{\mathrm{loc}}(Y) \cap L^\infty_{\mathrm{loc}}(Y,\mu)$,

$$\begin{aligned} &|\mathcal{E}_t^{\mathrm{sym}}(f^2\psi^2,1)| + |\mathcal{E}_t^{\mathrm{skew}}(f^2\psi^2,1)| + \left|\mathcal{E}_t^{\mathrm{skew}}(f,f\psi^2)\right| \\ &\qquad \le C_{11}\,\epsilon^{1/2} \int \psi^2\, d\Gamma(f,f) + (C_2 + C_3\Psi(r))\, \frac{C_1(\epsilon)}{\Psi(r)} \int_B f^2\, d\mu, \end{aligned}$$

where $B = B(x,R+r)$.

Assumption 2. There are constants C_4, C_5, $C_{11} \in [0,\infty)$ such that for all $t \in \mathbb{R}$, for any $\epsilon \in (0,1)$, any $0 < r < R \le R_0$, any ball $B(x,2R) \subset Y$, any cutoff function

$\psi \in \mathrm{CSA}(\Psi, \epsilon, C_0)$ for $B(x,R)$ in $B(x,R+r)$, and any $0 \leq f \in \mathcal{F}_{\mathrm{loc}}(Y)$ with $f + f^{-1} \in L^\infty_{\mathrm{loc}}(Y,\mu)$,

$$\left|\mathcal{E}_t^{\mathrm{skew}}(f, f^{-1}\psi^2)\right| \leq C_{11}\,\epsilon^{1/2} \int \psi^2\, d\Gamma(\log f, \log f) + (C_4 + C_5 \Psi(r)) \frac{C_1(\epsilon)}{\Psi(r)} \int_B d\mu,$$

where $B = B(x, R+r)$.

Remark 2.6. For simplicity, we may and will assume that the constants C_{11} in Assumption 1 and in Assumption 2 are the same.

2.5. Some preliminary computations

In the next three lemmas, we consider bilinear forms $(\mathcal{E}_t, \mathcal{F})$, $t \in \mathbb{R}$, which satisfy Assumptions 0 and 1 with respect to the reference form $(\mathcal{E}^*, \mathcal{F})$. Recall that Y is an open subset of X. For a non-negative function u and a positive integer n, let

$$u_n := u \wedge n.$$

Lemma 2.7. *Suppose Assumption 0 and Assumption 1 are satisfied. Let $p \in \mathbb{R}$, $\epsilon \in (0,1)$, $0 < r < R \leq R_0$, and $B(x,2R) \subset Y$. Let $\psi \in \mathrm{CSA}(\Psi, \epsilon, C_0)$ be a cutoff function for $B(x,R)$ in $B(x,R+r)$, and $0 \leq u \in \mathcal{F}_{\mathrm{loc}}(Y) \cap L^\infty_{\mathrm{loc}}(Y,\mu)$. Assume either of the following hypotheses:*

(i) *$p \geq 2$,*

(ii) *u is locally uniformly positive.*

Then $u u_n^q \in \mathcal{F}_{\mathrm{loc}}(Y)$, $u u_n^q \psi^2 \in \mathcal{F}_{\mathrm{c}}(Y)$, for any $q \geq 0$. Moreover, for any $k > 0$ it holds

$$\begin{aligned}
&(1-p)\,\mathcal{E}_t^{\mathrm{s}}(u, u u_n^{p-2}\psi^2) \\
&\quad\leq \left(8k\,\epsilon\, C_{10} + C\left(\frac{|1-p|^2}{k} + 1 - p\right)\right) \int \psi^2\, u_n^{p-2}\, d\Gamma(u,u) \\
&\qquad + \left(2k\,\epsilon\, C_{10}\,(p-2)^2 - C'\left((1-p)^2 + (1-p)\right)\right) \int \psi^2\, u_n^{p-2}\, d\Gamma(u_n,u_n) \\
&\qquad + 4k\, C_{10} \frac{C_0(\epsilon)}{\Psi(r)} \int \psi\, u^2\, u_n^{p-2}\, d\mu,
\end{aligned} \tag{2.11}$$

where $C = C_{10}$ if $|1-p|^2/k + 1 - p > 0$ and $C = 1/C_{10}$ otherwise, and $C' = 1/C_{10}$ if $(1-p)^2 + 1 - p > 0$ and $C' = C_{10}$ otherwise.

Proof. The first assertion follows from Lemma 1.3 in [19]. Moreover, by (2.2) and (2.1), we have for any $k > 0$ that

$$\begin{aligned}
&(1-p)\,\mathcal{E}_t^{\mathrm{s}}(u, u u_n^{p-2}\psi^2) \\
&\qquad\leq 4k \int u^2\, u_n^{p-2}\, d\Gamma_t(\psi,\psi) + \left(\frac{|1-p|^2}{k} + (1-p)\right) \int \psi^2\, u_n^{p-2}\, d\Gamma_t(u,u) \\
&\qquad\quad - \left((1-p)^2 + (1-p)\right) \int \psi^2\, u_n^{p-2}\, d\Gamma_t(u_n,u_n).
\end{aligned}$$

Hence (2.11) follows from applying (2.5) and (2.9). □

Lemma 2.8. *Suppose Assumptions 0 and 1 are satisfied. Let $p \in (-\infty, 1-\eta)$ for some small $\eta > 0$. Let $\epsilon \in (0,1)$, $0 < r < R \le R_0$, and $B(x,2R) \subset Y$. Let $\psi \in \mathrm{CSA}(\Psi, \epsilon, C_0)$ be a cutoff function for $B(x,R)$ in $B(x,R+r)$, and $0 \le u \in \mathcal{F}_{\mathrm{loc}}(Y) \cap L^\infty_{\mathrm{loc}}(Y,\mu)$. Assume us is locally uniformly positive and locally bounded. Then, for any $k > 0$, it holds*

$$\mathcal{E}^{\mathrm{s}}_t(u, u^{p-1}\psi^2) \le \Big(\frac{2\,C_{10}\,\epsilon}{\eta} p^2 + \frac{1}{C_{10}}\,(p - (1-\eta/2))\Big) \int \psi^2\, u^{p-2}\, d\Gamma(u,u)$$

$$+ \frac{8\,C_{10}}{\eta}\,\frac{C_0(\epsilon)}{\Psi(r)} \int \psi\, u^p\, d\mu. \tag{2.12}$$

For the proof, simply choose $k = \frac{2}{\eta}(1-p)$ in the proof of Lemma 2.7.

Lemma 2.9. *Suppose Assumptions 0 and 1 are satisfied. Let $p \in \mathbb{R}$, $\epsilon \in (0,1)$, $0 < r < R \le R_0$, and $B(x,2R) \subset Y$. Let $\psi \in \mathrm{CSA}(\Psi, \epsilon, C_0)$ be a cutoff function for $B(x,R)$ in $B = B(x,R+r)$, and $0 \le u \in \mathcal{F}_{\mathrm{loc}}(Y) \cap L^\infty_{\mathrm{loc}}(Y,\mu)$. Assume either of the following hypotheses:*

(i) $p \ge 2$,

(ii) $p \ne 0$ *and u is locally uniformly positive.*

Then,

$$|\mathcal{E}^{\mathrm{sym}}_t(u^2\, u_n^{p-2}\,\psi^2, 1)|$$
$$\le 2\,C_{11}\,\epsilon^{1/2} \int u_n^{p-2}\,\psi^2\, d\Gamma(u,u) + \frac{(p-2)^2}{2}\, C_{11}\,\epsilon^{1/2} \int u_n^{p-2}\,\psi^2\, d\Gamma(u_n,u_n)$$
$$+ (C_2 + C_3\Psi(r))\,\frac{C_1(\epsilon)}{\Psi(r)} \int_B u^2\, u_n^{p-2}\, d\mu,$$

and

$$|\mathcal{E}^{\mathrm{skew}}_t(u, u\, u_n^{p-2}\,\psi^2)| \le 2\,C_{11}\,\epsilon^{1/2} \int u_n^{p-2}\,\psi^2\, d\Gamma(u,u)$$
$$+ C_{11}\,\epsilon^{1/2}\Big(\frac{(p-2)^2}{2} + \frac{|p(p-2)|}{4}\Big) \int u_n^{p-2}\,\psi^2\, d\Gamma(u_n,u_n)$$
$$+ \big(C_2 + C_3\Psi(r)\big)\,\frac{C_1(\epsilon)}{\Psi(r)} \int_B u^2\, u_n^{p-2}\, d\mu$$
$$+ (C_2 + C_3\Psi(r))\,\frac{|p-2|}{|p|}\,\frac{C_1(\epsilon)}{\Psi(r)} \int_B u_n^p\, d\mu.$$

Proof. We will prove the assertion for $u \in \mathcal{D}$. Then the general case follows by approximation, using Assumption 0 (i), the locality of $\mathcal{E}_t$, and the fact that $\mathcal{D}$ is dense in $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$. First consider the case when u is uniformly positive on the support of ψ. By strong locality, (2.3) and (2.2), we have

$$\int \psi^2\, d\Gamma(u\, u_n^{(p-2)/2},\, u\, u_n^{(p-2)/2})$$
$$\le 2 \int u_n^{p-2}\,\psi^2\, d\Gamma(u,u) + \frac{(p-2)^2}{2} \int u_n^{p-2}\,\psi^2\, d\Gamma(u_n,u_n). \tag{2.13}$$

The first assertion follows easily from Assumption 1 and (2.13). By Lemma 2.13 in [19], we have

$$\mathcal{E}_t^{\text{skew}}(u, u u_n^{p-2}\psi^2) = \mathcal{E}_t^{\text{skew}}(u u_n^{(p-2)/2}, u u_n^{(p-2)2}\psi^2) + \frac{2-p}{p}\mathcal{E}_t^{\text{skew}}(u_n^{p/2}, u_n^{p/2}\psi^2)$$

$$+ \frac{2-p}{p}\mathcal{E}_t^{\text{skew}}(u_n^p\psi^2, 1). \tag{2.14}$$

Hence, by Assumption 1, (2.3) and (2.13), we have

$$\begin{aligned}|\mathcal{E}_t^{\text{skew}}(u, u u_n^{p-2}\psi^2)| &\le 2\, C_{11}\epsilon^{1/2}\int u_n^{p-2}\psi^2\, d\Gamma(u,u)\\ &\quad + C_{11}\,\epsilon^{1/2}\Big(\frac{(p-2)^2}{2} + \frac{|p(p-2)|}{4}\Big)\int u_n^{p-2}\psi^2\, d\Gamma(u_n,u_n)\\ &\quad + \big(C_2 + C_3\Psi(r)\big)\frac{C_1(\epsilon)}{\Psi(r)}\int_B u^2 u_n^{p-2}\, d\mu\\ &\quad + (C_2 + C_3\Psi(r))\,\frac{|p-2|}{|p|}\,\frac{C_1(\epsilon)}{\Psi(r)}\int_B u_n^p\, d\mu.\end{aligned}$$

In the case when u is not uniformly positive on the support of ψ, repeat the proof with $u+\varepsilon$ in place of u. If $p \ge 2$, then we can let ε tend to 0 at the end of the proof. □

For $\varepsilon > 0$, let

$$u_\varepsilon := u + \varepsilon.$$

Lemma 2.10. *Suppose Assumptions 0 and 1 are satisfied. Let $p \in \mathbb{R}$, $\epsilon \in (0,1)$, $0 < r < R \le R_0$, and $B(x,2R) \subset Y$. Let $\psi \in \mathrm{CSA}(\Psi,\epsilon,C_0)$ be a cutoff function for $B(x,R)$ in $B(x,R+r)$, and $0 \le u \in \mathcal{F}_{\mathrm{loc}}(Y)\cap L^\infty_{\mathrm{loc}}(Y,\mu)$. Then, for any $k \ge 1$,*

$$\begin{aligned}&|\mathcal{E}_t(\varepsilon, u_\varepsilon^{p-1}\psi^2)|\\ &\qquad \le C_{11}\,\epsilon^{1/2}\,\frac{(p-1)^2}{4}\int \psi^2 u_\varepsilon^{p-2}\, d\Gamma(u_\varepsilon,u_\varepsilon) + (C_2 + C_3\Psi(r))\,\frac{C_1(\epsilon)}{\Psi(r)}\int_B u_\varepsilon^p\, d\mu,\end{aligned}$$

where $B = B(x, R+r)$.

Proof. We apply Assumption 1 and (2.2). Then,

$$\begin{aligned}&|\mathcal{E}_t(\varepsilon, u_\varepsilon^{p-1}\psi^2)|\\ &\le \varepsilon\,|\mathcal{E}_t^{\text{skew}}(1, u_\varepsilon^{p-1}\psi^2)| + \varepsilon\,|\mathcal{E}_t^{\text{sym}}(1, u_\varepsilon^{p-1}\psi^2)|\\ &\le \varepsilon\, C_{11}\,\epsilon^{1/2}\int \psi^2\, d\Gamma(u_\varepsilon^{(p-1)/2}, u_\varepsilon^{(p-1)/2}) + \varepsilon\,(C_2 + C_3\Psi(r))\frac{C_1(\epsilon)}{\Psi(r)}\int_B u_\varepsilon^{p-1}\, d\mu\\ &\le C_{11}\,\epsilon^{1/2}\,\frac{(p-1)^2}{4}\int \varepsilon u_\varepsilon^{p-3}\psi^2\, d\Gamma(u_\varepsilon,u_\varepsilon) + (C_2 + C_3\Psi(r))\frac{C_1(\epsilon)}{\Psi(r)}\int_B \varepsilon u_\varepsilon^{p-1}\, d\mu.\end{aligned}$$

Applying $\varepsilon \le u_\varepsilon$ completes the proof. □

3. Sobolev and Poincaré inequalities

3.1. Weak, strong, and weighted Poincaré inequalities

In this section we consider Sobolev and Poincaré inequalities for the symmetric reference form $(\mathcal{E}^*, \mathcal{F})$ defined in Section 2.1. We fix an open connected set $Y \subset X$ and $R_0 > 0$.

For the rest of the paper we suppose that

$$\text{(A2-}Y\text{)} \qquad \begin{array}{c} \text{if } B(x, 2R) \subset Y \text{ with } 0 < r < R \leq R_0, \\ \text{then } B(x, R+r) \text{ is relatively compact.} \end{array}$$

Note that any open set Y such that $\overline{Y}$ is complete in (X, d) satisfies (A2-Y), see, e.g., Lemma 1.1 (i) in [30].

Definition 3.1. The *volume doubling property* is satisfied on Y up to scale R_0 if there exists a constant $C_{\text{VD}} \in (1, \infty)$ such that, for every ball $B(x, 2R) \subset Y$ and for $0 < r < R \leq R_0$,

$$\text{(VD)} \qquad V(x, R+r) \leq C_{\text{VD}}\, V(x, R),$$

where $V(x, R) = \mu(B(x, R))$ denotes the volume of $B(x, R)$.

Lemma 3.2. *If* VD *is satisfied on* Y *up to scale* R_0*, then for* $\nu = \log_2(C_{\text{VD}})$,

$$\frac{\mu(B(x, R))}{\mu(B(y, s))} \leq C_{\text{VD}}^2 \left(\frac{R}{s}\right)^{\nu},$$

for all $0 < s < R \leq R_0$ *and* $y \in B(x, R)$ *with* $B(y, 2R) \subset Y$.

Proof. See Lemma 5.2.4 in [28]. □

Definition 3.3. $(\mathcal{E}^*, \mathcal{F})$ satisfies the (*strong*) *Poincaré inequality* PI(Ψ) on Y up to scale R_0, if there exists a constant $C_{\text{PI}} \in (0, \infty)$ such that for any $0 < r < R \leq R_0$ and $B(x, 2R) \subset Y$,

$$\text{(PI}(\Psi)\text{)} \qquad \forall f \in \mathcal{F}_{\text{loc}}(Y), \quad \int_B |f - f_B|^2 \, d\mu \leq C_{\text{PI}}\, \Psi(R+r) \int_B d\Gamma(f, f),$$

where $f_B = \frac{1}{V(x,R+r)} \int_{B(x,R+r)} f \, d\mu$ is the mean of f over $B = B(x, R+r)$.

Assumption 3. The reference form $(X, d, \mu, \mathcal{E}^*, \mathcal{F})$ satisfies A2-Y, VD, PI(Ψ) and CSA(Ψ) on Y up to scale R_0.

Theorem 3.4. *Suppose* Assumption 3 *is satisfied. Then* $(\mathcal{E}^*, \mathcal{F})$ *satisfies a weighted Poincaré inequality on* Y *up to scale* R_0*. That is, there exists a constant* $C_{\text{wPI}} \in (0, \infty)$ *such that for any* $0 < r < R \leq R_0$*, any* $B(x, 2R) \subset Y$*, and for every* $\epsilon \in (0, 1)$*, there exists a cutoff function* $\psi \in \text{CSA}(\Psi, \epsilon, C_0)$ *for* $B(x, R)$ *in* $B(x, R+r)$ *such that*

$$(3.1) \qquad \forall f \in \mathcal{F}_{\text{loc}}(Y), \quad \int |f - f_\psi|^2 \, \psi^2 \, d\mu \leq C_{\text{wPI}}\, \Psi(R+r) \int \psi^2 \, d\Gamma(f, f),$$

where

$$f_\psi = \frac{\int f\psi^2\,d\mu}{\int \psi^2\,d\mu}.$$

The constant C_{wPI} *depends only on* C_0, C_{VD}, C_{PI}.

Proof. Let $\epsilon \in (0,1)$. Let

$$\psi = \sum_{n=1}^{\infty} (b_{n-1} - b_n)\,\psi_n \tag{3.2}$$

be the cutoff function constructed in Lemma 2.3. In particular, for each nonnegative integer n, $b_n = e^{-n\lambda}$ for some $\lambda = \lambda(\epsilon)$, and $\psi_n \in \mathrm{CSA}(\Psi)$ is a cutoff function for B_{n-1} in B_n, where $B_n = B(x, R + r_n)$ and the sequence $r_n \uparrow r' < r$ is defined by (2.6). By Lemma 2.3, we have $\psi \in \mathrm{CSA}(\Psi, \epsilon, C_0)$ for a suitable choice of $\lambda(\epsilon)$. We will prove the weighted Poincaré inequality (3.1) for the weight ψ given by (3.2). By the triangle inequality,

$$\int |f - f_\psi|^2 \psi^2\,d\mu \le \int |f - f_{B_0}|^2 \psi^2\,d\mu + \int |f_{B_0} - f_\psi|^2 \psi^2\,d\mu.$$

The second integral on the right-hand side can be estimated by

$$\int |f_{B_0} - f_\psi|^2 \psi^2\,d\mu = \int \left| \frac{\int (f - f_{B_0})\psi^2\,d\mu}{\int \psi^2\,d\mu} \right|^2 \psi^2\,d\mu \le \int |f - f_{B_0}|^2 \psi^2\,d\mu,$$

where we used the definition of f_ψ and the Cauchy–Schwarz inequality. Thus, it suffices to show that there exists a constant $C \in (0,\infty)$ such that

$$\forall f \in \mathcal{F}_{\mathrm{loc}}(Y), \quad \int |f - f_{B_0}|^2 \psi^2\,d\mu \le C\,\Psi(R + r) \int \psi^2\,d\Gamma(f,f).$$

By (3.2) and the fact that ψ_n vanishes outside B_n and $0 \le \psi_n \le 1$, we have

$$\begin{aligned}
\int |f - f_{B_0}|^2 \psi^2\,d\mu &= \sum_n \sum_m (b_{n-1} - b_n)(b_{m-1} - b_m) \int |f - f_{B_0}|^2 \psi_n \psi_m\,d\mu \\
&\le \sum_n \sum_m (b_{n-1} - b_n)(b_{m-1} - b_m) \int_{B_n \cap B_m} |f - f_{B_0}|^2\,d\mu \\
&\le I_1 + I_2,
\end{aligned}$$

where we applied the triangle inequality with

$$I_1 := 2 \sum_n \sum_m (b_{n-1} - b_n)(b_{m-1} - b_m) \int_{B_n \cap B_m} |f - f_{B_n \cap B_m}|^2\,d\mu$$

and

$$I_2 := 2 \sum_n \sum_m (b_{n-1} - b_n)(b_{m-1} - b_m) \int_{B_n \cap B_m} |f_{B_n \cap B_m} - f_{B_0}|^2\,d\mu.$$

Observe that

$$b_{n-1} - b_n = e^{\lambda}(b_n - b_{n+1}).$$

Applying the strong Poincaré inequality on the ball $B_n \cap B_m = B_{n\wedge m}$, and using the fact that $\psi_{n+1} = 1$ on B_n, we obtain

$$
\begin{aligned}
I_1 &\le C_{\mathrm{PI}} \sum_n \sum_m (b_{n-1} - b_n)(b_{m-1} - b_m)\Psi(R + (r_n \wedge r_m)) \int_{B_n \cap B_m} d\Gamma(f,f) \\
&\le C_{\mathrm{PI}}\, \Psi(R+r) \sum_n \sum_m (b_{n-1} - b_n)(b_{m-1} - b_m) \int \psi_{n+1}\, \psi_{m+1}\, d\Gamma(f,f) \\
&\le C_{\mathrm{PI}}\, \Psi(R+r) \sum_n \sum_m e^{2\lambda}(b_n - b_{n+1})(b_m - b_{m+1}) \int \psi_{n+1}\, \psi_{m+1}\, d\Gamma(f,f) \\
&\le C_{\mathrm{PI}}\, e^{2\lambda}\, \Psi(R+r) \int \psi^2\, d\Gamma(f,f).
\end{aligned}
$$

Now we estimate I_2. Note that $|f_{B_n\cap B_m} - f_{B_0}|$ is constant and $\mu(B_n \cap B_m) \le V(x, R+r) \le C_{\mathrm{VD}}\mu(B_0)$ by the volume doubling property. Applying the triangle inequality, and then the Poincaré inequality on the balls $B_n \cap B_m$ and B_0, yields

$$
\begin{aligned}
I_2 &\le 2\, C_{\mathrm{VD}} \sum_n \sum_m (b_{n-1} - b_n)(b_{m-1} - b_m) \int_{B_0} |f_{B_n\cap B_m} - f_{B_0}|^2\, d\mu \\
&\le 4\, C_{\mathrm{VD}} \sum_n \sum_m (b_{n-1} - b_n)(b_{m-1} - b_m) \int_{B_n\cap B_m} |f_{B_n\cap B_m} - f|^2\, d\mu \\
&\quad + 4\, C_{\mathrm{VD}} \sum_n \sum_m (b_{n-1} - b_n)(b_{m-1} - b_m) \int_{B_0} |f - f_{B_0}|^2\, d\mu \\
&\le 8\, C_{\mathrm{VD}}\ C_{\mathrm{PI}} \sum_n \sum_m (b_{n-1} - b_n)(b_{m-1} - b_m)\Psi(R + (r_n \wedge r_m)) \int_{B_n\cap B_m} d\Gamma(f,f) \\
&\le 8\, C_{\mathrm{VD}}\ C_{\mathrm{PI}}\, e^{2\lambda}\, \Psi(R+r) \int \psi^2\, d\Gamma(f,f).
\end{aligned}
$$

□

Definition 3.5. $(\mathcal{E}^*, \mathcal{F})$ satisfies the *weak Poincaré inequality* weak-PI(Ψ) on Y up to scale R_0, if there exist constants $\kappa \in (0,1)$ and $C(\kappa) \in (0,\infty)$ such that for any $0 < r < \kappa R < R \le R_0$ and any ball $B(x, 2R) \subset Y$,

$$
\forall f \in \mathcal{F}_{\mathrm{loc}}(Y), \qquad \int_B |f - f_B|^2\, d\mu \le C(\kappa)\, \Psi(2R) \int_{B(x,2R)} d\Gamma(f,f),
$$

where $B = B(x, R+r)$.

Remark 3.6. If (A2-Y) and VD hold on Y up to scale R_0 and if the metric d is geodesic, then the weak Poincaré inequality PI(Ψ) on Y up to scale R_0 implies the strong Poincaré inequality on Y up to scale R_0. This is immediate from a weighted Poincaré inequality with weight function $\psi = 1_{B(x,R)}$, see Corollary 5.3.5 in [28]. The weighted Poincaré inequality with weight $\psi = 1_{B(x,R)}$ can be proved using a Whitney covering and chaining arguments that are applicable when the metric is geodesic. See Sections 5.3.2–5.3.5 of [28].

Lemma 3.7. *Assume that* $(\mathcal{E}^*, \mathcal{F})$ *satisfies* A2-Y *and* VD, PI(Ψ) *on* Y *up to scale* R_0*. Then the pseudo-Poincaré inequality holds: there is a constant* $C = C(\beta_1, \beta_2, C_\Psi, C_{\mathrm{VD}}, C_{\mathrm{PI}}) \in (0,\infty)$ *such that for any ball* $B(x, 2R) \subset Y$ *with* $0 < R \leq R_0$*, and any* $f \in \mathcal{F}_{\mathrm{c}}(B(x,R))$*,*

$$\int |f - f_s|^2 \, d\mu \leq C\, \Psi(s) \int d\Gamma(f,f), \quad \forall s \in (0,R),$$

where $f_s(y) := \frac{1}{V(y,s)} \int_{B(y,s)} f d\mu$*. If, in addition,* $f \in \mathcal{F}_{\mathrm{c}}(B(x,R/4))$ *and* $B(x,R) \neq Y$*, then*

$$\int f^2 \, d\mu \leq C\, \Psi(R) \int d\Gamma(f,f).$$

Proof. The proof is as in the classical case $\Psi(r) = r^2$, with the obvious changes regarding the use of $\Psi(r)$. The idea is to cover $B(x,R)$ with balls $2B_i$, where each B_i has radius $s/10$, and to apply the Poincaré inequality to each of the balls $4B_i$. For details, see Lemmas 5.3.2 and 5.2.5 in [28]. □

3.2. Localized Sobolev inequality

Definition 3.8. $(\mathcal{E}^*, \mathcal{F})$ satisfies the localized *Sobolev inequality* SI(Ψ) on Y up to scale R_0, if there exist constants $\kappa > 1$ and $C_{\mathrm{SI}} \in (0,\infty)$ such that for any ball $B(x,4R) \subsetneq B(x,8R) \subset Y$ with $0 < R \leq R_0/4$, and all $f \in \mathcal{F}_{\mathrm{c}}(B(x,R))$, we have

$$\left(\int_{B(x,R)} |f|^{2\kappa} \, d\mu \right)^{1/\kappa} \leq \frac{C_{\mathrm{SI}}}{V(x,R)^{1-1/\kappa}} \Psi(R) \int_{B(x,R)} d\Gamma(f,f). \tag{3.3}$$

Theorem 3.9. *If* A2-Y *and* VD, PI(Ψ) *are satisfied on* Y *up to scale* R_0*, then* $(\mathcal{E}^*, \mathcal{F})$ *satisfies* SI(Ψ) *on* Y *up to scale* R_0*. The Sobolev constant* C_{SI} *depends only on* β_1, β_2, C_Ψ*,* C_{VD} *and* C_{PI}*. The constant* κ *satisfies* $1 - 1/\kappa = \beta_1 / \log_2(C_{\mathrm{VD}})$*.*

Proof. We follow Theorem 5.2.3 in [28]. It suffices to proof the assertion for non-negative f. For any $y \in B = B(x,R)$, $0 < s < R$, we have by Lemma 3.2 that

$$|f_s(y)| \leq \frac{1}{\mu(B(y,s))} \int_{B(y,s)} |f| \, d\mu \leq \frac{C_{\mathrm{VD}}^2}{\mu(B)} \left(\frac{R}{s}\right)^\nu \|f\|_1,$$

where $\nu = \log_2(C_{\mathrm{VD}})$. For $0 \leq f \in \mathcal{F}_{\mathrm{c}}(B)$ and $\lambda \geq 0$, write

$$\mu(\{f \geq \lambda\}) \leq \mu(\{|f - f_s| \geq \lambda/2\} \cap B) + \mu(\{f_s \geq \lambda/2\} \cap B)$$

and consider two cases.

Case 1. If λ is such that

$$\frac{\lambda}{4} > \frac{C_{\mathrm{VD}}^2}{\mu(B)} \|f\|_1,$$

then pick $s \in (0,R)$ depending on λ in such a way that

$$\frac{\lambda}{4} = \frac{C_{\mathrm{VD}}^2}{\mu(B)} \left(\frac{R}{s}\right)^\nu \|f\|_1.$$

For this choice of s,

$$\mu(\{f_s \geq \lambda/2\} \cap B) = 0.$$

By (2.4), we then have for κ satisfying $1 - 1/\kappa = \beta_1/\nu$ that

$$\lambda^{1-1/\kappa} \leq C \frac{\Psi(R)}{\Psi(s)} \Big(\frac{\|f\|_1}{\mu(B)}\Big)^{1-1/\kappa}, \tag{3.4}$$

where C denotes a positive constant that may change from line to line and depends only on $\beta_1, \beta_2, C_\Psi, C_{\mathrm{VD}}, C_{\mathrm{PI}}$. Applying the pseudo-Poincaré inequality of Lemma 3.7 and (3.4), we obtain

$$\begin{aligned}\mu(\{f \geq \lambda\}) &\leq \mu(\{|f - f_s| \geq \lambda/2\} \cap B) \leq \frac{4}{\lambda^2} \int |f - f_s|^2 \, d\mu \\ &\leq C \frac{4}{\lambda^2} \Psi(s) \int d\Gamma(f,f) \leq \frac{C}{\lambda^{3-1/\kappa}} \Big(\frac{\|f\|_1}{\mu(B)}\Big)^{1-1/\kappa} \Psi(R) \int d\Gamma(f,f).\end{aligned}$$

Case 2. If λ is such that

$$\frac{\lambda}{4} \leq \frac{C_{\mathrm{VD}}^2}{\mu(B)} \, \|f\|_1,$$

then it follows from the second part of Lemma 3.7 that

$$\int f^2 \, d\mu \leq C \, \Psi(R) \int d\Gamma(f,f).$$

Hence,

$$\mu(\{f \geq \lambda\}) \leq \frac{1}{\lambda^2} \int f^2 \, d\mu \leq \frac{C}{\lambda^2} \Psi(R) \int d\Gamma(f,f).$$

We obtain that

$$\lambda^{3-1/\kappa} \, \mu(\{f \geq \lambda\}) \leq C \Big(\frac{\|f\|_1}{\mu(B)}\Big)^{1-1/\kappa} \Psi(R) \int d\Gamma(f,f) \tag{3.5}$$

holds in both cases. Now the proof can be completed easily by following the reasoning in Theorem 3.2.2 and Lemma 3.2.3 of [28]. □

4. The Moser iteration technique

For the rest of the paper, we fix a reference form $(\mathcal{E}^*, \mathcal{F})$ as in Section 2.1 and an open set $Y \subset X$. We assume $(\mathcal{E}^*, \mathcal{F})$ satisfies A2-Y, VD, CSA(Ψ) on Y up to scale $R_0 > 0$. Let $(\mathcal{E}_t, \mathcal{F})$, $t \in \mathbb{R}$, be a family of bilinear forms that satisfy Assumption 0 and Assumption 1.

4.1. Local weak solutions to the heat equation

We recall the notion of very weak solutions introduced in [19]. For an open time interval I and a separable Hilbert space H, let $L^2(I \to H)$ be the Hilbert space of those functions $v \colon I \to H$ such that

$$\|v\|_{L^2(I\to H)} := \Big(\int_I \|v(t)\|_H^2 \, dt\Big)^{1/2} < \infty.$$

It is well known that $L^2(I \to L^2(X,\mu))$ can be identified with $L^2(I \times X, dt \times d\mu)$. Indeed, continuous functions with compact support in $I \times X$ are dense in both spaces and the two norms coincide on these functions.

Let $L^2_{\mathrm{loc}}(I \to \mathcal{F}; U)$ be the space of all functions $u\colon I \times U \to \mathbb{R}$ such that for any open interval J relatively compact in I, and any open subset A relatively compact in U, there exists a function $u^\sharp \in L^2(I \to \mathcal{F})$ such that $u^\sharp = u$ a.e. in $J \times A$.

Definition 4.1. Define

$$D(L_t) = \{ f \in \mathcal{F} : g \mapsto \mathcal{E}_t(f,g) \text{ is continuous w.r.t. } \|\cdot\|_2 \text{ on } \mathcal{F}_{\mathrm{c}} \}.$$

For $f \in D(L_t)$, let $L_t f$ be the unique element in $L^2(X)$ such that

$$-\int L_t f g \, d\mu = \mathcal{E}_t(f,g) \quad \text{for all } g \in \mathcal{F}_{\mathrm{c}}.$$

Then we say that $(L_t, D(L_t))$ is the infinitesimal generator of $(\mathcal{E}_t, \mathcal{F})$ on X. See, e.g., [21].

Definition 4.2. Let I be an open interval and $U \subset X$ open. Set $Q = I \times U$. A function $u\colon Q \to \mathbb{R}$ is a *local very weak solution* of the heat equation $\frac{\partial}{\partial t} u = L_t u$ in Q, if

(i) $u \in L^2_{\mathrm{loc}}(I \to \mathcal{F}; U)$,

(ii) For almost every $a, b \in I$,

$$\forall \phi \in \mathcal{F}_{\mathrm{c}}(U), \quad \int u(b,\cdot)\phi \, d\mu - \int u(a,\cdot)\phi \, d\mu + \int_a^b \mathcal{E}_t(u(t,\cdot),\phi)\, dt = 0. \tag{4.1}$$

Definition 4.3. Let I be an open interval and $U \subset X$ open. Set $Q = I \times U$. A function $u\colon Q \to \mathbb{R}$ is a *local very weak subsolution* of $\frac{\partial}{\partial t} u = L_t u$ in Q, if

(i) $u \in L^2_{\mathrm{loc}}(I \to \mathcal{F}; U)$,

(ii) For almost every $a, b \in I$ with $a < b$, and any non-negative $\phi \in \mathcal{F}_{\mathrm{c}}(U)$,

$$\int u(b,\cdot)\phi \, d\mu - \int u(a,\cdot)\phi \, d\mu + \int_a^b \mathcal{E}_t(u(t,\cdot),\phi)\, dt \le 0. \tag{4.2}$$

A function u is called a *local very weak supersolution* if $-u$ is a local very weak subsolution.

Note that a local very weak solution is not required to have a weak time-derivative. A function $u\colon Q \to \mathbb{R}$ is a *local weak solution* in the classical sense if and only if u is a local very weak solution and $u \in \mathcal{C}_{\mathrm{loc}}(I \to L^2(U))$, where $\mathcal{C}_{\mathrm{loc}}(I \to L^2(U))$ is the space of measurable functions $u\colon I \times U \to \mathbb{R}$ such that for any open interval J relatively compact in I and any open subset A relatively compact in U, there exists a continuous function $u^\sharp\colon I \to L^2(U)$ such that $u = u^\sharp$ on $J \times A$. See Proposition 7.8 in [19].

4.2. Estimates for sub- and supersolutions

Let $B = B(x, r) \subset Y$ and $a \in \mathbb{R}$. For $\sigma, \delta \in (0, 1]$, set

$$\begin{aligned} &\delta B = B(x, \delta r),\\ &I^- = (a - \Psi(r), a), \quad I^+ = (a, a + \Psi(r)),\\ &I^-_\sigma = (a - \sigma\Psi(r), a), \quad I^+_\sigma = (a, a + \sigma\Psi(r)),\\ &Q^-(x, a, r) = I^- \times B(x, r), \quad Q^+(x, a, r) = I^+ \times B(x, r),\\ &Q^-_{\sigma,\delta} = I^-_\sigma \times \delta B, \quad Q^+_{\sigma,\delta} = I^+_\sigma \times \delta B. \end{aligned}$$

Let $0 < \sigma' < \sigma \le 1$ and $\hat\sigma := \sigma - \sigma'$. Let χ be a smooth function of the time variable t such that $0 \le \chi \le 1$, $\chi = 0$ in $(-\infty, a - \sigma\Psi(r))$, $\chi = 1$ in $(a - \sigma'\Psi(r), \infty)$ and

$$0 \le \chi' \le \frac{2}{\hat\sigma\Psi(r)}.$$

Let $0 < \delta' < \delta < 1$ and $\hat\delta := \delta - \delta'$. Let $d\bar\mu = d\mu \times dt$.

Lemma 4.4. *Let $p \ge 2$. Then there exists a cutoff function $\psi \in \mathrm{CSA}(\Psi, C_0)$ for $B(x, \delta' r)$ in $B(x, \delta' r + \hat\delta r)$ and constants $a_1 \in (0, 1)$, $A_1, A_2 \in [0, \infty)$ depending on C_0, C_{10}, C_{11} such that*

$$\begin{aligned} &\sup_{t\in I^-_{\sigma'}} \int u^p \psi^2\, d\mu + a_1 \int_{I^-_{\sigma'}} \int \psi^2\, d\Gamma(u^{p/2}, u^{p/2})\, dt\\ &\qquad \le \left(\left(A_1(1 + C_2)\frac{1}{\Psi(\hat\delta r)} + A_2\, C_3\right)p^{\beta_2} + \frac{2}{\hat\sigma\Psi(r)}\right) \int_{Q^-_{\sigma,\delta}} u^p\, d\bar\mu \end{aligned} \tag{4.3}$$

holds for any non-negative local very weak subsolution u of the heat equation for L_t in $Q = Q^-(x, a, r)$ which satisfies $\int_{I^-_\sigma} \int_{\delta B} u^p\, d\mu\, dt < \infty$.

Proof. We follow the line of reasoning in the proof of Theorem 3.11 in [19]. We pick $k = 2(p-1)$ and $\epsilon = c^*/p^2$ for some sufficiently small $c > 0$ that will be chosen later. By Lemma 2.7, we have for any $s \in I^-$, any cutoff function $\psi \in \mathrm{CSA}(\Psi, \epsilon, C_0)$ for $B(x, \delta' r)$ in $B(x, \delta' r + \hat\delta r)$, and any non-negative function $f \in \mathcal{F} \cap \mathcal{C}_{\mathrm{c}}(X)$, $f_n := f \wedge n$, that

$$\begin{aligned} -\mathcal{E}^{\mathrm{s}}_s(f, f f_n^{p-2}\psi^2) \le{}& \left(16 C_{10}\,\epsilon - \frac{1}{2\, C_{10}}\right) \int \psi^2 f_n^{p-2}\, d\Gamma(f, f)\\ &+ \left(4\, C_{10}\,\epsilon\,(p-2)^2 - \frac{p-2}{C_{10}}\right) \int \psi^2 f_n^{p-2}\, d\Gamma(f_n, f_n)\\ &+ 8 C_{10}\, \frac{C_0(\epsilon)}{\Psi(\hat\delta r)} \int \psi f^2 f_n^{p-2}\, d\mu. \end{aligned}$$

By Lemma 2.9, we have

$$
\begin{aligned}
&|\mathcal{E}_s^{\mathrm{skew}}(f, f f_n^{p-2}\psi^2)| + |\mathcal{E}_s^{\mathrm{sym}}(f^2 f_n^{p-2}\psi^2, 1)| \\
&\qquad \le 4\, C_{11}\, \epsilon^{1/2} \int f_n^{p-2}\, \psi^2\, d\Gamma(f,f) \\
&\qquad\quad + C_{11}\, \epsilon^{1/2}\Big((p-2)^2 + \frac{p(p-2)}{4}\Big) \int f_n^{p-2}\, \psi^2\, d\Gamma(f_n, f_n) \\
&\qquad\quad + 4\big(C_2 + C_3 \Psi(\hat{\delta} r)\big) \frac{C_1(\epsilon)}{\Psi(\hat{\delta} r)} \int_{\delta B} f^2 f_n^{p-2}\, d\mu.
\end{aligned}
$$

Combining the two estimates, we get

$$
\begin{aligned}
-\mathcal{E}_s(f, f f_n^{p-2}\psi^2) \le{}& \Big(16 C_{10}\,\epsilon - \frac{1}{2\,C_{10}} + 4\, C_{11}\,\epsilon^{1/2}\Big) \int \psi^2 f_n^{p-2}\, d\Gamma(f,f) \\
&+ \Big(4\, C_{10}\,\epsilon (p-2)^2 - \frac{1}{C_{10}}(p-2) + C_{11}\,\epsilon^{1/2}\Big((p-2)^2 + \frac{p(p-2)}{4}\Big)\Big) \\
&\cdot \int \psi^2 f_n^{p-2}\, d\Gamma(f_n, f_n) \\
&+ \big[8\, C_{10} + 4\big(C_2 + C_3\Psi(\hat{\delta} r)\big)\big] \frac{C_1(\epsilon)}{\Psi(\hat{\delta} r)} \int_{\delta B} f^2 f_n^{p-2}\, d\mu, \qquad (4.4)
\end{aligned}
$$

for any non-negative $f \in \mathcal{F} \cap \mathcal{C}_{\mathrm{c}}(X)$. By the regularity of the reference form, Assumption 0 and Lemma 2.12 in [19], we can, for any $t \in I^-$, approximate the very weak subsolution $u(t,\cdot)$ by functions in $\mathcal{F} \cap \mathcal{C}_{\mathrm{c}}(X)$, so that (4.4) holds with $u(t,\cdot)$ in place of f. On each side of the inequality, we take the Steklov average at t. Notice that, in fact, the right-hand side does not depend on s. Writing u for $u(t,\cdot)$ and u_n for $u_n(t,\cdot)$, we obtain

$$
\begin{aligned}
&-\frac{1}{h}\int_t^{t+h} \mathcal{E}_s(u, u u_n^{p-2}\psi^2)\, ds \\
&\qquad \le \Big(16\, C_{10}\,\epsilon - \frac{1}{2\,C_{10}} + 4\, C_{11}\,\epsilon^{1/2}\Big) \int \psi^2\, u_n^{p-2}\, d\Gamma(u,u) \\
&\qquad\quad + \Big(4\, C_{10}\,\epsilon (p-2)^2 - \frac{1}{C_{10}}(p-2) + C_{11}\,\epsilon^{1/2}\Big((p-2)^2 + \frac{p(p-2)}{4}\Big)\Big) \\
&\qquad\quad \cdot \int \psi^2\, u_n^{p-2}\, d\Gamma(u_n, u_n) \\
&\qquad\quad + \big[8\, C_{10} + 4\big(C_2 + C_3\Psi(\hat{\delta} r)\big)\big] \frac{C_1(\epsilon)}{\Psi(\hat{\delta} r)} \int_{\delta B} u^2\, u_n^{p-2}\, d\mu \qquad (4.5)
\end{aligned}
$$

This is the analog of Step 1 in the proof of Theorem 3.11 in [19].

For a positive integer n, let $u_n := u \wedge n$, and define a function $\mathcal{H}_n : \mathbb{R} \to \mathbb{R}$ by

$$
\mathcal{H}_n(v) := \begin{cases} \frac{1}{p}\, v^2 (v \wedge n)^{p-2}, & \text{if } v \le n, \\ \frac{1}{2}\, v^2 (v \wedge n)^{p-2} + n^p(1/p - 1/2), & \text{if } v > n. \end{cases}
$$

Then $\mathcal{H}_n'(v) = v(v \wedge n)^{p-2}$. For a small real number $h > 0$, let

$$u_h(t) := \frac{1}{h}\int_t^{t+h} u(s)ds, \quad t \in (a-\Psi(r), a-h),$$

be the Steklov average of u. In this proof, the subscript of the Steklov average will always be denoted as h, and u_h should not be confused with the bounded approximation u_n.

We will write $u_h(t,\cdot)$ for $u_h(t)$. Note that $u_h \in L^1((a-\Psi(r), a-h) \to \mathcal{F})$, and $\mathcal{H}_n(u(t,\cdot)), \mathcal{H}_n(u_h(t,\cdot)) \in \mathcal{F}_{\mathrm{loc}}$ at almost every t. The Steklov average u_h has a strong time-derivative

$$\frac{\partial}{\partial t}u_h(t,x) = \frac{1}{h}\big(u(t+h,x) - u(t,x)\big).$$

Let $s_0 = a - \frac{1+\sigma}{2}\Psi(r)$. Following the proof of Theorem 3.11 in [19] line by line, we obtain that for a.e. $t_0 \in I^-_{\sigma'}$, for h sufficiently small so that $t_0 + h < a$, and for $J := (s_0, t_0)$,

$$\begin{aligned}
(4.6)\quad & \int_X \mathcal{H}_n(u_h(t_0,\cdot))\psi^2\,d\mu \\
&\le -\int_J\int_X \frac{\partial u_h(t,\cdot)}{\partial t}\mathcal{H}_n'(u_h(t,\cdot))\psi^2\chi(t)\,d\mu\,dt + \int_J\int_X \mathcal{H}_n(u_h)\psi^2\chi'\,d\mu\,dt \\
&\le -\int_J \frac{1}{h}\int_t^{t+h} \mathcal{E}_s(u(s,\cdot), \mathcal{H}_n'(u_h(t,\cdot))\psi^2)\chi(t)ds\,dt \\
&\quad + \int_J\int_X \mathcal{H}_n(u_h)\psi^2\chi'\,d\mu\,dt \\
(4.7)\quad &\le -\int_J \frac{1}{h}\int_t^{t+h} \mathcal{E}_s(u(s,\cdot), [\mathcal{H}_n'(u_h(t,\cdot)) - \mathcal{H}_n'(u(t,\cdot))]\psi^2)ds\,\chi(t)\,dt \\
(4.8)\quad &\quad -\int_J \frac{1}{h}\int_t^{t+h} \mathcal{E}_s(u(s,\cdot) - u(t,\cdot), \mathcal{H}_n'(u(t,\cdot))\psi^2)ds\,\chi(t)\,dt \\
(4.9)\quad &\quad -\int_J \frac{1}{h}\int_t^{t+h} \mathcal{E}_s(u(t,\cdot), \mathcal{H}_n'(u(t,\cdot))\psi^2)ds\,\chi(t)\,dt \\
(4.10)\quad &\quad + \int_J\int_X \mathcal{H}_n(u_h)\psi^2\chi'\,d\mu\,dt.
\end{aligned}$$

We will take the limit as $h \to 0$ on both sides of the inequality. As in Step 2 of the proof of Theorem 3.11 in [19], it can be seen that (4.7) and (4.8) go to 0 as $h \to 0$. As in Step 3 of the proof of Theorem 3.11 in [19], it can be seen that

$$\lim_{h\to 0}\int_X \mathcal{H}_n(u_h(t_0,\cdot))\psi^2\,d\mu = \int_X \mathcal{H}_n(u(t_0,\cdot))\psi^2\,d\mu,$$

and

$$\lim_{h\to 0}\int_J\int_X \mathcal{H}_n(u_h)\psi^2\chi'\,d\mu\,dt = \int_J\int_X \mathcal{H}_n(u)\psi^2\chi'\,d\mu\,dt.$$

We have already estimated the Steklov average in (4.9) in inequality (4.5). Thus, taking the limit as $h \to 0$ in (4.6)–(4.10), we get

$$
\begin{aligned}
&\int_X \mathcal{H}_n(u(t_0,\cdot))\,\psi^2\,d\mu \\
&\quad - \Big(16C_{10}\,\epsilon - \frac{1}{2\,C_{10}} + 4\,C_{11}\,\epsilon^{1/2}\Big)\int_J\int \psi^2\,u_n^{p-2}\,d\Gamma(u,u)\,\chi(t)\,dt \\
&\quad - \Big(4\,C_{10}\,\epsilon(p-2)^2 - \frac{1}{C_{10}}(p-2) + C_{11}\,\epsilon^{1/2}\Big((p-2)^2 + \frac{p(p-2)}{4}\Big)\Big) \\
&\quad \cdot \int_J\int \psi^2\,u_n^{p-2}\,d\Gamma(u_n,u_n)\,\chi(t)\,dt \\
&\le \big[8C_{10} + 4\big(C_2 + C_3\Psi(\hat\delta r)\big)\big]\,\frac{C_1(\epsilon)}{\Psi(\hat\delta r)}\int_J\int_{\delta B} u^2\,u_n^{p-2}\,d\mu\,\chi(t)\,dt \\
&\quad + \int_J\int_X \mathcal{H}_n(u)\,\psi^2\,\chi'\,d\mu\,dt.
\end{aligned}
$$

Finally, we take the supremum over all $t_0 \in I^-_{\sigma'}$ on both sides of the above inequality, and then we let n tend to infinity. This is where we use the assumption that $\int_{I^-_\sigma}\int_{\delta B} u^p\,d\mu\,dt < \infty$. Multiplying both sides by p and setting $\epsilon = c/p^2$ for some sufficiently small $c > 0$ completes the proof. □

Lemma 4.5. *Let $p \in (1+\eta, 2]$ for some small $\eta > 0$. Then there exists a cutoff function $\psi \in \mathrm{CSA}(\Psi, C_0)$ for $B(x,\delta' r)$ in $B(x,\delta' r + \hat\delta r)$ and constants $a_1 \in (0,1)$, $A_1, A_2 \in [0,\infty)$ depending on η, C_0, C_{10}, C_{11} such that*

$$
\begin{aligned}
&\sup_{t\in I^-_{\sigma'}}\int u^p\psi^2\,d\mu + a_1\int_{I^-_{\sigma'}}\int \psi^2\,d\Gamma(u^{p/2},u^{p/2})\,dt \\
&\qquad \le \Big(\Big(A_1(1+C_2)\frac{1}{\Psi(\hat\delta r)} + A_2\,C_3\Big)p^{\beta_2} + \frac{2}{\hat\sigma\Psi(r)}\Big)\int_{Q^-_{\sigma,\delta}} u^p\,d\bar\mu. \qquad (4.11)
\end{aligned}
$$

holds for any locally bounded, non-negative local very weak subsolution u of the heat equation for L_t in $Q = Q^-(x,a,r)$.

We omit the proof of Lemma 4.5 because it is analogous to the proofs of Lemma 4.4 and Lemma 4.6. See also the proof of Lemma 3.12 in [19].

Let $\varepsilon \in (0,1)$ and $u_\varepsilon := u + \varepsilon$.

Lemma 4.6. *Let $0 \neq p \in (-\infty, 1-\eta)$ for some $\eta \in (0,1/2)$. Then there exists a cutoff function $\psi \in \mathrm{CSA}(\Psi, C_0)$ for $B(x,\delta' r)$ in $B(x,\delta' r + \hat\delta r)$ such that the following holds for any locally bounded, non-negative local very weak supersolution u of the heat equation for L_t in Q.*

(i) *Let $Q = Q^-(x,a,r)$. If $p < 0$, then there are $a_1 \in (0,1)$ and $A_1, A_2 \in [0,\infty)$*

depending on C_0*,* C_{10}*,* C_{11} *such that*

$$\sup_{t\in I^-_{\sigma'}} \int u_\varepsilon^p \psi^2\, d\mu + a_1 \int_{I^-_{\sigma'}} \int \psi^2\, d\Gamma(u_\varepsilon^{p/2}, u_\varepsilon^{p/2})\, dt$$

$$\le \Big(\Big(A_1(1+C_2)\frac{1}{\Psi(\hat\delta r)} + A_2\, C_3\Big)|p|(1+|p|^{\beta_2-1}) + \frac{2}{\hat\sigma \Psi(r)}\Big) \int_{Q^-_{\sigma,\delta}} u_\varepsilon^p\, d\bar\mu. \tag{4.12}$$

(ii) *Let* $Q = Q^+(x,a,r)$*. If* $p \in (0, 1-\eta)$*, then there are* $a_1 \in (0,1)$ *and* $A_1, A_2 \in [0,\infty)$ *depending on* η*,* C_0*,* C_{10}*,* C_{11} *such that*

$$\sup_{t\in I^+_{\sigma'}} \int u_\varepsilon^p \psi^2\, d\mu + a_1 \int_{I^+_{\sigma'}} \int \psi^2\, d\Gamma(u_\varepsilon^{p/2}, u_\varepsilon^{p/2})\, dt$$

$$\le \Big(\Big(A_1(1+C_2)\frac{1}{\Psi(\hat\delta r)} + A_2\, C_3\Big)p(1+p^{\beta_2-1}) + \frac{2}{\hat\sigma \Psi(r)}\Big) \int_{Q^+_{\sigma,\delta}} u_\varepsilon^p\, d\bar\mu. \tag{4.13}$$

Proof. First, consider the case $p \in (-\infty, 0)$. Let $\epsilon \in (0,1)$ be small (to be chosen later). Let $\psi \in \mathrm{CSA}(\Psi, \epsilon, C_0)$ a cutoff function for $B(x, \delta' r)$ in $B(x, \delta' r + \hat\delta r)$. By Lemma 2.8, we have for small $\varepsilon > 0$ and for large $k \sim (1-p)$, that

$$\mathcal{E}_t^{\mathrm{s}}(u_\varepsilon, u_\varepsilon^{p-1}\psi^2)$$
$$\le \Big(\frac{2\, C_{10}\, \epsilon}{\eta} p^2 + \frac{1}{C_{10}}\,(p-(1-\eta/2))\Big) \int \psi^2 u_\varepsilon^{p-2}\, d\Gamma(u_\varepsilon, u_\varepsilon) + \frac{8\, C_{10}}{\eta} \frac{C_0(\epsilon)}{\Psi(r)} \int \psi\, u_\varepsilon^p\, d\mu.$$

By (2.14) and Assumption 1, we have for $C = 1 + |2-p|/|p|$,

$$|\mathcal{E}_t^{\mathrm{sym}}(u_\varepsilon^p \psi^2, 1)| + |\mathcal{E}_t^{\mathrm{skew}}(u_\varepsilon, u_\varepsilon^{p-1}\psi^2)|$$
$$\le C\, C_{11}\, \epsilon^{1/2} \frac{p^2}{4} \int \psi^2 u_\varepsilon^{p-2}\, d\Gamma(u_\varepsilon, u_\varepsilon) + C\big(C_2 + C_3 \Psi(\hat\delta r)\big) \frac{C_1(\epsilon)}{\Psi(\hat\delta r)} \int_{\delta B} u_\varepsilon^p\, d\mu.$$

By Lemma 2.10, we have

$$|\mathcal{E}_t(\varepsilon, u_\varepsilon^{p-1}\psi^2)|$$
$$\le C_{11}\, \epsilon^{1/2} \frac{(p-1)^2}{4} \int \psi^2 u_\varepsilon^{p-2}\, d\Gamma(u_\varepsilon, u_\varepsilon) + \big(C_2 + C_3 \Psi(\hat\delta r)\big) \frac{C_1(\epsilon)}{\Psi(\hat\delta r)} \int_{\delta B} u_\varepsilon^p\, d\mu.$$

If $p < -(1-\eta)$, then we choose $\epsilon = c\eta/p^2$ for a sufficiently small constant $c > 0$. Otherwise, we let $\epsilon = c\eta^2$. Then the proof for the case $p \in (-\infty, 0)$ can be completed similarly to the proof of Lemma 4.4, see also Lemma 3.13 in [19].

For the case $p \in (0, 1-\eta)$, let χ be such that $0 \le \chi \le 1$, $\chi = 0$ in $(a + \sigma\Psi(r), \infty)$, $\chi = 1$ in $(-\infty, a + \sigma'\Psi(r))$, and

$$0 \ge \chi' \ge -\frac{2}{\hat\sigma \Psi(r)}.$$

The proof of (4.13) can be now completed similarly to the case $p \in (-\infty, 0)$, we skip the details. □

It is clear from the proofs that in the above lemmas the cutoff functions ψ can be chosen to be in $\mathrm{CSA}(\Psi, c(p^{-2} \wedge 1), C_0)$ for a small enough constant $c = c(\eta) > 0$.

4.3. Mean value estimates

In addition to the assumptions made at the beginning of Section 4, we assume here that the reference form $(\mathcal{E}^*, \mathcal{F})$ satisfies the localized Sobolev inequality SI(Ψ) on Y up to scale R_0. Let a_1 be small enough and A_1, A_2 large enough so that the estimates of Section 4.2 hold with these constants. Set $A_1' := A_1(1+C_2)/a_1$ and $A_2' := A_2 C_3/a_1$. Define δB, I^-, I^+, I^-_σ, I^+_σ, $Q^-_{\sigma,\delta}$, $Q^+_{\sigma,\delta}$ as in Section 4.2. In addition, assume that $2B \subset Y$.

Theorem 4.7. *Suppose Assumptions 0 and 1, A2-Y, VD, CSA(Ψ) and SI(Ψ) are satisfied on Y up to scale R_0. Let $p > 1+\eta$ for some $\eta > 0$. Fix a ball $B = B(x,r)$, $0 < r \leq R_0/4$, with $B(x,4r) \subsetneq B(x,8r) \subset Y$. Then there exists a constant A, depending only on η, β_1, β_2, C_Ψ, κ, C_{SI}, C_{VD}, C_0, C_{10} and C_{11}, such that, for any $a \in \mathbb{R}$, any $0 < \sigma' < \sigma \leq 1$, $0 < \delta' < \delta \leq 1$, and any non-negative local very weak subsolution u of the heat equation for L_t in $Q = Q^-(x,a,r)$, we have*

$$\sup_{Q^-_{\sigma',\delta'}} \{u^p\} \leq \left[\big(A_1' + A_2'\Psi((\delta-\delta')r)\big)(\delta-\delta')^{-\beta_2} p^{\beta_2} + (\sigma-\sigma')^{-1}\right]^{\frac{2\kappa-1}{\kappa-1}} \cdot \frac{A}{\Psi(r)\mu(B)} \int_{Q^-_{\sigma,\delta}} u^p \, d\bar{\mu}. \tag{4.14}$$

Proof. First, consider the case $p \geq 2$. For a ball $B_R = B(x,R)$, let $E(B_R) = C_{\text{SI}}\Psi(R)V(x,R)^{-1+1/\kappa}$ be the prefactor in the Sobolev inequality (3.3). Consider $0 \leq v \in \mathcal{F}_{\text{loc}}(B)$ and let $v_n = v \wedge n$. By Lemma 2.7, we have $v_n^q \in \mathcal{F}_{\text{loc}}(B)$ for all $q \geq 1$.

Let $0 < \delta_1 < \delta_0 \leq 1$ and $\hat{\delta}_0 := \delta_0 - \delta_1$. Let $\psi \in \text{CSA}(\Psi, \epsilon, C_0)$ be the cutoff function for $B(x,\delta_1 r)$ in $B(x, \delta_1 r + \hat{\delta}_0 r)$ provided by Lemma 4.4. We now apply the Hölder inequality, the Sobolev inequality on $B_{\delta_0 r}$ with $f = \psi v_n$, (2.3) and CSA(Ψ, ϵ, C_0). We get

$$\begin{aligned}
\int_{B(x,\delta_1 r)} v_n^{2(2-1/\kappa)} \, d\mu &\leq \Big(\int_{B(x,\delta_1 r)} v_n^{2\kappa} \, d\mu\Big)^{1/\kappa} \Big(\int_{B(x,\delta_1 r)} v_n^2 \, d\mu\Big)^{1-1/\kappa} \\
&\leq E(B(x,\delta_0 r)) \Big(\int_{B(x,\delta_0 r)} d\Gamma(\psi v_n, \psi v_n)\Big) \Big(\int_{B(x,\delta_1 r)} v_n^2 \, d\mu\Big)^{1-1/\kappa} \\
&\leq E(B(x,\delta_0 r)) \Big(2\int_{B(x,\delta_0 r)} \psi^2 \, d\Gamma(v_n, v_n) + 2\int_{B(x,\delta_0 r)} v_n^2 \, d\Gamma(\psi,\psi)\Big) \\
&\quad \cdot \Big(\int_{B(x,\delta_1 r)} v_n^2 \, d\mu\Big)^{1-1/\kappa} \\
&\leq 2E(B(x,\delta_0 r)) \Big((1+\epsilon)\int_{B(x,\delta_0 r)} \psi^2 \, d\Gamma(v_n,v_n) + \frac{C_0(\epsilon)}{\Psi(\hat{\delta}_0 r)} \int_{B(x,\delta_0 r)} v_n^2 \, d\mu\Big) \\
&\quad \cdot \Big(\int_{B(x,\delta_1 r)} v_n^2 \, d\mu\Big)^{1-1/\kappa}.
\end{aligned}$$

Letting $n \to \infty$, we obtain

$$\begin{aligned} \int_{B(x,\delta_1 r)} v^{2(2-1/\kappa)}\,d\mu &\le 2E(B(x,\delta_0 r))\Big((1+\epsilon)\int_{B(x,\delta_0 r)} \psi^2\,d\Gamma(v,v) + \frac{C_0(\epsilon)}{\Psi(\hat{\delta}_0 r)}\int_{B(x,\delta_0 r)} v^2\,d\mu\Big) \\ &\quad \cdot \Big(\int_{B(x,\delta_1 r)} v^2\,d\mu\Big)^{1-1/\kappa}. \end{aligned} \tag{4.15}$$

Now let $u \in L^2_{\rm loc}(I \to \mathcal{F}; B)$ be a non-negative local very weak subsolution of the heat equation in Q. Then for almost every $t \in I$, $v := u(t,\cdot)$ is in $\mathcal{F}_{\rm loc}(B)$ and satisfies (4.15). Let $0 < \sigma_1 < \sigma_0 \le 1$ and integrate (4.15) over $I^-_{\sigma_1}$. Applying then the Hölder inequality to the time integral yields

$$\begin{aligned} \int_{I^-_{\sigma_1}}\int_{\delta_1 B} u^{2\theta}\,d\mu\,dt &\le 2E(\delta_0 B)\Big((1+\epsilon)\int_{I^-_{\sigma_1}}\int_{\delta_0 B} \psi^2\,d\Gamma(u,u)dt + \frac{C_0(\epsilon)}{\Psi(\hat{\delta}_0 r)}\int_{I^-_{\sigma_1}}\int_{\delta_0 B} u^2\,d\mu\,dt\Big) \\ &\quad \cdot \sup_{t\in I^-_{\sigma_1}}\Big(\int_{\delta_0 B}\psi^2 u^2\,d\mu\Big)^{1-1/\kappa}, \end{aligned} \tag{4.16}$$

where $\theta = 2 - 1/\kappa$. Note that the right-hand side of (4.16) is finite by Lemma 4.4 (applied with $p = 2$). Hence the left hand side is finite and this means that u^θ is in $L^2(I^-_{\sigma_1} \times \delta_1 B)$, which is the prerequisite to apply Lemma 4.4 with $p = 2\theta$ in the next step.

Let $0 < \sigma_2 < \sigma_1$ and $0 < \delta_2 < \delta_1$. Applying Lemma 4.4 with $p = 2\theta$, we obtain that there exists a cutoff function in CSA(Ψ, C_0) for $B(x, \delta_2 r + (\delta_1 - \delta_2)r)$ in $\delta_1 B$ with which we can repeat the argument above to obtain that $u^{\theta\cdot\theta} \in L^2(I^-_{\sigma_2} \times \delta_2 B)$. Iteratively, we obtain that, for any strictly decreasing sequences (σ_i), $0 < \sigma_{i+1} < \sigma_i \le 1$, and (δ_i), $0 < \delta_{i+1} < \delta_i \le 1$, we have $\int_{Q^-_{\sigma_{i+1},\delta_{i+1}}} u^{2\theta^{i+1}}\,d\mu\,dt < \infty$. Therefore, for p as in Theorem 4.7,

$$\int_{Q^-_{\sigma,\delta}} u^{pq}\,d\mu\,dt < \infty, \qquad \text{for arbitrary } \sigma,\delta \in (0,1), \text{ and } q \ge 1. \tag{4.17}$$

Now we pick specific sequences (σ_i), (δ_i) and (q_i) with the aim of applying (4.16), Lemma 4.4 and CSA(Ψ) iteratively. Let $\sigma', \sigma, \delta', \delta$ be as in the Theorem. Set $\hat{\sigma}_i = (\sigma - \sigma')2^{-i-1}$ so that $\sum_{i=0}^\infty \hat{\sigma}_i = \sigma - \sigma'$. Set also $\sigma_0 = \sigma$, $\sigma_{i+1} = \sigma_i - \hat{\sigma}_i = \sigma - \sum_{j=0}^i \hat{\sigma}_j$. Set $\hat{\delta}_i = (\delta - \delta')2^{-i-1}$ so that $\sum_{i=0}^\infty \hat{\delta}_i = \delta - \delta'$. Set also $\delta_0 = \delta$, $\delta_{i+1} = \delta_i - \hat{\delta}_i = \delta - \sum_{j=0}^i \hat{\delta}_j$.

By Lemma 3.2 and (2.4),

$$\Big(\frac{\mu(B)}{\mu(\delta_i B)}\Big)^{1-1/\kappa} \le C\,\frac{\Psi(r)}{\Psi(\delta_i r)}. \tag{4.18}$$

Let $\psi_i \in \mathrm{CSA}(\Psi,\epsilon,C_0)$ be the cutoff function for $B(x,\delta_{i+1}r)$ in $B(x,\delta_{i+1}r+\hat{\delta}_i r)$ that is given by Lemma 4.4. Here, $\epsilon = c(p\theta^i)^{-2}$ for some small fixed constant $c>0$ that depends at most on C_{10} and C_{11}.

Similar to how we obtained (4.16) but with $u^{p\theta^i/2}$ in place of u, we get

$$\begin{aligned}
&\iint_{Q^-_{\sigma_{i+1},\delta_{i+1}}} u^{p\theta^{i+1}}\,d\bar{\mu}\\
&\le 2E(\delta_i B)\Big((1+\epsilon)\int_{I^-_{\sigma_{i+1}}}\int_{\delta_i B}\psi_i^2\,d\Gamma(u^{p\theta^i/2},u^{p\theta^i/2})\,dt + \frac{C_0(\epsilon)}{\Psi(\hat{\delta}_i r)}\int_{I^-_{\sigma_{i+1}}}\int_{\delta_i B} u^{p\theta^i}\,d\bar{\mu}\Big)\\
&\quad\cdot\Big(\sup_{t\in I^-_{\sigma_{i+1}}}\int_{\delta_i B}\psi_i^2 u^{p\theta^i}\,d\mu\Big)^{1-1/\kappa}.
\end{aligned}$$

By Lemma 4.4 together with (4.17), and by (4.18), the right-hand side is no more than

$$\begin{aligned}
&\frac{C\,\Psi(\delta_i r)}{\mu(\delta_i B)^{1-1/\kappa}}\Big(\Big[\Big(\Big(A_1'\frac{1}{\Psi(\hat{\delta}_i r)}+A_2'\Big)(p\theta^i)^{\beta_2}+\frac{2}{\hat{\sigma}_i\Psi(r)}\Big)+\frac{C_0(\epsilon)}{\Psi(\hat{\delta}_i r)}\Big]\iint_{Q^-_{\sigma_i,\delta_i}} u^{p\theta^i}\,d\bar{\mu}\Big)^{\theta}\\
&\le \frac{C}{[\Psi(r)\mu(B)]^{1-1/\kappa}}\\
&\quad\cdot\Big(\Big[\Big(\Big((A_1'+A_2'\Psi(\hat{\delta}_i r))\frac{\Psi(r)}{\Psi(\hat{\delta}_i r)}\Big)(p\theta^i)^{\beta_2}+\frac{2}{\hat{\sigma}_i}\Big)+\frac{C_0(\epsilon)\Psi(r)}{\Psi(\hat{\delta}_i r)}\Big]\iint_{Q^-_{\sigma_i,\delta_i}} u^{p\theta^i}\,d\bar{\mu}\Big)^{\theta}\\
&\le \frac{1}{[\Psi(r)\mu(B)]^{1-1/\kappa}}\Big(C^{i+1}\Big(\big(A_1'+A_2'\Psi(\hat{\delta}r)\big)\hat{\delta}^{-\beta_2}p^{\beta_2}+\hat{\sigma}^{-1}\Big)\iint_{Q^-_{\sigma_i,\delta_i}} u^{p\theta^i}\,d\bar{\mu}\Big)^{\theta},
\end{aligned}$$

where the constant $C\in(0,\infty)$ (which may change from line to line) depends at most on θ, β_1, β_2, κ, C_Ψ, C_{SI}, C_{VD}, C_0, C_{10}, C_{11}. Hence,

$$\begin{aligned}
&\Big(\iint_{Q^-_{\sigma_{i+1},\delta_{i+1}}} u^{p\theta^{i+1}}\,d\bar{\mu}\Big)^{\theta^{-i-1}}\\
&\qquad\le \Big(\frac{1}{[\Psi(r)\mu(B)]^{1-1/\kappa}}\Big)^{\sum\theta^{-1-j}} C^{\sum(j+1)\theta^{-j}}\\
&\qquad\quad\cdot\Big[\Big(\big(A_1'+A_2'\Psi(\hat{\delta}r)\big)\hat{\delta}^{-\beta_2}p^{\beta_2}+\hat{\sigma}^{-1}\Big)\Big]^{\sum\theta^{-j}}\int_{Q^-_{\sigma,\delta}} u^p\,d\bar{\mu},
\end{aligned}$$

where all the summations are taken from $j=0$ to $j=i$. Letting i tend to infinity, we obtain

$$\sup_{Q^-_{\sigma',\delta'}}\{u^p\}\le\Big[\Big(\big(A_1'+A_2'\Psi(\hat{\delta}r)\big)\hat{\delta}^{-\beta_2}p^{\beta_2}+\hat{\sigma}^{-1}\Big)\Big]^{\frac{2\kappa-1}{\kappa-1}}\frac{C}{\Psi(r)\mu(B)}\int_{Q^-_{\sigma,\delta}} u^p\,d\bar{\mu}.$$

This yields (4.14).

At this stage of the proof, Corollary 4.8 already follows. Thus, in the case $1+\eta<p<2$ the assertion can be proved similarly, by using Lemma 4.5 and Corollary 4.8. □

Corollary 4.8. *Under the same hypotheses as Theorem 4.7. Then any non-negative local very weak subsolution u for L_t in Q is locally bounded. In particular, any local very weak solution of u for L_t in Q is locally bounded.*

Proof. The first statement follows from the proof of Theorem 4.7. By Proposition 3.4 in [19], for any local very weak solution u of the heat equation, $|u|$ is a non-negative local very weak subsolution. □

Theorem 4.9. *Suppose Assumptions 0 and 1,* A2-Y*,* VD*,* CSA(Ψ) *and* SI(Ψ) *are satisfied on Y up to scale R_0. Let $0 < p < 2$. Fix a ball $B = B(x,r)$, $0 < r \le R_0/4$, with $B(x,4r) \subsetneq B(x,8r) \subset Y$. Then there exists a constant A, depending only on C_Ψ, β_1, β_2, κ, C_{SI}, C_{VD}, C_0, C_{10}, C_{11}, such that, for any $a \in \mathbb{R}$, any $0 < \sigma' < \sigma \le 1$, $0 < \delta' < \delta \le 1$, and any non-negative local very weak subsolution u of the heat equation for L_t in $Q = Q^-(x,a,r)$, we have*

$$\sup_{Q^-_{\sigma',\delta'}} \{u^p\} \le C\, \frac{A}{\Psi(r)\mu(B)} \int_{Q^-_{\sigma,\delta}} u^p \, d\bar{\mu},$$

where

$$C = \left(\frac{4}{3}\right)^{\beta_2 \frac{2\kappa-1}{2(\kappa-1)} 4/p} \left[\big(A_1' + A_2'\Psi((\delta-\delta')r)\big)(\delta-\delta')^{-\beta_2} 2^{\beta_2} + (\sigma-\sigma')^{-1}\right]^{\frac{2\kappa-1}{\kappa-1}}.$$

Proof. We follow Theorems 2.2.3 and 5.2.9 in [28]. Let $D_1 := 2^{\beta_2}\big(A_1' + A_2'\Psi((\delta - \delta')r)\big)$. By (4.14) with $p = 2$, we have for any $0 < \sigma' < \sigma \le 1$, $0 < \delta' < \delta \le 1$,

$$\begin{aligned}
\sup_{Q^-_{\sigma',\delta'}} u &\le \left[D_1(\delta-\delta')^{-\beta_2} + (\sigma-\sigma')^{-1}\right]^{\frac{2\kappa-1}{2(\kappa-1)}} \left(\frac{A}{\Psi(r)\mu(B)}\right)^{1/2} \left(\int_{Q^-_{\sigma,\delta}} u^2 \, d\bar{\mu}\right)^{1/2} \\
&\le \left[D_1(\delta-\delta')^{-\beta_2} + (\sigma-\sigma')^{-1}\right]^{\frac{2\kappa-1}{2(\kappa-1)}} J \sup_{Q^-_{\sigma,\delta}} u^{(2-p)/2},
\end{aligned}$$

where $J := \big(\frac{A}{\Psi(r)\mu(B)}\big)^{1/2}\big(\int_{Q^-_{\sigma,\delta}} u^p \, d\bar{\mu}\big)^{1/2}$.

Set $\delta_0 := \delta'$, $\delta_{i+1} := \delta_i + (\delta - \delta_i)/4$. Then $(\delta - \delta_i) = \left(\frac{3}{4}\right)^i (\delta - \delta')$. Similarly, we set $\sigma_0 := \sigma'$, $\sigma_{i+1} := \sigma_i + (\sigma - \sigma_i)/4$. Applying the above inequality for each i yields

$$\sup_{Q^-_{\sigma_i,\delta_i}} u \le \left(\frac{4}{3}\right)^{i\beta_2 \frac{2\kappa-1}{2(\kappa-1)}} \left[D_1(\delta-\delta')^{-\beta_2} + (\sigma-\sigma')^{-1}\right]^{\frac{2\kappa-1}{2(\kappa-1)}} J \sup_{Q^-_{\sigma_{i+1},\delta_{i+1}}} u^{(2-p)/2},$$

Iterating this inequality, we get for $i = 1, 2, \ldots$,

$$\sup_{Q^-_{\sigma',\delta'}} u \le C \sup_{Q^-_{\sigma_i,\delta_i}} u^{(1-p/2)^i},$$

where

$$C = \left(\frac{4}{3}\right)^{\beta_2 \frac{2\kappa-1}{2(\kappa-1)} \sum_{j=0}^{i-1} j(1-p/2)^j} \left[\left[D_1(\delta-\delta')^{-\beta_2} + (\sigma-\sigma')^{-1}\right]^{\frac{2\kappa-1}{2(\kappa-1)}} J\right]^{\sum_{j=0}^{i-1}(1-p/2)^j}.$$

Letting $i \to \infty$, and noting that $\lim_{i\to\infty} \sup_{Q^-_{\sigma_i,\delta_i}} u^{(1-p/2)^i} = 1$, we get

$$\sup_{Q^-_{\sigma',\delta'}} u \leq \left(\frac{4}{3}\right)^{\beta_2 \frac{2\kappa-1}{2(\kappa-1)} 4/p^2} \left[\left[D_1(\delta-\delta')^{-\beta_2} + (\sigma-\sigma')^{-1} \right]^{\frac{2\kappa-1}{2(\kappa-1)}} J \right]^{2/p}.$$

Rasing each side to power p we get the desired inequality. □

The next theorem can be proved analogously to the proof of Theorem 4.7, by applying Lemma 4.6 instead of Lemma 4.4.

Theorem 4.10. *Suppose Assumptions 0 and 1,* A2-Y, VD, CSA(Ψ) *and* SI(Ψ) *are satisfied on Y up to scale R_0. Let $0 \neq p \in (-\infty, 1-\eta)$ for some small $\eta \in (0,1)$. Fix a ball $B = B(x,r)$, $0 < r \leq R_0/4$, with $B(x,4r) \subsetneq B(x,8r) \subset Y$. Let $a \in \mathbb{R}$. Let $u \in \mathcal{F}_{\mathrm{loc}}(Q)$ be any non-negative local very weak supersolution of the heat equation for L_t in Q. Suppose that u is locally bounded. Let $\varepsilon \in (0,1)$ and $u_\varepsilon := u + \varepsilon$. Let $0 < \sigma' < \sigma \leq 1$, $0 < \delta' < \delta \leq 1$.*

(i) *Let $Q = Q^-(x,a,r)$. If $p \in (-\infty, 0)$, then there exists a constant A, depending only on $\beta_1, \beta_2, C_\Psi, \kappa, C_{\mathrm{SI}}, C_{\mathrm{VD}}, C_0, C_{10}, C_{11}$, such that*

$$\sup_{Q^-_{\sigma',\delta'}} \{u_\varepsilon^p\} \leq \left[\left(A_1' + A_2' \Psi((\delta-\delta')r)\right)(\delta-\delta')^{-\beta_2}(1+|p|^{\beta_2}) + (\sigma-\sigma')^{-1} \right]^{\frac{2\kappa-1}{\kappa-1}} \cdot \frac{A}{\Psi(r)\mu(B)} \int_{Q^-_{\sigma,\delta}} u_\varepsilon^p \, d\bar{\mu}.$$

(ii) *Let $Q = Q^+(x,a,r)$. If $p \in (0, 1-\eta)$, then there exists a constant A, depending only on $\eta, \beta_1, \beta_2, C_\Psi, \kappa, C_{\mathrm{SI}}, C_{\mathrm{VD}}, C_0, C_{10}, C_{11}$, such that*

$$\sup_{Q^+_{\sigma',\delta'}} \{u_\varepsilon^p\} \leq \left[\left(A_1' + A_2' \Psi((\delta-\delta')r)\right)(\delta-\delta')^{-\beta_2}(1+p^{\beta_2}) + (\sigma-\sigma')^{-1} \right]^{\frac{2\kappa-1}{\kappa-1}} \cdot \frac{A}{\Psi(r)\mu(B)} \int_{Q^+_{\sigma,\delta}} u_\varepsilon^p \, d\bar{\mu}.$$

5. Parabolic Harnack inequality

5.1. The log lemma and an abstract lemma

Let $(X, d, \mu, \mathcal{E}^*, \mathcal{F})$ and $(\mathcal{E}_t, \mathcal{F})$, $t \in \mathbb{R}$, be as in Section 4. In this section, we suppose that Assumptions 0–3 are satisfied for an open subset $Y \subset X$.

Let a_1 be small and A_1, A_2 large enough so that the estimates of Section 4.2 hold with these constants. Recall that for $\varepsilon \in (0,1)$, $u_\varepsilon := u + \varepsilon$.

Lemma 5.1. *Suppose that* Assumptions 0–3 *are satisfied. Let $0 < \sigma < 1$, $0 < \delta < 1$ and $\hat{\delta} := 1 - \delta$. There exists a constant $C \in (0,\infty)$ such that, for any $a \in \mathbb{R}$, $0 < r \leq R_0$, $B = B(x,r) \subset Y$, and any non-negative, locally bounded function $u \in \mathcal{C}_{\mathrm{loc}}(I \to L^2(B))$ which is a local very weak supersolution of the heat equation for L_t in Q, there exists a constant $c \in (0,\infty)$ depending on $u(a,\cdot)$, such that*

(i)

$$\bar{\mu}(\{(t,z)\in K_+ : \log u_\varepsilon < -\lambda - c\}) \le C\,\Psi(r)\mu(B)\lambda^{-1}, \qquad \forall \lambda > 0,$$

where $Q = Q^+(x,a,r)$, $K_+ = (a, a+\sigma\Psi(r))\times \delta B$, *and*

(ii)

$$\bar{\mu}(\{(t,z)\in K_- : \log u_\varepsilon > \lambda - c\}) \le C\,\Psi(r)\mu(B)\lambda^{-1}, \qquad \forall \lambda > 0,$$

where $Q = Q^-(x,a,r)$, $K_- = (a-\sigma\Psi(r), a)\times \delta B$.

The constant C *depends on* C_{VD}, C_{PI}, C_Ψ, β_1, β_2, C_0, C_{10}, C_{11}, *and upper bounds on* $(1+C_2+C_4)+(C_3+C_5)\Psi(\hat{\delta}r)$, $1/\delta$ *and* $1/\hat{\delta}$.

Proof. For $h>0$, let

$$u_{\varepsilon,h}(t) := \frac{1}{h}\int_t^{t+h} u_\varepsilon(\tau)\,d\tau$$

be the Steklov average of u_ε. Let $\epsilon \in (0,1)$ (to be chosen later), and let $\psi \in \mathrm{CSA}(\Psi,\epsilon,C_0)$ be the cutoff function for $B(x,\delta r)$ in $B(x,r')$ given by Theorem 3.4, for some $r' \in (\delta r, r)$.

Using the fact that the Steklov average has a strong time-derivative and the assumption that u is local very weak supersolution, we obtain

$$\begin{aligned}
-\frac{d}{dt}\int \log u_{\varepsilon,h}(t)\psi^2\,d\mu &= -\frac{1}{h}\int [u(t+h)-u(t)]\frac{1}{u_{\varepsilon,h}(t)}\,\psi^2\,d\mu \\
&\le \frac{1}{h}\int_t^{t+h} \mathcal{E}_s\Big(u(s), \frac{1}{u_{\varepsilon,h}(t)}\,\psi^2\Big)\,ds \\
&= \frac{1}{h}\int_t^{t+h} \mathcal{E}_s\Big(u(s), \frac{1}{u_{\varepsilon,h}(t)}\,\psi^2 - \frac{1}{u_\varepsilon(t)}\,\psi^2\Big)\,ds \\
&\quad + \frac{1}{h}\int_t^{t+h} \mathcal{E}_s\Big(u(s)-u(t), \frac{1}{u_\varepsilon(t)}\,\psi^2\Big)\,ds \\
&\quad + \frac{1}{h}\int_t^{t+h} \mathcal{E}_s\Big(u_\varepsilon(t), \frac{1}{u_\varepsilon(t)}\,\psi^2\Big) - \mathcal{E}_s\Big(\varepsilon, \frac{1}{u_\varepsilon(t)}\,\psi^2\Big), ds \\
&= f_h(t) + \hat{f}_h(t) + g_h(t).
\end{aligned}$$

It can be shown that $f_h(t)$ and $\hat{f}_h(t)$ tend to 0 in $L^1((a,a+\sigma\Psi(r))\to\mathbb{R})$ as $h\to 0$. Next, we will estimate $g_h(t)$. We write $u_\varepsilon = u_\varepsilon(t)$. Applying (2.2), (2.1), (2.9) and $\mathrm{CSA}(\Psi,\epsilon,C_0)$, we have for any $k_0>0$ that

$$\begin{aligned}
\mathcal{E}_s^{\mathrm{s}}(u_\varepsilon, u_\varepsilon^{-1}\,\psi^2) &= \int 2\psi\,d\Gamma_s(\log(u_\varepsilon),\psi) - \int \psi^2\,d\Gamma_s(\log(u_\varepsilon),\log(u_\varepsilon)) \\
&\le 4k_0\int d\Gamma(\psi,\psi) - \Big(\frac{1}{C_{10}} - \frac{C_{10}}{k_0}\Big)\int \psi^2\,d\Gamma(\log(u_\varepsilon),\log(u_\varepsilon)) \\
&\le -\Big(\frac{1}{C_{10}} - \frac{C_{10}}{k_0}\Big)\int \psi^2\,d\Gamma(\log(u_\varepsilon),\log(u_\varepsilon)) + \frac{4k_0C_{10}C_0(\epsilon)}{\Psi(\hat{\delta}r)}\int_B d\mu.
\end{aligned}$$

By Assumption 1 and Assumption 2, we have

$$\begin{aligned}&\mathcal{E}_s^{\mathrm{skew}}(u_\varepsilon, u_\varepsilon^{-1}\psi^2) + \mathcal{E}_s^{\mathrm{sym}}(\psi^2, 1)\\ &\leq C_{11}\,\epsilon^{1/2}\int \psi^2\, d\Gamma(\log(u_\varepsilon), \log(u_\varepsilon)) + \big(C_2 + C_4 + (C_3 + C_5)\Psi(\hat{\delta} r)\big)\frac{C_1(\epsilon)}{\Psi(\hat{\delta} r)}\int_B d\mu.\end{aligned}$$

By Lemma 2.10,

$$\begin{aligned}&-\mathcal{E}_s(\varepsilon, u_\varepsilon^{-1}\psi^2)\\ &\qquad \leq C_{11}\,\epsilon^{1/2}\,\frac{1}{4}\int \psi^2\, d\Gamma(\log(u_\varepsilon), \log(u_\varepsilon)) + \big(C_2 + C_3\Psi(\hat{\delta} r)\big)\frac{C_1(\epsilon)}{\Psi(\hat{\delta} r)}\int_B d\mu.\end{aligned}$$

Hence, making a suitable choice of k_0 (large) and ϵ (small), we find that for sufficiently large $k > 1$ depending on C_0, C_{10}, C_{11} and an upper bound for $(1 + C_2 + C_4 + (C_3 + C_5)\Psi(\hat{\delta} r))$, we have

$$\begin{aligned}&-\frac{d}{dt}\int \log u_{\varepsilon,h}(t)\psi^2\, d\mu + \frac{1}{k}\int \psi^2\, d\Gamma(\log(u_\varepsilon), \log(u_\varepsilon))\\ &\qquad \leq f_h(t) + \hat{f}_h(t) + \big(1 + C_2 + C_4 + (C_3 + C_5)\,\Psi(\hat{\delta} r)\big)\frac{k}{\Psi(\hat{\delta} r)}\,\mu(B). \end{aligned}\tag{5.1}$$

Let

$$W(t) := -\frac{\int \log u_\varepsilon(t)\psi^2\, d\mu}{\int \psi^2\, d\mu} \quad \text{and} \quad W_h(t) := -\frac{\int \log u_{\varepsilon,h}(t)\psi^2\, d\mu}{\int \psi^2\, d\mu}.$$

By the weighted Poincaré inequality of Theorem 3.4, there is a constant $C_{\mathrm{wPI}} \in (0,1)$ such that, for a.e. $t \in I$,

$$\int |-\log u_\varepsilon(t) - W(t)|^2\, \psi^2\, d\mu \leq C_{\mathrm{wPI}}\,\Psi(r)\int \psi^2\, d\Gamma(\log u_\varepsilon(t), \log u_\varepsilon(t)).$$

The constant C_{wPI} depends only on C_{PI} and an upper bound on $\mu(B)/\mu(\delta B)$.

This and (5.1) yield

$$\begin{aligned}&\frac{d}{dt}W_h(t) + \frac{1}{C\,\Psi(r)\mu(B)}\int_{\delta B} |-\log u_\varepsilon(t) - W(t)|^2\psi^2\, d\mu\\ &\qquad \leq C'\,\frac{f_h(t) + \hat{f}_h(t)}{\int \psi^2\, d\mu} + C'\,\big(1 + C_2 + C_4 + (C_3 + C_5)\Psi(\hat{\delta} r)\big)\,\frac{k}{\Psi(\hat{\delta} r)},\end{aligned}$$

for some constants $C, C' \in (0,\infty)$ that depend only on k, C_{VD} C_{PI}, C_0 and an upper bound on $1/\delta$. Notice that by (2.4), $1/\Psi(\hat{\delta} r) \leq C''/\Psi(r)$ for some constant C'' depending only on C_Ψ, β_2, and on an upper bound on $1/\hat{\delta}$.

Now the proof can be completed easily by following Lemma 4.12 in [19] line by line, except for replacing r^2 by $\Psi(r)$ and applying (2.4) where needed. □

Let U_δ be a collection of measurable subsets of X such that $U_{\delta'} \subset U_\delta$ for any $0 < \delta' < \delta \leq 1$. Let J_σ be a collection of intervals in $\mathbb{R}$ such that $J_{\sigma'} \subset J_\sigma$ for any $0 < \sigma' < \sigma \leq 1$.

Lemma 5.2. *Fix $\sigma^*, \delta^* \in (0,1)$. Let f be a positive measurable function on $J_1 \times U_1$ which satisfies*

$$\sup_{J_{\sigma'} \times U_{\delta'}} f \leq \Big(\Big(\frac{C}{(\delta-\delta')^{\gamma_1}} + \frac{C}{(\sigma-\sigma')^{\gamma_2}} \Big) \frac{1}{|J_1|\mu(U_1)} \int_{J_\sigma} \int_{U_\delta} f^p \, d\mu \, dt \Big)^{1/p}$$

for all $\sigma^* \leq \sigma' < \sigma < 1$, $\delta^* \leq \delta' < \delta < 1$, $p \in (0, 1-\eta)$, for some $\gamma_1, \gamma_2 \geq 0$, $\eta \in (0,1)$, $C \in (0, \infty)$. Suppose further that

$$\bar{\mu}(\{\log f > \lambda\}) \leq C \, \frac{|J_1| \, \mu(U_1)}{\lambda}, \quad \forall \lambda > 0. \tag{5.2}$$

Then there is a constant $A_3 \in [1, \infty)$, depending only on $\sigma^*, \delta^*, \gamma_1, \gamma_2, C$ and a positive lower bound on η, such that

$$\sup_{J_{\sigma^*} \times U_{\delta^*}} f \leq A_3.$$

Proof. We follow the proof of Lemma 3 in [25] (see also the proof of Theorem 4 in [6], and Lemma 2.2.6 in [28]). Without loss of generality, assume for the proof that $|J_1|\mu(U_1) = 1$. Define

$$\phi = \phi(\sigma, \delta) := \sup_{J_\sigma \times U_\delta} f.$$

Decomposing $J_\sigma \times U_\delta$ into the sets where $\log f > \frac{1}{2}\log(\phi)$ and where $\log f \leq \frac{1}{2}\log(\phi)$, we get from (5.2) that

$$\int_{J_\sigma} \int_{U_\delta} f^p \, d\mu \, dt \leq \Big(\sup_{J_\sigma \times U_\delta} f^p \Big) \mu \left(\log f > \tfrac{1}{2} \log \phi \right) + \phi^{p/2} |J_\sigma| \mu(U_\delta) \leq \phi^p \frac{2C}{\log \phi} + \phi^{p/2}.$$

The two terms on the right-hand side are equal if

$$p = \frac{2}{\log \phi} \log \Big(\frac{\log \phi}{2C} \Big).$$

We have $p < 1 - \eta$ if ϕ is sufficiently large, that is, if

$$\phi \geq A_1 \tag{5.3}$$

for some A_1 depending only on η (note we can always take $C \geq 1$). Hence, for $\phi \geq A_1$, the first hypothesis of the lemma yields

$$\log \phi(\sigma', \delta') \leq \frac{1}{p} \log \Big(\frac{C}{(\delta-\delta')^{\gamma_1}} + \frac{C}{(\sigma-\sigma')^{\gamma_2}} \Big) + \frac{\log \phi}{2} + \frac{\log 2}{p}$$

$$\leq \frac{1}{p} \log \Big(\frac{2C}{(\delta-\delta')^{\gamma_1}} + \frac{2C}{(\sigma-\sigma')^{\gamma_2}} \Big) + \frac{\log \phi}{2} = \frac{\log \phi}{2} \Bigg[\frac{\log \left(\frac{2C}{(\delta-\delta')^{\gamma_1}} + \frac{2C}{(\sigma-\sigma')^{\gamma_2}} \right)}{\log \left(\frac{\log \phi}{2C} \right)} + 1 \Bigg].$$

If

$$\frac{\log \phi}{2C} \geq \Big(\frac{2C}{(\delta-\delta')^{\gamma_1}} + \frac{2C}{(\sigma-\sigma')^{\gamma_2}} \Big)^2, \tag{5.4}$$

then

$$\log \phi(\sigma', \delta') \leq \frac{3}{4} \log \phi.$$

On the other hand, if (5.4) or (5.3) is not satisfied, then

$$\log \phi(\sigma', \delta') \le \log \phi \le \log A_1 + 2C\Big(\frac{2C}{(\delta-\delta')^{\gamma_1}} + \frac{2C}{(\sigma-\sigma')^{\gamma_2}}\Big)^2.$$

In all cases, we obtain

$$\log \phi(\sigma', \delta') \le \frac{3}{4}\log \phi(\sigma, \delta) + A_2\Big(\frac{C^2}{(\delta-\delta')^{2\gamma_1}} + \frac{C^2}{(\sigma-\sigma')^{2\gamma_2}}\Big) \tag{5.5}$$

for some constant $A_2 \in (0, \infty)$ depending only on $\sigma^*, \delta^*, \gamma_1, \gamma_2, C$ and a positive lower bound on η. Let $\sigma_j = 1 - \frac{1-\sigma*}{1+j}$ and $\delta_j = 1 - \frac{1-\delta*}{1+j}$. Iterating (5.5), we get

$$\log \phi(\sigma^*, \delta^*) \le A_2 \sum_{j=0}^{\infty} \Big(\frac{3}{4}\Big)^j \Big(\frac{C^2}{(\delta_{j+1}-\delta_j)^{2\gamma_1}} + \frac{C^2}{(\sigma_{j+1}-\sigma_j)^{2\gamma_2}}\Big) =: A_3 < \infty. \quad \square$$

5.2. Parabolic Harnack inequalities

Let $(X, d, \mu, \mathcal{E}^*, \mathcal{F})$ and $(\mathcal{E}_t, \mathcal{F})$, $t \in \mathbb{R}$, be as in Section 4. In this section, we suppose that Assumptions 0 - 3 are satisfied for an open subset $Y \subset X$.

Let $B = B(x, r) \subset X$, $a \in \mathbb{R}$. Fix $\delta \in (0, 1)$ and let $0 < \tau_1 < \tau_2 < \tau_3 < \tau_4 \le 1$. Set

$$\begin{aligned} \delta B &= B(x, \delta r), \\ Q &= Q(x, a, r) = (a, a + \Psi(r)) \times B, \\ Q^- &= (a + \tau_1\Psi(r), a + \tau_2\Psi(r)) \times \delta B, \\ Q^+ &= (a + \tau_3\Psi(r), a + \tau_4\Psi(r)) \times \delta B. \end{aligned}$$

Theorem 5.3. *Suppose* Assumption 0–3 *are satisfied. Then the family* $(\mathcal{E}_t, \mathcal{F})$, $t \in \mathbb{R}$, *satisfies the parabolic Harnack inequality* PHI(Ψ) *on* Y *up to scale* R_0. *That is, there is a constant* $C_{\text{PHI}} \in (0, \infty)$ *such that for any* $a \in \mathbb{R}$, *any ball* $B(x, 4r) \subsetneq B(x, 8r) \subset Y$, $0 < r < R_0/4$, *and any non-negative local weak solution* u *of the heat equation for* L_t *in* $Q = Q(x, a, r)$, *we have*

$$\sup_{Q^-} u \le C_{\text{PHI}} \inf_{Q^+} u.$$

The constant C_{PHI} *depends only on* δ, $\tau_1, \tau_2, \tau_3, \tau_4$, C_Ψ, β_1, β_2, C_{VD}, C_{PI}, C_0, C_{10}, C_{11}, *and an upper bound on* $[(1 + C_2 + C_4) + (C_3 + C_5)\Psi((1-\delta)r)]$.

Proof. Let $\varepsilon \in (0, 1)$ and $u_\varepsilon := u + \varepsilon$. By Corollary 4.8, Theorem 4.10, and Theorem 5.1, we can apply Lemma 5.2 to u_ε on $(a, a + \tau_2\Psi(r)) \times \delta B$. We obtain that there is some c such that

$$\sup_{Q^-} u_\varepsilon e^c \le C.$$

Similarly, apply Lemma 5.2 to u_ε^{-1} on $(a + \tau_2\Psi(r), a + \tau_4\Psi(r)) \times \delta B$. We obtain that, for the same c as above,

$$\sup_{Q^+} (u_\varepsilon e^c)^{-1} \le C'.$$

Hence,

$$\sup_{Q^-} u_\varepsilon \le e^{-c} C \le C \frac{C'}{\sup_{Q^+} u_\varepsilon^{-1}} \le C\, C' \inf_{Q^+} u_\varepsilon .$$

Letting $\varepsilon \to 0$ on both sides finishes the proof. □

Corollary 5.4. *Suppose Assumptions 0–3 are satisfied globally on $Y = X$. If $C_3 = C_5 = 0$, then the family $(\mathcal{E}_t, \mathcal{F})$ satisfies the parabolic Harnack inequality $\mathrm{PHI}(\Psi)$ on X. That is, there is a constant C_{PHI} such that for any $a \in \mathbb{R}$ and any ball $B(x, 4r) \subsetneq X$, any non-negative local weak solution u of the heat equation for L_t in $Q = Q(x, a, r)$, we have*

$$\sup_{Q^-} u \le C_{\mathrm{PHI}} \inf_{Q^+} u.$$

The constant C_{PHI} depends only on δ, $\tau_1, \tau_2, \tau_3, \tau_4$, C_Ψ, β_1, β_2, C_{VD}, C_{PI}, C_0, C_{10}, C_{11}, C_2, C_4.

Corollary 5.5. *Suppose Assumptions 0–3 are satisfied and each $\mathcal{E}_t$ is left-strongly local. Let $\delta \in (0, 1)$. Then there exist $\alpha \in (0, 1)$ and $C \in (0, \infty)$ such that for any $a \in \mathbb{R}$, any ball $B(x, 4r) \subsetneq B(x, 8r) \subset Y$ with $0 < r < R_0/4$, any local weak solution u of the heat equation for L_t in $Q = Q(x, a, r)$ has a continuous version which satisfies*

$$\sup_{(t,y),(t',y') \in Q'} \left\{ \frac{|u(t, y) - u(t', y')|}{[\Psi^{-1}(|t - t'|) + d(y, y')]^\alpha} \right\} \le \frac{C}{r^\alpha} \sup_Q |u|$$

where $Q' = (a + \Psi((1 - \delta)r), a + \Psi(r)) \times \delta B$. The constant C depends only on δ, C_Ψ, β_1, β_2, C_{VD}, C_{PI}, C_0, C_{10}, C_{11}, and an upper bound on $[(1 + C_2 + C_4) + (C_3 + C_5)\Psi((1 - \delta)r)]$.

Proof. The proof is standard. For instance, the reasoning in the proof of Theorem 5.4.7 in [28] applies with only minor changes such as replacing r^2 by $\Psi(r)$. The left-strong locality is assumed because then constant functions are local weak solutions to the heat equation, a fact that is used in this proof. □

5.3. Characterization of the parabolic Harnack inequality in the symmetric strongly local case

It is known from the works of Grigor'yan [11] and Saloff-Coste [26] that on complete Riemannian manifolds, the parabolic Harnack inequality is characterized by the volume doubling condition together with the Poincaré inequality, as well as by two-sided Gaussian heat kernel bounds. For related results on fractal-type metric measure spaces with a symmetric strongly local regular Dirichlet form see, e.g., [3], [15], [4] and references therein.

The parabolic Harnack inequality $\mathrm{PHI}(\Psi)$ stated above is slightly different from the Harnack inequalities w-$\mathrm{PHI}(\Psi)$ or s-$\mathrm{PHI}(\Psi)$ introduced in [4] because, in defining Q, Q^-, Q^+, we used $\tau_i \Psi(r)$ rather than $\Psi(\tau_i r)$. Our choice is in accordance with the parabolic Harnack inequality stated in [16]. In order to clarify the relation between PHI(Ψ) and w-PHI(Ψ), let us define time-space cylinders $\hat{Q}$ as follows. For $0 < \sigma_1 < \sigma_2 < \sigma_3 < \sigma_4 < 1$, set

$$\begin{aligned} \hat{Q} &= \hat{Q}(x,a,r) = (a, a+\Psi(r)) \times B, \\ \hat{Q}^- &= (a+\Psi(\sigma_1 r), a+\Psi(\sigma_2 r)) \times \delta B, \\ \hat{Q}^+ &= (a+\Psi(\sigma_3 r), a+\Psi(\sigma_4 r)) \times \delta B. \end{aligned}$$

Let $\mathcal{F}'$ be the dual space of $\mathcal{F}$.

Definition 5.6. $(\mathcal{E}^*, \mathcal{F})$ satisfies the *weak parabolic Harnack inequality* w-PHI(Ψ) on X (for local weak solutions) if there is a constant $C \in (0,\infty)$ such that for any $a \in \mathbb{R}$, any ball $B(x,r) \subset X$, and any *bounded* local weak solution u of the heat equation for L_t in $\hat{Q} = \hat{Q}(x,a,r)$, it holds

$$\sup_{\hat{Q}^-} u \leq C \inf_{\hat{Q}^+} u.$$

Remark 5.7. In fact, [4] introduced the condition w-PHI(Ψ) for a space of so-called caloric functions. We show in Proposition 7.3 below that local weak solutions have all the properties that define a space of caloric functions.

Proposition 5.8. *Let $(X, d, \mu, \mathcal{E}^*, \mathcal{F})$ be a symmetric strongly local regular Dirichlet space. Assume that all metric balls in (X,d) are precompact and* VD *is satisfied. Let Ψ be as in* (2.4) *and consider*

(i) $(\mathcal{E}^*, \mathcal{F})$ *satisfies* PI(Ψ)*, and* CSA(Ψ) *on X,*
(ii) $(\mathcal{E}^*, \mathcal{F})$ *satisfies* PHI(Ψ) *on X,*
(iii) $(\mathcal{E}^*, \mathcal{F})$ *satisfies* w-PHI(Ψ) *on X (for local weak solutions),*
(iv) $(\mathcal{E}^*, \mathcal{F})$ *satisfies* weak-PI(Ψ)*, and* CSA(Ψ) *on X.*

The following implications hold:

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)}.$$

If, in addition, d is geodesic, then (iv) $\Rightarrow$ (i)*.*

Proof. The implication (i) to (ii) is the content of Corollary 5.4. To verify the implication (ii) to (iii), it suffices to find parameters τ_i and σ_i such that $\hat{Q}^- \subset Q^-$ and $\hat{Q}^+ \subset Q^+$. By (2.4), we have

$$\frac{\tau_4 \Psi(r)}{\Psi(\sigma_4 r)} \geq C_\Psi^{-1}\, \tau_4\, \sigma_4^{-\beta_1},$$

for any $\tau_4, \sigma_4 \in (0,1)$. We pick τ_4 and σ_4 such that the right-hand side is greater than 1. Applying (2.4) once again, we get

$$\frac{\tau_3 \Psi(r)}{\Psi(\sigma_3 r)} \leq C_\Psi\, \tau_3\, \sigma_3^{-\beta_2},$$

for any $\tau_3, \sigma_3 \in (0,1)$. We pick $\tau_3 < \tau_4$ and $\sigma_3 < \sigma_4$ such that the right-hand side is less than 1. Then $\hat{Q}^+ \subset Q^+$. Similarly, we find $0 < \tau_1 < \tau_2 < \tau_3$ and $0 < \sigma_1 < \sigma_2 < \sigma_3$ such that $\hat{Q}^- \subset Q^-$.

Under VD, condition w-PHI(Ψ) is equivalent to weak heat kernel estimates (w-HKE(Ψ) and w-LLE(Ψ)) by Theorem 3.1 in [4]. Under VD, these heat kernel estimates imply the weak Poincaré inequality weak-PI(Ψ) and CSA(Ψ) by Theorem 2.12 in [17] except for the continuity of the cutoff functions which follows from the Hölder continuity of the Dirichlet heat kernel, which is a consequence of the parabolic Harnack inequality; see also [1], [3]. This proves that (iii) implies (iv). For the implication (iv) $\Rightarrow$ (i) we refer to Remark 3.6. □

Definition 5.9. The reverse volume doubling property (RVD) holds if there are constants C_{RVD} and $\nu_0 \in [1, \infty)$ such that

$$\frac{\mu(B(x,R))}{\mu(B(y,s))} \geq C_{\text{RVD}} \left(\frac{R}{s}\right)^{\nu_0} \tag{5.6}$$

for any $0 < s \leq R$, $x \in X$, $y \in B(x,R)$ with $X \setminus B(x,R) \neq \emptyset$.

Remark 5.10. (i) Suppose, in addition to the hypotheses of Proposition 5.8, that RVD holds. Then condition CSA(Ψ) in (iv) can equivalently be replaced by the generalized capacity condition introduced in [14]. Moreover, under RVD, (iv) is equivalent to a weak upper bound and a weak near-diagonal lower bound for the heat kernel, see Theorem 1.2 in [14]. The weak heat kernel bounds imply (iii) by Theorem 3.1 in [4].

(ii) If the metric space (X, d) is not geodesic then (iii) may fail to imply (ii). See [4] for a counterexample on a non-geodesic space.

(iii) For the implication (iv) $\Rightarrow$ (i), the hypothesis that (X, d) is geodesic could be replaced by a chaining condition. Then the strong Poincaré inequality can be derived from the weak Poincaré inequality by a Whitney covering argument; see, e.g. [28].

Conjecture: The strong parabolic Harnack inequality PHI(Ψ) implies the strong Poincaré inequality PI(Ψ), that is, (ii) $\Leftrightarrow$ (i) in Proposition 5.8.

6. Estimates for the heat propagator

Let $(\mathcal{E}_t, \mathcal{F})$ be a family of bilinear forms that satisfies Assumptions 0, 1, and 2 globally on $Y = X$ with respect to the reference form $(\mathcal{E}^*, \mathcal{F})$. Observe that the bilinear forms $\hat{\mathcal{E}}_t(f,g) := \mathcal{E}_t(g,f)$ satisfy the same assumptions. In addition, we suppose that Assumption 3 is satisfied locally on X, that is, every point $x \in X$ has a neighborhood $Y_x = B(x, 8r_x)$ where Assumption 3 is satisfied with $Y = Y_x$ up to scale $R_0 = 4r_x$ and $B(x, 4r_x) \subsetneq Y_x$. Recall that α and c are positive constants introduced in Assumption 0 (vi).

Proposition 6.1. *Let $s < T \leq +\infty$. For every $f \in L^2(X)$ there exists a unique weak solution u to the heat equation for L_t on $(s,T) \times X$ satisfying the initial condition $u(s, \cdot) = f$.*

More precisely, there exists a unique $u \in L^2((s,T) \to \mathcal{F})$ *of the initial value problem*

$$(6.1)\quad \begin{aligned} &\int_s^T \left\langle \frac{\partial}{\partial t} u, \phi \right\rangle_{\mathcal{F}',\mathcal{F}} dt + \int_s^T \mathcal{E}_t(u,\phi)\,dt = 0, \qquad \text{for all } \phi \in L^2((s,T) \to \mathcal{F}),\\ &\lim_{t \downarrow s} u(t,\cdot) = f \qquad \text{in } L^2(X). \end{aligned}$$

In particular, u *has a weak time-derivative* $\frac{\partial}{\partial t} u \in L^2((s,T) \to \mathcal{F}')$ *and* $u \in C^0([s,T] \to L^2(X))$.

Proof. The proof for the case when $\mathcal{E}_t$ is non-negative definite is given in Chapter 3, Theorem 4.1 and Remark 4.3 of [20]. For the general case, it suffices to notice that $\mathcal{E}_t + \alpha\langle\cdot,\cdot\rangle$ is positive definite by Assumption 0 (vi), and u is a solution to the initial value problem for L_t if and only if $e^{-\alpha(t-s)}u$ is a solution to the initial value problem for $L_t - \alpha$. □

For $t > s$ we consider the *transition operator* associated with $L_t - \partial/\partial t$,

$$T_t^s : L^2(X) \to \mathcal{F}.$$

The transition operator assigns to every $f \in L^2(X)$ the function $u(t) = T_t^s f \in \mathcal{F}$, where $u\colon t \mapsto T_t^s f$ is the unique solution of the initial value problem (6.1) with $T = +\infty$ given by Proposition 6.3. We set $T_s^s f := \lim_{t\downarrow s} T_t^s f = f$. From Corollary 5.5, we obtain that $(t,y) \mapsto T_t^s f(y)$ has a jointly continuous version which we will denote by $P_t^s f(y)$. The transition operators satisfy

$$(6.2)\qquad T_t^r f = T_t^s \circ T_s^r f, \quad \forall r \le s \le t,\ f \in L^2(X).$$

This follows from the fact that both $t \mapsto T_t^r f$ and $t \mapsto T_t^s \circ T_s^r f$ are weak solutions of the heat equation on $(s,\infty) \times X$ and satisfy the initial condition $T_t^r f\big|_{t=s} = T_s^r f = T_t^s \circ T_s^r f\big|_{t=s}$, and because the weak solution to this initial value problem is unique by Proposition 6.1. Moreover, applying Assumption 0 (vi) it follows that

$$(6.3)\qquad \|T_t^s f\|_{L^2} \le e^{(\alpha-c)(t-s)}\|f\|_{L^2}, \quad \forall f \in L^2(X).$$

Proposition 6.2. *The transition operators* $T_t^s f$, $s \le t$, *are positivity preserving. That is, if* $f \in L^2(X)$, $f \ge 0$, *then* $T_t^s f \ge 0$.

Proof. Since $e^{-\alpha(t-s)}T_t^s f$ is the transition operator for $L_t - \alpha$, and $e^{-\alpha(t-s)}T_t^s f \ge 0$ if and only if $e^{-\alpha(t-s)}T_t^s f \ge 0$, and by Assumption 0 (vi), it suffices to give the proof for the case when $\mathcal{E}$ is non-negative definite.

Take $u = T_t^s f$ and $\phi = u - u^+$ in (6.1). Let $u^+ := \max\{u, 0\}$. By locality, $\mathcal{E}(u, u-u^+) \ge 0$. We also have $\left\langle \frac{\partial}{\partial t}(u-u^+), u\right\rangle_{\mathcal{F}',\mathcal{F}} \le 0$. Therefore,

$$\begin{aligned} 0 \ge \int_s^T \left\langle \frac{\partial}{\partial t} u, u - u^+ \right\rangle_{\mathcal{F}',\mathcal{F}} dt &\ge \int_s^T \frac{\partial}{\partial t} \left\langle u, u-u^+\right\rangle_{\mathcal{F}',\mathcal{F}} dt \\ &= \left\langle u(T), u(T) - u^+(T)\right\rangle_{\mathcal{F}',\mathcal{F}} - \left\langle u(s), u(s) - u^+(s)\right\rangle_{\mathcal{F}',\mathcal{F}}. \end{aligned}$$

Since $u(s) = f \ge 0$, we have $u(s) - u^+(s) = 0$ and thus $\langle u(T), u(T) - u^+(T)\rangle_{\mathcal{F}',\mathcal{F}} \le 0$. Therefore, $u(T) = u^+(T) \ge 0$. □

Similarly, there exist transition operators S_s^t, $s \le t$, corresponding to the heat equation for the adjoints $\hat{L}_s$ of the time-reversed generators L_s. It is immediate from (6.1) that S_s^t is the adjoint of T_t^s. Let $Q_s^t f$ be the continuous version of $S_s^t f$ which exists by Corollary 5.5.

Proposition 6.3. *There exists a unique integral kernel $p(t,y,s,x)$ with the following properties:*

(i) *$p(t,y,s,x)$ is non-negative and jointly continuous in $(t,y,x) \in (s,\infty)\times X\times X$.*

(ii) *For every fixed $s < t$ and $y \in X$, the maps $x \mapsto p(t,y,s,x)$ and $y \mapsto p(t,y,s,x)$ are in $L^2(X)$.*

(iii) *For every $s < t$, all $x, y \in X$ and every $f \in L^2(X)$,*

$$P_t^s f(y) = \int_X p(t,y,s,x) f(x)\mu(dx) \quad \textit{and} \quad Q_s^t f(x) = \int_X p(t,y,s,x) f(y)\mu(dy).$$

(iv) *There exists a constant $C \in (0,\infty)$ such that, for every $s < t$ and $x \in X$,*

$$p(t,x,s,x) \le e^{(\alpha-c)(t-s)} \frac{C}{V(x,\tau_x)},$$

where $\tau_x = r_x \wedge \Psi^{-1}(2(t-s))$, and and C depends at most on β_1, β_2, C_Ψ, C_0, C_{10}, C_{11}, C_{VD}, C_{PI}, and on an upper bound on $(1 + C_2 + C_3\Psi(\tau_x))$.

(v) *For every $s < r < t$ and all $x, y \in X$,*

$$p(t,y,s,x) = \int_X p(t,y,r,z)\, p(r,z,s,x)\, d\mu(z).$$

(vi) *For every $s < r$ and every fixed $x \in X$, the map $(t,y) \mapsto p(t,y,s,x)$ is a weak solution of the heat equation for L_t in $(r,\infty)\times X$.*

Proof. In the special case when $(\mathcal{E}_t, \mathcal{F})$ is a time-independent symmetric strongly local regular Dirichlet form, the proof is given in Section 4.3.3 of [4].

Let $f \in L^2(X)$, $f \ge 0$, and let $s < t$. Then $(t - \frac{1}{2}\Psi(\tau_y), t + \frac{1}{2}\Psi(\tau_y)) \subset (s, s+\Psi(r_y))$. By the mean value estimate of Theorem 4.7, the joint continuity of $P_t^s f(y)$ in (t,y), and by (6.3), we have

$$\begin{aligned} \left[P_t^s f(y)\right]^2 &\le \frac{C}{\Psi(\tau_y)V(y,\tau_y)} \int_{t-\frac{1}{2}\Psi(\tau_y)}^{t+\frac{1}{2}\Psi(\tau_y)} \int_{B(y,\tau_y)} \left[P_u^s f(z)\right]^2 d\mu(z)du \\ &\le e^{(\alpha-c)(t-s)} \frac{C}{V(y,\tau_y)} \|f\|_2^2, \end{aligned} \tag{6.4}$$

for some constant $C \in (0,\infty)$ that depends on y only through an upper bound on $C_3(\Psi(\tau_y))$. Considering f^+ and f^-, the displayed inequality extends to all $f \in L^2$. This shows that $f \mapsto P_t^s f(y)$ is a bounded linear functional. By the Riesz representation theorem, there exists a unique function $p_{t,y}^s \in L^2(X)$ such that, for every $y \in X$,

$$P_t^s f(y) = \int p_{t,y}^s(x) f(x)\, d\mu(x), \qquad \text{for all } f \in L^2(X), \tag{6.5}$$

and

$$\|p^s_{t,y}\|_2^2 \le \frac{Ce^{(\alpha-c)(t-s)}}{V(y,\tau_y)}. \tag{6.6}$$

By similar arguments, we obtain that there exists a function $q^t_{s,x} \in L^2(X)$ such that

$$Q^t_s f(x) = \int q^t_{s,x} f(y)\, d\mu(y), \qquad \text{for all } f \in L^2(X), \tag{6.7}$$

and

$$\|q^t_{s,x}\|_2^2 \le \frac{Ce^{(\alpha-c)(t-s)}}{V(x,\tau_x)}. \tag{6.8}$$

Since Q^t_s is the adjoint of P^s_t, we have $p^s_{t,y}(x) = q^t_{s,x}(y)$ for almost every $x, y \in X$. We define

$$p(t,y,s,x) := \int p^r_{t,y}(z)\, q^r_{s,x}(z)\, d\mu(z). \tag{6.9}$$

for some $r \in (s,t)$. Then

$$p(t,y,s,x) = \int p^r_{t,y}(z)\, p^s_{r,z}(x)\, d\mu(z) \quad \text{for a.e. } x \in X.$$

Proposition 6.2, together with (6.5) and (6.7), implies that $p^r_{t,y}$ and $q^r_{t,x}$ are non-negative almost everywhere, hence $p(t,y,s,x)$ is non-negative for all $x, y \in X$. Applying (6.2), we get for any $f \in L^2(X)$,

$$\begin{aligned} P^s_t f(y) = P^r_t \circ P^s_r f(y) &= \int p^r_{t,y}(x)\, P^s_r f(x)\, d\mu(x) = \int Q^r_s\, p^r_{t,y}(x)\, f(x)\, d\mu(x) \\ &= \int\!\!\int q^r_{s,x}(z)\, p^r_{t,y}(z)\, d\mu(z) f(x)\, d\mu(x) = \int p(t,y,s,x)\, f(x)\, d\mu(x). \end{aligned}$$

Similarly, we obtain $Q^t_s f(x) = \int p(t,y,s,x) f(y)\, d\mu(y)$. Combining with (6.5) and (6.7), we see that $p(t,y,s,\cdot) = p^s_{t,y} \in L^2(X)$ and $p(t,\cdot,s,x) = q^t_{s,x} \in L^2(X,\mu)$.

From a computation similar to the one above, we see that $p(t,y,s,x)$ is in fact independent of the choice of r, and the semigroup property (v) holds.

The upper bound (iv) follows from (6.9), the Cauchy–Schwarz inequality, as well as (6.6) and (6.8).

Since $p(r,\cdot,s,x)$ is in $L^2(X)$ when $s < r$, the semigroup property implies that $p(t,y,s,x) = P^r_t p(r,y,s,x)$ for almost every $x \in X$. Since $(t,y) \mapsto P^r_t p(r,y,s,x)$ a weak solution on $(r,\infty) \times X$, we have proved (vi).

It remains to show the joint continuity. It suffices to show that $p(t,y,s,x)$ is continuous in x locally uniformly in (t,y). Let $f \in L^2(X)$. We apply Corollary 5.5 to the weak solution $P^s_t f$ for L_t in $Q = Q(x,t,\tau_x) = (t - \frac{1}{2}\Psi(\tau_x), t + \frac{1}{2}\Psi(\tau_x)) \times B(x,\tau_x)$, Then,

$$|P^s_t f(x') - P^s_t f(x)| \le C \left(\frac{d(x,x')}{\tau_x}\right)^{\alpha} \sup_{(a,z)\in Q} |P^s_a f(z)|.$$

By (6.4),

$$\sup_{(a,z)\in Q} |P_a^s f(z)| \le e^{(\alpha-c)(t-s)} \frac{C'}{V(x,\tau_x)^{1/2}} \|f\|_2.$$

Here, C is a positive constant that may change from line to line. Now we set $f = p_{r,y}^s$ where $r = (s+t)/2$. Then $P_t^r f = p(t,y,s,\cdot)$ and $\|f\|_2^2 \le Ce^{(\alpha-c)(r-s)}/V(y,\tau_y)$ by (6.6). Hence,

$$|P_t^s f(x') - P_t^s f(x)| \le C \left(\frac{d(x,x')}{\tau_x}\right)^{\alpha} e^{(\alpha-c)2(t-s)} \frac{1}{V(x,\tau_x)^{1/2}} \frac{1}{V(y,\tau_y)^{1/2}}.$$

This shows that $p(t,y,s,x)$ is continuous in x locally uniformly in (t,y), and completes the proof of the joint continuity. □

The next lemma is immediate from CSA and (2.2), (2.3).

Lemma 6.4. *Let $\psi \in \mathrm{CSA}(\Psi,\epsilon,C_0)$ be a cutoff function for $B(x,R)$ in $B(x,R+r)$. Let $\phi = e^{M\psi}$ for some constant $M \in \mathbb{R}$. Let $A = B(x,R+r)\setminus B(x,R)$. Then*

$$\int f^2\, d\Gamma(\phi,\phi) \le \frac{2\epsilon}{1-2\epsilon} M^2 \int_A \phi^2\, d\Gamma(f,f) + \frac{C_0\epsilon^{1-\beta_2/2}}{(1-2\epsilon)\Psi(r)} M^2 \int_A \phi^2 f^2\, d\mu.$$

Assumption 4. There are constants $C_6, C_7, C_{11} \in [0,\infty)$ such that for all $t \in \mathbb{R}$, for any $\epsilon \in (0,1)$, any $0 < r < R \le R_0$, any ball $B(x,2R) \subset Y$, any $M \ge 1$, any cutoff function $\psi \in \mathrm{CSA}(\Psi,\epsilon,C_0)$ for $B(x,R)$ in $B(x,R+r)$, and any $0 \le f \in \mathcal{F}_{\mathrm{loc}}(Y) \cap L^\infty_{\mathrm{loc}}(Y,\mu)$,

$$\begin{aligned} &|\mathcal{E}_t^{\mathrm{sym}}(f^2\phi^2,1)| + \left|\mathcal{E}_t^{\mathrm{skew}}(f,f\phi^2)\right| \\ &\qquad \le C_{11}\,\epsilon^{1/2} M \int \phi^2\, d\Gamma(f,f) + (C_6 + C_7\Psi(r)) \frac{C_1(\epsilon)}{\Psi(r)} M \int f^2\phi^2\, d\mu, \end{aligned}$$

where $B = B(x,R+r)$, $\phi = e^{-M\psi}$.

We set $p(t,y,s,x) := \delta_x(y)$ whenever $t \le s$. Let

$$\Phi_{\beta_2}(R,t) := \sup_{r>0}\left\{\frac{R}{r} - \frac{t}{r^{\beta_2}} \frac{R^{\beta_2}}{\Psi(R)}\right\}.$$

Lemma 6.5. *Let $x,y \in X$. Suppose Assumption 4 is satisfied and* $\mathrm{CSA}(\Psi,C_0)$ holds locally on $B(x,d(x,y))$ up to scale $\frac{1}{2}d(x,y)$. *Let $f_1 \in L^2(X)$ with support in $B(x,d(x,y)/4)$, and let $f_2 \in L^2(X)$ with support in $B(y,d(x,y)/4)$. Then there is a constant $C' \in (0,\infty)$ such that, for any $s < t$,*

$$\int T_t^s f_1(x) f_2(x)\, d\mu(x) \le \|f_1\|_{L^2}\|f_2\|_{L^2} \exp\left(-\Phi_{\beta_2}(d(x,y),C'(t-s)) + (\alpha-c)(t-s)\right).$$

The constant C' depends at most on C_Ψ, β_1, β_2, C_0, C_{10}, C_{11}, and on an upper bound on $(C_6 + C_7\Psi(d(x,y)))$.

Proof. Set $R = d(y,x)$. Let $\psi \in \mathrm{CSA}(\Psi, \epsilon, C_0)$ be a cutoff function for $B(x, \frac{1}{4}R)$ in $B(x, \frac{3}{4}R)$. Let $\phi = e^{-M\psi}$ for some $M \geq 1$ that we will choose later. Let $u = T_t^s f_x$. Following Theorem 2 in [8], we get

$$\begin{aligned}\frac{1}{2}\frac{\partial}{\partial t}\|\phi u\|_2^2 = -\mathcal{E}_t(u, u\phi^2) &\leq -\int \phi^2\, d\Gamma_t(u,u) + \frac{1}{2}\int \phi^2\, d\Gamma_t(u,u) + 2\int u^2\, d\Gamma_t(\phi,\phi) \\ &\quad - \mathcal{E}_t^{\mathrm{sym}}(u^2\phi^2, 1) - \mathcal{E}_t^{\mathrm{skew}}(u, u\phi^2) \\ &\leq \Big(-1 + \frac{1}{2} + 2\frac{2\epsilon}{1-2\epsilon}M^2 + C_{11}\,\epsilon^{1/2}M\Big)\int \phi^2\, d\Gamma_t(u,u) \\ &\quad + \Big(\frac{M^2}{1-2\epsilon} + (C_6 + C_7\Psi(R))\epsilon^{-1/2}M\Big)\frac{C\epsilon^{1-\beta_2/2}}{\Psi(R)}\|\phi u\|_2^2,\end{aligned}$$

by Lemma 6.4 and Assumption 4. Here, C is a positive constants that depends at most on C_Ψ, β_1, β_2, C_0, C_{10}, C_{11}, and on an upper bound on $(C_6 + C_7\Psi(R))$. Choosing $\epsilon = \hat{c}/M^2$ for some small enough $\hat{c} = \hat{c}(C_{11})$, we get

$$\|\phi T_t^s f_1\|_2 \leq \exp\Big(\frac{C' M^{\beta_2}}{\Psi(R)}(t-s)\Big)\|\phi f_1\|_2.$$

If $(t-s) \geq \Psi(R)$, then $\Phi_{\beta_2}(R, C'(t-s))$ is bounded from above. In this case the desired estimate follows by the Cauchy–Schwarz inequality and (6.3). Indeed,

$$\int T_t^s f_1(x)\, f_2(x)\, d\mu(x) \leq \|T_t^s f_1\|_{L^2}\,\|f_2\|_{L^2} \leq e^{(\alpha-c)(t-s)}\,\|f_1\|_{L^2}\,\|f_2\|_{L^2}.$$

Similarly, if the supremum (in the definition of) $\Phi_{\beta_2}(R, C'(t-s))$ is attained at some $r > R$, then $\Phi_{\beta_2}(R, C'(t-s)) \leq R/r < 1$, and the assertion follows.

It remains to consider the case when $(t-s) < \Psi(R)$ and the supremum $\Phi_{\beta_2}(R, C'(t-s))$ is attained at some $r \leq R$. Then we choose $M := R/r \geq 1$. We get

$$M - \frac{C' M^{\beta_2}}{\Psi(R)}(t-s) = \frac{R}{r} - \frac{C'(t-s)R^{\beta_2}}{r^{\beta_2}\Psi(R)} = \Phi_{\beta_2}(R, C'(t-s)).$$

Hence,

$$\begin{aligned}\int T_t^s f_1(x)\, f_2(x)\, d\mu(x) &\leq \|\phi T_t^s f_1\|_{L^2}\,\|\phi^{-1} f_2\|_{L^2} \\ &\leq \exp\Big(\frac{C' M^{\beta_2}}{\Psi(R)}(t-s)\Big)\,\|\phi f_1\|_{L^2}\,\|\phi^{-1} f_2\|_{L^2} \\ &\leq \exp\Big(\frac{C' M^{\beta_2}}{\Psi(R)}(t-s)\Big)\,\Big(\sup_{B(x,R/4)}\phi\Big)\,\Big(\sup_{B(y,R/4)}\phi^{-1}\Big)\,\|f_1\|_{L^2}\,\|f_2\|_{L^2} \\ &\leq \exp\Big(\frac{C' M^{\beta_2}}{\Psi(R)}(t-s) - M\Big)\,\|f_1\|_{L^2}\,\|f_2\|_{L^2} \\ &\leq \exp\big(-\Phi_{\beta_2}(R, C'(t-s))\big)\,\|f_1\|_{L^2}\,\|f_2\|_{L^2}.\end{aligned}$$

□

By Theorem 4.9, there exists a constant $C \in (0,\infty)$ such that the following L^1-mean value estimate holds for any $0 < r \le r_y$ and any non-negative local very weak subsolution u of the heat equation for L_t in $(t - \frac{1}{2}\Psi(r), t + \frac{1}{2}\Psi(r)) \times B(y,r)$,

$$u(t,y) \le \frac{C}{\Psi(r)\,\mu(B(y,r))} \int_{t-\frac{1}{2}\Psi(r)}^{t+\frac{1}{2}\Psi(r)} \int_{B(y,r)} u \, d\mu \, dt, \tag{6.10}$$

where C depends on C_Ψ, β_1, β_2, C_0, C_{10}, C_{11}, C_{PI}, C_{VD}, and on an upper bound on $(1 + C_2 + C_3\Psi(r))$. Here, on the left hand side, we used the jointly continuous version of u that exists by Corollary 5.5.

Theorem 6.6. *Suppose Assumptions 0, 1, 2 and 4 are satisfied globally on X, and Assumption 3 is satisfied locally on X. Let $x, y \in X$. Suppose* $\mathrm{CSA}(\Psi, C_0)$ holds locally on $B(x, d(x,y))$ and on $B(y, d(x,y))$ up to scale $\frac{1}{2}d(x,y)$. *Then there exist constants $C, C' \in (0,\infty)$ such that, for all $s < t$,*

$$p(t,y,s,x) \le C\, \frac{\exp\left(-\Phi_{\beta_2}(d(x,y), C'(t-s)) + (\alpha - c)(t-s)\right)}{V(x,\tau_x)^{1/2}\, V(y,\tau_y)^{1/2}},$$

where $\tau_x = \Psi^{-1}(\frac{t-s}{2}) \wedge r_x$, $\tau_y = \Psi^{-1}(\frac{t-s}{2}) \wedge r_y$. The constants C, C' depend only on C_Ψ, β_1, β_2, C_0, C_{10}, C_{11}, $C_{\mathrm{VD}}(Y)$, $C_{\mathrm{PI}}(Y)$ for $Y = Y_x$ and for $Y = Y_y$, and on an upper bound on $(1 + C_2 + C_6 + C_3(\Psi(\tau_x) + \Psi(\tau_y)) + C_7\Psi(d(x,y)))$.

Proof. Applying the L^1-mean value estimate (6.10) to $(t,y) \mapsto p(t,y,s,x)$ and to $(s,x) \mapsto p(t',y',s,x)$, we get

$$\begin{aligned} &p(t,y,s,x) \\ &\quad\le \frac{C}{\Psi(\tau_y)V(y,\tau_y)} \int_{t-\frac{1}{2}\Psi(\tau_y)}^{t+\frac{1}{2}\Psi(\tau_y)} \int_{B(y,\tau_y)} p(t',y',s,x)\, d\mu(y')\, dt' \\ &\quad\le D \int_{t-\frac{1}{2}\Psi(\tau_y)}^{t+\frac{1}{2}\Psi(\tau_y)} \int_{B(y,\tau_y)} \int_{s-\frac{1}{2}\Psi(\tau_x)}^{s+\frac{1}{2}\Psi(\tau_x)} \int_{B(x,\tau_x)} p(t',y',s',x')\, d\mu(x') ds'\, d\mu(y')\, dt', \end{aligned}$$

where $D = \frac{C^2}{\Psi(\tau_x)\Psi(\tau_y)V(x,\tau_x)V(y,\tau_y)}$.

In the case $\tau_x \vee \tau_y \le d(x,y)/4$, Lemma 6.5 yields

$$\begin{aligned} &\int_{B(y,\tau_y)} \int_{B(x,\tau_x)} p(t',y',s',x')\, d\mu(x')\, d\mu(y') \\ &\qquad\le V(x,\tau_x)^{1/2}\, V(y,\tau_y)^{1/2} \exp\left(-\Phi_{\beta_2}(d(x,y), C'(t-s)) + (\alpha - c)(t-s)\right). \end{aligned}$$

In the case $\tau_x \vee \tau_y \ge d(x,y)/4$, $\Phi_{\beta_2}(d(x,y), C'(t-s))$ is bounded from above. By the Cauchy–Schwarz inequality and (6.3),

$$\begin{aligned} \int_{B(y,\tau_y)} \int_{B(x,\tau_x)} p(t',y',s',x')\, d\mu(x')\, d\mu(y') &= \int P_{t'}^{s'} 1_{B(x,\tau_x)}(y') 1_{B(y,\tau_y)}(y')\, d\mu(y') \\ &\le \|T_{t'}^{s'} 1_{B(x,\tau_x)}\|_2\, \|1_{B(y,\tau_y)}\|_2 \le e^{(\alpha-c)(t'-s')}\, V(x,\tau_x)^{1/2}\, V(y,\tau_y)^{1/2}. \end{aligned}$$

In both cases, we obtain the desired estimate. □

Definition 6.7. For an open set $U \subset X$, the time-dependent Dirichlet-type forms on U are defined by

$$\mathcal{E}^D_{U,t}(f,g) := \mathcal{E}_t(f,g), \quad f,g \in D(\mathcal{E}^D_U),$$

where, for each $t \in \mathbb{R}$, the domain $D(\mathcal{E}^D_{U,t}) := \mathcal{F}^0(U)$ is defined as the closure of $\mathcal{F} \cap \mathcal{C}_c(U)$ in $\mathcal{F}$ for the norm $\|\cdot\|_{\mathcal{F}}$. Let $T^D_U(t,s)$, $t \geq s$, be the associated transition operators with integral kernel $p^D_U(t,y,s,x)$.

Proposition 6.8. *Let $V \subset U \subset X$ be open subsets. For any $t > s$, $x, y \in V$,*

$$p^D_V(t,y,s,x) \leq p^D_U(t,y,s,x).$$

Proof. We may assume that each $\mathcal{E}_t$ is non-negative definite (if not, multiply the kernels by $e^{-\alpha(t-s)}$ and notice that the associated bilinear forms $\mathcal{E}_t + \alpha$ are non-negative definite by Assumption 0 (vi)). Let $r \in (s,t)$.

Let $f(z) = p^D_U(r,z,s,x)$. Then $p^D_U(t,\cdot,s,x) = P^D_U(t,r)f$ is a non-negative local weak solution of the heat equation in $(r,\infty) \times V$. As $t \downarrow r$, $P^D_U(t,r)f \to f$ in $L^2(U)$, and by non-negativity also in $L^2(V)$. Hence, by Corollary 7.2,

$$p^D_U(t,y,s,x) \geq P^D_V(t,r)\,p^D_U(r,\cdot,s,x) = \int_V p^D_V(t,y,r,z)\,p^D_U(r,z,s,x)\,d\mu(z).$$

Similarly, we have for $p^D_V(t,y,s,x) = Q^D_V(s,r)\,p^D_V(t,y,r,\cdot)(x)$ that

$$p^D_V(t,y,s,x) \leq Q^D_U(s,r)\,p^D_V(t,y,r,\cdot)(x) = \int_U p^D_U(r,z,s,x)\,p^D_V(t,y,r,z)\,d\mu(z).$$

Combining both inequalities finishes the proof. □

Theorem 6.9. *Suppose Assumptions 0, 1, 2, and 4 are satisfied globally on X, and Assumption 3 is satisfied locally on X. Let $a \in X$ and $B = B(a, r_a)$.*

(i) *For any fixed $\epsilon \in (0,1)$ there are constants $c', C' \in (0,\infty)$, such that for any $x \in B(a,(1-\epsilon)r_a)$ and $0 < \epsilon(t-s) \leq \Psi(r_a)$, the Dirichlet heat propagator p^D_B satisfies the near-diagonal lower bound*

$$p^D_B(t,y,s,x) \geq \frac{c'}{V(x, \Psi^{-1}(t-s) \wedge R_x)},$$

for any $y \in B(a,(1-\epsilon)r_a)$ with $d(y,x) \leq \epsilon\Psi^{-1}(t-s)$, where $R_x = d(x,\partial B)$. The constants c', C' depend at most on C_Ψ, β_1, β_2, C_0, C_{10}, C_{11}, on $C_{\mathrm{VD}}(Y_a)$ and $C_{\mathrm{PI}}(Y_a)$ for $Y_a = B(a, 8r_a)$, and on an upper bound on $(1 + C_2 + C_4 + (C_3 + C_5)\Psi(\tau_a))$.

(ii) *There exist constants $C, C' \in (0,\infty)$ such that for any $x,y \in B$, $t > s$, the Dirichlet heat propagator p^D_B satisfies the upper bound*

$$p^D_B(t,y,s,x) \leq C\,\frac{\exp\left(-\Phi_{\beta_2}(d(x,y), C'(t-s)) + (\alpha - c)(t-s)\right)}{V(x,\tau_a)^{1/2} V(y,\tau_a)^{1/2}},$$

where $\tau_a = \Psi^{-1}\left(\frac{t-s}{2}\right) \wedge r_a$.

The constants c', C, C' depend at most on C_Ψ, β_1, β_2, C_0, C_{10}, C_{11}, on $C_{\text{VD}}(Y_a)$ and $C_{\text{PI}}(Y_a)$ for $Y_a = B(a, 8r_a)$, and on an upper bound on $(1+C_2+C_6+C_3\Psi(\tau_a)+C_7\Psi(d(x,y)))$.

Proof. The on-diagonal estimate in (i) can be proved in the same way as in Theorem 5.6 of [19]. See also Theorem 5.4.10 in [28]. For the near-diagonal estimate, apply the parabolic Harnack inequality of Theorem 5.3.

(ii) is immediate from Theorem 6.6 and the set monotonicity of the heat propagator proved in Proposition 6.8. □

If (X, d) satisfies a chain condition as in [15], then we can apply the parabolic Harnack inequality repeatedly along chains to obtain an off-diagonal lower bound. In particular, if d is geodesic, then the lower bound in Proposition 6.9(i) can be improved to the following corollary. By Proposition 6.8, we obtain the same lower bound for the global heat propagator $p(t, y, s, x)$.

Let

$$\Phi(R,t) := \sup_{r>0}\left\{\frac{R}{r} - \frac{t}{\Psi(r)}\right\}.$$

Corollary 6.10. *Suppose d is geodesic. Then there are constants $C'', c', c'' \in (0,\infty)$ such that for any $a \in X$, all $x, y \in B(a, r_a/2)$, and $t > s$, the Dirichlet heat kernel on $B = B(a, r_a)$ satisfies the lower bound*

$$p_B^D(t,y,s,x) \geq \frac{c'}{V\big(x, \Psi^{-1}\big(\frac{t-s}{2}\big) \wedge r_a\big)} \exp\left(-C''\Phi(d(x,y), c''(t-s))\right),$$

The constants c', c'', C'' depend on C_Ψ, β_1, β_2, C_0, C_{10}, C_{11}, C_2, C_3, C_4, C_5, on $C_{\text{VD}}(Y)$ and $C_{\text{PI}}(Y)$ for $Y = B(a, 8r_a)$, and on an upper bound on $(1 + C_2 + C_4 + (C_3 + C_5)\Psi(r_a))$.

Proof. From Theorem 6.9 (i) we obtain an on-diagonal bound for $0 < \epsilon(t-s) < \Psi(r_a)$. The off-diagonal estimate (for any $t > s$) follows from the parabolic Harnack inequality. □

Corollary 6.11. *Suppose Assumptions 0, 1, 2 and* A2-Y, VD, PI(Ψ), CSA(Ψ) *are satisfied globally on $Y = X$. Suppose d is geodesic. If $C_3 = C_5 = 0$, then there are constants $C, C', c', c'', C'' \in (0,\infty)$ such that for any $x, y \in X$ and $t > s$, we have*

$$p(t,y,s,x) \geq c' \frac{\exp\left(-C''\Phi(d(x,y), c''(t-s))\right)}{V(x, \Psi^{-1}(t-s))},$$

$$p(t,y,s,x) \leq C \frac{\exp\left(-\Phi_{\beta_2}(d(x,y), C'(t-s)) + (\alpha - c)(t-s)\right)}{V(x, \Psi^{-1}(t-s))}.$$

The constants C, C', c', c'', C'' depend only on C_Ψ, β_1, β_2, C_0, C_{10}, C_{11}, C_2, C_4, $C_{\text{VD}}(X)$, $C_{\text{PI}}(X)$.

7. Parabolic maximum principle and caloric functions

Proposition 7.1 (Parabolic maximum principle). *Suppose* $(\mathcal{E}_t, \mathcal{F})$, $t \in \mathbb{R}$, *is a family of bilinear forms satisfying Assumption* 0. *Assume that* $\mathcal{E}_t^{\mathrm{sym}}(f, f) \geq 0$ *for all* $t \in \mathbb{R}$ *and* $f \in \mathcal{F}$. *Let* $I = (s, T)$ *for some* $-\infty < s < T \leq \infty$. *Let* $U \subset X$ *be an open subset. Let* $u \in \mathcal{C}_{\mathrm{loc}}(I \to L^2(U))$ *be a local very weak subsolution of the heat equation for* L_t *in* $I \times U$. *Assume that* $u^+(t, \cdot) \in \mathcal{F}^0(U)$ *for every* $t \in I$, *and* $u^+(t, \cdot) \to 0$ *in* $L^2(U)$ *as* $t \to s$. *Then* $u \leq 0$ *almost everywhere on* $I \times U$.

For weak subsolutions of the heat equation for symmetric regular Dirichlet forms, the parabolic maximum principle is proved in [13], Proposition 5.2 (see also Proposition 4.11 in [12]). Their proof makes explicit use of the Markov property of the Dirichlet form. Below we give a proof of Proposition 7.1 that relies on Steklov averages.

Proof of Proposition 7.1. Let u be as in the proposition. Then (4.2) extends to all $\phi \in \mathcal{F}^0(U)$ by an approximation argument together with the Cauchy–Schwarz inequality and Assumption 0. Thus, for any fixed t, we can take $\phi = (u^+)_h(t) \in \mathcal{F}^0(U)$ as test function in (4.2). Let $s < a < b < T$ and $h > 0$ be so small that $b + h < T$. Since u_h has the strong time-derivative $\frac{\partial}{\partial t}(u^+)_h(t) = \frac{1}{h}[u^+(t+h) - u^+(t)]$, we have

$$\begin{aligned}
(7.1)\qquad \int_U (u^+)_h^2(b)\, d\mu - \int_U (u^+)_h^2(a)\, d\mu &= \int_a^b \frac{d}{dt} \int_U (u^+)_h^2(t)\, d\mu\, dt \\
&= 2 \int_a^b \frac{1}{h} \int_U [u^+(t+h) - u^+(t)](u^+)_h(t)\, d\mu\, dt \\
&= 2 \int_a^b \frac{1}{h} \int_U [u(t+h) - u(t)](u^+)_h(t)\, d\mu\, dt \\
&\quad - 2 \int_a^b \frac{1}{h} \int_U [u^-(t+h) - u^-(t)](u^+)_h(t)\, d\mu\, dt \\
&\leq -2 \int_a^b \frac{1}{h} \int_t^{t+h} \mathcal{E}_s\big(u(s), (u^+)_h(t)\big)\, ds\, dt \\
&\quad - 2 \int_a^b \frac{1}{h} \int_U [u^-(t+h) - u^-(t)](u^+)_h(t)\, d\mu\, dt \\
(7.2)\qquad &\leq -2 \int_a^b \frac{1}{h} \int_t^{t+h} \mathcal{E}_s\big(u(s), (u^+)_h(t) - u^+(t)\big)\, ds\, dt \\
(7.3)\qquad &\quad - 2 \int_a^b \frac{1}{h} \int_t^{t+h} \mathcal{E}_s\big(u(s) - u(t), u^+(t)\big)\, ds\, dt \\
(7.4)\qquad &\quad - 2 \int_a^b \frac{1}{h} \int_t^{t+h} \mathcal{E}_s\big(u(t), u^+(t)\big)\, ds\, dt \\
(7.5)\qquad &\quad + \frac{2}{h} \int_a^b \int_U u^-(t)(u^+)_h(t)\, d\mu\, dt.
\end{aligned}$$

Letting h go to 0, we see that (7.2) and (7.3) tend to 0 by Assumption 0 and Lemma 3.8 and Corollary 3.10 in [19]. In (7.4), observe that $-\mathcal{E}_s(u(t), u^+(t)) = -\mathcal{E}_s(u^+(t), u^+(t)) \leq 0$ because $\mathcal{E}_s$ is local and its symmetric part is non-negative definite. The integrand in (7.5) converges to 0 pointwise almost everywhere. Hence (7.5) goes to 0 by the dominated convergence theorem. Thus, we obtain

$$\int_U (u^+)^2(b)\, d\mu - \int_U (u^+)^2(a)\, d\mu \leq 0.$$

for almost every $s < a < b < T$. The assumption that $u^+(t, \cdot) \to 0$ in $L^2(U)$ as $t \to s$ implies that we can make $\int_U (u^+)^2(a)\, d\mu$ arbitrarily small by choosing a sufficiently close to s. Hence,

$$\int_U (u^+)^2(b)\, d\mu \leq 0,$$

so $u^+(b) = 0$ μ-almost everywhere on U, for almost every $b \in I$. This proves that $u \leq 0$ almost everywhere on $I \times U$. □

Corollary 7.2 (Super-mean value inequality)**.** *Suppose $(\mathcal{E}_t, \mathcal{F})$, $t \in \mathbb{R}$, is a family of bilinear forms satisfying Assumption 0. Assume that $\mathcal{E}_t^{\mathrm{sym}}(f, f) \geq 0$ for all $t \in \mathbb{R}$ and $f \in \mathcal{F}$. Let $I = (s, T)$ for some $-\infty < s < T \leq \infty$. Let $f \in L^2(U)$, $f \geq 0$. Let $u \in \mathcal{C}_{\mathrm{loc}}(I \to L^2(U))$ be a non-negative local very weak supersolution of the heat equation for L_t in $(s, T) \times U$ such that $u(t, \cdot) \to f$ in $L^2(U)$ as $t \downarrow s$. Then, for every $t \in (s, T)$,*

$$u(t, x) \geq P_U^D(t, s)\, f(x) \qquad \text{for a.e. } x \in U.$$

Proof. Following Corollary 2.3 in [4], we apply the parabolic maximum principle to the local very weak subsolution $v(t, \cdot) = P_U^D(t, s) f - u(t, \cdot)$. Indeed, we have $v^+(t, \cdot) \in \mathcal{F}^0(U)$ for every $t \in I$ by Proposition 6.2 and Lemma 4.4 in [12]. Now Proposition 7.1 yields that $v \leq 0$ almost everywhere in $I \times U$. Continuity in t completes the proof of the super-mean value inequality. □

The properties listed in the next proposition are the defining properties of a space of caloric functions as defined in [4].

Proposition 7.3. *Suppose $(\mathcal{E}_t, \mathcal{F})$, $t \in \mathbb{R}$, is a family of left-strongly local bilinear forms satisfying Assumption 0. Assume that $\mathcal{E}_t^{\mathrm{sym}}(f, f) \geq 0$ for all $t \in \mathbb{R}$ and $f \in \mathcal{F}$. Let $I = (s, T)$ for some $T \leq \infty$. Let $U \subset X$ be open. Let $\mathcal{W}(I \times U)$ be the space of local weak solutions of the heat equation for L_t on $I \times U$. Then*

(i) *$\mathcal{W}(I \times U)$ is a linear space over $\mathbb{R}$.*

(ii) *If $I' \subset I$ and $U' \subset U$, then $\mathcal{W}(I \times U) \subset \mathcal{W}(I' \times U')$.*

(iii) *For any $f \in L^2(U)$, the function $(t, x) \mapsto P_{U,t}^D f(x)$ is in $\mathcal{W}(I \times U)$.*

(iv) *Any constant function in U is the restriction to U of a time-independent function in $\mathcal{W}(I \times U)$.*

(v) *For any non-negative* $u \in \mathcal{W}(I \times U)$ *and every* $s < r < t < T$,

$$u(t,x) \geq P_U^D(t,r)\,u(r,x) \quad \text{for a.e. } x \in U.$$

Proof. Properties (i) and (ii) are immediate from the definition of local weak solutions. Property (iii) is immediate from the definition of P_U^D. Property (iv) follows from the left-strong locality and the definition of local weak solutions. Property (v) follows from Corollary 7.2. □

8. Construction of non-symmetric local bilinear forms

In this section we dicuss the construction of non-symmetric bilinear forms on a given symmetric strongly local regular Dirichlet space $(X, d, \mu, \mathcal{E}^*, \mathcal{F})$. Let $Y \subset X$ be an open subset and $R_0 > 0$. Suppose Assumption 3 is satisfied.

Definition 8.1. Let $\mathcal{H}$ be the space of all non-negative functions $h \in \mathcal{F}_{\mathrm{b}}$ for which there exists a constant $C_h \in (0, \infty)$ such that

$$\forall f \in \mathcal{F}, \qquad \int f^2\, d\Gamma(h,h) \leq C_h\, \|f\|_{\mathcal{F}}^2. \tag{8.1}$$

For instance, $\mathcal{H}$ contains linear combinations of cutoff functions that satisfy CSA(Ψ).

Proposition 8.2. *Let* $h \in \mathcal{H}$. *For* $f, g \in \mathcal{F}_{\mathrm{b}}$, *set*

$$\mathcal{E}(f,g) := \mathcal{E}^*(f,g) + \int g\, d\Gamma(f,h) - \int f\, d\Gamma(g,h). \tag{8.2}$$

Then $\mathcal{E}$ *extends uniquely to a local bilinear form on* $\mathcal{F} \times \mathcal{F}$ *and* $(\mathcal{E}, \mathcal{F})$ *satisfies Assumptions 0, 1, and 2. In particular, the results of Section 4, Sections 5.1–5.2, and Section 6 apply to* $(\mathcal{E}, \mathcal{F})$*, provided that the reference form* $(\mathcal{E}^*, \mathcal{F})$ *satisfies Assumption 3 as required for these results (locally or globally).*

Proof. First we show that $\mathcal{E}$ extends uniquely to $\mathcal{F} \times \mathcal{F}$. For $g \in \mathcal{F}$, let (g_n) be a sequence in $\mathcal{F}_{\mathrm{b}}$ that converges to g in $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$. Passing to a subsequence, we may assume that g_n converges to g also quasi-everywhere. It is clear that for any $f \in \mathcal{F}$, $\int g_n d\Gamma(f,h)$ is well-defined. Applying (2.3) and (8.1),

$$\begin{aligned} \Big| \int g_n\, d\Gamma(f,h) - \int g_m\, d\Gamma(f,h) \Big| &\leq \Big(\int (g_n - g_m)^2 d\Gamma(h,h) \Big)^{1/2} \Big(\int d\Gamma(f,f) \Big)^{1/2} \\ &\leq C_h^{1/2}\, \|g_n - g_m\|_{\mathcal{F}} \Big(\int d\Gamma(f,f) \Big)^{1/2}. \end{aligned} \tag{8.3}$$

Since the right-hand side converges to 0 as $n \to \infty$, we see that $\int g_n d\Gamma(f,h)$ converges and we denote its limit formally by $\int g\, d\Gamma(f,h)$. The limit $\int g\, d\Gamma(f,h)$ does not depend on the approximating sequence (g_n).

Since $g_n \to g$ in $\mathcal{F}$ and quasi-everywhere, it is easy to show that

$$\lim_{n\to\infty} \int g_n \, d\Gamma(f,h) = \frac{1}{2}\int d\Gamma(fg,h) + \frac{1}{2}\int d\Gamma(f,gh) - \int d\Gamma(fh,g)$$

whenever $\int d\Gamma(fg,h) = \lim_{n\to\infty} \int d\Gamma(fg_n,h)$ exists and is finite. The other integrals on the right-hand side are well-defined and finite because gh and fh are in $\mathcal{F}$. This justifies denoting the limit by $\int g\, d\Gamma(f,h)$. Taking $g_m = 0$ in (8.3), we see that

$$\int g d\Gamma(f,h) \le C_h^{1/2} \|g\|_{\mathcal{F}} \|f\|_{\mathcal{F}}.$$

Interchanging f and g, we find that $\int f d\Gamma(g,h)$ is also well-defined. Thus, $\mathcal{E}$ extends to $\mathcal{F}\times\mathcal{F}$ and the extension is bilinear, local, and satisfies Assumption 0 (i). Below we will verify that $(\mathcal{E},\mathcal{F})$ satisfies Assumption 0(ii)–(vi), Assumption 1 and Assumption 2.

It is immediate from (8.2) that $(\mathcal{E},\mathcal{F})$ is a local bilinear form. The symmetric part of $\mathcal{E}$ is $\mathcal{E}^{\text{sym}} = \mathcal{E}^{\text{s}} = \mathcal{E}^*$. This follows easily from the definition of $\mathcal{E}^{\text{s}}$ and the strong locality of $(\mathcal{E}^*,\mathcal{F})$. Thus, part (ii), (iii) and (vi) of Assumption 0 are trivially satisfied. Observe that $\mathcal{L}(f,g) = \int g\, d\Gamma(f,h)$. Since Γ obeys the product rule and the chain rule, part (iv) and part (v) of Assumption 0 are verified.

Next, we show that $(\mathcal{E},\mathcal{F})$ satisfies Assumption 1. The estimate on $\mathcal{E}^{\text{sym}}$ is trivially satisfied. Let $\epsilon \in (0,1)$. Let $0 < r < R \le R_0$ and $B(x,2R) \subset Y$. Let $g \in \mathrm{CSA}(\Psi,\epsilon,C_0)$ be a cutoff function for $B(x,R)$ in $B = B(x,R+r)$. Let $0 \le f \in \mathcal{F}_{\text{loc}}(Y) \cap L^\infty_{\text{loc}}(Y,\mu)$. By (2.1), (8.1), (2.3), and $\mathrm{CSA}(\Psi,\varepsilon,C_0)$,

$$\begin{aligned}
\big|\mathcal{E}^{\text{skew}}(f^2g^2,1)\big| &= \Big|\int d\Gamma(f^2g^2,h)\Big| = \Big|\int f^2 g\, d\Gamma(g,h)\Big| \\
&\le \Big(\int f^2g^2 d\Gamma(h,h)\Big)^{1/2}\Big(\int f^2 d\Gamma(g,g)\Big)^{1/2} \le C_h^{1/2}\|fg\|_{\mathcal{F}}\Big(\int f^2 d\Gamma(g,g)\Big)^{1/2} \\
&\le C_h^{1/2}\Big((2+2\epsilon)\int g^2 d\Gamma(f,f) + \frac{C_0\epsilon^{1-\beta_2/2}}{\Psi(r)}\int_B f^2\, d\mu\Big)^{1/2} \\
&\quad \cdot \Big(\epsilon \int g^2 d\Gamma(f,f) + \frac{C_0\epsilon^{1-\beta_2/2}}{\Psi(r)}\int_B f^2\, d\mu\Big)^{1/2} \\
&\le 5\, C_h^{1/2}\,\epsilon^{1/2} \int g^2 d\Gamma(f,f) + 3\,\frac{C_0\epsilon^{(1-\beta_2)/2}}{\Psi(r)}\int_B f^2\, d\mu.
\end{aligned}$$

Furthermore, we have

$$\mathcal{E}^{\text{skew}}(f,fg^2) = -2\int f^2 g\, d\Gamma(g,h),$$

and by (2.1), (8.1), (2.3), and the cutoff Sobolev inequality (2.5),

$$\begin{aligned}
&2\Big|\int f^2 g\, d\Gamma(g,h)\Big| \\
&\le 2\Big(\int f^2 d\Gamma(g,g)\Big)^{1/2}\Big(\int (fg)^2 d\Gamma(h,h)\Big)^{1/2}
\end{aligned}$$

$$
\begin{aligned}
&\le 2\,C_h^{1/2}\Big(\int f^2 d\Gamma(g,g)\Big)^{1/2}\Big(2\int g^2 d\Gamma(f,f) + 2\int f^2 d\Gamma(g,g) + \int f^2 g^2 d\mu\Big)^{1/2}\\
&\le 2\,C_h^{1/2}\Big(\epsilon\int g^2 d\Gamma(f,f) + \frac{C_0(\epsilon)}{\Psi(r)}\int g f^2 d\mu\Big)^{1/2}\Big(2\int g^2 d\Gamma(f,f)\Big)^{1/2}\\
&\quad + 2\,C_h^{1/2}\Big(2\int f^2 d\Gamma(g,g) + \int f^2 g^2 d\mu\Big)\\
&\le 2\,C_h^{1/2}\Big[\frac{1}{\epsilon^{1/2}}\Big(\epsilon\int g^2 d\Gamma(f,f) + \frac{C_0(\epsilon)}{\Psi(r)}\int g f^2 d\mu\Big) + 2\epsilon^{1/2}\int g^2 d\Gamma(f,f)\Big]\\
&\quad + 2\,C_h^{1/2}\Big(2\epsilon\int g^2 d\Gamma(f,f) + \Big(2\frac{C_0(\epsilon)}{\Psi(r)}+1\Big)\int f^2 g d\mu\Big)\\
&\le C_{11}\epsilon^{1/2}\int g^2 d\Gamma(f,f) + (C_2 + C_3\Psi(r))\frac{\epsilon^{-1/2}C_0(\epsilon)}{\Psi(r)}\int_B f^2 d\mu,
\end{aligned}
$$

for some constants C_{11}, C_2 and C_3 depending only on C_h and C_0. This proves that $(\mathcal{E},\mathcal{F})$ satisfies Assumption 1. Similarly, one can verify that Assumption 4 is satisfied.

Next, we show that $(\mathcal{E},\mathcal{F})$ satisfies Assumption 2. Let g be as above and $0 \le f \in \mathcal{F}_{\text{loc}}(Y)$ with $f + f^{-1} \in L^\infty_{\text{loc}}(Y)$. By (2.2), (2.1), (8.1), and by the cutoff Sobolev inequality (2.5),

$$
\begin{aligned}
|\mathcal{E}^{\text{skew}}(f, f^{-1}g^2)| &= \Big|\int f^{-1}g^2 d\Gamma(f,h) - \int f d\Gamma(f^{-1}g^2,h)\Big|\\
&= \Big| -2\int g d\Gamma(g,h) + 2\int g^2 d\Gamma(\log f, h)\Big|\\
&\le 2\Big(\int d\Gamma(g,g)\Big)^{1/2}\Big(\int g^2 d\Gamma(h,h)\Big)^{1/2}\\
&\quad + 2\Big(\int g^2 d\Gamma(\log f,\log f)\Big)^{1/2}\Big(\int g^2 d\Gamma(h,h)\Big)^{1/2}\\
&\le 2\,C_h^{1/2}\Big(\int d\Gamma(g,g) + \int g^2 d\mu\Big)\\
&\quad + 2\,C_h^{1/2}\Big(\int g^2 d\Gamma(\log f,\log f)\Big)^{1/2}\Big(\int d\Gamma(g,g) + \int g^2 d\mu\Big)^{1/2}\\
&\le 2\,C_h^{1/2}\Big[\epsilon^{1/2}\int g^2 d\Gamma(\log f,\log f) + \Big(\frac{C_0}{\Psi(r)}+1\Big)(1+\epsilon^{-1/2})\int g d\mu\Big]\\
&\le C_{11}\epsilon^{1/2}\int g^2 d\Gamma(\log f,\log f) + (C_4 + C_5\Psi(r))\frac{\epsilon^{-1/2}C_0}{\Psi(r)}\int_B d\mu,
\end{aligned}
$$

for some constants C_{11}, C_4, C_5 depending only on C_h and C_0. □

Remark 8.3. We point out that the restricted bilinear form $(\mathcal{E},\mathcal{F}_{\text{b}})$ satisfies the inequality in Assumption 0 (i) for all $f,g \in \mathcal{F}_{\text{b}}$. For the proof of the parabolic Harnack inequality this is in fact sufficient. Indeed, by locality, the definition (8.2) makes sense for any pair (f,g) where $f \in \mathcal{F}_{\text{loc}}(Y)$ and $g = f\psi^2$ for some $\psi \in \mathcal{F}_{\text{c}} \cap L^\infty(Y,\mu)$. Moreover, if $(f_k) \subset \mathcal{F} \cap \mathcal{C}_{\text{c}}(X)$ converges to some $f \in \mathcal{F}$ in $(\mathcal{F}, \|\cdot\|_{\mathcal{F}})$

and quasi-everywhere, then one can easily verify that, for any positive integer n and any $p \geq 2$,

$$\lim_{k\to\infty} \mathcal{E}(f_k, f_k(f_k \wedge n)^{p-2}\psi^2) = \mathcal{E}(f, f(f \wedge n)^{p-2}\psi^2).$$

This is in fact sufficient to apply the argument of (4.4) and the paragraph thereafter, which is the only place where we have used Assumption 0(i) within Section 4 and Section 5.1–5.2.

However, the full Assumption 0 (i) (for general $f, g \in \mathcal{F}$) may be needed to ensure existence of weak solutions.

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