Smooth torus actions are described by a single vector field

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Abstract. Consider a smooth effective action of a torus $\mathbb{T}^n$ on a connected C^∞-manifold M. Assume that M is not a torus endowed with the natural action. Then we prove that there exists a complete vector field X on M such that the automorphism group of X equals $\mathbb{T}^n \times \mathbb{R}$, where the factor $\mathbb{R}$ comes from the flow of X and $\mathbb{T}^n$ is regarded as a subgroup of $\mathrm{Diff}(M)$. Thus one may reconstruct the whole action of $\mathbb{T}^n$ from a single vector field.

1. Introduction

In a previous work [7], and related to the so called inverse Galois problem, we raised the question of whether or not a given effective group action on a manifold is determined, or "described", by non-classic tensors in general, or more specifically, by vector fields. More precisely: consider an effective action of a Lie group G on an m-manifold M, thus we can think of G as a subgroup of the group $\mathrm{Diff}(M)$ of diffeomorphisms of M. Given a vector field X on M, we say X is a *describing vector field* for the G-action if the following hold:

(1) X is complete and its flow Φ_t commutes with the action of G; so $G \leq \mathrm{Aut}(X)$.

(2) The group homomorphism

$$\begin{array}{ccc} G \times \mathbb{R} & \to & \mathrm{Aut}(X) \\ (g, t) & \mapsto & g \circ \Phi_t \end{array}$$

is an isomorphism.

Notice that we compare $\mathrm{Aut}(X)$ with $G \times \mathbb{R}$ instead of G since we always have to take into account the flow of X (see Remark 1.1).

Within this setting, the main result in [7] shows that any finite group action on a connected manifold admits a describing vector field. Here we give the analogous result for torus actions.

Theorem A. *Consider an effective action of the torus $\mathbb{T}^n$ on a connected m-manifold M. Assume that $n \leq m - 1$. Then there exist a describing vector field for this action.*

The proof of Theorem A involves three steps. First, the result is established for free actions (Section 3) and then extended to effective ones (Section 4), in both cases assuming $m - n \geq 2$. Finally the case $m - n = 1$ is considered in Section 5.

Remark 1.1. Notice that according to Proposition 7.1, if the $\mathbb{T}^n$-action on M^m is effective then $n \leq m$.

On the one hand, when $n = m$, we can make the identification $M = \mathbb{T}^n$ endowed with the natural $\mathbb{T}^n$-action. In this case, if X is the fundamental vector field associated to a dense affine vector field on $\mathbb{T}^n$ (see Section 2), then $\mathrm{Aut}(X) = \mathbb{T}^n$.

On the other hand, when $n < m$, no complete vector field X on M satisfies $\mathrm{Aut}(X) = \mathbb{T}^n$. Indeed, assume $\mathrm{Aut}(X) = \mathbb{T}^n$, let $X_1, \ldots, X_n$ be a basis of the Lie algebra of fundamental vector fields, and let $f_1, \ldots, f_n$ be any $\mathbb{T}^n$-invariant functions. Then $[X_r, \sum_{j=1}^n f_j X_j] = 0$, $r = 1, \ldots, n$; so the flow of $\sum_{j=1}^n f_j X_j$ commutes with the action of $\mathbb{T}^n$ and, as every element of the flow of X belongs to $\mathrm{Aut}(X)$, with this flow too. That is to say the flow of $\sum_{j=1}^n f_j X_j$ is included in $\mathrm{Aut}(X)$. But since $n < m$ some of the functions $f_1, \ldots, f_n$ can be taken non constant, which implies that the flow of $\sum_{j=1}^n f_j X_j$ is not included in $\mathbb{T}^n$, so $\mathrm{Aut}(X) \neq \mathbb{T}^n$, *contradiction.*

Thus our result is "minimal" because the flow of X is always included in $\mathrm{Aut}(X)$.

Remark 1.2. Finally, notice that Theorem A cannot be extended to a general compact Lie group. In Section 6 we construct effective actions of SO(3) (Example 6.3), and of a non-connected compact group of dimension two (Example 6.4), for which there is no describing vector field.

Terminology. The reader is supposed to be familiarized with our previous paper [7]. All structures and objects considered are real C^∞ and manifolds are without boundary, unless otherwise stated. For general questions on differential geometry the reader is referred to [3], and for those on differential topology to [2].

2. Preliminary results on vector fields

In this section we collect some results on vector fields that are needed in the following sections. On $\mathbb{R}^k$ we set coordinates $x = (x_1, \ldots, x_k)$, and define $\xi = \sum_{j=1}^k x_j \partial/\partial x_j$.

Lemma 2.1. *For any function $g \colon \mathbb{R}^k \to \mathbb{R}$ with $g(0) = 0$, there is a function $f \colon \mathbb{R}^k \to \mathbb{R}$ such that $\xi \cdot f = g$.*

Proof. Although the proof of this result is routine, we outline it here. Let $\mathcal{T}$ denote the Taylor's series of g at $0 \in \mathbb{R}^k$. Then there exists a series $\mathcal{S}$ such that formally $\xi \cdot \mathcal{S} = \mathcal{T}$. According to Borel's theorem (see Theorem 1.5.4 in [4]), there is a function φ whose Taylor's series at origin equals $\mathcal{S}$, and therefore making $g - \xi \cdot \varphi$ we may suppose $\mathcal{T} = 0$.

Since ξ is hyperbolic at the origin, following Theorem 10 in page 38 of [5], there exists a function $\tilde{f} \colon \mathbb{R}^k \to \mathbb{R}$ such that $g - \xi \cdot \tilde{f}$ vanishes on a open neighborhood A of $0 \in \mathbb{R}^k$. Therefore it suffices to show the result when $g_{|A} = 0$.

But $\mathbb{R}^k - \{0\}$ can be identified to $S^{k-1} \times \mathbb{R}$ in such a way that $\xi = \partial/\partial t$, where t is the variable in $\mathbb{R}$ and $S^{k-1} \times (-\infty, 1)$ corresponds to $B_\varepsilon(0) - \{0\} \subset A$ for some radius $\varepsilon > 0$.

Finally set $f = \int_0^t g \mathrm{d}s$ on $S^{k-1} \times \mathbb{R}$ and $f(0) = 0$. □

Recall that a vector field T on $\mathbb{T}^n$, with flat coordinates $\theta = (\theta_1, \ldots, \theta_n)$, is named *affine* if $T = \sum_{r=1}^n a_r \partial/\partial\theta_r$, where $a_1, \ldots, a_n \in \mathbb{R}$. The trajectories of an affine vector field are dense if and only if $a_1, \ldots, a_n$ are rationally independent; *in this case we say that T is dense.*

Lemma 2.2. *On $\mathbb{R}^k \times \mathbb{T}^n$ with coordinates $(x, \theta) = (x_1, \ldots, x_k, \theta_1, \ldots, \theta_n)$ consider the vector field $X = \xi + T$, where T is dense. Then $\mathcal{L}_X$, the set of all vector fields on $\mathbb{R}^k \times \mathbb{T}^n$ which commute with X, is a Lie algebra of dimension $k^2 + n$ with basis*

$$\left\{ x_j \frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial \theta_r} \right\}, \quad j, \ell = 1, \ldots, k; \, r = 1, \ldots, n.$$

Proof. Let $Y \in \mathcal{L}_X$ be the vector field defined as $Y = \sum_{j=1}^k f_j(x, \theta)\partial/\partial x_j + \sum_{r=1}^n g_r(x, \theta)\partial/\partial\theta_r$, and let Φ_t denote its flow. Since $\{0\} \times \mathbb{T}^n$ is compact, Φ_t is defined on $(\{0\} \times \mathbb{T}^n) \times (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$.

Observe that $\{0\} \times \mathbb{T}^n$ is the set of all points whose X-trajectory has compact adherence. Therefore $\Phi_t(\{0\} \times \mathbb{T}^n) \subset \{0\} \times \mathbb{T}^n$ for any $t \in (-\varepsilon, \varepsilon)$, that implies that Y is tangent to $\{0\} \times \mathbb{T}^n$, and therefore $f_j(0, \theta) = 0$, for $j = 1, \ldots, k$.

A computation, taking into account that every $\partial/\partial\theta_r$ commutes with X, yields

$$[X, Y] = \widetilde{Y} + \sum_{r=1}^n (X \cdot g_r)\, \partial/\partial\theta_r \,,$$

where $\widetilde{Y}$ is a functional combination of $\partial/\partial x_i$'s. Therefore $X \cdot g_r = 0$, i.e., g_r is constant along the trajectories of X, for $r = 1, \ldots, n$. But the ω-limit of these trajectories is $\{0\} \times \mathbb{T}^n$ and g_r is constant on this set because $\{0\} \times \mathbb{T}^n$ is the adherence of the trajectory of any of its points, so each g_r is constant on $\mathbb{R}^k \times \mathbb{T}^n$.

By considering $Y - \sum_{r=1}^n g_r \partial/\partial\theta_r$ one may suppose $Y = \sum_{j=1}^k f_j(x, \theta)\partial/\partial x_j$, where each $f_j(\{0\} \times \mathbb{T}^n) = 0$. Now from $[X, Y] = 0$ follows $X \cdot f_j = f_j$, $j = 1, \ldots, k$.

On the other hand given $f \colon \mathbb{R}^k \times \mathbb{T}^n \to \mathbb{R}$ such that $f(\{0\} \times \mathbb{T}^n) = 0$ and $X \cdot f = f$, then f does not depend on θ and it is linear on x. Indeed if $k = 0$ it is obvious; assume $k = 1$ by the moment. Then $f = xg$ for some function g since f vanishes on $\{0\} \times \mathbb{T}^n$, and $X \cdot f = f$ becomes $X \cdot g = 0$. Therefore g is constant along the trajectories of X and, by the same reason as before, constant on $\mathbb{R} \times \mathbb{T}^n$.

Now suppose $k \geq 2$. Let E be any vector line in $\mathbb{R}^k$. As X is tangent to $E \times \mathbb{T}^n$ and the restriction of ξ to E is still the radial vector field, $f \colon E \times \mathbb{T}^n \to \mathbb{R}$ is independent of θ and linear on E (it is just the case $k = 1$). Since the union of all the vector lines E equals $\mathbb{R}^k$, it follows that f does not depend on θ and $f \colon \mathbb{R}^k \to \mathbb{R}$ is linear on each E. But, as it is well known, this last property implies that $f \colon \mathbb{R}^k \to \mathbb{R}$ is linear.

In short, every f_j is linear on x and independent of θ.   □

Corollary 2.3. *If* $F \colon \mathbb{R}^k \times \mathbb{T}^n \to \mathbb{R}^k \times \mathbb{T}^n$ *is an automorphism of* $X = \xi + T$ *and* T *is dense, then there exist an isomorphism* $\varphi \colon \mathbb{R}^k \to \mathbb{R}^k$, *and an element* $\lambda \in \mathbb{T}^n$, *such that* $F(x, \theta) = (\varphi(x), \theta + \lambda)$.

Proof. The diffeomorphism F induces an isomorphism on $\mathcal{L}_X$, the Lie algebra described in Lemma 2.2. Now observe that:

1) The only elements in $\mathcal{L}_X$ with singularities are elements $\sum_{j,\ell=1}^{k} a_{j\ell} x_j \partial/\partial x_\ell$, where $(a_{j\ell}) \in \mathrm{GL}(n, \mathbb{R})$. Besides they give rise to the foliation $d\theta_1 = \cdots = d\theta_n = 0$ (extend it to $\{0\} \times \mathbb{T}^n$ by continuity); so this foliation is an invariant of F.

2) The center of $\mathcal{L}_X$ is spanned by $\xi, \partial/\partial\theta_1, \ldots, \partial/\partial\theta_r$. But the adherences of the trajectories of a vector field $b\xi + b_1 \partial/\partial\theta_1 + \cdots + b_r \partial/\partial\theta_r$ are always tori if and only if $b = 0$, so F sends the Lie subalgebra spanned by $\partial/\partial\theta_1, \ldots, \partial/\partial\theta_r$ into itself.

These two facts imply that $F(x, \theta) = (\varphi(x), \psi(\theta))$, where $\psi \colon \mathbb{T}^n \to \mathbb{T}^n$ is an affine transformation of $\mathbb{T}^n$; that is,

$$\psi(\theta) = \Big(\sum_{j=1}^{n} c_{1j}\theta_j, \ldots, \sum_{j=1}^{n} c_{nj}\theta_j \Big) + \lambda \,,$$

where $(c_{\ell j}) \in \mathrm{GL}(n, \mathbb{Z})$ and $\lambda \in \mathbb{T}^n$. As $X = T$ on $\{0\} \times \mathbb{T}^n$, which is an invariant of F, it follows that $(a_1, \ldots, a_n)$ is an eigenvector of $(c_{\ell j})$ whose eigenvalue equals 1. But in this case $a_1, \ldots, a_n$ are rationally dependent unless $(c_{\ell j}) = \mathrm{Id}$. In short, $\psi(\theta) = \theta + \lambda$.

On the other hand, $\varphi \colon \mathbb{R}^k \to \mathbb{R}^k$ has to be an automorphism of ξ, which implies that φ is an isomorphism by Lemma 3.4 in [7].   □

Lemma 2.4. *On* $\mathbb{R}^k \times \mathbb{T}^n$, *consider the vector field* $\widetilde{X} = \tilde{\xi} + \widetilde{T}$, *where* $\tilde{\xi} = \sum_{j=1}^{k} \tilde{f}_j(x)\partial/\partial x_j$ *and* $\widetilde{T} = \sum_{i=1}^{n} \tilde{g}_r(x)\partial/\partial\theta_r$. *Assume that on* $\mathbb{R}^k$ *there hold:*

(a) $\tilde{\xi}$ *is complete,*

(b) $\tilde{\xi}(0) = 0$ *and its linear part at the origin is a positive multiple of identity,*

(c) *the outset of the origin equals* $\mathbb{R}^k$.

Then there exists a self-diffeomorphism of $\mathbb{R}^k \times \mathbb{T}^n$ *that commutes with the natural* $\mathbb{T}^n$-*action and transforms* $\widetilde{X}$ *into*

$$b\xi + \sum_{r=1}^{n} b_r \frac{\partial}{\partial\theta_r} \,,$$

where $b \in \mathbb{R}^+$ *and* $b_1, \ldots, b_n \in \mathbb{R}$.

Proof. By Sternberg's linearization theorem [6], see Proposition 2.1 in [7], there exists a diffeomorphism $f\colon \mathbb{R}^k \to \mathbb{R}^k$ transforming $\tilde{\xi}$ into $b\sum_{j=1}^{k} x_j\partial/\partial x_j$, with $b > 0$. Dividing by b we may suppose $\tilde{\xi} = \xi$.

By Lemma 2.1 there are functions $\varphi_1, \ldots, \varphi_n\colon \mathbb{R}^k \to \mathbb{R}$ such that $\xi \cdot \varphi_r = g_r - g_r(0)$, $r = 0, \ldots, n$. Now, if $\varphi = (\varphi_1, \ldots, \varphi_n)$ and $\tilde{\pi}\colon \mathbb{R}^n \to \mathbb{T}^n$ is the canonical covering, then the diffeomorphism $F\colon \mathbb{R}^k \times \mathbb{T}^n \to \mathbb{R}^k \times \mathbb{T}^n$ given by $F(x,\theta) = (x, \theta - \tilde{\pi} \circ \varphi)$ transforms $\widetilde{X}$ into $\xi + \sum_{r=1}^{n} g_r(0)\partial/\partial\theta_r$.   □

Remark 2.5. If $\widetilde{X}$ matches the hypotheses of Lemma 2.4 and $h\colon \mathbb{R}^k \to \mathbb{R}$ is a positive bounded function, then $h\widetilde{X}$ satisfies these hypotheses too (when h is regarded as a function on $\mathbb{R}^k \times \mathbb{T}^n$ in the obvious way). Therefore $h\widetilde{X}$ can be written as in Lemma 2.4 for a suitable choice of coordinates, and thus the control of its automorphisms becomes simple.

3. Free actions

In this section we assume there is a free $\mathbb{T}^n$-action on the connected m-manifold M, where $m - n \geq 2$. This gives rise to a principal fibre bundle $\pi\colon M \to B$ whose structure group is $\mathbb{T}^n$ and B is a connected manifold of dimension $k = m - n \geq 2$. Then, we construct a suitable vector field on B that is later on lifted to M by means of a connection.

In order to construct the vector field on B, we closely follow along the lines in Section 3 of [7] applied to the case of the trivial group action on B.

Consider a Morse function $\mu\colon B \to \mathbb{R}$ that is proper and non-negative. Denote by C the set of its critical points, which is closed and discrete, and therefore countable. As B is paracompact, there exists a locally finite family $\{A_p\}_{p\in C}$ of disjoint open sets such that $p \in A_p$, $p \in C$.

Following along the lines in Section 3 of [7], there exists a Riemannian metric $\tilde{g}$ on B such that the gradient vector field Y of μ is complete and, besides, around each $p \in C$ there are coordinates $(x_1, \ldots, x_k)$ with $p \equiv 0$ and $Y = \sum_{j=1}^{k} \lambda_j x_j \partial/\partial x_j$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R} - \{0\}$, where

(1) $\lambda_1 = \cdots = \lambda_k > 0$ if p is a source of Y, that is a minimum of μ,

(2) $\lambda_1 = \cdots = \lambda_k < 0$ if p is a sink of Y (a maximum of μ),

(3) some λ_j are positive and the others are negative if p is a saddle.

Note that these properties still hold when Y is multiplied by a positive bounded function, since they only depend on the Sternberg theorem.

Let I be the set of local minima of μ, that is the set of sources of Y, and S_i, $i \in I$, the outset of i relative to Y. Now Lemma 3.3 of [7] becomes:

Lemma 3.1. *The family $\{S_i\}_{i\in I}$ is locally finite and the set $\bigcup_{i\in I} S_i$ is dense in B.*

In what follows, and by technical reasons, we make use of the notion of *order of nullity* instead of *chain*. More precisely, for every $i \in I$ we choose a subset P_i of A_i

with $k + 1$ points close enough to i but different from it, in such a way that the linear α-limits of their trajectories are in general position (see pages 319 and 320 in [7] for definitions).

Set $P = \bigcup_{i \in I} P_i$. Consider an injective map $p \in P \mapsto n_p \in \mathbb{N} - \{0\}$. Let $\mathbb{N}'$ be the image of P.

By definition, a differentiable object has *order r at point q* if its $(r - 1)$-jet at this point vanishes but its r-jet does not; for instance Y has order one at sources, sinks and saddles.

Since $\{A_i\}_{i \in I}$ is still locally finite, one may construct a bounded function $\tau \colon B \to \mathbb{R}$ such that τ is positive on $B - P$ and has order $2n_p$ at every $p \in P$.

Put $Z = \tau Y$. Then $Z^{-1}(0) = Y^{-1}(0) \cup P$; that is, the singularities of Z are the sources, sinks and saddles of Y plus the points of P, that we call *artificial singularities*, and whose order is even ≥ 2. Note that two different artificial singularities have different orders.

Let R_i, $i \in I$, be the Z-outset of i. As $S_i - R_i$ is the union of $k + 1$ half-trajectories of Y, one has the following.

Lemma 3.2. *The family $\{R_i\}_{i \in I}$ is locally finite and the set $\bigcup_{i \in I} R_i$ is dense in B.*

On the principal fibre bundle $\pi \colon M \to B$ consider a connection $\mathcal{C}$ which is a product around every fibre $\pi^{-1}(p)$, $p \in C$ (i.e., there exist an open set $p \in A \subset B$ and a fiber bundle isomorphism between $\pi \colon \pi^{-1}(A) \to A$ and $\pi_1 \colon A \times \mathbb{T}^n \to A$ in such a way that $\mathcal{C}$, regarded on $\pi_1 \colon A \times \mathbb{T}^n \to A$, is given by $\mathcal{C}(q, \theta) = T_q A \times \{0\} \subset T_{(q,\theta)}(A \times \mathbb{T}^n))$. This kind of connection always exists because $\{A_p\}_{p \in C}$ is locally finite.

Let Y' denote the lift of Y to M by means of $\mathcal{C}$; that is, $Y'(u) \in \mathcal{C}(u)$ and $\pi_*(Y'(u)) = Y(\pi(u))$ for every $u \in M$. By construction, Y' is $\mathbb{T}^n$-invariant and $Y'(u) = 0$ if and only if $Y(\pi(u)) = 0$.

Let T be a dense affine vector field on $\mathbb{T}^n$, and let T' be the fundamental vector field, on M, associated to T through the action. As describing vector field we take $X' = (\tau \circ \pi)(Y' + T')$, which clearly is $\mathbb{T}^n$-invariant and complete.

The rest of this section is devoted to show that X' is a describing vector field. First we study the behavior of X' near some fibres. If p is a source of Y, there exist coordinates $(x_1, \ldots, x_k)$, about $p \in B$, with $p \equiv 0$ and $Y = a \sum_{j=1}^{k} x_j \partial/\partial x_j$, $a > 0$. As around $\pi^{-1}(p)$ the connection is a product, these coordinates can be prolonged to a system of coordinates (x, θ) on a product open set $A \times \mathbb{T}^n$, with the obvious identifications, while $\mathcal{C}$ is given by the first factor. In this case,

$$Y' + T' = a \sum_{j=1}^{k} x_j \frac{\partial}{\partial x_j} + T \,,$$

since T' is just T regarded as a vector field on $A \times \mathbb{T}^n$.

The same happens when p is a sink but $a < 0$. If p is a saddle then the model of the first part is $\sum_{j=1}^{k} \lambda_j x_j \partial/\partial x_j$, with some λ_j positive and the others negative.

Thus, when $p \in C$, the torus $\pi^{-1}(p)$ is the adherence of a trajectory of X', this vector field never vanishes on $\pi^{-1}(p)$ and, besides,

(a) if $p \in I$, then $\pi^{-1}(p)$ is the α-limit of some external trajectories but never the ω-limit;

(b) if p is a sink, then $\pi^{-1}(p)$ is the ω-limit of some external trajectories but never the α-limit;

(c) if p is a saddle, then $\pi^{-1}(p)$ is the α-limit of some external trajectories and the ω-limit of other ones.

On the other hand, $(X')^{-1}(0) = \pi^{-1}(P)$. Moreover, if $p \in P$, then X' has order $2n_p$ at each point of $\pi^{-1}(p)$.

If X' is multiplied by a positive, bounded and $\mathbb{T}^n$-invariant function $\rho \colon M \to \mathbb{R}$, then $\rho = \tilde{\rho} \circ \pi$ for some positive and bounded function $\tilde{\rho} \colon B \to \mathbb{R}$. Thus $\rho X' = ((\tilde{\rho}\tau) \circ \pi)(Y' + T')$. As τ and $\tilde{\rho}\tau$ have the same essential properties, the foregoing description still holds for $\rho X'$. In other words, this description is geometric and independent of how trajectories are parameterized.

Consider $i \in I$ and identify its Y-outset S_i to $\mathbb{R}^k$ in such a way that $i \equiv 0$ and $Y = a\sum_{\ell=1}^k x_\ell \partial/\partial x_\ell$, $a > 0$. As S_i is contractible, the fibre bundle $\pi \colon \pi^{-1}(S_i) \to S_i$ is trivial; so it can be regarded as $\pi_1 \colon \mathbb{R}^k \times \mathbb{T}^n \to \mathbb{R}^k$ while

$$Y' + T' = a\sum_{\ell=1}^k x_\ell \frac{\partial}{\partial x_\ell} + \sum_{r=1}^n g_r(x)\frac{\partial}{\partial \theta_r}.$$

Finally Lemma 2.4 allows us to suppose

$$Y' + T' = a\sum_{\ell=1}^k x_\ell \frac{\partial}{\partial x_\ell} + \sum_{r=1}^n a_r \frac{\partial}{\partial \theta_r},$$

with $a > 0$ and $a_1, \ldots, a_n \in \mathbb{R}$. Moreover, since $T' = \sum_{r=1}^n a_r \partial/\partial\theta_r$ at any point of $\{0\} \times \mathbb{T}^n$, scalars $a_1, \ldots, a_n$ are rationally independent (recall that $Y'(u) = 0$ whenever $Y(\pi(u)) = 0$).

Now is clear that, for every $p \in P_i$ and $u \in \pi^{-1}(p)$, there exists a trajectory of $X' = (\tau \circ \pi)(Y' + T')$ whose α-limit and ω-limit are $\pi^{-1}(i)$ and u respectively. As $n_p \neq n_{p'}$ when $p \neq p'$, the existence of this kind of trajectories shows that any automorphism of X' has to send $\pi^{-1}(i)$ in itself and, consequently, the X'-outset of $\pi^{-1}(i)$ in itself too (here outset means the set of points whose trajectory has its α-limit included in $\pi^{-1}(i)$).

The next question is to determine this outset. First we identify the outset R_i, of i with respect to $Z = \tau Y$, to $\mathbb{R}^k$ in such a way that $i \equiv 0$ and $Z = b\sum_{\ell=1}^k x_\ell \partial/\partial x_\ell$, $b > 0$. Again $\pi \colon \pi^{-1}(R_i) \to R_i$ is trivial and reasoning as before, taking into a account that X' is complete on $\pi^{-1}(R_i)$, which allows to apply Lemma 2.4, leads us to the case $\pi^{-1}(R_i) \equiv \mathbb{R}^k \times \mathbb{T}^n \to \mathbb{R}^k \equiv R_i$ and

$$X' = b\sum_{\ell=1}^k x_\ell \frac{\partial}{\partial x_\ell} + \sum_{r=1}^n b_r \frac{\partial}{\partial \theta_r}$$

where $b > 0$ and $b_1, \ldots, b_n \in \mathbb{R}$ are rationally independent. Therefore the X'-outset of $\pi^{-1}(i)$ equals $\pi^{-1}(R_i)$.

Let $F\colon M \to M$ be an automorphism of X'. Then $F\colon \pi^{-1}(R_i) \to \pi^{-1}(R_i)$ is a diffeomorphism. As the trajectories of $\sum_{r=1}^{n} b_r \partial/\partial\theta_r$ are dense, from Corollary 2.3 it follows that $F(x, \theta) = (\varphi(x), \theta + \lambda)$, where $\lambda \in \mathbb{T}^n$ and $\varphi\colon \mathbb{R}^k \to \mathbb{R}^k$ is an isomorphism.

For any $p \in P_i$ some trajectories of X' with α-limit $\pi^{-1}(i)$ have, as ω-limit, a point of $\pi^{-1}(p)$, that is a singularity of order $2n_p$. Since $n_p \neq n_{p'}$ when $p \neq p'$, the set of these trajectories has to be an invariant of F. Regarded on $R_i \subset B$, and taking into account that X' projects in Z, this fact implies that φ has to map the trajectory of $Z = b \sum_{\ell=1}^{k} x_\ell \partial/\partial x_\ell$ of α-limit i and ω-limit p into itself. Thus the direction vector of this curve is an eigenvector of φ with positive eigenvalue. But there are $k + 1$ eigenvector (as many as points in P_i) and they are in general position, so φ is a positive multiple of identity.

Therefore there exists t_i such that $\Phi_{t_i}(x, \theta) = (\varphi(x), \theta + \tilde{\lambda}_i)$, where Φ_t denotes the flow of X' and $\tilde{\lambda}_i \in \mathbb{T}^n$. Since Φ_t and the action of $\mathbb{T}^n$ commute, that shows the existence of $t_i \in \mathbb{R}$ and $\lambda_i \in \mathbb{T}^n$ such that $F = \lambda_i \circ \Phi_{t_i}$ on $\pi^{-1}(R_i)$.

It is easily seen that the family $\{\pi^{-1}(R_i)\}_{i \in I}$ is locally finite and the set $\bigcup_{i \in I} \pi^{-1}(R_i)$ is dense in M.

On each $\pi^{-1}(R_i)$ the actions of $\mathbb{T}^n$ and F commute, and hence so do they on M. Thus F induces a diffeomorphism $f\colon B \to B$ such that $f \circ \pi = \pi \circ F$. Besides, since Z is the projection of X', our f is an automorphism of Z. Now from the expression of $F\colon \pi^{-1}(R_i) \to \pi^{-1}(R_i)$ it follows that $f = \varphi_{t_i}$ on R_i, $i \in I$, where φ_t is the flow of Z.

Lemma 3.3. *All t_i's are equal and $f = \varphi_t$ for some $t \in \mathbb{R}$.*

Proof. Notice that X has no regular periodic trajectories (here dimension of $B \geq 2$ is needed). Then, the lemma follows from the proof of Lemma 3.7 in [7], when G is the trivial group and X becomes Z. $\square$

Therefore composing F with Φ_{-t} allows to suppose that f is the identity and $F = \lambda_i$ on each $\pi^{-1}(R_i)$, $i \in I$.

Consider a $\mathbb{T}^n$-invariant Riemannian metric on M. Then F is an isometry on every $\pi^{-1}(R_i)$ so, by continuity, on M. Take $i_0 \in I$; then the isometries F and λ_{i_0} agree on $\pi^{-1}(R_{i_0})$. But on connected manifolds, isometries are determined by their 1-jet at any point. Therefore $F = \lambda_{i_0}$ on M. In other words $(\lambda, t) \in \mathbb{T}^n \times \mathbb{R} \to \lambda \circ \Phi_t \in \mathrm{Aut}(X)$ is an epimorphism.

We now prove injectivity. Assume $\lambda \circ \Phi_t = \mathrm{Id}$, that is $\Phi_t = (-\lambda)$. Then $\varphi_t\colon B \to B$ equals the identity because $(-\lambda)$ induces this map on B. Since Z has no periodic regular trajectories, this implies $t = 0$ and, finally, $\lambda = 0$ because the action of $\mathbb{T}^n$ is effective. In short, X' *is a describing vector field for free actions.*

Remark 3.4. Note that if $\rho\colon M \to \mathbb{R}$ is a $\mathbb{T}^n$-invariant, positive and bounded function, then $\rho X'$ is a describing vector field too. Indeed, there exists a function $\tilde{\rho}\colon B \to \mathbb{R}$ such that $\rho = \tilde{\rho} \circ \pi$ and it suffices reasoning with $\tilde{\rho}\tau$ instead of τ.

4. Effective actions

Throughout this section we assume that $m - n \geq 2$ and the $\mathbb{T}^n$-action is effective. Our next goal is to construct a describing vector field on M. Let S be the set of those points of M whose isotropy (stabilizer) is non trivial. By Proposition 7.1, the set $M - S$ is dense, open, connected and $\mathbb{T}^n$-invariant. Moreover, the $\mathbb{T}^n$-action on $M - S$ is free, so we can consider on this set a describing vector field X' as in Section 3. Let φ be a function like in Proposition 7.2 for X' and $M - S$, and $\widehat{X}'$ be the vector field on M given by $\varphi X'$ on $M - S$ and zero on S.

Set $\psi = h \circ \varphi$, where $h \colon \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = 0$ if $t \leq 0$ and $h(t) = \exp(-1/t)$ if $t > 0$. The function ψ, which is $\mathbb{T}^n$-invariant, vanishes at order infinity at every point of S, i.e., all its r-jets vanish. Besides, it is bounded on M and positive on $M - S$.

Now put $\widetilde{X} = \psi \widehat{X}'$. Clearly $\widetilde{X}$ vanishes at order infinity at u if and only if $u \in S$; therefore S is an invariant of $\widetilde{X}$. Moreover, $\widetilde{X}$ is complete on M and $M - S$ respectively. By Remark 3.4, our $\widetilde{X}$ is a describing vector field on $M - S$ since $\widetilde{X} = (\psi\varphi)X'$ on this set.

If $F \colon M \to M$ is an automorphism of $\widetilde{X}$, then $F(S) = S$, and $F \colon M - S \to M - S$ is an automorphism of $\widetilde{X}$; so on $M - S$ one has $F = \lambda \circ \widetilde{\Phi}_t$, where $\widetilde{\Phi}_t$ is the flow of $\widetilde{X}$. By continuity, $F = \lambda \circ \widetilde{\Phi}_t$ everywhere, which implies that $\widetilde{X}$ is a describing vector field on M (the homomorphism injectivity is inherited from $M - S$).

5. The case $m - n = 1$

First assume the action is free, which gives rise to a principal fibre bundle $\pi \colon M \to B$ with B connected and of dimension one. Therefore B is $\mathbb{R}$ or S^1 and $\pi \colon M \to B$ can be identified to $\pi_1 \colon B \times \mathbb{T}^n \to B$. One will need the following result, whose proof is routine.

Lemma 5.1. *On a open set $0 \in A \subset \mathbb{R}$ consider a vector field X such that its $(r-1)$-jet at the origin vanishes but its r-jet does not, $r \geq 1$. Let φ_t be the flow of X. Given $f \colon A \to \mathbb{R}$ and $t_1, t_2 \in \mathbb{R}$, if $f = \varphi_{t_1}$ on $A \cap (0, \infty)$ and $f = \varphi_{t_2}$ on $A \cap (-\infty, 0)$, then $t_1 = t_2$.*

Set $Y = q(q^2 + 1)^{-1}\partial/\partial x$, where $q = x(x-1)(x-2)(x-3)(x-4)$ when $B = \mathbb{R}$, and $Y = \sin(3\alpha)\partial/\partial\alpha$ if $B = S^1$ endowed with the angular coordinate α. Clearly Y is complete. In the first case the sources are $0, 2, 4$ and the sinks $1, 3$, and in the second one $0, 2\pi/3, 4\pi/3$ and $\pi/3, \pi, 5\pi/3$, respectively.

When $\dim B \geq 2$ we have created new singularities called artificial. Now we increase the order of sinks instead (otherwise the non-singular set has too many components). Let $\tau \colon B \to \mathbb{R}$ be a bounded function which is positive outside sinks and has order two at 1 and $\pi/3$, order four at 3 and π and, finally, order six at $5\pi/3$. Set $Z = \tau Y$, which is a complete vector field.

This time the Z-outsets $\{R_i\}_{i \in I}$, where $I = \{0, 2\pi/3, 4\pi/3\}$ if $B = S^1$, or $I = \{0, 2, 4\}$ if $B = \mathbb{R}$, equal those of Y, and any of them is an invariant of Z

because the ω-limits of its trajectories have different orders or are empty; even more, every side of the outset has to be preserved.

On $M = B \times \mathbb{T}^n$ with coordinates (x, θ) or (α, θ), set $X' = \tau(Y + T)$, where T is a dense affine vector field and τ, Y, T are regarded on $M = B \times \mathbb{T}^n$ in the natural way. Now $(X')^{-1}(0) = (B - \bigcup_{i \in I} R_i) \times \mathbb{T}^n$. Therefore $(X')^{-1}(0)$ is the union of two $(B = \mathbb{R})$ or three $(B = S^1)$ fibres; moreover, points of different fibres have different orders.

Let $F \colon M \to M$ be an automorphism of X'; reasoning as in the case $\dim B \geq 2$ shows that $F = \lambda_i \circ \Phi_{t_i}$ on $R_i \times \mathbb{T}^n$ for each $i \in I$, where Φ_t is the flow of X'. Thus the induced automorphism $f \colon B \to B$ of Z equals φ_{t_i} on R_i, where φ_t is the flow of Z. Now Lemma 5.1 implies that $t_i = t_j$ if R_i and R_j are contiguous. In short, $f = \varphi_t$ for some $t \in \mathbb{R}$. The rest of the proof is similar to that of $\dim B \geq 2$.

Finally notice that $\rho X'$ is a describing vector field too if $\rho \colon M \to \mathbb{R}$ is $\mathbb{T}^n$-invariant, positive and bounded; therefore the result can be extended to the case of an effective action following the lines in Section 4.

6. Examples

We start this section by giving two examples of effective torus actions. The first one is a general construction on the Lie groups. In the second example a describing vector field is constructed for the usual action of $\mathbb{T}^3$ on S^5.

On the other hand, two examples more show that the main theorem fails for general compact Lie groups. More precisely, for effective actions of SO(3) (Example 6.3) and for effective actions of a non-connected compact group, of dimension two, with abelian Lie algebra (Example 6.4).

Example 6.1. Let G be a connected Lie group, with center ZG, and two (non necessary maximal) tori $H, \widetilde{H} \leq G$. Then there exists a $(H \times \widetilde{H})$-action on G given by $(h, \tilde{h}) \cdot g = hg\tilde{h}^{-1}$, whose kernel K equals $\{(h, h) \mid h \in H \cap \widetilde{H} \cap ZG\}$. Thus an effective action of the torus $(H \times \widetilde{H})/K$ on G is induced.

Now suppose that G is compact with rank r and ZG finite, that is the center of the Lie algebra of G is zero. Let H be a maximal torus of G; set $\widetilde{H} = H$. Then one obtains an effective action of $\mathbb{T}^{2r} \equiv (H \times H)/K$ on G. Moreover the isotropy group of any $g \in G$ has two or more elements if and only if $(gHg^{-1}) \cap H$ is not included in ZG; by Proposition 7.1 this happens for almost no $g \in G$.

Example 6.2. On $S^5 = \{y \in \mathbb{R}^6 \mid y_1^2 + \cdots + y_6^2 = 1\}$ consider the action of $\mathbb{T}^3$ given by the fundamental vector fields $U_j = -y_{2j}\partial/\partial y_{2j-1} + y_{2j-1}\partial/\partial y_{2j}$, $j = 1, 2, 3$. In order to construct a describing vector field for this action we follow along the lines of Sections 3 and 4 up to some minor changes. First observe that the singular set for this action is $S = \{y \in S^5 \mid (y_1^2 + y_2^2)(y_3^2 + y_4^2)(y_5^2 + y_6^2) = 0\}$, so the action of $\mathbb{T}^3$ on $S^5 - S$ is free.

Let $\pi \colon S^5 \to \mathbb{R}^2$ be the map given by $\pi(y) = (y_1^2 + y_2^2, y_3^2 + y_4^2)$, and B be the interior of the triangle of vertices $(0,0), (1,0), (0,1)$. Then $\pi \colon S^5 - S \to \mathbb{B}$ is the $\mathbb{T}^3$-principal bundle associated to the action of $\mathbb{T}^3$. A connection $\mathcal{C}$ for

this principal bundle is defined by $\operatorname{Ker}(\alpha_1 \wedge \alpha_2 \wedge \alpha_3)$, where each $\alpha_j = (y_{2j-1}^2 + y_{2j}^2)^{-1}(-y_{2j}dy_{2j-1} + y_{2j-1}dy_{2j})$, which is flat since $d\alpha_1 = d\alpha_2 = d\alpha_3 = 0$.

The vector fields

$$V_r = (y_5^2 + y_6^2)\Big(y_{2r-1}\frac{\partial}{\partial y_{2r-1}} + y_{2r}\frac{\partial}{\partial y_{2r}}\Big) - (y_{2r-1}^2 + y_{2r}^2)\Big(y_5\frac{\partial}{\partial y_5} + y_6\frac{\partial}{\partial y_6}\Big),$$

$r = 1, 2$, are tangent to $\mathcal{C}$ and project in

$$2(1 - x_1 - x_2)\frac{\partial}{\partial x_r},$$

$r = 1, 2$, on B endowed with coordinates $x = (x_1, x_2)$.

Set

$$Y = 2(1 - x_1 - x_2)x_1 x_2 \Big[\Big(x_1 - \frac{1}{4}\Big)\frac{\partial}{\partial x_1} + \Big(x_2 - \frac{1}{4}\Big)\frac{\partial}{\partial x_2}\Big],$$

whose lifted vector field through $\mathcal{C}$ is

$$Y' = (y_1^2 + y_2^2)(y_3^2 + y_4^2)\left((y_1^2 + y_2^2 - 1/4)V_1 + (y_3^2 + y_4^2 - 1/4)V_2\right).$$

Note that Y' extends in a natural way to S^5. Moreover, Y' vanishes on S.

Clearly Y on B and Y' on S^5 and $S^5 - S$ are complete. Besides, $(1/4, 1/4)$ is the only source of Y.

Let $\tau\colon \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$\tau(x) = \rho(x)\left((x_1 - 1/8)^2 + (x_2 - 1/8)^2\right)$$
$$\left((x_1 - 1/8)^2 + (x_2 - 1/4)^2\right)^2 \left((x_1 - 1/4)^2 + (x_2 - 1/8)^2\right)^3,$$

where $\rho(x) = x_1^{10}x_2^{10}(1 - x_1 - x_2)^{10}$, whose zeros on in B are $(1/8, 1/8)$ with order two, $(1/8, 1/4)$ with order four and $(1/4, 1/8)$ with order six.

Now $X' = (\tau \circ \pi)(Y' + U_1 + eU_2 + e^2 U_3)$, defined on S^5, is a describing vector field for the action of $\mathbb{T}^3$.

Indeed, the only difference with respect to the construction of Sections 3 and 4 is that every point of S is a singularity of X' with order ≥ 10 instead of infinity, but this is not important because the order of the remaining singularities of X' is always ≤ 6.

Example 6.3. Let H be a closed subgroup of a connected Lie group G and G/H be the (quotient) homogeneous space associated to the equivalence relation $g_1 \mathcal{R} g_2$ if and only of $g_2 = g_1 h$ for some $h \in H$. As it is well known, G acts on G/H by setting $g \cdot \overline{g'} = \overline{gg'}$.

Now assume H discrete; then the canonical projection $\pi\colon G \to G/H$ is a covering. Moreover, a vector field V on G/H commutes with the action of G, that is to say with every fundamental vector field, if and only if its lifted vector field V', on G, is at the same time left G-invariant and right H-invariant.

If one supposes that V' is left G-invariant this property is equivalent to say that $V'(e)$, where e is the identity of G, is invariant by (the adjoint action of) H. Therefore if no element of $T_e G - \{0\}$ is invariant by H, then any vector field on G/H which commutes with the action of G vanishes identically.

Set $G = \mathrm{SO}(3)$ and let H be the subgroup of order four consisting of the identity plus the three rotations of angle π around any of the coordinates axes (i.e., H is the Klein four-group). Then no element of $T_e\mathrm{SO}(3) - \{0\}$ is invariant by H.

Consider on $M = \mathbb{R}^k \times (\mathrm{SO}(3)/H)$, $k \geq 1$, the action of $\mathrm{SO}(3)$ given by $g \cdot (x, \overline{g'}) = (x, \overline{gg'})$. *This action is effective but it does not have any describing vector field.* Indeed, take a vector field U on M which commutes with the action of $\mathrm{SO}(3)$. Then U has to respect the foliation defined by the orbits of $\mathrm{SO}(3)$, so $U = \sum_{j=1}^{k} f_j(x)\partial/\partial x_j + V$, where V is a vector field tangent to the second factor and $x = (x_1, \ldots, x_k)$ are the canonical coordinates in $\mathbb{R}^k$.

Since U and $\sum_{j=1}^{k} f_j(x)\partial/\partial x_j$ commute with the action, V has to commute with the fundamental vector fields on each orbit, hence $V = 0$. In other words, $U = \sum_{j=1}^{k} f_j(x)\partial/\partial x_j$.

On the other hand, if $\varphi \colon \mathrm{SO}(3)/H \to \mathrm{SO}(3)/H$ is a diffeomorphism, then

$$F \colon \mathbb{R}^k \times (\mathrm{SO}(3)/H) \to \mathbb{R}^k \times (\mathrm{SO}(3)/H)$$

given by $F(x, \overline{g}) = (x, \varphi(\overline{g}))$ is an automorphism of U, so $\mathrm{Aut}(U)$ is strictly greater than $\mathrm{SO}(3) \times \mathbb{R}$.

Another possibility is to consider the action of $\mathrm{SO}(3)$ on the sphere S^2. Then for each $p \in S^2$ there exists a fundamental vector field X such that p is an isolated singularity of X. Therefore if V commutes with X then $V(p) = 0$; consequently, if V commutes with the action of $\mathrm{SO}(3)$ on S^2 necessarily $V = 0$.

By the same reason as before, the action of $\mathrm{SO}(3)$ on $\mathbb{R}^k \times S^2$, $k \geq 1$, defined by $g \cdot (x, p) = (x, g \cdot p)$, has no describing vector field.

Example 6.4. Let G be the group of affine transformations $\varphi \colon \mathbb{T}^2 \to \mathbb{T}^2$ defined by $\varphi(\theta) = a\theta + \lambda$, where $a = \pm 1$ and $\lambda \in \mathbb{T}^2$. The fundamental vector fields of the natural action of G on $\mathbb{T}^2$ are $b_1\partial/\partial\theta_1 + b_2\partial/\partial\theta_2$, $b_1, b_2 \in \mathbb{R}$. If V is a vector field on $\mathbb{T}^2$ which commutes with the action of G, it has to commute with the fundamental vector fields, therefore $V = c_1\partial/\partial\theta_1 + c_2\partial/\partial\theta_2$, $c_1, c_2 \in \mathbb{R}$.

But, at the same time, V commutes with $\tilde{\varphi}(\theta) = -\theta$ since $\tilde{\varphi} \in G$, which implies $V = 0$. Now reasoning as in Example 6.3 shows that the effective action of G on $\mathbb{R}^k \times \mathbb{T}^2$, $k \geq 1$, defined by $\varphi \cdot (x, \theta) = (x, \varphi(\theta))$ has no describing vector field.

Observe that G is a compact Lie group with abelian Lie algebra, but it is not connected.

7. Two auxiliary results

Here we include two complementary results which were needed before. The first one is a straightforward consequence of the principal orbit theorem on the structure of the orbits of the action of a compact Lie group (see Theorem IV.3.1 in [1]).

Proposition 7.1. *Consider an effective action of $\mathbb{T}^n$ on a connected m-manifold M. Let S be the set of those points of M whose isotropy group has two or more elements, i.e., the isotropy group is non trivial. Then the set $M - S$ is connected, dense, open and $\mathbb{T}^n$-invariant. Moreover, $n \leq m$.*

The second result shows how, for connected compact Lie group actions, locally defined invariant vector fields give rise to invariant vector fields defined on the whole manifold.

Proposition 7.2. *Consider an action of a connected compact Lie group G on a m-manifold M. Given a vector field X on an open set A of M, both of them G-invariant, then there exists a G-invariant bounded function $\varphi : M \to \mathbb{R}$, which is positive on A and vanishes on $M - A$, such that the vector field $\hat{X}$ on M defined by $\hat{X} = \varphi X$ on A and $\hat{X} = 0$ on $M - A$ is differentiable and G-invariant.*

First let us recall some elementary facts on actions. Let Z be a vector field on M and $\widetilde{Z}$ the vector field depending on a parameter $g \in G$ given by $\widetilde{Z}(g, p) = (g_*)^{-1}(Z(g \cdot p))$. On G consider a bi-invariant volume form whose integral equals 1 and the measure associated to it, that is, the Haar measure. Then since $\widetilde{Z}(\{p\} \times G) \subset T_p M$, the formula

$$Z'(p) = \int_G \widetilde{Z}(g, p)$$

defines a G-invariant vector field Z' on M. Moreover, if $Z = \rho U$, where U is a G-invariant vector field, then $Z' = \psi_\rho U$, where ψ_ρ is the G-invariant function constructed from ρ in the usual way, that is,

$$\psi_\rho(p) = \int_G \rho(g \cdot p) \,.$$

In order to prove Proposition 7.2, we start considering, for X and A, a function $\varphi \colon M \to \mathbb{R}$ like in Proposition 5.5, p. 329, in [7], and the vector field $\hat{X}$ defined by $\hat{X} = \varphi X$ on A and $\hat{X} = 0$ on $M - A$.

Now observe that $\hat{X}' = \psi_\varphi X$ on A because on this open set X is G-invariant and $\hat{X} = \varphi X$. Thus ψ_φ is the required function (up to the name) since ψ_φ and $\hat{X}'$ vanish on $M - A$; so Proposition 7.2 is proved.

References

[1] Bredon, G. E.: *Introduction to compact transformation groups.* Pure and Applied Mathematics 46, Academic Press, New York-London, 1972.

[2] Hirsch, M. W.: *Differential topology.* Graduate Texts in Mathematics 33, Springer-Verlag, New York-Heidelberg, 1976.

[3] Kobayashi, S. and Nomizu, K.: *Foundations of differential geometry. Vol. I.* Interscience Publishers, a division of John Wiley & Sons, New York-London 1963.

[4] Narasimhan, R.: *Analysis on real and complex manifolds.* North-Holland Mathematical Library 35, North-Holland Publishing Co., Amsterdam, 1985.

[5] Roussarie, R.: *Modèles locaux de champs et de formes.* Astérisqûe 30, Société Mathématique de France, Paris, 1975.

[6] Sternberg, S.: On the structure of local homeomorphisms of euclidean n-space. II. *Amer. J. Math.* **80** (1958), 623–631.

[7] Turiel, F.-J. and Viruel, A.: Finite C^∞-actions are described by a single vector field. *Rev. Mat. Iberoam.* **30** (2014), no. 1, 317–330.

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