On a paper of Berestycki–Hamel–Rossi and its relations to the weak maximum principle at infinity, with applications

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Abstract. The aim of this paper is to study a new equivalent form of the weak maximum principle for a large class of differential operators on Riemannian manifolds. This new form has been inspired by the work of Berestycki, Hamel and Rossi for trace operators, and allows us to shed new light on it and to introduce a new sufficient bounded Khas'minskii type condition for its validity. We show its effectiveness by applying it to obtain some uniqueness results in a geometric setting.

1. Introduction

Maximum principles have numerous interesting applications in analytic and geometric contexts and have therefore been the object of intensive study for many decades. Of particular interest for us are their versions "at infinity", in the spirit of the original works of Omori [17], Yau [27] and Pigola, Setti and the third author [19]. Their geometric applications range, for instance, from the theory of submanifolds in Riemannian and Lorentzian spaces to Ricci and mean curvature solitons. A detailed introduction to the different formulations of the maximum principle and its usefulness in Geometric Analysis can be found in [3] and [19].

This paper is devoted to the study of a new equivalent form of the weak maximum principle at infinity, WMP for short, first introduced in [18], for a broad family of differential operators on Riemannian manifolds (see Definition 2.1 below). This class includes, for instance, the p-Laplacian, the mean curvature operator and others that naturally arise in geometric and analytic settings. The origin of the present work lies in the investigation of a maximum principle formulated in [6], which we now recall. Let L be an elliptic linear differential operator in trace form

on $\mathbb{R}^m$ written, in the standard basis $\{\partial_i\}$, in the form

$$Lu = a^{ij}u_{ij} + b^i u_i, \quad a^{ij}, b^i \in C^{0,\alpha}_{\text{loc}}(\mathbb{R}^m).$$

We recall that the usual strong maximum principle states that for each $c \in L^\infty_{\text{loc}}(\mathbb{R}^m)$, any non-positive C^2 solution of $Lu + c(x)u \geq 0$ on an open, connected set Ω cannot attain the value zero at an interior point unless $u \equiv 0$. This relevant result is due to E. Hopf, and can be found in Theorem 2.1.1 of [24] (see also Theorem 3.5 in [11], and [20]). When Ω has non-compact closure and zero is not attained, one can ask whether or not u may approach zero as x diverges in Ω. The maximum principle in Lemma 2.1 of [6] is stated in the following form: given the open set $\Omega \subseteq \mathbb{R}^m$, for each $\varepsilon > 0$ and each solution of

$$\begin{cases} Lu + c(x)u \geq \varepsilon & \text{on } \Omega \subset \mathbb{R}^m, \\ u \leq 0 & \text{on } \Omega, \end{cases}$$

with $\sup_{\partial\Omega} u < 0$ if $\partial\Omega \neq \varnothing$ and $c \in C^0(\Omega) \cap L^\infty(\Omega)$, we have

$$(1.1) \qquad\qquad\qquad\qquad \sup_\Omega u < 0.$$

Note that this statement refers to a certain Ω.

Clearly, the validity of this property is not granted for each L and, when Ω is unbounded, depends on the behavior at infinity of a^{ij}, b^i as well as on the growth-decay of the ellipticity constants of $\{a^{ij}\}$. The search for sharp conditions ensuring (1.1) is an interesting problem, that has been considered, for instance, in [6], see Lemmas 2.1 and 3.3 therein.

In Theorem 2.2 below, we will prove that a generalized version of the property above on a Riemannian manifold M, here called property (P) (see Definition 2.3 below) and related to the entire family of open sets $\Omega \subset M$ with $\partial\Omega \neq \varnothing$, is indeed equivalent to the WMP for all the linear and nonlinear operators defined below on M. As a consequence, the theory developed in [3] and [19] for the WMP can be used to deduce very general conditions e.g., involving only the volume growth of M, for the validity of property (P), see for instance Theorem 2.3 below.

The study of the equivalence between the WMP and property (P) presented here also enables us to introduce a bounded Khas'minskii-type assumption, sufficient to guarantee the validity of the weak maximum principle. Again, we refer to [3] and [15] for a thorough presentation of "unbounded" Khas'minskii-type conditions in the linear and nonlinear case, and to [14] for recent improvements.

The terminology is due to the fact that, as proved in [18], the WMP for the Laplace–Beltrami operator is equivalent to the stochastic completeness of the Brownian motion determined by Δ on M, that is, the property that Brownian paths have infinite lifetime almost surely. We also remark that the stochastic completeness of Δ on M does not require $(M, \langle\,,\,\rangle)$ to be geodesically complete and, vice versa, the latter does not imply the former.

To show the effectiveness of this new form of the WMP contained in Theorem 2.2 below we prove a comparison, and a companion uniqueness, result for

positive bounded solutions of certain differential equations. As a special, remarkable case, we deduce uniqueness for the Lichnerowicz-type equation (see [2] for a detailed introduction)

$$(1.2) \qquad \Delta u + a(x)u - b(x)u^{\sigma} + c(x)u^{\tau} = 0, \quad \tau < 1 < \sigma$$

on M. More precisely, we have the following.

Theorem 1.1. *Let $(M, \langle \ , \ \rangle)$ be a stochastically complete manifold, $a(x)$, $b(x)$, $c(x) \in C^0(M) \cap L^{\infty}(M)$ and satisfying $b(x) \geq 0, c(x) \geq 0$ and either*

$$\inf_M b(x) > 0 \quad or \quad \inf_M c(x) > 0.$$

Then equation (1.2) *has at most one C^2 solution u such that, for some constant $C > 0$,*

$$(1.3) \qquad C^{-1} \leq u \leq C \quad on \ M.$$

Remark 1.2. The above theorem is a consequence of the comparison result in Theorem 2.10 below. A similar, but not overlapping, comparison result can be found in Theorem 3.4 of [2], where the assumptions on the coefficients are:

$$b(x) > 0 \quad \text{and} \quad c(x) \geq 0,$$

$$\sup_M \frac{a_-(x)}{b(x)} < +\infty, \quad \sup_M \frac{c(x)}{b(x)} < +\infty.$$

Remark 1.3. The two-sided bound in (1.3) is automatically granted under very general conditions on a, b and c. See for instance Theorem 3.9 of [2], where the authors give some sufficient conditions.

We note that the condition $c(x) \geq 0$ is quite natural thinking of the physical meaning of the coefficient. Indeed, in the analysis of Einstein's field equations in General Relativity, the initial data have to satisfy the Einstein constraint conditions that can be expressed in a geometric form as follows. Let $(M^n, \hat{g})$ be a Riemannian manifold of dimension $n \geq 3$ and $\hat{K}$ a symmetric 2-covariant tensor field on M. Then $(M, \hat{g})$ is said to satisfy the Einstein constraint equations with non-gravitational energy density $\hat{\rho}$ and non-gravitational momentum density $\hat{J}$ if

$$(1.4) \qquad \begin{cases} S_{\hat{g}} - |\hat{K}|_{\hat{g}}^2 + (\mathrm{tr}_{\hat{g}}\, \hat{K})^2 = \hat{\rho} \\ \mathrm{div}_{\hat{g}}\, \hat{K} - \mathrm{d}(\mathrm{tr}_{\hat{g}}\, \hat{K}) = \hat{J}, \end{cases}$$

where $S_{\hat{g}}$ is the scalar curvature of $(M, \hat{g})$.

A procedure to look for solutions of the above is, according to Lichnerowicz [12], York and Choquet-Bruhat [28], [29] and [8], first choosing a conformal data, that is, a Riemannian manifold (M^n, g), a symmetric 2-covariant tensor σ for which $\mathrm{tr}_g\, \sigma = 0$; a scalar function τ, a nonnegative scalar function ρ and a 1-form J. Letting Δ, S_g denote the Laplace–Beltrami operator and the scalar curvature of g,

one then looks for a function $u > 0$ and a vector field W that solve, in the case of an Einstein-scalar field, the system

$$(1.5) \quad \begin{cases} \Delta u - c_m S_g u + c_m \left[|\sigma + \mathring{\mathcal{L}}_W g|_g^2 + \rho \right] u^{-2^*-1} - b_m \, \tau^2 \, u^{2^*-1} = 0, \\ \operatorname{div}_g(\sigma + \mathring{\mathcal{L}}_W g) = J + \dfrac{n-1}{n} \, u^{2^*} \, \mathrm{d}\tau, \end{cases}$$

where $\mathring{\mathcal{L}}_W g$ is the traceless Lie derivative of g in the direction W, and

$$2^* = \frac{2m}{m-2}, \quad c_m = \frac{m-2}{4(m-1)}, \quad b_m = \frac{m-2}{4m}.$$

If (u, W) is a solution of (1.5), then setting

$$\hat{g} = u^{\frac{4}{n-2}} g, \quad \hat{K} = u^{-2}(\sigma + \mathring{\mathcal{L}}_W g) + \frac{\tau}{n} \, u^{\frac{4}{n-2}} g, \quad \hat{\rho} = \rho \, u^{-2 \cdot 2^*}, \quad \hat{J} = J u^{-2^*}$$

the 4-tuple $(\hat{g}, \hat{K}, \hat{\rho}, \hat{J})$ solves (1.4). For details we refer to [7]. Evidently, the scalar equation in (1.5) is of the form (1.2) with $c(x) \geq 0$, $b(x) \geq 0$.

Further comments on Theorem 1.1, for instance on assumption (1.3), will be given in Section 3.

As a second example, let us consider the operator

$$\Delta_f = \Delta - \langle \nabla f, \nabla \, \rangle, \quad f \in C^\infty(M).$$

This operator naturally appears in the study of gradient Ricci solitons, that is, Riemannian manifolds $(M, \langle \, , \, \rangle)$ with a potential function f, if any, satisfying the equation

$$(1.6) \quad \operatorname{Ricc} + \operatorname{Hess}(f) = \lambda \langle \, , \, \rangle$$

for some constant $\lambda \in \mathbb{R}$. Solitons generate self-similar solutions of the Ricci flow and they are said to be expanding, steady or shrinking respectively when $\lambda < 0$, $\lambda = 0$, $\lambda > 0$. We have the following.

Theorem 1.4. *Let $(M, \langle \, , \, \rangle)$ be a complete manifold with a fixed origin o and set $r(x) = \operatorname{dist}(x, o)$. Let $f \in C^\infty(M)$ and suppose that*

$$(1.7) \quad \liminf_{r \to +\infty} \frac{\log \int_{B_r} e^{-f}}{r^2} < +\infty.$$

Let $a(x), b(x) \in C^0(M) \cap L^\infty(M)$ satisfy

$$\inf_M b(x) > 0.$$

Then, for some $C > 0$, the equation

$$\Delta_f u + a(x)u - b(x)u \log u = 0$$

admits at most one solution $u \in C^2(M)$ satisfying $C^{-1} \leq u \leq C$ on M.

Proof of Theorem 1.4. We note that under assumption (1.7) we have the validity of the WMP for the operator Δ_f ([3], Chapter 4). The remaining assumptions guarantee the applicability of Theorem 2.10, from which the result follows at once.   □

Corollary 1.5. *Let* $(M, \langle \ , \ \rangle)$ *be a stochastically complete manifold,* $a, b \in \mathbb{R}$ *with* $b > 0$. *Then the only solution* u *of*

$$\Delta u + au - bu \log u = 0$$

satisfying $C^{-1} \leq u \leq C$ *on* M, *for some* $C > 0$, *is* $u = e^{a/b}$.

Suppose now that $(M, \langle \ , \ \rangle, f)$ is a gradient soliton. By Hamilton's identity,

$$S + |\nabla f|^2 - 2\lambda f = \Lambda$$

for some constant $\Lambda \in \mathbb{R}$ and where S is the scalar curvature of M. Tracing (1.6),

$$S + \Delta f = m\lambda, \quad m = \dim M,$$

so that putting the two equations together we obtain

$$\Delta f - |\nabla f|^2 + 2\lambda f = m\lambda - \Lambda.$$

We set $u = e^{-f}$ and the above becomes

$$\Delta u + (m\lambda - \Lambda)u + 2\lambda u \log u = 0.$$

Remark 1.6. We have been informed that a more general version of Theorem 1.4 and Corollary 1.5 appeared in Theorem 3.9 in [1], where less restrictive assumptions are imposed on the coefficients a and b and on the solution.

Corollary 1.7. *Let* $(M, \langle \ , \ \rangle)$ *be a stochastically complete manifold with non constant scalar curvature. Then there are no expanding Ricci soliton structures on* M *with bounded potential.*

Remark 1.8. If $(M, \langle \ , \ \rangle)$ is geodesically complete, by Proposition 8.7 and 8.12 of [3] a gradient Ricci soliton structure on $(M, \langle \ , \ \rangle)$ automatically implies that M is stochastically complete. As a matter of fact, the full Omori–Yau principle holds both for Δ and for $\Delta_f = \Delta - \langle \nabla f, \nabla \ \rangle$, as shown in [10].

The paper is organized as follows: Section 2 is devoted to the introduction of the family of differential operators, the weak maximum principle and the statement of the aforementioned conditions for its validity, while Section 3 contains the proofs of the results stated in the previous section.

2. The weak maximum principle

In what follows we denote by $(M, \langle \ , \ \rangle)$ a connected Riemannian manifold of dimension $m \geq 2$. Let X be a smooth vector field on M and T be a symmetric, 2-covariant, positive semidefinite tensor field. With $t : TM \to TM$ we indicate the corresponding endomorphism defined by

$$t(Y) = T(Y, \ )^{\sharp}$$

where $^{\sharp} : T^*M \to TM$ is the musical isomorphism.

We let $\varphi\colon M \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy

$$(\varphi 1) \quad \begin{aligned} &\varphi(\ ,s) \in C^0(M) \quad \forall\, s \in \mathbb{R}_0^+; \qquad \varphi(x,\ ) \in C^0(\mathbb{R}_0^+) \quad \forall\, x \in M; \\ &\varphi(x,0) = 0 \qquad \forall\, x \in M; \qquad \varphi(x,s) > 0 \qquad\quad \text{on } M \times \mathbb{R}^+. \end{aligned}$$

More requirements will be added in due time. We define the operator $Q = Q_{\varphi,X,T}$ acting on $C^1(M)$ by

$$(2.1) \qquad Qu = \operatorname{div}\left(|\nabla u|^{-1}\varphi(x,|\nabla u|)t(\nabla u)\right) - \langle X, \nabla u \rangle$$

in the appropriate weak sense. Of course the definition can be extended to a larger class of functions, but in what follows we restrict ourselves to $C^1(M)$. Note that for $\varphi(x,s) = s$ and $u \in C^2(M)$ the previous operator can also be put into the form

$$(2.2) \qquad Qu = \operatorname{tr}\left(t \circ \operatorname{hess}(u)\right) + \langle (\operatorname{div} T)^\sharp - X, \nabla u \rangle,$$

where $\operatorname{hess}(u)$ is the $(1,1)$-version of the Hessian of u. Thus, with the appropriate choice of X, linear trace operators are included in the family (2.2) (at least when T is C^1). Other interesting examples are obtained with the choices

$$\begin{cases} (\varphi,t,X) = (s^{p-1}, I, 0), p > 1 & \text{the } p\text{-Laplace operator,} \\ (\varphi,t,X) = \left(\dfrac{s}{\sqrt{1+s^2}}, I, 0\right) & \text{the mean curvature operator,} \\ (\varphi,t,X) = (s, I, X) & \text{the } X\text{-Laplacian.} \end{cases}$$

The last example is ubiquitous in the theory of Ricci solitons, especially in the gradient case $X = \nabla f$ for some potential function $f \in C^\infty(M)$. Moreover, the same kind of operator is also key in the study of optimal transportation ([26]) and solitons for the mean curvature flow ([9]).

Another example deserves special attention. Let $f\colon M \to N$ be a two-sided immersed hypersurface with a chosen, global unit normal vector field ν. Set A to denote the second fundamental tensor in the direction of ν. For $0 \leq k \leq m$, Newton's operators $P_k\colon TM \to TM$ are inductively defined by setting

$$P_0 = I, \quad P_k = S_k I - A \circ P_{k-1},$$

where S_k is the k-th symmetric function of the eigenvalues of A with the agreement that $S_0 = 1$. The associated differential operators acting on $C^2(M)$,

$$L_k u = \operatorname{tr}\left(P_k \circ \operatorname{hess}(u)\right),$$

are of the form (2.2) with the choice $X = \operatorname{div} P_k$. Note that the ellipticity of L_k is not automatically granted, but there exist various sufficient conditions of a geometric nature; see for instance the discussion in Section 3.2 of [4] and the references therein.

We begin by recalling the definition of the weak maximum principle.

Definition 2.1. We say that the weak maximum principle, WPM, holds for Q on M if for each $u \in C^1(M)$ with $u^* = \sup_M u < +\infty$ and for each $\gamma < u^*$ we have

$$(2.3) \qquad \inf_{\Omega_\gamma} Qu \le 0,$$

where

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\}.$$

Of course, (2.3) has to be interpreted in the appropriate weak sense, that is, for each $\varepsilon > 0$ there exists $\psi \in C_c^\infty(\Omega_\gamma)$, $\psi \ge 0$, $\psi \not\equiv 0$, such that

$$-\int |\nabla u|^{-1} \varphi(x, |\nabla u|) T(\nabla u, \nabla \psi) + \langle X, \nabla u \rangle \psi \le \varepsilon \int \psi.$$

Before defining property (P) for the operator Q in possibly unbounded domains, it is worth to comment on the quasilinear version of E. Hopf's strong maximum principle recalled in the Introduction. To the best of our knowledge, a strong maximum principle for inequalities of the type

$$(2.4) \qquad Qu \ge B(x, u, \nabla u) \quad \text{on } \Omega \subset M,$$

that is, the statement that each non-constant, C^1-solutions $u \le 0$ of (2.4) cannot attain the value zero, first appeared in a generality close to the one of the present paper in Theorem 8.1 of [22] (see also [23]). The conditions placed on Q and B are mild, in particular, they do not require B to be increasing in the variable u. Hence, they correspond in the linear case $Lu + c(x)u \ge 0$ to coefficients $c(x)$ which might be positive somewhere. For related results in a manifold setting, see Theorems 5.5 and 5.6 in [21]. Section 5.4 in [24] contains a thorough discussion of the strong maximum principle for (2.4), together with various interesting comments.

We now proceed to give the definition of property (P), mentioned in the introduction, and to make explicit its relation to the WMP.

Definition 2.2. A pair of functions $(c(x), h(s)) \in L^\infty_{\text{loc}}(M) \times C^0(\mathbb{R})$ is called admissible for property (P) below if $h(0) = 0$.

Definition 2.3. We say that Q satisfies property (P) for the admissible pair (c, h) if, for each open set $\Omega \subset M$ with $\partial\Omega \ne \varnothing$, for each $0 < \beta(s) \in C^0(\mathbb{R})$, each solution $v \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ of

$$(2.5) \qquad \begin{cases} Qv + c(x)h(v) \ge \beta(v) & \text{on } \Omega \\ v \le 0 & \text{on } \Omega \\ \sup_{\partial\Omega} v < 0, \end{cases}$$

satisfies $\sup_\Omega v < 0$.

Remark 2.1. Clearly, the maximum principle in [6], with a fixed $\Omega \subseteq \mathbb{R}^m$, can be recovered if T has components $\{a^{ij}\}$, $X = X^i \partial_i$ with $X^i = T^i_{kk} - b^i$, $\varphi(x, s) = s$, $h(s) = s$ and $\beta(s) = \varepsilon > 0$. However, we note that $h(s)$ does not need to have the same sign of s.

We are now in a position to state our first main result:

Theorem 2.2. *Let $(M, \langle \ , \ \rangle)$ be a Riemannian manifold, let $\varphi \colon M \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy $(\varphi 1)$, and let X and T be as above. Consider the operator Q defined in (2.1). Then the following statements are equivalent:*

   i) *Q satisfies the WMP on M.*

   ii) *Q satisfies (P) for each admissible pair (c, h) with $c_- \in L^\infty(M)$ and $sh(s) \geq 0$ on $\mathbb{R}$.*

   iii) *Q satisfies (P) for each admissible pair (c, h) with $c \in L^\infty(M)$.*

   iv) *Q satisfies (P) for some admissible pair (c, h) with $c \in L^\infty(M)$.*

Here, as usual, $c_-(x) = -\min\{0, c(x)\}$ is the negative part of c.

Despite the simplicity of the proof, the above equivalences are particularly interesting since there are various sufficient conditions for Q to satisfy the WMP. Among them, a very general criterion is the following one, that can be deduced from Theorem 4.1 of [3]: suppose that $X \equiv 0$ and

$$(\varphi 2) \qquad\qquad \varphi(x, s) \leq A(x) s^\delta \quad \text{on } M \times \mathbb{R}_0^+$$

for some $\delta > 0$ and $A(x) \in C^0(M)$ with $A(x) > 0$ on M. Fix an origin $o \in M$ and set $r(x) = \operatorname{dist}(x, o)$. Let B_r denote the geodesic ball of radius r centered at o. Let T be a 2-covariant, symmetric tensor field on M satisfying

$$(2.6) \qquad \text{there exist } T_\pm \in C^0(\mathbb{R}_0^+) \text{ such that } 0 < T_-(r) \leq T(Y, Y) \leq T_+(r)$$

for each $Y \in T_x M$, $|Y| = 1$, $x \in \partial B_r$. According to the value of δ introduced in $(\varphi 2)$, define

$$T_\delta(r) = \begin{cases} T_+(r) & \text{if } 0 < \delta \leq 1, \\ T_-(r)^{\frac{1-\delta}{2}} T_+(r)^{\frac{1+\delta}{2}} & \text{if } \delta > 1, \end{cases} \quad \text{and} \quad T_\delta^*(r) = \max_{[0,r]} T_\delta(s).$$

Theorem 2.3. *Let $(M, \langle \ , \ \rangle)$ be a complete Riemannian manifold with a fixed origin o, and let $\varphi \colon M \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ satisfy $(\varphi 1)$, $(\varphi 2)$. With $A(x)$ and δ as in $(\varphi 2)$, assume that*

$$A(x) \leq \mathscr{A}(r(x)),$$

where $\mathscr{A} \colon \mathbb{R}_0^+ \to \mathbb{R}^+$ is continuous, non decreasing. Let T satisfy (2.6) and suppose

$$\lim_{r \to +\infty} \frac{T_\delta^*(r) \mathscr{A}(r)}{r^{1+\delta}} = 0, \quad \text{and} \quad \liminf_{r \to +\infty} \frac{T_\delta^*(r) \mathscr{A}(r)}{r^{1+\delta}} \log\left(\int_{B_r} A(x) \right) < +\infty.$$

Then the operator Q defined in (2.1) with $X \equiv 0$ satisfies each of i)–iv) *in Theorem 2.2.*

Example 2.4. If Q is a linear operator with $X \equiv 0$ (for which $\varphi(x, s) = s$ and thus $\mathscr{A}(r) \equiv 1$), and the biggest eigenvalue of T satisfies $T_+(r) \leq C(1 + r)^\mu$ for some $C > 0$, $\mu \geq 0$, then property (P) is met whenever $\mu < 2$ and

$$\liminf_{r \to +\infty} \frac{\log \operatorname{Vol}(B_r)}{r^{2-\mu}} < +\infty.$$

This last condition is certainly satisfied, for instance, on $\mathbb{R}^m$ for each $m \geq 1$. Consequently, a self-adjoint, linear operator on $\mathbb{R}^m$ as in (2.2) with $X \equiv 0$ satisfies the WMP (hence, the principle in [6] for every fixed $\Omega \subseteq \mathbb{R}^m$) whenever $T_+(r) \leq C(1+r)^\mu$ and $\mu < 2$.

We would like to note that there are many natural geometric contexts where ($\varphi 2$) is satisfied. For instance, if $\left(M, \langle\ ,\ \rangle, e^{-f}dV_M\right)$, $f \in C^\infty(M)$ is a weighted Riemannian manifold and $\varphi \colon \Sigma \to M$ is an isometric immersion, one defines the weighted mean curvature vector H_f by setting $H_f = H + (\overline{\nabla}f)^\perp$, where $\overline{\nabla}$ is the gradient on M and H is the usual, non-normalized, mean curvature vector. Given a relatively compact domain $\Omega \subset \Sigma$, the weighted volume of Ω is given by

$$\mathrm{Vol}_f(\Omega) = \int_\Omega e^{-f}dV_\Sigma.$$

Then Σ is f-minimal, that is, a critical point of the weighted volume functional with respect to compactly supported variations if and only if $H_f \equiv 0$.

The Jacobi operator L_f of an f-minimal hypersurface is given by

$$e^{-f}L_f u = \mathrm{div}\left(e^{-f}\nabla u\right) + e^{-f}\left(|A|^2 + \overline{\mathrm{Ricc}}_f(\nu,\nu)\right)u,$$

where A is the second fundamental tensor of the immersion in the direction of the unit normal vector ν determining the orientation of Σ and $\overline{\mathrm{Ricc}}_f = \overline{\mathrm{Ricc}} + \mathrm{Hess}(f)$, is the Bakry–Emery Ricci tensor of M. In this case

$$\varphi(x,s) = e^{-f(x)}s,$$

that clearly satisfies ($\varphi 2$) with $\delta = 1$, $A(x) = e^{-f(x)} > 0$.

We observe that the above example includes that of (for instance) self-shrinkers in $\mathbb{R}^n$, that is, immersions $\varphi \colon M^m \to \mathbb{R}^n$ for which the (not normalized) mean curvature vector satisfies

$$H = -\frac{1}{2}\varphi^\perp.$$

To see this it is enough to choose $f(x) = |x|^2/4$ on $\mathbb{R}^n$. We refer the interested reader to [13] for an introduction to the mean curvature flow and the role of self-shrinkers as models for its singularities, and to [9] for deep results that exploit the parallelism between self-shrinkers and f-minimal hypersurfaces.

Vice versa, a sufficient condition for Q to satisfy (P) will provide a sufficient condition for Q to satisfy the WMP on M. In this way we are able to introduce a "bounded" Khas'minskii-type condition in Definition 2.4 below. We observe that a sufficient Khas'minskii-type assumption via a function γ (in the linear case) or a family of functions $\{\gamma_\varepsilon\}_{\varepsilon>0}$ (in the nonlinear case) on M have already been given for linear and nonlinear operators, and we refer to [3] and [15] for a thorough discussion. However, the validity of the latter results rests on the property that

$$\gamma(x),\, \gamma_\varepsilon(x)\ \to +\infty \quad \text{as } x \to \infty \text{ in } M.$$

In the assumptions we are going to present here the function γ is bounded. Indeed, we introduce the following condition that we name the "bounded Khas'minskii" property, (BK) for short.

Definition 2.4. We say that Q satisfies property (BK) for some admissible pair (c, h) if, for each $p \in M$ and each $\varepsilon > 0$ there exist constants $B_{1,2} = B_{1,2}(\varepsilon) > 0$ and a function $\gamma = \gamma_{p,\varepsilon} \in C^1(M)$ with the following properties:

$$
\begin{cases}
\text{i)} & \gamma(p) = 0 \\
\text{ii)} & 0 \leq \gamma \leq B_1 \text{ on } M \\
\text{iii)} & Q\gamma + c(x)h(\gamma) \leq \varepsilon \\
\text{iv)} & \liminf_{x \to \infty} \gamma(x) \geq B_2.
\end{cases}
$$

We suggest that the reader consult the recent [14] for a detailed study of the interplay between (BK) and previous Khas'minskii conditions. The aforementioned paper deals with classes of operators that include most (though not all) of those considered in the present work, and the main problems are tackled via the use of viscosity solutions.

In the next results we shall need a few more technical requirements to guarantee the ellipticity of Q:

$(\varphi 3)$ $\qquad \varphi(x, \ ) \in C^0(\mathbb{R}_0^+) \cap C^1(\mathbb{R}^+) \quad$ for each $x \in M$;

$(\varphi 4)$ $\qquad \dfrac{\partial \varphi}{\partial s}(x, \ ) > 0 \quad$ on $\mathbb{R}^+$ for each $x \in M$;

$(\varphi 5)$ $\qquad \dfrac{\varphi(x, s)}{s} \in L^1(0^+) \quad$ for each $x \in M$;

$(T1)$ $\qquad$ for each $x \in M, 0 \neq \xi \in T_x M$, the (symmetric) bilinear form

$$
\frac{1}{|\xi|^2} \left\{ \frac{\partial \varphi}{\partial s}(x, |\xi|) - \frac{\varphi(x, |\xi|)}{|\xi|} \right\} \langle \xi, \ \rangle \odot T(\xi, \ ) + \frac{\varphi(x, |\xi|)}{|\xi|} T(\ , \ )
$$

$\qquad$ is positive definite.

Here $\odot$ denotes the symmetric tensor product

$$
a \odot b = \frac{1}{2}(a \otimes b + b \otimes a).
$$

Remark 2.5. In the linear case $\varphi(x, s) = s$, $(T1)$ implies that T be positive definite.

We are now ready to state our second main result, which relates condition (BK) to the WMP and property (P).

Theorem 2.6. *Let $(M, \langle \ , \ \rangle)$ be a Riemannian manifold and φ, X, T be as above. Assume the validity of $(\varphi 1)$, $(\varphi 3)$, $(\varphi 4)$, $(\varphi 5)$, $(T1)$ and of (BK) for an admissible pair (c, h) with*

$$
c_- \in L^\infty(M), \quad sh(s) \geq 0 \text{ on } \mathbb{R}.
$$

Then (P) holds for that pair. In particular, if also $c \in L^\infty(M)$ or (BK) holds for the pair $(0, 0)$, then Q satisfies the WMP and (P) holds for each admissible pair (c, h) with $c \in L^\infty(M)$.

Conditions ($\varphi3$), ($\varphi4$), ($\varphi5$), ($T1$) are needed to ensure the validity of a comparison theorem for the divergence part Q_0 of Q, that is, for the operator

$$(2.7) \qquad Q_0 u = \operatorname{div}\left(|\nabla u|^{-1}\varphi(x,|\nabla u|)t(\nabla u)\right).$$

Since the result is interesting in its own, we report its statement here referring to Section 3 for its proof. We stress that the extra assumptions ($\varphi3$), ($\varphi4$), ($\varphi5$), and ($T1$) are mild, making the comparison very general and, to the best of our knowledge, new.

Proposition 2.7. *Assume that* ($\varphi1$), ($\varphi3$), ($\varphi4$), ($\varphi5$), ($T1$) *hold, and let* $\Omega \subset M$ *be a relatively compact domain. Let* $u,v \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ *solve*

$$(2.8) \qquad \begin{cases} Q_0 u \geq Q_0 v & on\ \Omega, \\ u \leq v & on\ \partial\Omega, \end{cases}$$

where Q_0 *is the operator defined in* (2.7). *Then,* $u \leq v$ *on* Ω.

Remark 2.8. It shall be noticed that Theorems 5.2 and 5.3 in [21] give sharp comparison results for inequalities of the type

$$Q_0 u \geq B(x,u,\nabla u), \quad Q_0 v \leq B(x,v,\nabla v),$$

where B,u,v satisfy suitable assumptions. However, the results are skew with Proposition 2.7. The monograph [24] contains a thorough investigation of comparison principles for a variety of differential operators, including operators in divergence form (although under more restrictive assumptions), and is by now a standard reference for the interested reader. See also [22] and [5] for improvements and other related results.

When the comparison property for Q_0 in the form of Proposition 2.7 above is available, in order to prove the validity of the WMP it is enough to check property (P) on domains Ω with non-compact closure.

Proposition 2.9. *Let* φ *and* T *satisfy* ($\varphi1$), ($\varphi3$), ($\varphi4$), ($\varphi5$) *and* ($T1$), *let* $X \in \mathfrak{X}(M)$ *and define* Q *as in* (2.1). *Then the equivalent conditions in Theorem* 2.2 *are also equivalent to the following:*

  v) *Q satisfies (P) for some admissible pair (c,h) with $c \in L^\infty(M)$, where Ω in (P) has non-compact closure.*

We close this paragraph with an application of the weak maximum principle to prove a comparison result for subsolutions and supersolutions of semilinear differential equations.

Theorem 2.10. *Let* Q *be the linear differential operator defined in* (2.2) *and let* $u,v \in C^2(M)$ *satisfy*

$$u,v \in L^\infty(M) \cap C^2(M), \quad u \geq 0, \quad \inf_M v > 0$$

and

$$Qu \geq f(x, u), \quad Qv \leq f(x, v) \quad on \ M,$$

where $f \colon M \times \mathbb{R}_0^+ \to \mathbb{R}$ *is locally Lipschitz in the variable* s, *uniformly in* $x \in M$, *and such that*

(2.9) $$\forall s_1, s_2, \quad 0 < s_1 < s_2, \quad \inf_{x \in M} \left(\frac{f(x, s_2)}{s_2} - \frac{f(x, s_1)}{s_1} \right) > 0.$$

If the WMP *holds for* Q *on* M, *then* $u \leq v$ *on* M.

The above comparison result is very general and can be applied to a wide variety of differential equations. As an example, we consider the Lichnerowicz-type equation

$$Qu + a(x)u - b(x)u^\sigma + c(x)u^\tau = 0,$$

where Q is as in (2.2) and linear (that is, $\varphi(x, s) = s$), $a(x), b(x), c(x)$ are continuous, $\sigma > 1$ and $\tau < 1$.

We note that, for $f(x, s) = -a(x)s + b(x)s^\sigma - c(x)s^\tau$, the assumptions of the theorem are satisfied provided $a(x), b(x), c(x)$ are bounded, $b(x) \geq 0$, $c(x) \geq 0$ and either $\inf_M b > 0$ or $\inf_M c > 0$. In particular, the assumption on the local Lipschitz continuity of the function f in the variable s is satisfied even though $\tau < 1$ since, as it will be apparent from the proof, this assumption is only necessary for s in the range of the subsolution and supersolution, which, for the Lichnerowicz equation, is assumed bounded below by a positive constant.

3. Proofs of the analytic results

The proof of Theorem 2.2 is quite straightforward once we recall the validity of the next result (see Theorem 4.6 of [3], or [4]).

Theorem 3.1. *The* WMP *holds for* Q *if and only if the open weak maximum principle, for short* OWMP, *holds for* Q *on* M, *that is, for each* $g \in C^0(\mathbb{R})$, *for each open set* $\Omega \subset M$ *with* $\partial \Omega \neq \varnothing$, *each solution* $v \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ *of*

$$\begin{cases} Qv \geq g(v) & on \ \Omega \\ \sup_\Omega v < +\infty \end{cases}$$

satisfies either

$$\sup_\Omega v = \sup_{\partial \Omega} v \quad or \quad g\left(\sup_\Omega v \right) \leq 0.$$

We are now ready for the:

Proof of Theorem 2.2. We prove the chain of implications i) $\Rightarrow$ ii) $\Rightarrow$ iii) $\Rightarrow$ iv) $\Rightarrow$ i).

i) $\Rightarrow$ ii). Suppose that Q satisfies the WMP and choose an admissible pair (c, h) satisfying

(3.1) $$c_- \in L^\infty(M), \quad sh(s) \geq 0 \text{ on } \mathbb{R}$$

and a function $0 < \beta \in C^0(\mathbb{R})$. Let $\Omega \subset M$ be an open set with $\partial\Omega \neq \varnothing$ and $v \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ be a solution of (2.5). From the differential inequality in (2.5) and since v and $h(v)$ have the same sign, we deduce

$$Qv \geq \beta(v) + c_-(x)\,h(v) \geq \beta(v) + \|c_-\|_{L^\infty(M)}\,h(v).$$

By Theorem 3.1 applied with the choice $g(s) = \beta(s) + \|c_-\|_{L^\infty(M)}h(s)$, we deduce that either

$$\sup_\Omega v = \sup_{\partial\Omega} v < 0 \quad \text{or} \quad g\Big(\sup_\Omega v\Big) \leq 0.$$

In the latter case, since $\beta > 0$ and $h(0) = 0$ imply $g(0) = \beta(0) > 0$, necessarily it must be $\sup_\Omega v < 0$. Thus in both cases we have the validity of property (P).

ii) $\Rightarrow$ iii). Let v satisfy (2.5) for an admissible pair (c, h) with $c \in L^\infty(M)$ and, by contradiction, suppose that $\sup_\Omega v = 0$. Choose $\gamma < 0$ sufficiently close to zero in such a way that

$$\|c\|_{L^\infty(M)}\|h\|_{L^\infty([\gamma,0])} < \inf_{[\gamma,0]} \beta.$$

This is possible since $\beta, h \in C^0(\mathbb{R})$, $\beta > 0$ and $h(0) = 0$. Since v cannot be constant, we can also choose γ in such a way that the open set

$$\Omega_\gamma = \{x \in \Omega : v(x) > \gamma\}$$

satisfies $\overline{\Omega}_\gamma \subset \Omega$. In particular we have

$$\begin{cases} Qv \geq \beta(v) - c(x)h(v) \geq \inf_{[\gamma,0]} \beta - \|c\|_{L^\infty(M)}\|h\|_{L^\infty([\gamma,0])} = \bar\beta > 0 & \text{on } \Omega_\gamma, \\ \sup_{\partial\Omega_\gamma} v = \gamma < 0. \end{cases}$$

In other words

$$\begin{cases} Qv \geq \bar\beta > 0 & \text{on } \Omega_\gamma, \\ \sup_{\partial\Omega_\gamma} v = \gamma < 0. \end{cases}$$

Thus, by ii) applied on Ω_γ with the pair $(0, 0)$, which is admissible and satisfies (3.1), we have $\sup_{\Omega_\gamma} v < 0$, a contradiction.

iii) $\Rightarrow$ iv) is obvious.

iv) $\Rightarrow$ i). Consider the admissible pair (c, h) granted by iv) with $c \in L^\infty(M)$. By contradiction, let $u \in C^1(M)$ be such that $u^* = \sup_M u < +\infty$ and let $\gamma < u^*$, $\Lambda > 0$ be such that $Qu \geq \Lambda$ on

$$\Omega_\gamma = \{x \in M : u(x) > \gamma\}.$$

Using the fact that $Q(u+c) = Q(u)$ for each constant c, possibly adding a constant to u we can suppose that $\gamma > 0$ and possibly increasing γ we can ensure that $\partial\Omega_\gamma \neq \varnothing$ and furthermore that

$$\|c\|_{L^\infty(M)}\|h\|_{L^\infty([\gamma-u^*,0])} < \Lambda.$$

On $\overline{\Omega}_\gamma$ we define $v = u - u^*$. For $x \in \Omega_\gamma$, $v(x) \in [\gamma - u^*, 0]$, hence we have

$$Qv + c(x)h(v) \geq Qv - \|c\|_{L^\infty(M)}\|h\|_{L^\infty([\gamma-u^*,0])}$$
$$\geq \Lambda - \|c\|_{L^\infty(M)}\|h\|_{L^\infty([\gamma-u^*,0])} = \beta > 0.$$

By property (P) for the admissible pair (c,h) applied to v on Ω_γ and with the choice of $\beta(s) = \beta > 0$ defined above we have $\sup_{\Omega_\gamma} v < 0$, a contradiction. □

We now turn to the proof of the comparison principle stated in Proposition 2.7. To this end, we let Q_0 be as defined in (2.7). The next lemma provides the necessary monotonicity condition on Q_0 for the validity of a comparison result.

First let φ satisfy $(\varphi 1)$, $(\varphi 3)$, $(\varphi 4)$ and, for $x \in M$ and $\xi \in T_x M$ fixed, define the function

$$g_{x,\xi} : T_x M \setminus \{0\} \longrightarrow \mathbb{R}$$

by setting

$$g_{x,\xi}(\eta) = T\big(|\eta|^{-1}\varphi(x,|\eta|)\eta, \xi\big).$$

Since

$$|g_{x,\xi}(\eta)| \leq |T|_x \varphi(x,|\eta|)\,|\xi|,$$

the validity of $(\varphi 1)$ implies that

$$g_{x,\xi}(\eta) \to 0 \text{ as } \eta \to 0.$$

This allows us to extend $g_{x,\xi} : T_x M \to \mathbb{R}$ continuously by setting $g_{x,\xi}(0) = 0$.

Lemma 3.2. *Assume the validity of $(\varphi 1)$, $(\varphi 3)$, $(\varphi 4)$ and $(\varphi 5)$ and define $g_{x,\xi}$ for $x \in M$, $\xi \in T_x M$ as above. Let $\nabla u, \nabla v \in T_x M$ and set $X_t = t\nabla u + (1-t)\nabla v$ for $t \in [0,1]$. Suppose that $|\nabla u| + |\nabla v| > 0$ and let T be a 2-covariant tensor field on M. Then at x we have*

$$\bar{g}(x) = g_{x,\nabla u - \nabla v}(\nabla u) - g_{x,\nabla u - \nabla v}(\nabla v) =$$

$$(3.2) \quad = \int_0^1 \left\{ \frac{\varphi(x,|X_t|)}{|X_t|} T(\nabla u - \nabla v, \nabla u - \nabla v) \right.$$

$$\left. + \frac{1}{|X_t|^2}\Big[\frac{\partial \varphi}{\partial s}(x,|X_t|) - \frac{\varphi(x,|X_t|)}{|X_t|}\Big]\langle X_t, \nabla u - \nabla v\rangle T(X_t, \nabla u - \nabla v) \right\} dt.$$

Furthermore, if $(T1)$ holds, then $\bar{g}(x) \geq 0$ and $\bar{g}(x) = 0$ if and only if $\nabla u = \nabla v$.

Proof. First assume that $X_t \neq 0$ on $[0,1]$. Let $\gamma : [0,1] \to M$ be the constant curve $\gamma(t) = x$ for $t \in [0,1]$ and consider X_t as a vector field along γ. To simplify notations, we set $Y = \nabla u - \nabla v$. Let $\{e_i\}$ be a local orthonormal frame around x satisfying $(\nabla_{e_j} e_i)(x) = 0$ for all $i,j = 1,\ldots,m$. Using the latter, jointly with the properties of covariant differentiation $\frac{D}{dt}$ along a curve, the fact that $\dot{\gamma} \equiv 0$ on $[0,1]$

and $X_t \neq 0$ on $[0,1]$, we have

$$
\begin{aligned}
(3.3) \quad & \frac{d}{dt}\langle |X_t|^{-1}\varphi(x,|X_t|)T(X_t, \ )^\sharp, Y\rangle = \Big\langle \frac{D}{dt}\big(|X_t|^{-1}\varphi(x,|X_t|)T(X_t, \ )^\sharp\big), Y\Big\rangle \\
& = \Big\langle -|X_t|^{-3}\Big\langle \frac{D}{dt}X_t, X_t\Big\rangle \varphi(x,|X_t|)T(X_t, \ )^\sharp, Y\Big\rangle \\
& \quad + \Big\langle |X_t|^{-2}\frac{\partial\varphi}{\partial s}(x,|X_t|)\Big\langle \frac{D}{dt}X_t, X_t\Big\rangle T(X_t, \ )^\sharp, Y\Big\rangle \\
& \quad + \Big\langle |X_t|^{-1}\varphi(x,|X_t|)\frac{D}{dt}T(X_t, \ )^\sharp, Y\Big\rangle \\
& = -|X_t|^{-3}\varphi(x,|X_t|)\langle X_t, Y\rangle T(X_t, Y) \\
& \quad + |X_t|^{-2}\frac{\partial\varphi}{\partial s}(x,|X_t|)\langle X_t, Y\rangle T(X_t, Y) \\
& \quad + |X_t|^{-1}\varphi(x,|X_t|)\Big\langle \frac{D}{dt}\big(tT(\nabla u, \ )^\sharp + (1-t)T(\nabla v, \ )^\sharp\big), Y\Big\rangle \\
& = \frac{\varphi(x,|X_t|)}{|X_t|}T(Y,Y) + \frac{1}{|X_t|^2}\Big[\frac{\partial\varphi}{\partial s}(x,|X_t|) - \frac{\varphi(x,|X_t|)}{|X_t|}\Big]\langle X_t, Y\rangle T(X_t, Y).
\end{aligned}
$$

Thus in this case the result follows immediately by integration. Now suppose there exists $t_0 \in [0,1]$ with $X_{t_0} = 0$. Then, $X_t = (t - t_0)Y$ and necessarily $Y \neq 0$ (otherwise, $\nabla u = \nabla v = 0$), whence t_0 is unique. The cases $t_0 = 0$, $t_0 = 1$ are simpler, so let us assume $t_0 \in (0,1)$. Let I be the integrand in (3.2). For $\varepsilon > 0$ sufficiently small, integrating (3.3) on the intervals $[0, t_0 - \varepsilon]$ and $[t_0 + \varepsilon, 1]$, we get

$$
\int_0^{t_0-\varepsilon} I + \int_{t_0+\varepsilon}^1 I = g_{x,\nabla u - \nabla v}(X_{t_0-\varepsilon}) - g_{x,\nabla u - \nabla v}(\nabla v) \\
+ g_{x,\nabla u - \nabla v}(\nabla u) - g_{x,\nabla u - \nabla v}(X_{t_0+\varepsilon}).
$$

By the continuity of $g_{x,\xi}$, its linearity in ξ and since $X_{t_0} = 0$, the RHS of the above converges to $\bar{g}(x)$ as $\varepsilon \to 0^+$. On the other hand, because of $(\varphi 1)$ and $(\varphi 4)$ we get

$$
\frac{\partial\varphi}{\partial s}(x,s) \in L^1(0^+).
$$

Coupling with $(\varphi 5)$ and using $X_t = (t - t_0)Y$, the LHS converges to $\int_0^1 I$. Under the validity of $(T1)$, the fact that $\bar{g}(x) \geq 0$ and $\bar{g}(x) = 0$ if and only if $\nabla u = \nabla v$ follows immediately from (3.2). $\qquad\square$

Proof of Proposition 2.7. Suppose by contradiction that, for some $\varepsilon > 0$, the relatively compact open set $\Omega_\varepsilon = \{x \in \Omega : u(x) > v(x) + \varepsilon\}$ is nonempty. Note that $\overline{\Omega}_\varepsilon \subset \Omega$. Let $\alpha \in C^1(\mathbb{R})$ satisfy

$$
\alpha \equiv 0 \text{ on } (-\infty, \varepsilon], \quad \alpha' > 0 \text{ on } (\varepsilon, +\infty).
$$

We test the differential inequality in (2.8) with the test function $w = \alpha(u-v) \in C_c^1(\Omega)$ to get

$$
\int_{\Omega_\varepsilon} \alpha'(u-v)\, T\big(|\nabla u|^{-1}\varphi(x,|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(x,|\nabla v|)\nabla v, \nabla u - \nabla v\big) \leq 0.
$$

However, by Lemma 3.2,

$$\bar{g}(x) = T\big(|\nabla u|^{-1}\varphi(x,|\nabla u|)\nabla u - |\nabla v|^{-1}\varphi(x,|\nabla v|)\nabla v, \nabla u - \nabla v\big) \geq 0.$$

Since $\alpha'(u-v) > 0$ on Ω_ε, it follows that $\bar{g} \equiv 0$ on Ω_ε. Again by Lemma 3.2, $\nabla u \equiv \nabla v$ on $\Omega_\varepsilon \cap \{|\nabla u| + |\nabla v| > 0\}$, hence on Ω_ε. Integrating, $u - v$ is constant on each connected component of Ω_ε, contradicting its very definition.   □

With this preparation we are ready to prove Proposition 2.9.

Proof of Proposition 2.9. Since item iv) in Theorem 2.2 implies item v) in Proposition 2.9, it is enough to prove that, in our assumptions, (P) automatically holds for (c,h) if Ω has compact closure. Therefore, pick a solution $v \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ of

$$\begin{cases} Qv + c(x)h(v) \geq \beta(v) & \text{on } \Omega \subset\subset M, \\ v \leq 0 & \text{on } \Omega, \\ \sup_{\partial\Omega} v < 0, \end{cases}$$

and by contradiction suppose that $\sup_\Omega v = 0$. Then, by compactness, 0 is attained at an interior point of $\overline{\Omega}$ that is, at a point of Ω. Define the compact set $\Gamma = \{x \in \Omega : v(x) = 0\}$. Fix $\gamma < 0$ sufficiently close to zero in order that

$$\Omega_\gamma = \{x \in \Omega : v(x) > \gamma\} \subset\subset \Omega.$$

On Ω_γ we have

$$(3.4) \qquad Q_0 v = Qv + \langle X, \nabla v \rangle \geq \beta(v) - c(x)h(v) + \langle X, \nabla v \rangle$$
$$\geq \beta(v) - \|X\|_{L^\infty(\Omega)}\|\nabla v\|_{L^\infty(\Omega_\gamma)} - \|c\|_{L^\infty(M)}\|h\|_{L^\infty([\gamma,0])}$$
$$\geq \inf_{[\gamma,0]}\beta - \|X\|_{L^\infty(\Omega)}\|\nabla v\|_{L^\infty(\Omega_\gamma)} - \|c\|_{L^\infty(M)}\|h\|_{L^\infty([\gamma,0])}.$$

Since Ω_γ shrinks to Γ as $\gamma \uparrow 0$ and $v \in C^1(\Omega)$ satisfies $\nabla v = 0$ on Γ, we can make $\|\nabla v\|_{L^\infty(\Omega_\gamma)}$ as small as we wish up to choosing γ close enough to zero. The same happens for $\|h\|_{L^\infty([\gamma,0])}$ since $h(0) = 0$ and h is continuous. Since $0 < \beta \in C^0(\mathbb{R})$, from (3.4) we deduce that

$$Q_0 v \geq \frac{1}{2}\beta(0) > 0 \quad \text{on } \Omega_\gamma$$

if γ is close enough to zero. Since the constant γ satisfies $Q_0\gamma = 0$ and $v = \gamma$ on $\partial\Omega_\gamma$, by Proposition 2.7 we deduce that $v \leq \gamma$ on Ω_γ; contradiction.   □

Proof of Theorem 2.6. Let (c,h) be an admissible pair as in (BK) with

$$c_- \in L^\infty(M), \quad sh(s) \geq 0 \text{ on } \mathbb{R}.$$

By contradiction suppose the existence of an open set $\Omega \subset M$ with $\partial\Omega \neq \varnothing$, of $0 < \beta \in C^0(\mathbb{R})$ and $v \in C^0(\overline{\Omega}) \cap C^1(\Omega)$ solution of

$$(3.5) \qquad \begin{cases} Qv + c(x)h(v) \geq \beta(v) & \text{on } \Omega, \\ v \leq 0 & \text{on } \Omega, \\ \sup_{\partial\Omega} v < 0, \end{cases}$$

for which $\sup_\Omega v = 0$. Set $\beta_0 = \inf_{[-1,0]} \beta > 0$ and choose ε and σ satisfying

$$(3.6) \qquad \varepsilon \in \left(0, \frac{\beta_0}{8}\right), \quad 0 < \sigma < \min\left\{B_1, B_2, \frac{1}{2}, -\sup_{\partial\Omega} v\right\},$$

where $B_j = B_j(\varepsilon)$ are as in (BK). Furthermore, we require σ to be small enough that if $s \in [-B_1, B_1]$ and $|\bar{s}| < 2\sigma$, then

$$(3.7) \qquad |h(s + \bar{s}) - h(s)| \leq \frac{\beta_0}{4\|c_-\|_{L^\infty(M)}},$$

with the agreement that this is automatically satisfied if $c_- \equiv 0$. Note that (3.7) is possible since $h(s) \in C^0(\mathbb{R})$.

Since $\sup_\Omega v = 0$, the set

$$\Omega_{-\sigma} = \{x \in \Omega : v(x) > -\sigma\}$$

is not empty and we choose $x_0 \in \Omega_{-\sigma}$. Next we consider the function $\gamma = \gamma_{x_0,\varepsilon}$ granted by (BK) and we let $a > 0$ sufficiently small to satisfy $\sigma < B_2 - a$. Because of property i) in Definition 2.4,

$$x_0 \in U = \{x \in M : \gamma_{x_0,\varepsilon}(x) < B_2 - a\}.$$

By property iv) of Definition 2.4, U is relatively compact by the Hopf–Rinow theorem, since M is complete. Set

$$w(x) = v(x) - \gamma_{x_0,\varepsilon}(x) \quad \text{on } \overline{\Omega}.$$

By properties i) and ii) of Definition 2.4, (3.6), the fact that $x_0 \in \Omega_{-\sigma}$ and the inequalities $\sigma < B_2 - a$, $v \leq 0$ on $\overline{\Omega}$, we deduce

$$(3.8) \qquad \begin{aligned} &\text{i)} \quad w \leq 0 \quad \text{on } \Omega, \\ &\text{ii)} \quad \begin{aligned}[t] w(x) &= v(x) - \gamma_{x_0,\varepsilon}(x) \leq -\gamma_{x_0,\varepsilon}(x) < v(x_0) \\ &= v(x_0) - \gamma_{x_0,\varepsilon}(x_0) = w(x_0) \quad \text{on } \partial U \cap \overline{\Omega}. \end{aligned} \end{aligned}$$

Let $w^* \leq 0$ denote the maximum of w on the compact set $\overline{U \cap \Omega}$ and let x_1 be a maximum point that, by (3.8) ii), lies in $U \cap \overline{\Omega}$. Using (3.6), $v(x_0) > -\sigma$ and i) of Definition 2.4, we have

$$(3.9) \qquad 0 \geq w^* = w(x_1) \geq w(x_0) = v(x_0) > -\sigma > \sup_{\partial\Omega} v \geq \sup_{\partial\Omega} w,$$

so that $x_1 \notin \partial\Omega$. It follows that

$$x_1 \in U \cap \Omega.$$

Now consider the function

$$u(x) = \gamma_{x_0,\varepsilon}(x) + w^* \quad \text{on } \overline{\Omega}.$$

Note that, on $\overline{U \cap \Omega}$, $u - v = w^* - w$, hence

$$(3.10) \qquad \text{i) } u - v \geq 0 \quad \text{on } \overline{U \cap \Omega}, \quad \text{ii) } (u - v)(x_1) = 0$$

and by (3.9) and (3.8),

(3.11) $u - v = w^* - w > 0$ on $\partial\Omega \cup (\partial U \cap \overline{\Omega}) \supset \partial(U \cap \Omega).$

Fix $\bar{\varepsilon} > 0$ and let
$$\Omega_{\bar{\varepsilon}} = \left\{ x \in \overline{U \cap \Omega} \ : \ u(x) - v(x) < \bar{\varepsilon} \right\}.$$

In view of (3.10) and (3.11), $x_1 \in \Omega_{\bar{\varepsilon}}$ for each $\bar{\varepsilon} > 0$; furthermore, for $\bar{\varepsilon}$ sufficiently small, say $\bar{\varepsilon} \leq \varepsilon_0$ for some fixed ε_0, $\overline{\Omega}_{\bar{\varepsilon}} \subset U \cap \Omega$, which follows immediately from (3.11).

As $\bar{\varepsilon} \downarrow 0^+$, $\Omega_{\bar{\varepsilon}}$ shrinks to the compact set

$$\{x \in U \cap \Omega : u(x) = v(x)\},$$

where, by (3.10), $\nabla u = \nabla v$. Taking into account that $u, v \in C^1(\Omega_{\bar{\varepsilon}})$, up to choosing $\bar{\varepsilon}$ small enough we can guarantee that

(3.12) $|\nabla u - \nabla v| < \dfrac{\beta_0}{4\|X\|_{L^\infty(\Omega_{\varepsilon_0})}}$ on $\Omega_{\bar{\varepsilon}}$,

where X is the vector field in the definition of Q, and we agree that (3.12) is automatically satisfied if $X \equiv 0$. Up to shrinking $\bar{\varepsilon}$ further, we can also assume that

(3.13) $0 < \bar{\varepsilon} < \min\left\{\dfrac{1}{2}, \sigma\right\}.$

Define
$$\bar{u}(x) = u(x) - \bar{\varepsilon} = \gamma_{x_0,\varepsilon}(x) + w^* - \bar{\varepsilon}$$

on $\overline{\Omega}_{\bar{\varepsilon}}$ and observe that, by the definition of $\Omega_{\bar{\varepsilon}}$ and by (3.12) we have

i) $\Omega_{\bar{\varepsilon}} = \{x \in \Omega \cap U : \bar{u}(x) < v(x)\}$;   ii) $|\nabla\bar{u} - \nabla v| < \dfrac{\beta_0}{4\|X\|_{L^\infty(\Omega_{\varepsilon_0})}}$ on $\Omega_{\bar{\varepsilon}}$.

Now because of iii) of Definition 2.4 and the definition of Q,

(3.14) $Q\bar{u} = Q(w^* - \bar{\varepsilon} + \gamma_{x_0,\varepsilon}) = Q\gamma_{x_0,\varepsilon} \leq \varepsilon - c(x)h(\gamma_{x_0,\varepsilon})$

on $\Omega_{\bar{\varepsilon}}$. Since $\gamma_{x_0,\varepsilon} \geq 0$ on M, $\bar{u} = u - \bar{\varepsilon} \leq v \leq 0$ on $\Omega_{\bar{\varepsilon}}$ and $h(s)$ has the same sign of s, we infer that
$$h(\gamma_{x_0,\varepsilon}) \geq 0, \quad h(\bar{u}) \leq 0 \quad \text{on } \Omega_{\bar{\varepsilon}}.$$

Hence, from (3.14) we deduce that on $\Omega_{\bar{\varepsilon}}$

$\quad Q\bar{u} \leq \varepsilon + c_-(x)h(\gamma_{x_0,\varepsilon}) = \varepsilon + c_-(x)\{h(\gamma_{x_0,\varepsilon}) - h(\gamma_{x_0,\varepsilon} + w^* - \bar{\varepsilon})\} + c_-(x)h(\bar{u})$

(3.15) $\leq \varepsilon + \|c_-\|_{L^\infty(M)}|h(\gamma_{x_0,\varepsilon}) - h(\gamma_{x_0,\varepsilon} + w^* - \bar{\varepsilon})|.$

From ii) in Definition 2.4 we deduce that $0 \leq \gamma_{x_0,\varepsilon} \leq B_1$, and coupling (3.9) and (3.13) we get $|w^* - \bar{\varepsilon}| < 2\sigma$. Using (3.7), we therefore obtain

$$|h(\gamma_{x_0,\varepsilon}) - h(\gamma_{x_0,\varepsilon} + w^* - \bar{\varepsilon})| < \frac{\beta_0}{4\|c_-\|_{L^\infty(M)}}.$$

Inserting into (3.15) we obtain

(3.16) $$Q\bar{u} \leq \varepsilon + \frac{\beta_0}{4} \quad \text{on } \Omega_{\bar{\varepsilon}}.$$

On the other hand, on $\Omega_{\bar{\varepsilon}}$,

$$0 \geq v > \bar{u} = u - \bar{\varepsilon} = \gamma_{x_0,\varepsilon} + w^* - \bar{\varepsilon} \geq w^* - \bar{\varepsilon} \geq -2\sigma > -1.$$

Hence, by the definition of β_0,

$$\beta(v) \geq \beta_0 \quad \text{on } \Omega_{\bar{\varepsilon}}.$$

Using (3.5) together with $|v| < 2\sigma$, $h(0) = 0$ and (3.7) again, we get on $\Omega_{\bar{\varepsilon}}$

(3.17) $$Qv \geq \beta_0 - c(x)h(v) \geq \beta_0 - \|c_-\|_{L^\infty(M)}|h(v)| \geq \frac{3}{4}\beta_0.$$

Putting together (3.6), (3.16), (3.17) and (3.12) on $\Omega_{\bar{\varepsilon}} = \{x \in U \cap \Omega : v(x) > u(x)\}$ we get

$$\begin{aligned} Q_0 v - Q_0 \bar{u} &= Qv - Q\bar{u} - \langle X, \nabla v - \nabla \bar{u} \rangle \\ &\geq \frac{3}{4}\beta_0 - \varepsilon - \frac{\beta_0}{4} - \|X\|_{L^\infty(\Omega_{\varepsilon_0})}\|\nabla v - \nabla \bar{u}\|_{L^\infty(\Omega_{\bar{\varepsilon}})} \geq \frac{\beta_0}{8} \end{aligned}$$

and $v = \bar{u}$ on $\partial\Omega_{\bar{\varepsilon}}$. In our assumptions we are able to use the comparison result in Proposition 2.7 on $\Omega_{\bar{\varepsilon}}$ to deduce that $v \leq \bar{u}$ on $\Omega_{\bar{\varepsilon}}$, contradiction. This completes the proof of Theorem 2.6. □

We now turn to Theorem 2.10. The proof follows from Lemma 3.4 in [6], but we reproduce it here for the sake of completeness. To this end, we state the following.

Lemma 3.3. *Let $u_1, u_2 \in C(M)$ be two positive, bounded functions satisfying*

$$\inf_M u_1 > 0, \quad \inf_M (u_2 - u_1) > 0.$$

If f is locally Lipschitz continuous in the variable $s \in \mathbb{R}_0^+$, uniformly in $x \in M$, and (2.9) holds, then there exists $\varepsilon > 0$ such that

$$\forall x \in M, \quad \frac{u_2(x)}{u_1(x)} f(x, u_1(x)) \leq f(x, u_2(x)) - \varepsilon.$$

The proof of this lemma is elementary and can be found in [6].

The next proof is similar, in spirit, to that of Theorem 4.3 of [2] or that of Theorem 3.1 in [25]. However, the new form of the WMP enables us to apply directly the argument in [6].

Proof of Theorem 2.10. Since $\inf_M v > 0$, there exists $k > 0$ such that $kv \geq u$ on M. Set

$$k^* = \inf\{k > 0 : kv - u \geq 0 \text{ on } M\}$$

and suppose $k^* > 1$. The function $w = u - k^* v$ is nonpositive, of class C^2 and, by the linearity of Q, it satisfies

$$Qw \geq f(x, u) - k^* f(x, v) \quad \text{on } M.$$

Since
$$\inf_M (k^* v - v) = (k^* - 1) \inf v > 0,$$
we can apply Lemma 3.3 to the functions $u_1 = v$ and $u_2 = k^* v$, so there exists $\varepsilon > 0$ such that
$$\frac{k^* v}{v} f(x, v) \le f(x, k^* v) - \varepsilon,$$
which yields
$$Qw \ge f(x, u) - f(x, k^* v) + \varepsilon \quad \text{on } M.$$
Setting
$$c(x) = \begin{cases} \dfrac{f(x, u(x)) - f(x, k^* v(x))}{u(x) - k^* v(x)} & \text{if } k^* v(x) \ne u(x), \\ 0 & \text{if } k^* v(x) = u(x), \end{cases}$$
we see that $c \in L^\infty(M)$ since $f(x, \cdot)$ is locally Lipschitz. Moreover, w satisfies
$$Qw - c(x)w \ge \varepsilon.$$
We can therefore apply the WMP to w and deduce that $\sup_M w < 0$. Thus we can choose $\delta > 0$ so small that $w + \delta v \le 0$ on M, implying, by the definition of w, that $(k^* - \delta)v \ge u$, contradicting the very definition of k^*. □

Remark 3.4. As already pointed out, if also $\inf_M u > 0$, then we can relax the Lipschitz assumption to

$f \colon M \times \mathbb{R}^+ \to \mathbb{R}$ is locally Lipschitz in the variable $s \in \mathbb{R}^+$, uniformly in $x \in M$,

that is, f is not required to be Lipschitz in a neighbourhood of $s = 0$.

From Theorem 2.10 we immediately obtain the validity of the next:

Corollary 3.5. *Let Q be the linear differential operator defined in (2.2). Let $f \colon M \times \mathbb{R}^+ \to \mathbb{R}$ be locally Lipschitz in the variable s, uniformly in $x \in M$ and such that (2.9) holds. Assume the validity of the WMP for Q on M. Then the equation*
$$Qu = f(x, u)$$
has at most one C^2 solution u satisfying
$$C^{-1} \le u \le C \quad \text{on } M$$
for some constant $C > 0$.

Now Theorem 1.1 follows directly from the above corollary. We conclude with a remark.

Remark 3.6. We first stress that the lower bound $u \ge C^{-1}$ in (1.3) is indeed necessary for the validity of the Theorem. Indeed, suppose $c(x) = 0$ and consider the Yamabe equation

$$(3.18) \qquad \Delta u + \frac{m(m-2)}{2} u - u^{\frac{m+2}{m-2}} = 0, \quad m \ge 3$$

on $\mathbb{H}^m$, the m-dimensional hyperbolic space of constant sectional curvature -1. Here the assumptions of Theorem 1.1 are satisfied, but as shown in Section 5.2 of [16], having fixed an origin $o \in \mathbb{H}^m$ and set $r(x) = \text{dist}(x, o)$, the family of distinct functions

$$u_a(x) = \frac{1}{m(m-2)a^2} \Big(a^2 - \tanh^2 \frac{r(x)}{2} \Big)^{-(m-2)/2} \Big[2 \cosh^2 \frac{r(x)}{2} \Big]^{-(m-2)/2}, \quad a > 1,$$

is a family of solutions of (3.18) satisfying $u_a(x) \to 0$ as $r(x) \to \infty$.

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