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# Norm convolution inequalities in Lebesgue spaces

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**Abstract.** We obtain upper and similar lower estimates of the $(L_p, L_q)$ norm for the convolution operator. The upper estimate improves on known convolution inequalities. The technique to obtain lower estimates is applied to study boundedness problems for oscillatory integrals.

## 1. Introduction

Let $1 \leqslant p \leqslant \infty$, $L_p \equiv L_p(\mathbb{R})$, and let the convolution operator be given by

$$(Af)(x) = (K * f)(x) = \int_{\mathbb{R}} K(x-y) f(y) dy, \quad K \in L_{\mathrm{loc}}. \tag{1.1}$$

The Young convolution inequality

$$\|A\|_{L_p \to L_q} \leqslant \|K\|_{L_r}, \quad 1 + \frac{1}{q} = \frac{1}{p} + \frac{1}{r}, \quad 1 \leqslant p \leqslant q \leqslant \infty,$$

plays a very important role both in Harmonic Analysis and PDE (see, e.g., Chapter 4, §2, 4 in [2], [5], [12]). Hardy and Littlewood (see, e.g., [28]) extended this result to include the kernels $K(x) = |x|^{-1/r}$, which correspond to the fractional integration theorem. Hörmander [10] has weakened the condition $K \in L_r$ by giving a strictly larger class which includes the Hardy–Littlewood kernels.

Young's estimates were generalized by O'Neil [23], who showed that for $1 < p < q < \infty$ and $1/r = 1 - 1/p + 1/q$,

$$\|A\|_{L_p \to L_q} \leqslant C \|K\|_{L_{r,\infty}} := C \sup_{t>0} \; t^{1/r} K^*(t), \tag{1.2}$$

where $K^*(t) = \inf\big\{\sigma : \mu\{x \in \Omega : |f(x)| > \sigma\} \leqslant t\big\}$ is the decreasing rearrangement of $K$. Note that inequality (1.2) unlike (1.1) gives the Hardy–Littlewood–Sobolev fractional integration theorem.

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There are several generalizations of both Young and O'Neil's inequalities for various function spaces (weighted $L_p$ spaces, classical and weighted Lorentz spaces, weighted Besov and Hardy spaces, Wiener spaces, Orlicz spaces; see, e.g., [3], [6], [9], [13], [16], [17], [18], [21], [24], [32] and references therein). We also remark that the sharp Young convolution inequality was obtained in [1] and [4].

Another extension of Young's convolution inequality was shown using the Wiener amalgam space $W\big(L_{r,\infty}[-1,1],\, l_{r,\infty}(\mathbb{Z})\big)$ (see, e.g., [8]): for $1 < p < q < \infty$ and $1/r = 1 - 1/p + 1/q$ one has

$$\|A\|_{L_p \to L_q} \leqslant C\, \|K\|_{W(L_{r,\infty}[-1,1],\, l_{r,\infty}(\mathbb{Z}))}, \tag{1.3}$$

$$\|A\|_{L_p \to L_q} \leqslant C\, \|K\|_{W(l_{r,\infty}(\mathbb{Z}),\ L_{r,\infty}[-1,1])}, \tag{1.4}$$

where

$$\|K\|_{W(L_{r,\infty}[-1,1],\, l_{r,\infty}(\mathbb{Z}))} := \sup_{n\in\mathbb{N}}\ n^{1/r} \Big( \sup_{0\leqslant t\leqslant 2}\ t^{1/r} \tilde{K}^*(t,\cdot)\Big)^*_n,$$

$$\|K\|_{W(l_{r,\infty}(\mathbb{Z}),\ L_{r,\infty}[-1,1])} := \sup_{0\leqslant t\leqslant 2}\ t^{1/r} \Big( \sup_{n\in\mathbb{N}}\ n^{1/r}\, \tilde{K}^*(\cdot,n)\Big)^*(t),$$

and

$$\tilde{K}(x,m) := K(m+x), \quad m\in\mathbb{Z},\ x\in[-1,1].$$

Inequality (1.4) was proved by Stepanov [31]. To make the paper self-contained, we provide the proof of inequality (1.3) in Section 3.

The goal of this paper is to improve both O'Neil and Stepanov-type upper estimates of $\|A\|_{L_p\to L_q}$, i.e., inequalities (1.2), (1.3), and (1.4), and to obtain the lower estimate of the same form as the upper estimate. As a corollary, we get a characterization of $\|A\|_{L_p\to L_q}$ for some regular kernels. Moreover, the technique that we use to obtain lower estimates is applied to study boundedness problems for oscillatory integrals.

To formulate our main results, we will need the following definitions. Let $d > 0$ and let

- $M_1$ be the set of intervals of length $\leqslant d$;
- $M_2$ be the set of measurable sets $e \subset [-d,d]$ such that $\operatorname{diam}(e) = \sup_{x,y\in e}|x-y| \leqslant d$;
- $W_1$ be the set of all finite arithmetic progressions of integer numbers;
- $W_2$ be the set of finite sets $w\subset\mathbb{Z}$ such that $\min_{i,j\in w}|i-j| \geqslant 2$.

Now we define the sets $\mathfrak{L}_d$, $\mathfrak{U}_d$, and $\mathfrak{V}_d$ as follows:

$$\mathfrak{L}_d = \Big\{E = \bigcup_{k\in w}(e+kd) :\ e\in M_1,\ \ w\in W_1\Big\},$$

$$\mathfrak{U}_d = \Big\{E = \bigcup_{k\in w}(e_k+kd) :\ e_k\in M_2,\ \ w\in W_2,\ |e_k| = |e_j|,\quad k,j\in w\Big\},$$

$$\mathfrak{V}_d = \Big\{E = \bigcup_{x\in e}(x+w(x)d) :\ \ e\in M_2,\ w(x)\in W_2,\ |w(x)| = |w(y)|,\ x,y\in e\Big\},$$

where $|e|$ is a measure of the set $e \in M_i$ and $|w|$ is a number of elements of $w \in W_i$. Note that $\mathfrak{L}_d \subset \mathfrak{U}_d \cap \mathfrak{V}_d$. If $E \in \mathfrak{L}_d$, then $|E| = |e||w|$, where $e \in M_1$ and $w \in W_1$. Similarly, this property holds for $E \in \mathfrak{U}_d$ and $E \in \mathfrak{V}_d$.

**Theorem 1.1.** *Let $1 < p < q < \infty$. If for some $d > 0$ we have either*

$$\sup_{E \in \mathfrak{U}_d} \frac{1}{|E|^{1/p-1/q}} \int_E |K(x)|\, dx \leqslant D, \tag{1.5}$$

*or*

$$\sup_{E \in \mathfrak{V}_d} \frac{1}{|E|^{1/p-1/q}} \int_E |K(x)|\, dx \leqslant D, \tag{1.6}$$

*then the operator $Af = K * f$ is bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$, and*

$$\|A\|_{L_p \to L_q} \leqslant C(p,q)\, D, \tag{1.7}$$

*where $C(p,q)$ depends on $p$ and $q$.*

Next, we investigate the lower bounds of $\|K * f\|_{L_p \to L_q}$.

**Remark 1.2.** Let $1 < p \leqslant q < \infty$. If the operator $Af = K * f$ is bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$, then

$$\sup_{|E|>0, E \in \mathfrak{M}} \frac{1}{|E-E|^{1/p}|E|^{-1/q}} \left| \int_E K(x)\, dx \right| \leqslant \|A\|_{L_p \to L_q}, \tag{1.8}$$

where $\mathfrak{M} = \mathfrak{M}_+ \cup \mathfrak{M}_-$,

$$\mathfrak{M}_+ = \big\{E \subset \mathbb{R} : E - E + y \subset \{x : K(x) \geqslant 0\} \quad \text{for any} \quad y \in E\big\}, \tag{1.9}$$

and

$$\mathfrak{M}_- = \big\{E \subset \mathbb{R} : E - E + y \subset \{x : K(x) < 0\} \quad \text{for any} \quad y \in E\big\}. \tag{1.10}$$

Note that if $(x_0 - \delta, x_0 + \delta) \subset \{x : K(x) \geqslant 0\}$, then any $E \subset (x_0 - \frac{\delta}{3}, x_0 + \frac{\delta}{3})$ belongs to $\mathfrak{M}_+$. Setting $\mathfrak{N}(B) = \{E : |E - E| \leqslant B|E|\}$, estimate (1.8) in particular implies that for non-negative kernels $K$ we have

$$\sup_{E \in \mathfrak{N}(B)} \frac{1}{|E|^{1/p-1/q}} \int_E K(x)\, dx \leqslant B^{1/p}\, \|A\|_{L_p \to L_q}. \tag{1.11}$$

Moreover, for certain regular kernels $K$ the upper and lower bounds in (1.5) and (1.8) coincide, that is, we get the equivalent relation for $\|A\|_{L_p \to L_q}$. More precisely, we say that a locally integrable function $K(x)$ is *weak monotone* if there exists a constant $C > 0$ such that for any $x \in \mathbb{R} \setminus \{0\}$,

$$|K(x)| \leqslant \frac{C}{|x|} \left| \int_0^x K(t) dt \right|. \tag{1.12}$$

Note that if an even nonnegative function $K(\cdot)$ is monotone decreasing on $\mathbb{R}_+$ or, more generally, quasi-monotone[1], then $K(\cdot)$ is weak monotone. On the other hand, there are weak monotone functions which are not quasi-monotone, for example,

$$K(x) = \frac{|\cos|x|^\beta|}{|x|^\alpha}, \qquad \alpha < 1 \leqslant \alpha + \beta.$$

**Corollary 1.3.** *Let $1 < p < q < \infty$ and $K(x) \geqslant 0$ be a weak monotone function. Hence, a necessary and sufficient condition for the operator $Af = K * f$ to be bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ is*

$$\sup_{|x|>0} |x|^{1/p'+1/q} K(x) < \infty.$$

*Moreover,*

$$C_1(p,q) \sup_{|x|>0} |x|^{1/p'+1/q} K(x) \leqslant \|A\|_{L_p \to L_q} \leqslant C_2(p,q) \sup_{|x|>0} |x|^{1/p'+1/q} K(x).$$

We note that the upper and lower bounds in Theorem 1.1 and Corollary 1.3 do not distinguish the operators with kernels $K$ and $|K|$. Therefore, it is important to obtain a lower bound for non-regular operators, where the operator $(Af)(x) = \int_{-\infty}^{\infty} K(x,y) f(x) dx$ is non-regular for $(L_p, L_q)$ if it is bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ and $(\tilde{A}f)(x) = \int_{-\infty}^{\infty} |K(x,y)| f(x) dx$ is not bounded. The next result provides lower bounds for such operators.

**Theorem 1.4.** *Let $1 < p < q < \infty$, $d > 0$, and the operator $Af = K * f$ be bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$. If for any $B > 0$ we have*

$$\sup_{\substack{E \in \mathfrak{L}_d \\ |E| \leqslant B}} \frac{1}{|E|^{1/p-1/q}} \left| \int_E K(x)\, dx \right| \leqslant C(B) < \infty, \tag{1.13}$$

*then*

$$\sup_{E \in \mathfrak{L}_d} \frac{1}{|E|^{1/p-1/q}} \left| \int_E K(x)\, dx \right| \leqslant C(p,q) \, \|A\|_{L_p \to L_q}. \tag{1.14}$$

For bounded kernels $K$, condition (1.13) holds for any $d > 0$, and we have:

**Corollary 1.5.** *Let $1 < p < q < \infty$ and $|K(x)| \leqslant C$. Then*

$$\sup_{E \in \mathfrak{L}} \frac{1}{|E|^{1/p-1/q}} \left| \int_E K(x)\, dx \right| \leqslant C(p,q) \, \|A\|_{L_p \to L_q}, \tag{1.15}$$

*where $\mathfrak{L} = \bigcup_{d>0} \mathfrak{L}_d$.*

[1] A function $K(\cdot)$ on $\mathbb{R}_+$ is quasi-monotone if there exists $\tau > 0$ such that $f(x)/x^\tau$ is monotone decreasing.

In particular, the convolution operator with the kernel $K(x) = K_n(x)$, where $K_n$ is a non-trivial trigonometric polynomial of degree at most $n$, or $K(x) = |\sin x^2|$ is not bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ for $1 < p < q < \infty$ since the left-hand side of (1.15) is not finite (see Section 6).

**Remark 1.6.** The statement of Theorem 1.4 does not hold in the case $p = q$. For example, take

$$K(x) = \sum_{n\neq 0} (\operatorname{sgn} n)\chi_{D_n}(x), \quad D_n = \left[n - \frac{1}{2|n|}; n + \frac{1}{2|n|}\right],$$

then $Af = f * K$ is bounded from $L_p$ to $L_p$, $1 < p < \infty$ (see [30]), but $\sup_{E\in\mathfrak{L}_1} \left|\int_E K(x)\,dx\right| = \infty$ since for $E_0 = \bigcup_{k=1}^s ([0,t]+k) \in \mathfrak{L}_1$, $0 < t < 1/2$, $s \in \mathbb{N}$, we get

$$\sup_{E\in\mathfrak{L}_1} \left|\int_E K(x)\,dx\right| \geqslant \int_{E_0} K(x)\,dx = \sum_{k=1}^{s} \int_0^t K(x+k)\,dx \geqslant \sum_{k=1/2t}^{s} \int_0^t K(x+k)\,dx$$
$$\geqslant \sum_{k=1/2t}^{s} \frac{1}{2k} \to \infty \quad \text{as } s \to \infty.$$

By $C, C_i$ we will denote positive constants that may be different on different occasions. We write $F \asymp G$ if $F \leqslant C_1 G$ and $G \leqslant C_2 F$ for some positive constants $C_1$ and $C_2$ independent of essential quantities involved in the expressions $F$ and $G$. By $\chi_E(x)$ we define the characteristic function of the set $E$. Let $|E|$ be the Lebesgue measure of $E$.

The paper is organized as follows. In Section 2 we obtain a required version of the Riesz lemma for rearrangements (see, e.g., [28]). Sections 3 and 4 are devoted to the estimates of $\|A\|_{L_p\to L_q}$ from above (Theorem 1.1) and below (Remark 1.2, Corollary 1.3, Theorem 1.4), respectively. In Section 5, we show that the right-hand side estimate in (1.7) implies (1.2), (1.3), and (1.4) but the reverse does not hold in general. We conclude with Section 6, where we obtain several $(L_p, L_q)$-boundedness results for a convolution with oscillating kernels. In particular, we obtain sharp necessary conditions on $a$ and $b$ for the operator $Af = K * f$ with $K(x) = e^{i|x|^a}/|x|^b$ to be bounded from $L_p$ to $L_q$.

Finally, we remark that some results from this paper were announced in the note [22].

## 2. Rearrangement inequalities

First, we denote the decreasing rearrangement of $f$ on $\mathbb{Z}$ by $f^*$. We also denote $f^{**}(n) := \frac{1}{n}\sum_{k=1}^n f^*(k)$. The convolution of functions $f$ and $K$ on $\mathbb{Z}$ is defined by

$$(K * f)(k) = \sum_{n\in\mathbb{Z}} K(k-n)f(n).$$

The following results are inspired by [23], [21].

**Lemma 2.1.** *Let functions $f, g$, and $K$ be defined on $\mathbb{Z}^n$. Then*

$$\sum_{k\in\mathbb{Z}} g(k)(K*f)(k) \leqslant 2\sum_{r=1}^{\infty} r\, g^{**}(r)\, f^{**}(r)\, K^{**}(r). \tag{2.1}$$

*Proof.* From

$$f^{**}(n) = \sup_{\substack{|e|=n\\ e\subset\mathbb{Z}}} \frac{1}{|e|}\sum_{s\in e}|f(s)|$$

(see Chapter 2, §3, in [2]) and the Hardy–Littlewood inequality ([2], p. 44), we write

$$\begin{aligned}
\sum_{k\in\mathbb{Z}} g(k)(K*f)(k) &\leqslant \sum_{r=1}^{\infty} g^*(r)(K*f)^{**}(r)\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}} \sum_{m\in\mathbb{Z}} |f(m)|\frac{1}{|e|}\sum_{s\in e}|K(s-m)|\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}} \sum_{m=1}^{\infty} f^*(m)\Big(\frac{1}{|e|}\sum_{s\in e}|K(s-\cdot)|\Big)^{**}(m)\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}} \sum_{m=1}^{\infty} f^*(m)\Big(\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}} \frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sum_{s\in e}|K(s-t)|\Big)\\
&\leqslant \sum_{r=1}^{\infty} g^*(r) \sum_{m=1}^{\infty} f^*(m)\Big(\sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}} \frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sum_{s\in e}|K(s-t)|\Big).
\end{aligned}$$

We consider

$$\Phi(r,m) = \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}} \frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sum_{s\in e}|K(s-t)|.$$

If $r\leqslant m$, then

$$\Phi(r,m) \leqslant \sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sum_{s\in e}\sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}} \frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}|K(s-t)| = K^{**}(m)$$

and if $m\leqslant r$, then

$$\Phi(r,m) \leqslant \sup_{\substack{|\omega|=m\\ \omega\subset\mathbb{Z}}} \frac{1}{|e|}\frac{1}{|\omega|}\sum_{t\in w}\sup_{\substack{|e|=r\\ e\subset\mathbb{Z}}}\sum_{s\in e}|K(s-t)| = K^{**}(r).$$

Hence, we get

$$\Phi(r,m) \leqslant K^{**}(\max\{r,m\}).$$

Therefore,

$$
\begin{aligned}
\sum_{k\in\mathbb{Z}}^{\infty} g(k)(K*f)(k) &\leqslant \sum_{r=1}^{\infty} g^*(r) \sum_{m=1}^{\infty} f^*(m)\, K^{**}(\max\{r,m\}) \\
&= \sum_{r=1}^{\infty} g^*(r)K^{**}(r) \sum_{m=1}^{r} f(m)^* + \sum_{r=1}^{\infty} g^*(r) \sum_{m=r+1}^{\infty} f^*(m)\, K^{**}(m) \\
&= \sum_{r=1}^{\infty} r\, g^*(r)\, K^{**}(r)\, f^{**}(r) + \sum_{m=1}^{\infty} f^*(m)\, K^{**}(m) \sum_{r=1}^{m} g^*(r) \\
&\leqslant 2 \sum_{r=1}^{\infty} r\, g^{**}(r)\, K^{**}(r)\, f^{**}(r).
\end{aligned}
$$

□

The continuous analogue of the previous lemma is the following result.

**Lemma 2.2.** *Let $f$ and $g$ be measurable functions on $[0,d]$ and $K$ be measurable on $[-d,d]$. Then*

$$
\int_0^d g(y) \int_0^d f(x)K(y-x)\, dx\, dy \leqslant 2 \int_0^d t g^{**}(t) f^{**}(t) \Bigg( \sup_{\substack{|e|=t \\ e\in M_2}} \frac{1}{|e|} \int_e |K(x)|dx \Bigg) dt. \tag{2.2}
$$

*Proof.* Similarly to the proof of Lemma 2.1, we have

$$
\begin{aligned}
\int_0^d g(y)\,(K*f)\,(y)dy &\leqslant \int_0^d g^*(s) \int_0^d f^*(t) \sup_{\substack{|e|=s \\ e\subset[0,d]}} \sup_{\substack{|\omega|=t \\ \omega\subset[0,d]}} \frac{1}{|e|}\frac{1}{|\omega|} \int_e \int_\omega |K(y-x)|\; dxdy \\
&= \int_0^d g^*(s) \int_0^d f^*(t)\Phi(s,t)\, dt\, ds.
\end{aligned}
$$

Further, for $s \leqslant t$, we get

$$
\Phi(s,t) \leqslant \sup_{\substack{e\subset[0,d] \\ |e|=s}} \frac{1}{|e|} \int_e \sup_{\substack{|\omega|=t \\ \omega\subset[0,d]}} \frac{1}{|\omega|} \int_\omega |K(y-x)|\; dx\, dy = \sup_{\substack{|w|=t \\ w\in M_2}} \frac{1}{|\omega|} \int_\omega |K(x)|\, dx,
$$

and for $s \geqslant t$,

$$
\Phi(s,t) \leqslant \sup_{\substack{|e|=s \\ e\in M_2}} \frac{1}{|e|} \int_e |K(y)|\, dy.
$$

Finally, as in the proof of Lemma 2.1, we have

$$
\int_0^d g(y)\,(K*f)\,(y)\, dy \leqslant 2 \int_0^d t\, g^{**}(t)\, f^{**}(t) \sup_{\substack{|e|=t \\ e\in M_2}} \frac{1}{|e|} \int_e |K(x)|\, dx.
$$

□

## 3. Proofs of upper bounds

*Proof of inequality* (1.3). By Minkowski's inequality, we get

$$\begin{aligned}\|K*f\|_{L_q(\mathbb{R})} &= \Big(\sum_{k\in\mathbb{Z}}\int_0^1\Big|\sum_{m\in\mathbb{Z}}\int_0^1 f(x+m)K((y-x)+(k-m))\,dx\Big|^q dy\Big)^{1/q}\\ &\leqslant \Big(\sum_{k\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\Big(\int_0^1\Big(\int_0^1 f(x+m)\,K((y-x)+k-m)\,dx\Big)^q dy\Big)^{1/q}\Big)^q\Big)^{1/q}.\end{aligned}$$

Using O'Neil's inequality (1.2) and then its discrete analogue, we have

$$\begin{aligned}\|K*f\|_{L_q(\mathbb{R})} &\leqslant C\Big(\sum_{k\in\mathbb{Z}}\Big(\sum_{m\in\mathbb{Z}}\|f(\cdot+m)\|_{L_p(0,1)}\|K(\cdot+k-m)\|_{L_{r,\infty}(-1,1)}\Big)^q\Big)^{1/q}\\ &\leqslant C\Big(\sum_{k\in\mathbb{Z}}\|f(\cdot+m)\|^p_{L_p(0,1)}\Big)^{1/p}\Big\|\|K(\cdot+n)\|_{L_{r,\infty}(-1,1)}\Big\|_{l_{r,\infty}(\mathbb{Z})}\\ &= C\,\|f\|_{L_p(\mathbb{R})}\Big\|\|K(\cdot+n)\|_{L_{r,\infty}(-1,1)}\Big\|_{l_{r,\infty}(\mathbb{Z})}.\end{aligned}$$

□

*Proof of Theorem* 1.1. Suppose (1.5) holds. Let $d>0$ for $k\in\mathbb{Z}$ and $x\in[0,d]$, we denote

$$\widetilde{f}(x,k):=f(x+kd),\quad \widetilde{g}(x,k):=g(x+kd),\quad\text{and}\quad \widetilde{K}(x,k):=K(x+kd).$$

We are going to estimate the following quantity:

$$J:=\int_{\mathbb{R}}g(y)\int_{\mathbb{R}}f(x)K(y-x)\,dx\,dy.$$

Let us write it as follows:

$$\begin{aligned}J &= \sum_{k\in\mathbb{Z}}\int_0^d g(y+kd)\sum_{m\in\mathbb{Z}}\int_0^d f(x+md)\,K\big((y-x)+(k-m)d\big)\,dx\,dy\\ &= \sum_{k\in\mathbb{Z}}\int_0^d \tilde{g}(y,k)\sum_{m\in\mathbb{Z}}\int_0^d \tilde{f}(x,m)\,\tilde{K}(y-x,k-m)\,dx\,dy. \qquad (3.1)\end{aligned}$$

To estimate this, we first use Lemma 2.2:

$$\begin{aligned}J &\leqslant 2\sum_{k\in\mathbb{Z}}\sum_{m\in\mathbb{Z}}\int_0^d t\,\tilde{f}^{(**)_1}(t,m)\,\tilde{g}^{(**)_1}(t,k)\sup_{\substack{|e|=t\\ e\in M_2}}\frac{1}{|e|}\Big|\int_e\tilde{K}(x,k-m)\,dx\Big|\,dt\\ &= 2\int_0^d t\Big(\sum_{k\in\mathbb{Z}}\tilde{g}^{(**)_1}(t,k)\sum_{m\in\mathbb{Z}}\tilde{f}^{(**)_1}(t,m)\sup_{\substack{|e|=t\\ e\in M_2}}\frac{1}{|e|}\int_e|\tilde{K}(x,k-m)|\,dx\Big)\,dt,\end{aligned}$$

where

$$\tilde f^{(**)_1}(t,m)=\frac{1}{t}\int_0^t \tilde f^{*_1}(u,m)\,du,\quad m\in\mathbb{Z},$$
$$\tilde g^{(**)_1}(t,k)=\frac{1}{t}\int_0^t \tilde g^{*_1}(u,k)\,du,\quad k\in\mathbb{Z},$$

and $\tilde f^{*_1}(t,m),\tilde g^{*_1}(t,k)$ are decreasing rearrangements of $\tilde f(x,m),\tilde g(x,k)$ with respect to $x$ and with fixed $m$ and $k$, correspondingly.

Applying now Lemma 2.1, we get

$$\begin{aligned}J&\leqslant 4\int_0^d t\sum_{s=1}^{\infty} s\,\tilde f^{**}(t,s)\,\tilde g^{**}(t,s)\Big(\sup_{\substack{|\omega|=s\\ \omega\subset\mathbb{Z}}}\frac{1}{|\omega|}\sum_{m\in w}\sup_{\substack{|e|=t\\ e\in M_2}}\frac{1}{|e|}\int_e|\tilde K(x,m)|\,dx\Big)\,dt\\ &=:4\int_0^d t\sum_{s=1}^{\infty} s\,\tilde f^{**}(t,s)\,\tilde g^{**}(t,s)\,F_d(t,s;K)\,dt,\end{aligned}$$

where

$$\tilde f^{**}(t,s)=\frac{1}{s}\sum_{l=1}^{s}\big(\tilde f^{(**)_1}(t,\cdot)\big)_l^{*_2}\quad\text{and}\quad \tilde g^{**}(t,s)=\frac{1}{s}\sum_{l=1}^{s}\big(\tilde g^{(**)_1}(t,\cdot)\big)_l^{*_2}.$$

Then writing

$$\begin{aligned}&(ts)\,\tilde g^{**}(t,s)\tilde f^{**}(t,s)F_d(t,s;K)\\ &\qquad\leqslant\big((ts)^{1/p-1/q}\tilde f^{**}(t,s)\big)\,\big(\tilde g^{**}(t,s)\big)\,\Big(\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{Z}}}(ts)^{1/r}\,F_d(t,s;K)\Big)\end{aligned}$$

and using Hölder's inequality with parameters $q$ and $q'$ and the fact that $L_{p,q}\hookrightarrow L_{p,q_1}$ for $q\leqslant q_1$, we get

$$\begin{aligned}\int_0^d\sum_{s=1}^{\infty}t\,s\,\tilde f^{**}(t,s)\,\tilde g^{**}(t,s)\,F_d(t,s;K)dt&\leqslant 4\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{Z}}}(ts)^{1-(1/p-1/q)}F_d(t,s;K)\\ &\cdot\Big(\sum_{s\in\mathbb{N}}\int_0^d\big(\tilde g^{**}(t,s)\big)^{q'}dt\Big)^{1/q'}\Big(\sum_{s\in\mathbb{N}}\int_0^d\big(\tilde f^{**}(t,s)\big)^{p}dt\Big)^{1/p}.\end{aligned}$$

Note that Hardy's inequalities of the type

$$\Big\|\frac{1}{t}\int_0^t f\Big\|_{L_p}\leqslant(p')^p\|f\|_{L_p}\quad\text{and}\quad\Big\|\frac{1}{n}\sum\nolimits_{k=1}^{n}f_n\Big\|_{l_p}\leqslant(p')^p\|f_n\|_{l_p}$$

imply

$$\sum_{s\in\mathbb{N}}\int_0^d \big(\tilde f^{**}(t,s)\big)^p\,dt \leqslant \big(p'\big)^p\int_0^d\sum_{s\in\mathbb{N}}\Big(\big(\tilde f^{(**)_1}(t,\cdot)\big)_s^{*_2}\Big)^p\,dt$$
$$= \big(p'\big)^p\sum_{s\in\mathbb{Z}}\int_0^d\big(\tilde f^{(**)_1}(t,s)\big)^p\,dt \leqslant \big(p'\big)^{2p}\sum_{s\in\mathbb{Z}}\int_0^d(\tilde f^{*_1}(t,s))^p\,dt$$
$$= \big(p'\big)^{2p}\sum_{s\in\mathbb{Z}}\int_0^d(\tilde f(t,s))^p\,dt = \big(p'\big)^{2p}\|f\|_{L_p}^p.$$

This yields the following inequality:

$$\Big|\int_{\mathbb{R}} g(y)\int_{\mathbb{R}} f(x)K(y-x)\,dxdy\Big| \leqslant C\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{Z}}}(ts)^{1-(1/p-1/q)}F_d(t,s;K)\|f\|_{L_p}\|g\|_{L_{q'}}.$$

Thus,

$$\|A\|_{L_p\to L_q}\leqslant C\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{Z}}}(ts)^{1-(1/p-1/q)}F_d(t,s;K), \tag{3.2}$$

where

$$\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{Z}}}(ts)^{1-(1/p-1/q)}F_d(t,s;K)$$
$$=\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{Z}}}\sup_{|\omega|=s}\frac{1}{(ts)^{1/p-1/q}}\sum_{m\in\omega}\sup_{\substack{|e|=t,\,\mathrm{diam}\,(e)\leqslant d\\ e\subset[-d,d]}}\int_e|\tilde K(x,m)|\,dx.$$

Thus, if condition (1.5) holds, we get

$$\|A\|_{L_p\to L_q}\leqslant C(p,q)\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{Z}}}\sup_{|\omega|=s}\frac{1}{(ts)^{1/p-1/q}}\sum_{m\in\omega}\sup_{\substack{|e|=t\\ e\in M_2}}\int_e|\tilde K(x,m)|\,dx.$$

Similarly, in the case when condition (1.6) holds we have

$$\|A\|_{L_p\to L_q}\leqslant C(p,q)\sup_{\substack{0<t\leqslant d\\ s\in\mathbb{Z}}}\frac{1}{(ts)^{1/p-1/q}}\sup_{\substack{|e|=t\\ e\in M_2}}\int_e\sup_{\substack{|w|=s\\ w\in W_2}}\sum_{m\in\omega}|\tilde K(x,m)|\,dx.$$

This can be proved as above but in this case we first apply Lemma 2.1 and then Lemma 2.2.

To finish the proof of Theorem 1.1, it is sufficient to show the following.

**Lemma 3.1.** *Let $d>0$, $0<\gamma\leqslant 1$, $t>0$, and $s\in\mathbb{N}$.*

(A) *Let $w\subset\mathbb{Z}$ and $|w|=s$. Then*

$$\frac{1}{(ts)^\gamma}\sum_{m\in\omega}\sup_{\substack{|e|=t\\ e\in M_2}}\int_e|\tilde K(x,m)|\,dx\leqslant 4\sup_{E\in\mathfrak{U}_d}\frac{1}{|E|^\gamma}\int_E|K(x)|\,dx,$$

*where $\tilde K(x,m)=K(x+md)$.*

(B) *Let* $e \in M_2$ *and* $|e| = t$. *Then*

$$\frac{1}{(ts)^\gamma} \int_e \sup_{\substack{|w|=s \\ w\subset\mathbb{Z}}} \sum_{m\in w} |\tilde{K}(x,m)|\,dx \leqslant 4 \sup_{E\in\mathfrak{V}_d} \frac{1}{|E|^\gamma} \int_E |K(x)|\,dx.$$

*Proof.* For any $m \in w$ and $t \in (0,d]$ we find $e_m \in M_2$, i.e., $e_m \subset [-d,d]$, $\operatorname{diam}(e_m) \leqslant d$ such that $|e_m| = t$ and

$$\begin{aligned}\sup_{|e|=t} \int_e |\tilde{K}(x,m)|\,dx &\leqslant 2 \int_{e_m} |\tilde{K}(x,m)|\,dx \\ &= 2 \int_{e_m} |K(x+md)|\;dx = 2 \int_{e_m+md} |K(x)|\,dx.\end{aligned}$$

We have

$$\frac{1}{(ts)^\gamma} \sum_{m\in\omega} \sup_{\substack{|e|=t,\,\operatorname{diam} e\leqslant d \\ e\subset[-d,d]}} \int_e |\tilde{K}(x,m)|\,dx \leqslant \frac{2}{(ts)^\gamma} \sum_{m\in w} \int_{e_m+md} |K(x)|\,dx.$$

Since any set $w \subset \mathbb{Z}$ can be represented as a union of $w_1$ and $w_2$ from $W_2$, i.e., such that $\min_{k,m\in w_i} |k-m| \geqslant 2$, we get

$$\begin{aligned}\frac{1}{(ts)^\gamma} \sum_{m\in\omega} \sup_{\substack{|e|=t,\,\operatorname{diam} e\leqslant d \\ e\subset[-d,d]}} \int_e |\tilde{K}(x,m)|\,dx &\leqslant \frac{2}{(ts)^\gamma} \Big( \sum_{m\in w_1} + \sum_{m\in w_2} \Big) \int_{e_m+md} |K(x)|\,dx \\ &\leqslant \frac{2}{(ts)^\gamma} \Big( \int_{\bigcup_{m\in w_1}(e_m+md)} |K(x)|\,dx + \int_{\bigcup_{m\in w_2}(e_m+md)} |K(x)|\,dx \Big) \\ &\leqslant 4 \sup_{E\in\mathfrak{U}_d} \frac{1}{|E|^\gamma} \int_E |K(x)|\,dx,\end{aligned}$$

where in the last inequality we used that $\big|\bigcup_{m\in w_1}(e_m+md)\big| = |w_1||e_m| \leqslant ts$ and similarly for $w_2$.

The proof of the inequality from the part (B) is similar. First,

$$\frac{1}{(ts)^\gamma} \int_e \sup_{\substack{|w|=s \\ w\subset\mathbb{Z}}} \sum_{m\in w} |\tilde{K}(x,m)|\,dx \leqslant \frac{2}{(ts)^\gamma} \int_e \sum_{m\in w(x)} |\tilde{K}(x,m)|\,dx$$

for some $w(x) \in \mathbb{Z}$ such that $|w(x)| = s$. Then

$$\begin{aligned}\frac{1}{(ts)^\gamma} \int_e \sup_{\substack{|w|=s \\ w\subset\mathbb{Z}}} \sum_{m\in w} |\tilde{K}(x,m)|\,dx &\leqslant \frac{2}{(ts)^\gamma} \int_{\bigcup_{x\in e}(x+w_1(x)d)} |K(x)|\,dx \\ &\quad + \frac{2}{(ts)^\gamma} \int_{\bigcup_{x\in e}(x+w_2(x)d)} |K(x)|\,dx \leqslant 4 \sup_{E\in\mathfrak{V}_d} \frac{1}{|E|^\gamma} \int_E |K(x)|\,dx.\end{aligned}$$

$\square$

## 4. Proof of lower bounds

*Proof of Remark* 1.2. Let $E \in \mathfrak{M}$ and $f_0(x) = \chi_{E-E}(x)$. Then we get

$$\|K*f_0\|_{L_q} = \Big(\int_{-\infty}^{\infty}\Big|\int_{E-E} K(y-x)\,dx\Big|^q dy\Big)^{1/q} \geqslant \Big(\int_E\Big|\int_{E-E+y} K(x)\,dx\Big|^q dy\Big)^{1/q}.$$

Since for any $y \in E$ we have $E - E + y \supset E$, using the fact that $K$ keeps its sign for any $x \in E$, we get

$$\Big|\int_{E+E-y} K(x)\,dx\Big| \geqslant \Big|\int_E K(x)\,dx\Big|.$$

Therefore,

$$\|K * f_0\|_{L_q} \geqslant |E|^{1/q}\Big|\int_E K(x)\,dx\Big|$$

and

$$\|A\|_{L_p\to L_q} \geqslant \frac{|E|^{1/q}}{|E-E|^{1/p}}\Big|\int_E K(x)\,dx\Big|. \qquad \square$$

*Proof of Corollary* 1.3. From the definition of weak monotone function, we get

$$\begin{aligned}
\sup_{E\in\mathfrak{U}_d}\frac{1}{|E|^{1/p-1/q}}\int_E K(t)\,dt &\leqslant C\sup_{E\in\mathfrak{U}_d}\frac{1}{|E|^{1/p-1/q}}\int_E\Big(\frac{1}{t}\int_0^t K(y)dy\Big)\,dt\\
&\leqslant C\sup_{t\neq 0}\frac{1}{|t|^{1/p-1/q}}\Big|\int_0^t K(t)\,dt\Big|\ \sup_{E\in\mathfrak{U}_d}\frac{1}{|E|^{1/p-1/q}}\int_E\frac{dt}{|t|^{1-1/p+1/q}}\\
&\leqslant C\sup_{t\neq 0}\frac{1}{|t|^{1/p-1/q}}\Big|\int_0^t K(x)\,dx\Big|\ \sup_{s>0}\frac{1}{s^{1/p-1/q}}\int_{-s}^{s}\frac{dt}{|t|^{1-1/p+1/q}}\\
&\leqslant C\sup_{t\neq 0}\frac{1}{|t|^{1/p-1/q}}\Big|\int_0^t K(s)ds\Big|
\end{aligned}$$

and

$$C_1\sup_{|t|>0}|t|^{1/p'+1/q}K(t) \leqslant \sup_{t\neq 0}\frac{1}{|t|^{1/p-1/q}}\Big|\int_0^t K(s)ds\Big| \leqslant C_2\sup_{|t|>0}|t|^{1/p'+1/q}K(t).$$

Therefore, if $\sup_{|t|>0}|t|^{1/p'+1/q}K(t) < \infty$, Theorem 1.1 implies that the operator $A$ is bounded from $L_p$ to $L_q$.

On the other hand, by (1.11) with $B = 2$, we get

$$\|A\|_{L_p\to L_q} \geqslant 2^{-1/p}\sup_{E\in\mathfrak{N}(2)}\frac{1}{|E|^{1/p-1/q}}\int_E K(s)\,ds.$$

Since $\mathfrak{N}(2)$ contains all intervals $[0,t)$, we estimate

$$\|A\|_{L_p\to L_q} \geqslant 2^{-1/p}\sup_{t\neq 0}\frac{1}{t^{1/p-1/q}}\Big|\int_0^t K(s)\,ds\Big| \geqslant C\sup_{|t|>0}|t|^{1/p'+1/q}K(t). \qquad \square$$

*Proof of Theorem* 1.4. Suppose that $B > 0$ and

$$\alpha := \sup_{\substack{E\in\mathfrak{L}_d\\ |E|\leqslant B}} \frac{1}{|E|^{1/r'}}\left|\int_E K(x)\,dx\right| < \infty.$$

Then we consider $E_0 \in \mathfrak{L}_d, |E_0| \leqslant B$, such that

$$\frac{1}{|E_0|^{1/r'}}\left|\int_{E_0} K(x)\,dx\right| \geqslant \frac{\alpha}{2}.$$

Since the convolution is translation invariant, we assume that $E_0$ is of form

$$E_0 = \bigcup_{i=0}^{m} \left([0,b] + ird\right),$$

where $b \leqslant d$, $m, r \in \mathbb{N}$.

Let us take $0 < \delta < 1/2$ to be specified later. We define the following sets $E_{1+\delta}$ and $E_\delta$:

$$E_{1+\delta} = \bigcup_{i=0}^{[(1+\delta)m]} \left([0,(1+\delta)b] + ird\right) \quad \text{and} \quad E_\delta = \bigcup_{i=0}^{[\delta m]} \left([0,\delta b] + ird\right).$$

Then taking $f_0 = \chi_{E_{1+\delta}}$, the boundedness of the operator $A$ implies

$$\begin{aligned}\|K * f_0\|_{L_q} &\leqslant \|A\|_{L_p\to L_q}\|f_0\|_{L_p} = \|A\|_{L_p\to L_q}|E_{1+\delta}|^{1/p}\\ &\leqslant 2\,\|A\|_{L_p\to L_q}(1+\delta)^{2/p}|E_0|^{1/p}.\end{aligned} \tag{4.1}$$

On the other hand,

$$\begin{aligned}\|K * f_0\|_{L_q} &= \left(\int_{-\infty}^{\infty}\left|\int_{E_{1+\delta}} K(x-y)\,dx\right|^q dy\right)^{1/q}\\ &= \left(\sum_{j\in\mathbb{Z}}\int_0^d\left|\sum_{i=0}^{(1+\delta)m}\int_0^{b(1+\delta)} K((ir-j)d + (x-y))\,dx\right|^q dy\right)^{1/q}\\ &\geqslant \left(\sum_{j=0}^{[\delta m]}\int_0^{\delta b}\left|\sum_{i=0}^{(1+\delta)m}\int_0^{(1+\delta)b} K((i-j)rd + (x-y))\,dx\right|^q dy\right)^{1/q}\\ &= \left(\sum_{j=0}^{[\delta m]}\int_0^{\delta b}\left|\sum_{i=-j}^{(1+\delta)m-j}\int_{-y}^{(1+\delta)b-y} K(ird + x)\,dx\right|^q dy\right)^{1/q}.\end{aligned}$$

Dividing the inner sum into five terms, we estimate

$$\|K * f_0\|_{L_q} \geqslant \Big( \sum_{j=0}^{[\delta m]} \int_0^{\delta b} \Big[ \Big| \sum_{i=0}^{m} \int_0^b K(ird + x) dx \Big| - \Big| \sum_{i=-j}^{-1} \int_{-y}^{(1+\delta)b-y} K(ird + x) dx \Big|$$
$$- \Big| \sum_{i=m+1}^{[(1+\delta)m]-j} \int_{-y}^{(1+\delta)b-y} K(ird + x)\, dx \Big| - \Big| \sum_{i=0}^{m} \int_{-y}^{0} K(ird + x)\, dx \Big|$$
$$- \Big| \sum_{i=0}^{m} \int_{b}^{(1+\delta)b-y} K(ird + x)\, dx \Big| \Big]^q dy \Big)^{1/q}$$
$$= \Big( \int_{E_\delta} \Big[ \Big| \int_{E_0} K(x)\, dx \Big| - \sum_{i=1}^{4} \Big| \int_{E_i} K(x)\, dx \Big| \Big]^q dy \Big)^{1/q},$$

where

$$E_1 = \bigcup_{i=-j}^{-1} ([-y, (1+\delta)b - y] + ird), \quad E_2 = \bigcup_{i=m+1}^{[(1+\delta)m]-j} ([-y, (1+\delta)b - y] + ird),$$
$$E_3 = \bigcup_{i=0}^{m} ([-y, 0] + ird), \quad E_4 = \bigcup_{i=0}^{m} ([b, (1+\delta)b - y] + ird).$$

Note that $E_i \in \mathfrak{L}_d$ and $|E_i| \leqslant 2\delta |E_0| \leqslant B$, $i = 1, 2, 3, 4$.

Now we set $\delta = (2\,(16^{r'}))^{-1} < 1/2$. Then

$$\frac{1}{|E_0|^{1/r'}} \Big| \int_{E_0} K(x)\, dx \Big| \geqslant \frac{\alpha}{2} \geqslant \frac{1}{2|E_i|^{1/r'}} \Big| \int_{|E_i|} K(x)\, dx \Big|$$

and therefore

$$\Big| \int_{E_i} K(x)\, dx \Big| \leqslant \frac{2|E_i|^{1/r'}}{|E_0|^{1/r'}} \Big| \int_{E_0} K(x)\, dx \Big|.$$

Taking into account $|E_i| \leqslant 2\delta |E_0|$, we get

$$\|K * f_0\|_{L_q} \geqslant \Big( \int_{E_\delta} \Big[ \Big| \int_{E_0} K(x)\, dx \Big| \Big( 1 - 2 \sum_{i=1}^{4} \Big( \frac{|E_i|}{|E_0|} \Big)^{1/r'} \Big) \Big]^q dy \Big)^{1/q}$$
$$\geqslant |E_\delta|^{1/q} \Big| \int_{E_0} K(x)\, dx \Big| \big( 1 - 8(2\delta)^{1/r'} \big).$$

Since $|E_\delta| \geqslant |E_0|\delta^2/2$ and $1 - 8(2\delta)^{1/r'} = 1/2$,

$$\|K * f_0\|_{L_q} \geqslant \frac{1}{2}\, \delta^{2/q}\, |E_0|^{1/q} \Big| \int_{E_0} K(x)\, dx \Big|.$$

Using (4.1), we have

$$\|A\|_{L_p \to L_q} \geqslant C(p, q) \frac{1}{|E_0|^{1/r'}} \Big| \int_{E_0} K(x)\, dx \Big| \geqslant \frac{C(p, q)}{2} \sup_{\substack{E \in \mathfrak{L}_d \\ |E| \leqslant B}} \frac{1}{|E|^{1/r'}} \Big| \int_{E} K(x)\, dx \Big|.$$

Since $B > 0$ can be chosen arbitrarily, we conclude the proof of Theorem 1.4. □

We draw attention to the fact that attempts have already been made at proving the lower estimate for the convolution operator in [19], although they require stronger hypotheses than those used here. Moreover, for kernels which are so-called weakly oscillating, some necessary conditions so that the convolution maps $L_p$ into $L_q$ were proved in [14].

## 5. Comparison new upper bounds with O'Neil and Stepanov-type inequalities

Let us first show that estimate (1.7) in Theorem 1.1 implies (1.2), (1.3), and (1.4). Indeed, it is known ([2], Chapter 2, §3) that

$$\sup_{t>0} t^{1/r} K^*(t) \asymp \sup_{t>0} t^{1/r} K^{**}(t) \asymp \sup_{|E|>0} \frac{1}{|E|^{1/r'}} \int_E |K(x)|\, dx, \tag{5.1}$$

and therefore

$$\max\Big\{ \sup_{E\in\mathfrak{U}_d} \frac{1}{|E|^{1/r'}} \int_E |K(x)|\,dx,\ \sup_{E\in\mathfrak{V}_d} \frac{1}{|E|^{1/r'}} \int_E |K(x)|\,dx\Big\} \leqslant C(r) \sup_{t>0} t^{1/r} K^*(t), \tag{5.2}$$

where $1/r = 1-(1/p-1/q) < 1$, $r' = r/(r-1)$.

Let $d=1$. Assume that $E\in\mathfrak{U}_d$, that is, $E=\bigcup_{i\in w}(e_i+i)$. Let $|w|=s$, $|e_i|=t$. Then

$$\begin{aligned}
\frac{1}{|E|^{1/r'}} \int_E |K(x)|\,dx &= \frac{1}{(st)^{1/r'}} \sum_{i\in w} \int_{e_i+i} |K(x)|\,dx = \frac{1}{(st)^{1/r'}} \sum_{i\in w} \int_{e_i} |K(x+i)|\,dx \\
&= \frac{1}{(st)^{1/r'}} \sum_{i\in w} \int_{e_i} |\tilde K(x,i)|\,dx \leqslant \frac{1}{(st)^{1/r'}} \sum_{i\in w} \int_0^t \tilde K^*(\xi,i) d\xi \\
&\leqslant \frac{1}{(st)^{1/r'}} \sum_{i\in w} \sup_{0<\xi\leqslant 1} \xi^{1/r} \tilde K^*(\xi,i) \int_0^t \xi^{-1/r} d\xi \leqslant (r')^2 \sup_{n\in\mathbb{N}} s^{1/r} \Big( \sup_{0<t\leqslant 1} t^{1/r} \tilde K^*(t,\cdot)\Big)^*_s.
\end{aligned}$$

Let $E\in\mathfrak{V}_d$, $d=1$, that is, $E=\bigcup_{x\in e}(x+w(x))$. Let $|e|=t$ and $|w(x)|=s$. Then, similarly as above,

$$\begin{aligned}
\frac{1}{|E|^{1/r'}} \int_E |K(x)|\,dx &= \frac{1}{(st)^{1/r'}} \int_e \sum_{i\in w(x)} |K(x+i)|\,dx \\
&= \frac{1}{(st)^{1/r'}} \int_e \sum_{i\in w(x)} |\tilde K(x,i)|\,dx \leqslant \frac{1}{(st)^{1/r'}} \int_e \sum_{i=1}^{s} \tilde K^*(x,i)\,dx \\
&\leqslant \frac{1}{(st)^{1/r'}} \int_e \sup_{k\in\mathbb{N}} k^{1/r} \tilde K^*(x,k) \sum_{i=1}^{s} i^{-1/r}\,dx \leqslant (r')^2 \sup_{0<\xi\leqslant 1} \xi^{1/r} \Big(\sup_{k\in\mathbb{N}} k^{1/r} \tilde K^*(\cdot,k)\Big)^*_\xi.
\end{aligned}$$

Thus, (1.7) refines estimates (1.2), (1.3), and (1.4) since

$$UpBo := \min\Big\{ \sup_{E\in\mathfrak{U}_d} \frac{1}{|E|^{1/r'}} \int_E |K(x)|\,dx,\ \sup_{E\in\mathfrak{B}_d} \frac{1}{|E|^{1/r'}} \int_E |K(x)|\,dx \Big\}$$

$$\leqslant C(r)\min\big\{\|K\|_{L_{r,\infty}}, \|K\|_{W(L_{r,\infty}[-1,1],\,l_{r,\infty}(\mathbb{Z}))}, \|K\|_{W(l_{r,\infty}(\mathbb{Z}),\ L_{r,\infty}[-1,1])}\big\}. \tag{5.3}$$

We now give examples capturing the difference between these estimates. To construct an example of $K$ showing that there is no constant in the inequality reverse to (5.3), it is sufficient to take the sum for $K$'s from Examples 5.1 and 5.2 below.

**Example 5.1.** *Let $1 < p < q < \infty$ and $1/r = 1-(1/p-1/q)$. Define the function $K(x)$ on $\mathbb{R}$ as follows:*

$$K(x) = \begin{cases} 2^{k/r}, & \text{for } x\in[-k,-k+2^{-k}], \quad k\in\mathbb{N};\\ 1, & \text{for } x\in[k,k+1/k), \quad k\in\mathbb{N};\\ 0, & \text{otherwise.}\end{cases}$$

*This function satisfies*

$$\sup_{E\in\mathfrak{U}_1} \frac{1}{|E|^{1/r'}} \Big|\int_E K(x)\,dx\Big| < \infty, \tag{5.4}$$

*but*

$$\|K\|_{L_{r,\infty}} = \infty \tag{5.5}$$

*and*

$$\|K\|_{W(L_{r,\infty}[-1,1],\,l_{r,\infty}(\mathbb{Z}))} = \sup_{n\in\mathbb{N}} n^{1/r}\Big(\sup_{0\leqslant t\leqslant 2} t^{1/r}\tilde{K}^*(t,\cdot)\Big)^*_n = \infty. \tag{5.6}$$

Indeed, let us show (5.4). Let $K_+(x) = K(x)\chi_{[0,\infty)}(x)$, $K_-(x) = K(x)\chi_{(-\infty,0)}(x)$, then $K_+(x)+K_-(x) = K(x)$ and therefore,

$$\sup_{E\in\mathfrak{U}_1} \frac{1}{|E|^{1/r'}} \int_E K(x)\,dx \leqslant \sup_{E\in\mathfrak{U}_1} \frac{1}{|E|^{1/r'}} \int_E K_+(x)\,dx + \sup_{E\in\mathfrak{U}_1} \frac{1}{|E|^{1/r'}} \int_E K_-(x)\,dx.$$

Let $e\in\mathfrak{U}_1$. Then $E = \cup_{k\in w} e_k + k$, where $|e_k| = t < 1$ and $w\subset\mathbb{Z}$, $|w| = s$. We have

$$\begin{aligned}\frac{1}{|E|^{1/r'}} \int_E K_+(x)\,dx &= \frac{1}{(st)^{1/r'}} \sum_{k\in w}\int_{e_k} K_+(x+k)\,dx\\ &\leqslant \frac{1}{(st)^{1/r'}} \sum_{k=1}^{s}\int_0^t K_+(x+k)\,dx\\ &= \frac{1}{(st)^{1/r'}}\Big(\sum_{k=1}^{1/t}\int_0^t K_+(x+k)\,dx + \sum_{k=1/t}^{m}\int_0^{1/k} K_+(x+k)\,dx\Big)\\ &= \frac{1}{(st)^{1/r'}}\Big(\sum_{k=1}^{1/t} t + \sum_{k=1/t}^{s}\frac{1}{k}\Big) \leqslant \frac{2}{(st)^{1/r'}}\big(1+\ln(st)\big) \leqslant 2r'.\end{aligned}$$

Further,

$$\frac{1}{|E|^{1/r'}}\int_E K_-(x)\,dx = \frac{1}{|s|^{1/r'}}\frac{1}{t^{1/r'}}\sum_{k\in w}\int_{e_k} K_-(x+k)\,dx$$

$$\leqslant \frac{1}{(st)^{1/r'}}\sum_{k\in w}\int_0^t K_-(x+k)\,dx$$

$$\leqslant \frac{1}{(st)^{1/r'}}\Bigg(\sum_{\substack{k\in w\\ |k|<\log_2(1/t)}}\int_0^t K_-(x-|k|)\,dx + \sum_{\substack{k\in w\\ |k|\geqslant\log_2(1/t)}}\int_0^{2^{-|k|}} K_-(x-|k|)\,dx\Bigg)$$

$$= \frac{1}{(st)^{1/r'}}\Bigg(\sum_{\substack{k\in w\\ |k|<\log_2(1/t)}} 2^{|k|/r}t + \sum_{\substack{k\in w\\ |k|\geqslant\log_2(1/t)}} 2^{-|k|/r'}\Bigg)$$

$$\leqslant \frac{C(r)}{(st)^{1/r'}}\big(t^{1/r'}+t^{1/r'}\big) \leqslant C(r).$$

Combining these estimates, we get

$$\sup_{E\in\mathfrak{U}_1}\frac{1}{|E|^{1/r'}}\left|\int_E K(x)\,dx\right| \leqslant C(r).$$

To show (5.5), we note that $K_+^*(t)\equiv 1$. Hence,

$$\sup_{t>0}\ t^{1/r}K^*(t) \geqslant \sup_{t>0}\ t^{1/r}K_+^*(t) = \infty.$$

To show (5.6), we note

$$\begin{aligned}
&\|K\|_{W(L_{r,\infty}[-1,1],\,l_{r,\infty}(\mathbb{Z}))} \geqslant \|K_-\|_{W(L_{r,\infty}[0,1],\,l_{r,\infty}(\mathbb{Z}))}\\
&= \sup_{n\in\mathbb{N}} n^{1/r}\Big(\sup_{0<t\leqslant 1}\ t^{1/r}\big(\widetilde{K}_-(t,n)\big)_n^*\Big) = \sup_{n\in\mathbb{N}} n^{1/r}\Big(\sup_{0<t\leqslant 2^{-n}}\ t^{1/r}2^{n/r}\Big) = \sup_{n\in\mathbb{N}} n^{1/r} = \infty.
\end{aligned}$$

□

**Example 5.2.** *Let $1<p<q<\infty$ and $1/r = 1-(1/p-1/q)$, and $m\in\mathbb{N}$. Define the function $K_m(x)$ on $\mathbb{R}$ as follows:*

$$K_m(x) = \begin{cases} 2^{(m-k)/r}, & \text{for } x\in\left[\frac{k-1}{2^m},\frac{k}{2^m}\right)+n,\ \ 2^k\leqslant n<2^{k+1},\ 1\leqslant k\leqslant 2^m;\\ 0, & \text{otherwise.}\end{cases}$$

*This function satisfies*

$$\|K_m\|_{W(l_{r,\infty}(\mathbb{Z}),\,L_{r,\infty}[-1,1])} = \sup_{0\leqslant t\leqslant 2}\ t^{1/r}\Big(\sup_{n\in\mathbb{N}} n^{1/r}\widetilde{K}_m^*(\cdot,n)\Big)_t^* = 2^{(m+1)/r} \tag{5.7}$$

*but*

$$\|K\|_{L_{r,\infty}} < 4 \tag{5.8}$$

*and therefore, $UpBo \leqslant C(r)$.*

Let $x \in [0,1)$ and $k \in \mathbb{N}$ such that $x \in [(k-1)/2^m, k/2^m)$. Then

$$\big(\widetilde{K}_m(x,\cdot)\big)_n^* = \begin{cases} 2^{(m-k)/r}, & \text{for } 1 \leqslant n < 2^k, \\ 0, & \text{for } 2^k \leqslant n, \end{cases}$$

and therefore,

$$\sup_{n\in\mathbb{N}} n^{1/r}\big(\widetilde{K}_m(x,\cdot)\big)_n^* = 2^{(m-k)/r}\, 2^{k/r} = 2^{m/r},$$

which yields (5.7).

Let us assume that $0 < t \leqslant 2^{m/r}$ and that the integer $k$ satisfies $2^{(m-k)/r} < t \leqslant 2^{(m-k+1)/r}$. Then

$$\begin{aligned} &\big|x : K_m(x) \geqslant t\big| \leqslant \big|x : K_m(x) \geqslant 2^{(m-k)/r}\big| \\ &\leqslant \sum_{j=0}^{k-1} \big|x : 2^{(m-k+j+1)/r} > K_m(x) \geqslant 2^{(m-k+j)/r}\big| = \sum_{j=0}^{k-1} \frac{2^{k-j}}{2^m} \leqslant 2^{k-m+1} \leqslant 4t^{-r}. \end{aligned}$$

Hence,

$$\|K\|_{L_{r,\infty}} = \sup_{t>0} t\big|x : K_m(x) \geqslant t\big|^{1/r} \leqslant 4$$

and by (5.2), we get $UpBo \leqslant C(r)$.

Let us now give a direct proof of the fact that $UpBo \leqslant C(r)$. Let $E \in \mathfrak{V}_1$. Then $E = \cup_{x\in e} x + w(x)$, $|e| = t < 1$ and $|w(x)| = s$. Let also $k_s : 2^{k_s} \leqslant s < 2^{k_s+1}$. Then for $x \in \big[\frac{k_s+n-1}{2^m}, \frac{k_s+n}{2^m}\big)$

$$\begin{aligned} \frac{1}{s^{1/r'}} \sum_{i\in w(x)} K_m(x+i) &\leqslant 2 \begin{cases} 2^{\frac{m-(k_s+n)}{r}}\, s^{1/r}, & n \geqslant 1 \\ 2^{\frac{m-(k_s+n)}{r}+k_s+n}\, s^{-1/r'}, & n \leqslant 0 \end{cases} \\ &\leqslant 2 \begin{cases} 2^{(m-n)/r}, & n \geqslant 1, \\ 2^{m/r+n/r'}, & n \leqslant 0. \end{cases} \end{aligned}$$

Then

$$\begin{aligned} &\frac{1}{|E|^{1/r'}} \int_E K_m(x)\,dx = \frac{1}{s^{1/r'}} \frac{1}{t^{1/r'}} \int_e \sum_{i\in w(x)} K_m(x+i)\,dx \\ &= \frac{1}{s^{1/r'}} \frac{1}{t^{1/r'}} \sum_{k=1}^{2^m} \int_{[\frac{k-1}{2^m},\frac{k}{2^m})\cap e} \sum_{i\in w(x)} K_m(x+i)\,dx \\ &= \frac{1}{t^{1/r'}} \sum_{k=1}^{k_s} \int_{[\frac{k-1}{2^m},\frac{k}{2^m})\cap e} \frac{1}{s^{1/r'}} \sum_{i\in w(x)} K_m(x+i)\,dx \\ &\qquad + \frac{1}{t^{1/r'}} \sum_{k=k_s+1}^{2^m} \int_{[\frac{k-1}{2^m},\frac{k}{2^m})\cap e} \frac{1}{s^{1/r'}} \sum_{i\in w(x)} K_m(x+i)\,dx \\ &= \frac{2\cdot 2^{m/r}}{t^{1/r'}} \Bigg( \sum_{k=1}^{k_s} \Big|\Big[\frac{k-1}{2^m}, \frac{k}{2^m}\Big) \cap e\Big| 2^{-(k_s-k)/r'} + \sum_{k=k_s+1}^{2^m} \Big|\Big[\frac{k-1}{2^m}, \frac{k}{2^m}\Big) \cap e\Big| 2^{-(k-k_s)/r} \Bigg). \end{aligned}$$

Since $\left|\left[\frac{k-1}{2^m}, \frac{k}{2^m}\right) \cap e\right| \leqslant \min\{2^{-m}, t\}$, we have

$$\frac{1}{|E|^{1/r'}} \int_E K_m(x)\,dx \leqslant 2\big((1-2^{-1/r'})^{-1}+(1-2^{-1/r})^{-1}\big)\frac{2^{m/r}\min\{2^{-m},t\}}{t^{1/r'}} \leqslant C(r),$$

i.e., $UpBo \leqslant \sup\limits_{E\in\mathfrak{V}_1} \frac{1}{|E|^{1/r'}} \int_E |K(x)|\,dx \leqslant C(r)$ follows. □

## 6. Convolution with oscillating kernels

In this section we discuss the $L_p$ to $L_q$ mapping properties of oscillatory integrals with the kernels $K(x) = k(x)e^{i\varphi(x)}$. The $(L_p, L_q)$ mapping problem for $K$, that is to determine all pairs of $(p, q)$ for which $\|K * f\|_q \leqslant C\|f\|_p$, has a long history (see [7], [10], [11], [15], [25], [26], [29] and references therein) and comes about in studying convergence for Fourier series, in solving boundary value problems for PDE's (see [12], [27], [29]). The particular case of

$$\mathcal{K}(x) = \frac{e^{i|x|^a}}{|x|^b}$$

is of special importance and has been studied in several papers. The $L_p$-boundedness of the operator $Af = \mathcal{K} * f$ was studied for the first time in [24]. The $L_p$-boundedness of $\mathcal{K} * f$ was completely characterized by P. Sjölin [25], [26] and independently by W. Jurkat and G. Sampson [15]: *if* $0 < a \neq 1$ *and* $1 - a/2 \leqslant b < 1$, *the operator* $\mathcal{K} * f$ *is bounded in* $L_p$ *if and only if*

$$p_0 := \frac{a}{a-1+b} \leqslant p \leqslant p_0'.$$

*Moreover* (see [25]), *if* $b < 1 - a/2$, *then boundedness of* $\mathcal{K} * f$ *is false for any* $1 \leqslant p \leqslant \infty$. Note that the condition $1 - a/2 \leqslant b$ guarantees that $p_0 \leqslant p_0'$. For certain values of $a$ the result was also proved independently by C. Fefferman, [26].

The boundedness of $\mathcal{K} * f$ from $L_p$ to $L_q$, $1 < p < q < \infty$, was studied by V. Drobot, A. Noparstek, and G. Sampson in [24]. They derived the following result: if $0 < a \neq 1$, $b \leqslant \lambda$, and $\frac{a}{2}\lambda + b - \lambda > 0$, where $\lambda = 1 - (1/p - 1/q)$, and

$$\frac{a}{\lambda(a-1)+b} < q < \frac{a}{\lambda - b}, \tag{6.1}$$

then $\mathcal{K} * f$ is bounded from $L_p$ to $L_q$. If $q > a/(\lambda - b)$, then $\mathcal{K} * f$ is not bounded from $L_p$ to $L_q$. One particular goal of this section is to show that the left-hand bound of $q$ in (6.1) is also sharp; see Corollary 6.5 below.

For the case of $a = 1$ the boundedness of $\mathcal{K} * f$ on $L_p$ was studied in detail in [26]. We give the following simple corollary of Remark 1.2 that corresponds to the $(L_p, L_q)$ mapping problem of $\mathcal{K}$ with $a = 1$.

**Corollary 6.1.** *Let $1 < p, q < \infty$, let $T$ be a continuous periodic function such that $T(0) \neq 0$, and let*

$$K(x) = \frac{T(x)}{|x|^b}.$$

*Then the operator $A = K * f$ is bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ if and only if $p < q$ and $b = 1 - (1/p - 1/q)$.*

*Proof.* Sufficiency follows from

$$|K(x)| \leqslant \|T\|_{L_\infty} |x|^{-b}.$$

Let $p < q$. To show necessity, suppose that $T(0) > 0$ and $d$ is the period of $T$. There exist $\xi > 0$ and $0 < \alpha < d/2$ such that $T(x) > \xi$ for $x \in (-\alpha, \alpha)$. Considering $E_0 = \bigcup_{k=0}^{s}((-\delta, \delta) + kd)$, where $\delta < \alpha/4$, we have $E_0 - E_0 = \bigcup_{k=-s}^{s}((-2\delta, 2\delta) + kd)$ and $E_0 - E_0 + y \subset \bigcup_{k \in \mathbb{Z}}((-\alpha, \alpha) + kd)$, $y \in E_0$, and then the function $K$ keeps its sign on $E_0 - E_0 + y$, $y \in E_0$. Therefore, $E_0 - E_0 + y \in \mathfrak{M} = \mathfrak{M}_+ \cup \mathfrak{M}_-$; see(1.9) and (1.10). Applying Remark 1.2 gives

$$\begin{aligned}
\|A\|_{L_p \to L_q} &\geqslant \sup_{E \in \mathfrak{M}} \frac{1}{|E|^{1/p}|E - E|^{-1/q}} \left| \int_E K(x)\, dx \right| \\
&\geqslant \frac{1}{|E_0|^{1/p}|E_0 - E_0|^{-1/q}} \left| \int_{E_0} K(x)\, dx \right| \\
&\geqslant (2\delta(s+1))^{-1/p} (4\delta(2s+1))^{1/q} \sum_{k=0}^{s} \int_{-\delta + kd}^{\delta + kd} \frac{\xi}{|x|^b}\, dx \\
&\geqslant C(b,\xi)\, (\delta s)^{1/q - 1/p} \left( \delta^{1-b} + \sum_{k=1}^{s} \frac{2\delta}{(k+1)^b d^b} \right) \qquad (6.2) \\
&\geqslant C(b,\xi) \left( s^{1/q-1/p}\, \delta^{1-b+1/q-1/p} + d^{-b}\, \delta^{1+1/q-1/p}\, s^{1-b+1/q-1/p} \right).
\end{aligned}$$

Since $0 < \delta < \alpha/2$, and since $s \in \mathbb{N}$ can be chosen arbitrarily, we arrive at $b = 1 - (1/p - 1/q)$.

If $p = q$, then (6.2) implies

$$\|A\|_{L_p \to L_p} \geqslant C(b,\xi) \sum_{k=1}^{s} \frac{2\delta}{(k+1)^b d^b} \geqslant C(b,\xi,d)\delta \ln s \to \infty \quad \text{as } s \to \infty$$

for fixed $\delta$, i.e., $A = K * f$ is not bounded in $L_p$.

Finally, in the case of $p > q$ we get

$$\|A\|_{L_p \to L_q} \geqslant C(b,\xi)\delta^{1/q-1/p+1-b} s^{1/q-1/p} \to \infty \quad \text{as } s \to \infty$$

for fixed $\delta$. $\square$

Our next two theorems provide several necessary conditions for $K * f$ to be bounded from $L_p$ to $L_q$.

**Theorem 6.2.** *Assume that $1 < p \leqslant q < \infty$ and the operator $Af = K * f$ is bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$. Then*

$$\sup_{E\in\mathfrak{M}} \frac{1}{\langle E\rangle^{1/p'}|E|^{1/p-1/q}}\left|\int_E K(x)\,dx\right| \leqslant C(p,q)\,\|A\|_{L_p\to L_q},$$

$$\sup_{E\in\mathfrak{M}} \frac{1}{\langle E\rangle^{1/q}|E|^{1/p-1/q}}\left|\int_E K(x)\,dx\right| \leqslant C(p,q)\,\|A\|_{L_p\to L_q},$$

*where $\mathfrak{M} = \mathfrak{M}_+ \cup \mathfrak{M}_-$,*

$$\mathfrak{M}_\pm = \Big\{E : E = \bigcup_{k\in\omega}[-\gamma,\gamma]+x_k,\ \ x_k\in\mathbb{R},\ \ \gamma < d/3,$$
$$|x_k - x_j| \geqslant 2d,\ \ k,j\in\omega\subset\mathbb{Z}\ \text{ such that }\ E' = \bigcup_{k\in\omega}[-3\gamma,3\gamma]+x_k \subset D_\pm\Big\},$$

*$\langle E\rangle$ is the cardinality of $\omega$, and*

$$D_+ = \{x : K(x) \geqslant 0\},\quad D_- = \{x : K(x)\leqslant 0\}.$$

*Proof.* Suppose that $E_0 \in \mathfrak{M}$. Then $E_0 = \bigcup_{k\in\omega}[-\gamma,\gamma]+x_k \subset E_0'$ such that either $E_0' \subset D_+$ or $E_0'\subset D_-$. By Corollary 4.1 in [20] we get

$$\begin{aligned}\|A\|_{L_p\to L_q} &\geqslant C\|A\|_{L_{p,1}\to L_{q,\infty}} \asymp \sup_{\substack{|E|>0\\|W|>0}} \frac{1}{|E|^{1/q'}}\frac{1}{|W|^{1/p}}\left|\int_E\int_W K(y-x)\,dx\,dy\right|\\ &\geqslant \frac{1}{|E_0|^{1/q'}\big|[-2\gamma,2\gamma]\big|^{1/p}}\left|\int_{E_0}\int_{[-2\gamma,2\gamma]} K(y-x)\,dx\,dy\right|\\ &= \frac{1}{|E_0|^{1/q'}(4\gamma)^{1/p}}\left|\sum_{k\in\omega}\int_{[-\gamma,\gamma]+x_k}\int_{[-2\gamma,2\gamma]+y} K(x)\,dx\,dy\right|.\end{aligned}$$

Since $[-\gamma,\gamma]+x_k \subset [-2\gamma,2\gamma]+y\subset[-3\gamma,3\gamma]+x_k\subset D_\pm$ for any $y\in[-\gamma,\gamma]+x_k$, the last expression can be estimated from below by

$$\frac{2\gamma}{|E_0|^{1/q'}(4\gamma)^{1/p}}\left|\sum_{k\in\omega}\int_{[-\gamma,\gamma]+x_k} K(x)\,dx\right| = \frac{2^{-1/p}}{|\omega|^{1/p'}|E_0|^{1/p-1/q}}\left|\int_{E_0} K(x)\,dx\,dy\right|.$$

Noting that $\langle E\rangle = |\omega|$ we conclude the proof of the first statement of the theorem.

To show the second inequality, we take $E = [-2\gamma,2\gamma]$ and $W = E_0$. $\square$

As application of Theorem 6.2, we consider convolutions with oscillating kernels.

**Theorem 6.3.** *Let $1 < p\leqslant q<\infty$, $\lambda = 1-(1/p-1/q)$, and $\beta = \min(1/p,1/q')$. Let $u$ and $v$ be positive monotone functions on $(0,\infty)$ such that $u\in C^1(0,\infty)$ and $u'$ is strictly monotone.*

*The operator $Af = K * f$, where*

$$K(x) = \frac{\cos(2\pi u^{-1}(|x|))}{v(|x|)},$$

*is not bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$ if either:*

(i) $|u'(x)|$ *is decreasing and*

$$\lim_{N\to+\infty} \frac{|u'(N)|^\lambda}{N^\beta} \sum_{k=1}^{N-2} \frac{1}{(v\circ u)(k)} = \infty, \tag{6.3}$$

*or*

(ii) $|u'(x)|$ *is increasing and*

$$\lim_{N\to+\infty} \frac{|u'(N)|^\lambda}{N^\beta} \sum_{k=N+2}^{2N} \frac{1}{(v\circ u)(k)} = \infty. \tag{6.4}$$

*Proof.* Note that $\cos(2\pi u^{-1}(x)) \geqslant \sqrt{2}/2$ for $x \in \Delta_k = [\min\{u(-1/4+k), u(1/4+k)\}, \max\{u(-1/4+k), u(1/4+k)\}]$, $k \in \mathbb{N}$. Define

$$\delta := \delta(N) := \min\left\{\frac{|u(\pm 1/4+N) - u(N)|}{3}, |u(\pm 1/8+N) - u(N)|\right\}.$$

Therefore, $\delta$ is such that $(-3\delta, 3\delta) + u(N) \subset \Delta_N$ and $\cos(2\pi u^{-1}(x)) \geqslant \sqrt{2}/2$ for $x \in [-\delta, \delta] + u(k)$.

First, let us assume that $|u'(x)|$ is decreasing. If $\gamma$ is such that $0 < |\gamma| < 1/2$, the mean value theorem yields

$$|u(\gamma + k) - u(k)| = |u'(\theta_k + k)|\,|\gamma|,$$

where $|\theta_k| \leqslant |\gamma| < 1/2$. This gives, for $k < N$,

$$|u(\gamma+k) - u(k)| = |\gamma||u'(\theta_k + k)| \leqslant |\gamma||u'(\theta_N + N)| = |u(\gamma + N) - u(N)|.$$

Hence, for $1 \leqslant k < N$,

$$\delta \leqslant \min\left\{\frac{|u(\pm 1/4+k) - u(k)|}{3}, |u(\pm 1/8+k) - u(k)|\right\}. \tag{6.5}$$

Therefore,

$$E_N := \bigcup_{k=1}^{N}([-\delta, \delta] + u(k)) \subset E'_N := \bigcup_{k=1}^{N}([-3\delta, 3\delta] + u(k)) \subset D_+ := \{x : K(x) \geqslant 0\},$$

and, moreover, $\cos(2\pi u^{-1}(x)) \geqslant \sqrt{2}/2$ for $x \in E_N$. This implies $E_N \in \mathfrak{M}$. By Theorem 6.2, we obtain

$$\begin{aligned} C\,\|A\|_{L_p\to L_q} &\geqslant \sup_{E\in\mathfrak{M}} \frac{1}{\langle E\rangle^{1/p'}|E|^{1/p-1/q}}\left|\int_E K(x)\,dx\right| + \sup_{E\in\mathfrak{M}} \frac{1}{\langle E\rangle^{1/q}|E|^{1/p-1/q}}\left|\int_E K(x)\,dx\right| \\ &\geqslant \left(\frac{1}{\langle E_N\rangle^{1/p'}|E_N|^{1/p-1/q}} + \frac{1}{\langle E_N\rangle^{1/q}|E_N|^{1/p-1/q}}\right)\left|\int_{E_N} K(x)\,dx\right|. \end{aligned}$$

Further, using (6.5) and monotonicity of $u$ and $v$, we get

$$\int_{[-\delta,\delta]+u(k)} \frac{1}{v(x)}\,dx \geqslant 2\delta \min\Big\{\frac{1}{(v\circ u)(\pm 1/4+k)}\Big\},$$

which gives

$$\begin{aligned}\frac{1}{|E_N|^{1/p-1/q}}\int_{E_N} K(x)\,dx &\geqslant \frac{\sqrt{2}}{2}\frac{1}{|E_N|^{1/p-1/q}}\sum_{k=1}^{N}\int_{[-\delta,\delta]+u(k)}\frac{1}{v(x)}\,dx\\ &\geqslant C\,\frac{\delta^{\lambda}}{N^{1/p-1/q}}\sum_{k=1}^{N}\min\Big\{\frac{1}{(v\circ u)(\pm 1/4+k)}\Big\}.\end{aligned}$$

Using the definition of $\delta$, we have

$$\langle E_N\rangle = N,\quad |E_N| = 2N\delta \geqslant \frac{N}{6}\,|u'(1/4+N)|.$$

Hence,

$$\begin{aligned}\frac{1}{|E_N|^{1/p-1/q}}\int_{E_N} K(x)\,dx &\geqslant C\,\frac{|u'(1/4+N)|^{\lambda}}{N^{1/p-1/q}}\sum_{k=1}^{N}\min\Big\{\frac{1}{(v\circ u)(\pm 1/4+k)}\Big\}\\ &\geqslant C\,\frac{|u'(N+1)|^{\lambda}}{N^{1/p-1/q}}\sum_{k=2}^{N-2}\frac{1}{(v\circ u)(k)}\end{aligned}$$

and

$$\begin{aligned}C\,\|A\|_{L_p\to L_q} &\geqslant \Big(\frac{1}{N^{1/p'}}+\frac{1}{N^{1/q}}\Big)\frac{|u'(N+1)|^{\lambda}}{N^{1/p-1/q}}\sum_{k=2}^{N-2}\frac{1}{(v\circ u)(k)}\\ &= \Big(\frac{1}{N^{1/q'}}+\frac{1}{N^{1/p}}\Big)|u'(N+1)|^{\lambda}\sum_{k=2}^{N-2}\frac{1}{(v\circ u)(k)}.\end{aligned}$$

Letting $N$ tend to infinity we arrive at the statement of the theorem in the case of decreasing $|u'|$.

If $|u'(x)|$ is increasing, then $|u(\gamma+k)-u(k)| \geqslant |u(\gamma+N)-u(N)|$ for $k<N$. In this case we define

$$E_N := E_{N,M} := \bigcup_{k=N}^{M}([-\delta,\delta]+u(k)),\qquad M>N.$$

Since $E_N \in \mathfrak{M}$, Theorem 6.2 gives

$$\begin{aligned}C\,\|A\|_{L_p\to L_q} &\geqslant \Big(\frac{1}{(M-N)^{1/p}}+\frac{1}{(M-N)^{1/q'}}\Big)(u'(N-1))^{\lambda}\sum_{k=N+2}^{M}\frac{1}{(v\circ u)(k)}\\ &\geqslant \frac{(u'(N-1))^{\lambda}}{(M-N)^{\beta}}\sum_{k=N+2}^{M}\frac{1}{(v\circ u)(k)}.\qquad (6.6)\end{aligned}$$

Taking $M=2N$ gives condition (6.4). □

**Remark 6.4.** Taking $N = 2$ in (6.6), one can see that condition (6.4) in the part (ii) can be replaced by the following condition:

$$\lim_{M\to+\infty} \frac{1}{M^\beta} \sum_{k=1}^{M} \frac{1}{(v\circ u)(k)} = \infty. \tag{6.7}$$

In the next result we apply Theorem 6.3 to study the $(L_p, L_q)$ mapping problem for $\mathcal{K}(x) = e^{i|x|^a}/|x|^b$.

**Corollary 6.5.** *Let $1 < p \leqslant q < \infty$ and $\lambda = 1 - (1/p - 1/q)$. Let also*

$$\mathcal{K}(x) = \frac{e^{i|x|^a}}{|x|^b},$$

*where $a \neq 0$, $a \neq 1$, and $b \neq \lambda$. If*

$$\max(q, p') > \frac{a}{\lambda - b} > 0,$$

*then the operator $Af = \mathcal{K} * f$ is not bounded from $L_p$ to $L_q$.*

**Remark 6.6.** (i) Note that this result for certain values of $q$, $a$, and $b$ was shown in [24].

(ii) The positive result for the $(L_p, L_q)$ mapping problem with $p < q$ reads as follows (see [24]). Let $1 < p < q < \infty$. If $a > 0$, $a \neq 1$, $b \leqslant \lambda$, and $2 \leqslant \frac{a}{\lambda-b}$, then the operator $Af = \mathcal{K} * f$ is bounded from $L_p$ to $L_q$ provided that $\max(q, p') < \frac{a}{\lambda-b}$.

(iii) The case when $p = q$ can be written similarly to the result of Corollary 6.5 (note that in this case $\lambda = 1$): Assume that $p = q$ and $a > 0$, $a \neq 1$. If $b < \lambda$ and $2 \leqslant a/(\lambda - b)$, then the operator $Af = \mathcal{K} * f$ is bounded in $L_p$ if and only if $\max(p, p') \leqslant a/(\lambda - b)$. Moreover, if $2 > a/(\lambda - b)$, then the operator $Af = \mathcal{K} * f$ is not bounded in $L_p$ for any $1 \leqslant p \leqslant \infty$. This is an equivalent statement of the result from [25].

(iv) We note that Corollary 6.5 also holds, with the same proof, for the kernel

$$\mathcal{K}_0(x) = \frac{e^{i|x|^a}}{(1 + |x|)^b}, \quad a > 0, \quad a \neq 1;$$

see for example [15] for the boundedness properties of $\mathcal{K}_0 * f$ in $L_p$.

*Proof of Corollary* 6.5. We use Theorem 6.3 with $u(t) = t^{1/a}$, $v(t) = t^b$, $t > 0$. Note that the conditions $a \neq 0, a \neq 1$ imply that $u$ is strictly monotone.

If either $a > 1$ or $a < 0$, then $|u'(t)| = t^{1/a-1}/|a|$, $t > 0$, is decreasing. For $N \in \mathbb{N}$ we have

$$\begin{aligned}\frac{|u'(N)|^\lambda}{N^\beta} \sum_{k=1}^{N-2} \frac{1}{(v\circ u)(k)} &\asymp \frac{(N^{1/a-1})^\lambda}{N^\beta} \sum_{k=1}^{N} \frac{1}{k^{b/a}} \\ &\geqslant \max\left(\frac{1}{N^{1/p}}, \frac{1}{N^{1/q'}}\right) N^{\lambda(1/a-1)+1-b/a} \\ &= \max\left(N^{\lambda/a-b/a-1/q}, N^{\lambda/a-b/a-1/p'}\right) \to +\infty\end{aligned}$$

as $N \to +\infty$ provided that either $p' > \frac{a}{\lambda-b}$ or $q > \frac{a}{\lambda-b}$.

Suppose now that $0 < a < 1$. Then $|u'(t)| = t^{1/a-1}/a$ is increasing and

$$\begin{aligned}\frac{(u'(N))^\lambda}{N^\beta} \sum_{k=N+2}^{2N} \frac{1}{(v \circ u)(k)} &\asymp \frac{(N^{1/a-1})^\lambda}{N^\beta} \sum_{k=N}^{2N} \frac{1}{k^{b/a}} \\ &\asymp N^{(1/a-1)\lambda} \max\left(\frac{1}{N^{1/p}}, \frac{1}{N^{1/q'}}\right) N^{1-b/a} \\ &= \max\left(N^{\lambda/a-b/a-1/q}, N^{\lambda/a-b/a-1/p'}\right) \to +\infty\end{aligned}$$

as $N \to +\infty$, provided that $\max(p', q) > \frac{a}{\lambda-b}$. □

We finish this section by highlighting the following applications of Corollary 6.1. First, we note that Corollary 6.1 implies that the operator $A = K * f$, where $K$ is a continuous non-trivial periodic function, is not bounded from $L_p(\mathbb{R})$ to $L_q(\mathbb{R})$. This also follows from Corollary 1.5 since

$$\sup_{\mathfrak{L}_d} \frac{1}{|E|^\gamma} \left| \int_E K(x)\, dx \right| = \infty, \quad 0 < \gamma < 1, \quad K \not\equiv 0.$$

Another application of Corollary 6.1 is the following.

**Example 6.7.** *Let $0 < \gamma < 1$. We have*

$$\sup_{\mathfrak{L}_d} \frac{1}{|E|^\gamma} \int_E |\sin x^2|\, dx = \infty. \tag{6.8}$$

*In particular, the operator $K * f$ with $K(x) = |\sin x^2|$ is not bounded from $L_p$ to $L_q$.*

Indeed, since $|\sin x^2| \geqslant (\sin x^2)^2 = (1 - \cos 2x^2)/2$, we have

$$\sup_{\mathfrak{L}_d} \frac{1}{|E|^\gamma} \int_E |\sin x^2|\, dx \geqslant \sup_{\mathfrak{L}_d} \frac{1}{|E|^\gamma} \int_E \frac{dx}{2} - \sup_{\mathfrak{L}_d} \frac{1}{|E|^\gamma} \int_E \cos 2x^2\, dx.$$

It is clear that

$$\sup_{\mathfrak{L}_d} \frac{1}{|E|^\gamma} \int_E \frac{dx}{2} = \sup_{\mathfrak{L}_d} \frac{1}{2} |E|^{1-\gamma} = \infty.$$

Therefore, it is enough to show that

$$\sup_{E \in \mathfrak{L}_d} \frac{1}{|E|^\gamma} \left| \int_E \cos x^2\, dx \right| < \infty.$$

Let $E \in \mathfrak{L}_d$, that is, $E = \bigcup_{k \in \omega} ([0, b] + kd)$, where $\omega$ is a finite arithmetic progression, $r \in \mathbb{N}$, and $0 < b \leqslant d$. Let $\omega = \{kr\}_{k=k_0}^{k_0+m}$, $k_0 \geqslant 0$. Then

$$\int_E \cos x^2\, dx = \sum_{k=k_0}^{k_0+m} \int_{krd}^{b+krd} \cos x^2\, dx$$

and

$$\int_{krd}^{b+krd} \cos x^2\,dx = \frac{1}{2}\int_{(krd)^2}^{(b+krd)^2} \frac{\cos x}{\sqrt{x}}\,dx$$
$$= \frac{1}{2}\Big(\frac{\sin(b+krd)^2}{b+krd} - \frac{\sin(krd)^2}{krd} + \frac{1}{2}\int_{(krd)^2}^{(b+krd)^2} \frac{\sin x}{x^{3/2}}\,dx\Big).$$

Assume that $mb > 1$. We obtain

$$\frac{1}{|E|^\gamma}\Big|\int_E \cos x^2\,dx\Big| = \frac{1}{(bm)^\gamma}\Big|\sum_{k=k_0}^{k_0+[1/b]}\int_{krd}^{b+krd}\cos x^2\,dx + \sum_{k=k_0+[1/b]+1}^{k_0+m}\int_{krd}^{b+krd}\cos x^2\,dx\Big|$$
$$\leqslant \frac{1}{(bm)^\gamma}\Big(2 + \sum_{k=k_0+[1/b]+1}^{k_0+m}\Big|\frac{\sin(b+krd)^2}{b+krd} - \frac{\sin(krd)^2}{krd}\Big| + \sum_{k=k_0+[1/b]+1}^{k_0+m}\int_{(krd)^2}^{(b+krd)^2}\frac{dx}{x^{3/2}}\Big)$$
$$\leqslant \frac{2}{(bm)^\gamma}\Big(1 + \sum_{k=k_0+[1/b]+1}^{k_0+m}\frac{1}{krd} + \sum_{k=k_0+[1/b]+1}^{k_0+m}\frac{b}{(krd)^2}\Big)$$
$$\leqslant \frac{2}{(bm)^\gamma}\Big(1 + \sum_{k=[1/b]+1}^{m}\frac{1}{krd} + \frac{b^2}{(rd)^2}\Big) \leqslant \frac{4}{(bm)^\gamma} + \frac{2}{d}\frac{\ln mb}{(mb)^\gamma} \leqslant C(d,\gamma),$$

since $mb > 1$.

Let now $mb \leqslant 1$, i.e., $|E| \leqslant 1$. Then

$$\frac{1}{|E|^\gamma}\Big|\int_E \cos x^2\,dx\Big| \leqslant \frac{|E|}{|E|^\gamma} = |E|^{1-\gamma} \leqslant 1.$$

Thus,

$$\sup_{\mathfrak{L}_d}\frac{1}{|E|^\gamma}\Big|\int_E \cos x^2\,dx\Big| \leqslant C(d,\gamma). \qquad \square$$

It is interesting to note that

$$\sup_{\mathfrak{L}_d}\frac{1}{|E|^\gamma}\Big|\int_E \sin x^2\,dx\Big| < \infty, \qquad 0<\gamma<1,$$

cf. (6.8). Also, note that $K*f$, where $K(x) = \sin x^2$, is bounded in $L_2$ since

$$\mathfrak{F}(\sin x^2)(y) = 2\int_0^\infty \sin x^2 \cos xy\,dx = \sqrt{\frac{\pi}{2}}\Big(\cos\frac{y^2}{4} - \sin\frac{y^2}{4}\Big),$$

which is bounded.

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