Rev. Mat. Iberoam. **34** (2018), no. 2, 853–878
DOI 10.4171/RMI/1006

Topological entropy of irregular sets

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Abstract. For expansive continuous maps with the specification property, we compute the topological entropy of the irregular set for the Birkhoff averages of a continuous function. This is the set of points for which the Birkhoff averages do not converge. The entropy is expressed in terms of a conditional variational principle. We also consider the general case of irregular sets obtained from ratios of Birkhoff averages of continuous functions. Moreover, we obtain a conditional variational principle for the topological entropy of the family of subsets of the irregular set formed by the points such that the set of accumulation points of the ratio of Birkhoff averages is a given interval. As nontrivial applications, we obtain conditional variational principles for the topological entropy of the level sets of local entropies, pointwise dimensions and Lyapunov exponents both on repellers and hyperbolic sets.

1. Introduction

This work is a contribution to the theory of *multifractal analysis*, an important subfield of the dimension theory of dynamical systems. Roughly speaking, multifractal analysis studies the complexity of the level sets of the invariant local quantities obtained from a dynamical system, such as the Birkhoff averages, the Lyapunov exponents, the pointwise dimensions and the local entropies. Since these functions are usually only measurable, it is appropriate to use quantities such as the topological entropy or the Hausdorff dimension to measure the complexity of their level sets. The concept of multifractal analysis was suggested in [11]. The first rigorous approach was obtained by Collet, Lebowitz and Porzio in [6] for a class of measures that are invariant under one-dimensional Markov maps. In [14], Lopes considered the measure of maximal entropy for a hyperbolic Julia set and in [18], Rand studied invariant Gibbs measures on a class of repellers. We refer the reader to the books [1], [2] and [15] for further references and for detailed expositions of various parts of the theory of multifractal analysis.

Mathematics Subject Classification (2010): 37B10.
Keywords: Irregular sets, topological entropy, specification property.

Our main aim is to compute the topological entropy of the irregular set for the ratios of Birkhoff averages of a continuous function, that is, the set of points for which the ratios of Birkhoff averages do not converge, in terms of a conditional variational principle. More precisely, let $f\colon X \to X$ be an expansive continuous map of a compact metric space. We always assume in the paper that f has the specification property (see Section 2 for the definition). Given a continuous function $\varphi\colon X \to \mathbb{R}$, we consider its Birkhoff averages

$$\varphi_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \varphi(f^i(x)).$$

By Birkhoff's ergodic theorem, if f preserves a finite measure μ on X, that is, $\mu(f^{-1}A) = \mu(A)$ for any measurable set $A \subset X$, then the limit

$$\psi(x) = \lim_{n\to\infty} \varphi_n(x) \tag{1.1}$$

exists for μ-almost every $x \in X$. Thus, at least from the point of view of ergodic theory, the irregular set

$$X_\varphi = \Big\{x \in X : \liminf_{n\to\infty} \varphi_n(x) < \limsup_{n\to\infty} \varphi_n(x)\Big\} \tag{1.2}$$

can be discarded, since it has zero measure with respect to any finite invariant measure. Remarkably, the set X_φ may be very large from other points of view. In particular, it was shown by Barreira and Schmeling in [3] that if X is a conformal repeller for a topologically mixing $C^{1+\varepsilon}$ map f and φ is a Hölder continuous function that is not cohomologous to a constant, then

$$h(f|X_\varphi) = h(f|X) \quad \text{and} \quad \dim_H X_\varphi = \dim_H X, \tag{1.3}$$

where $h(f|Z)$ is the topological entropy of f on the set Z and $\dim_H Z$ is the Hausdorff dimension of Z (the first identity in (1.3) also holds for nonconformal repellers). We recall that a function φ is said to be cohomologous to a constant if it can be written in the form

$$\varphi = \chi \circ f - \chi + c$$

for some bounded function χ and some constant c. In other words, the set X_φ is as large as the whole space X from the points of view of the topological entropy and of the Hausdorff dimension. Corresponding results were also obtained in [3] for locally maximal hyperbolic sets of $C^{1+\varepsilon}$ diffeomorphisms and for topological Markov chains.

The first identity in (1.3) was first established by Pesin and Pitskel in [16] for the full shift on two symbols. In a related direction, Shereshevsky [19] showed that for a generic C^2 surface diffeomorphism with a locally maximal hyperbolic set X and an equilibrium measure μ of a Hölder continuous C^0-generic function, the set of points for which the pointwise dimension does not exist has positive Hausdorff dimension, that is,

$$\dim_H\left\{x \in X : \liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} < \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}\right\} > 0.$$

The identities in (1.3) also hold for topologically mixing topological Markov chains, although in this case the two are equivalent since up to a multiplicative constant the topological entropy and the Hausdorff dimension coincide. For topological Markov chains, the result for the topological entropy was extended by Fan, Feng and Wu in [8] to arbitrary continuous functions. Moreover, for conformal repellers for $C^{1+\varepsilon}$ maps, the second identity in (1.3) was extended by Feng, Lau and Wu in [10] to arbitrary continuous functions. More recently, for maps with the specification property and continuous functions, it was shown by Chen, Tassilo and Shu in [5] that the irregular set X_φ has full topological entropy and it was shown by Thompson in [22] that it has full topological pressure.

To our best knowledge, with the exception of the argument in [16], which does not seem possible to generalize even to the full shift on tree symbols, all other arguments showing that a certain irregular set has full topological entropy or full Hausdorff dimension are based on the theory of multifractal analysis. We explain briefly the idea. As we mentioned earlier, multifractal analysis studies the complexity of the level sets of the invariant local quantities obtained from a dynamical system. For example, consider the function

$$F(\alpha) = h\big(f|\{x \in X : \psi(x) = \alpha\}\big),$$

with ψ as in (1.1). It is called the *multifractal spectrum* for the Birkhoff averages of φ. It turns out that if the dynamics $f|X$ has a certain degree of hyperbolicity, then for a generic continuous function φ the function F is analytic. Moreover, the level sets

$$K_\alpha = \big\{x \in X : \psi(x) = \alpha\big\}$$

have associated special measures μ_α (equilibrium or Gibbs measures) that are supported on them and that have metric entropy $h_{\mu_\alpha}(f) = h(f|K_\alpha)$. One can use these measures, or at least weak versions of them (such as weak Gibbs measures), to show that set of points $x \in X$ approximating successively two different level sets K_α and K_β, for which clearly $\psi(x)$ is not defined, has positive topological entropy. In other words, the irregular set formed by all points whose Birkhoff averages oscillate between α and β, with $\alpha \neq \beta$, has positive topological entropy. One can then exhaust the irregular set X_φ with these subsets and conclude from the properties of the multifractal spectrum that X_φ has full topological entropy. We refer the reader to the books [1] and [2] for full details.

More generally, in addition to the irregular sets X_φ in (1.2):

(1) We consider irregular sets obtained from ratios of Birkhoff averages of continuous functions. This includes the sets X_φ as a particular case and yields nontrivial applications to the level sets of local entropies, pointwise dimensions and Lyapunov exponents both for repellers and for hyperbolic sets.

(2) We obtain a conditional variational principle for the topological entropy of a family of subsets of the irregular set formed by those points for which the set of accumulation points of the ratio of Birkhoff averages of two continuous functions is a given interval.

We refer the reader to Sections 2 and 3 for full details. Instead, here we describe briefly some technical aspects, including the specific relation of our work to multifractal analysis, as well as the relation to former work in the literature.

The proofs are rather involved and once more are based on the theory of multifractal analysis. We note that a multifractal analysis for continuous maps with the specification property and Birkhoff averages of continuous functions, expressed in terms of a conditional variational principle, was claimed by Takens and Verbitskiy in [20]. However, a gap in the proof (related to choosing simultaneously various quantities sufficiently small) was pointed out by Pfister and Sullivan in [17], who also established a stronger result, using a different method. More recently, Thompson [21] obtained a conditional variational principle for the topological pressure of the level sets of the Birkhoff averages of a continuous function, thus including the result claimed by Takens and Verbitskiy as a particular case. His approach is closer in spirit to that in [20] although now using quantities and arguments that are more natural from the internal point of view of the thermodynamic formalism.

Instead, we go back to the original argument in [20] and find how to circumvent the problem for expansive maps and prove something weaker that is still sufficient for our purposes. More precisely, we do not require in our work the full force of multifractal analysis. We emphasize that by no means our main aim is to correct the problem in the proof of [20] while using analogous methods to prove their claim in the smaller class of expansive maps. Indeed, as described above, we are primarily interested in a large class of subsets of the irregular set, none of which (even the irregular set itself) were considered in [20].

Our approach has a major advantage: we are able to obtain a conditional variational principle for the topological entropy of a large family of subsets of the irregular set, as described above, and this is in fact the main contribution of our work. A particular case of this result (for Birkhoff averages and not for ratios of Birkhoff averages) was claimed earlier by Li and Wu in [13] without assuming expansiveness, although they use arguments and results from [20]. Our work shows that the result claimed in [13] is true at least for expansive maps (on the other hand, we think that their arguments cannot be used even assuming expansiveness).

2. Preliminaries

Let $X = (X, d)$ be a compact metric space and let $f\colon X \to X$ be an expansive continuous map. We recall that a map f is said to be *expansive* if there exists $\zeta > 0$ such that if

$$d(f^n(x), f^n(y)) < \zeta \quad \text{for all } n \geq 0,$$

then $x = y$. Given continuous functions $\varphi, \psi\colon X \to \mathbb{R}$ with $\psi > 0$, we consider the level sets

$$X(\alpha) = \Big\{x \in X : \lim_{n\to\infty} S(x, n) = \alpha\Big\},$$

where

$$S(x,n)=\frac{\sum_{i=0}^{n-1}\varphi(f^i(x))}{\sum_{i=0}^{n-1}\psi(f^i(x))}.$$

Moreover, given a set $I\subset\mathbb{R}$, we define

$$X_I=\{x\in X : A(x)=I\}, \tag{2.1}$$

where $A(x)$ is the set of accumulation points of the sequence $n\mapsto S(x,n)$. For example, if $I=\{\alpha\}$, then $X_I=X(\alpha)$.

Now we write

$$\mathcal{L}=\bigl\{\alpha\in\mathbb{R} : X(\alpha)\neq\emptyset\bigr\}.$$

Denoting by $\mathcal{M}$ the set of all f-invariant probability measures on X, for each $\alpha\in\mathcal{L}$ we define

$$H(\alpha)=\sup\left\{h_\mu(f):\mu\in\mathcal{M}\ \text{ and }\ \frac{\int_X\varphi\,d\mu}{\int_X\psi\,d\mu}=\alpha\right\},$$

where $h_\mu(f)$ is the metric entropy of f with respect to μ.

The map f is said to have the *specification property* if for each $\varepsilon>0$ there exists $m=m(\varepsilon)\in\mathbb{N}$ such that given intervals $I_j=[a_j,b_j]$ with $a_j,b_j\in\mathbb{N}$, for $j=1,\ldots,k$, such that

$$\operatorname{dist}(I_i,I_j)\geq m(\varepsilon),\quad i\neq j, \tag{2.2}$$

and given points $x_1,\ldots,x_k\in X$ there exists $x\in X$ such that

$$d(f^{p+a_j}(x),f^p(x_j))<\varepsilon$$

for $p=0,\ldots,b_j-a_j$ and $j=1,\ldots,k$. We note that any probability measure that is invariant under a map with the specification property has a generic point (see, for example, [7]) and hence,

$$\mathcal{L}=\left\{\frac{\int_X\varphi\,d\mu}{\int_X\psi\,d\mu}:\mu\in\mathcal{M}\right\}.$$

Since $\mathcal{M}$ is compact and connected, and the map $\mu\mapsto\int_X\varphi\,d\mu/\int_X\psi\,d\mu$ is continuous, when f has the specification property the set $\mathcal{L}$ is a closed interval.

Finally, we recall the notion of topological entropy on an arbitrary set. For each $n\in\mathbb{N}$ we define a distance d_n on X by

$$d_n(x,y)=\max\bigl\{d(f^k(x),f^k(y)):k=0,\ldots,n-1\bigr\},$$

where d is the original distance, and we denote by $B_n(x,\varepsilon)$ the d_n-ball of radius ε centered at x. A countable collection $\Gamma=\{B_{n_i}(x_i,\varepsilon)\}_{i\in J}$ is said to *cover* a set $Z\subset X$ if $Z\subset\bigcup_{i\in J}B_{n_i}(x_i,\varepsilon)$. Given $s\geq 0$, we define

$$m(Z,s,\varepsilon)=\lim_{N\to\infty}\inf_\Gamma\sum_{i\in J}\exp(-sn_i),$$

where the infimum is taken over all collections Γ covering Z such that $n_i\geq N$ for $i\in J$. The *topological entropy* $h(f|Z)$ of f on Z is defined by

$$h(f|Z)=\lim_{\varepsilon\to 0}\inf\bigl\{s\geq 0:m(Z,s,\varepsilon)=0\bigr\}.$$

3. Main result and applications

The following is our main result. It expresses the topological entropy of the irregular set X_I in terms of the function $H(\alpha)$.

Theorem 3.1. *Let $f\colon X \to X$ be an expansive continuous map with the specification property on a compact metric space and let $\varphi, \psi\colon X \to \mathbb{R}$ be continuous functions with $\inf \psi > 0$.*

1. *If I is not a closed subinterval of $\mathcal{L}$, then $X_I = \emptyset$.*
2. *If I is a closed subinterval of $\mathcal{L}$, then*

$$h(f|X_I) = \inf_{\alpha \in I} H(\alpha).$$

The proof of Theorem 3.1 is given in Section 4. Here we present several applications to repellers and hyperbolic sets.

We start with the case of repellers. Let $f\colon M \to M$ be a C^1 map on a smooth manifold and let $J \subset M$ be a compact f-invariant set such that $f|J$ is topologically mixing. We say that f is *expanding* on J and that J is a *repeller* for f if there exist $c > 0$ and $\tau > 1$ such that

$$\|d_x f^n v\| \geq c\,\tau^n\,\|v\|$$

for $x \in J$, $v \in T_x M$ and $n \in \mathbb{N}$. We shall always assume that there exists an open neighborhood U of J such that $J = \bigcap_{n=0}^{\infty} f^{-n} U$. Given continuous functions $\varphi, \psi\colon J \to \mathbb{R}$ with $\psi > 0$, we consider the sets X_I in (2.1) for $X = J$.

Since $f|J$ is expansive and has the specification property, the following result is an immediate consequence of Theorem 3.1.

Theorem 3.2. *If J is a repeller and $\varphi, \psi\colon J \to \mathbb{R}$ are continuous functions with $\inf \psi > 0$, then*

$$h(f|X_I) = \inf_{\alpha \in I} H(\alpha)$$

for any closed interval $I \subset \mathcal{L}$.

Now let μ be an equilibrium measure of a continuous function φ on the repeller. This means that μ attains the supremum in the variational principle for the topological pressure

$$P(\varphi) = \sup_{\nu \in \mathcal{M}} \Big(h_\nu(f) + \int_X \varphi \, d\nu \Big).$$

Without loss of generality, one can always assume that $P(\varphi) = 0$ (simply replace φ by $\varphi - P(\varphi)$). If μ is a Gibbs measure (this happens for example if $f|J$ is topologically mixing and φ is Hölder continuous), then

$$h_\mu(x) := \lim_{n\to\infty} -\frac{1}{n} \log \mu(B_n(x,\varepsilon)) = -\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x))$$

for μ-almost every $x \in X$, where

$$B_n(x,\varepsilon) = \bigcap_{k=0}^{n-1} f^{-k} B(f^k(x),\varepsilon)$$

for any sufficiently small fixed $\varepsilon > 0$. The number $h_\mu(x)$ (when defined) is called the local entropy of μ at the point x (with respect to f). The following result is an immediate consequence of Theorem 3.2.

Theorem 3.3. *If J is a repeller and μ is a Gibbs measure of a continuous function φ, then for any closed interval*

$$[a,b] \subset \Big\{ -\int_X \varphi\, d\nu : \nu \in \mathcal{M} \Big\},$$

we have

$$h(f|Y) = \inf_{\alpha\in[a,b]} \sup \Big\{ h_\nu(f) : \nu \in \mathcal{M} \text{ and } -\int_X \varphi\, d\nu = \alpha \Big\},$$

where Y is the set of all points $x \in X$ such that

$$\liminf_{n\to\infty} -\frac{1}{n}\log \mu(B_n(x,\varepsilon)) = a \quad \text{and} \quad \limsup_{n\to\infty} -\frac{1}{n}\log \mu(B_n(x,\varepsilon)) = b.$$

Now we consider the Lyapunov exponents. Let J be a conformal repeller of f (this means that $d_x f$ is a multiple of an isometry for every $x \in J$). We define the Lyapunov exponent of f at a point $x \in J$ by

$$\lambda(x) = \limsup_{n\to\infty} \frac{1}{n}\log \|d_x f^n\|.$$

It follows from Birkhoff's ergodic theorem that for any f-invariant finite measure on J, the number $\lambda(x)$ is in fact a limit for μ-almost every $x \in J$. Again, the following result is an immediate consequence of Theorem 3.2.

Theorem 3.4. *If J is a conformal repeller, then for any closed interval*

$$[a,b] \subset \Big\{ \int_X \log\|df\|\, d\nu : \nu \in \mathcal{M} \Big\},$$

we have

$$h(f|Z) = \inf_{\alpha\in[a,b]} \sup \Big\{ h_\nu(f) : \nu \in \mathcal{M} \text{ and } \int_X \log\|df\|\, d\nu = \alpha \Big\},$$

where Z is the set of all points $x \in X$ such that

$$\liminf_{n\to\infty} \frac{1}{n}\log \|d_x f^n\| = a \quad \text{and} \quad \limsup_{n\to\infty} \frac{1}{n}\log \|d_x f^n\| = b.$$

Finally, we consider the pointwise dimensions. Let J be a conformal repeller and let μ be a Gibbs measure of a continuous function $\varphi\colon J \to \mathbb{R}$. The limit

$$d_\mu(x) := \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = -\lim_{n\to\infty} \frac{\sum_{i=0}^{n-1} \varphi(f^i(x))}{\sum_{i=0}^{n-1} \log \|d_{f^i(x)} f\|}$$

is called the pointwise dimension of μ at the point x, whenever it exists. A consequence of Theorem 3.2 is the following.

Theorem 3.5. *If J is a conformal repeller and μ is a Gibbs measure of a continuous function φ, then for any closed interval*

$$[a,b] \subset \left\{ -\frac{\int_X \varphi\,d\nu}{\int_X \log \|df\|\,d\nu} : \nu \in \mathcal{M} \right\},$$

we have

$$h(f|W) = \inf_{\alpha\in[a,b]} \sup \left\{ h_\nu(f) : \nu \in \mathcal{M} \text{ and } -\frac{\int_X \varphi\,d\nu}{\int_X \log \|df\|\,d\nu} = \alpha \right\},$$

where W is the set of all points $x \in X$ such that

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = a \quad \text{and} \quad \limsup_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = b.$$

We also describe briefly corresponding results for locally maximal hyperbolic sets. Let $f\colon M \to M$ be a C^1 diffeomorphism of a smooth manifold M and let $\Lambda \subset M$ be a compact f-invariant set such that $f|\Lambda$ is topologically mixing. We say that f is a *hyperbolic set* for f if there exist $\tau \in (0,1)$, $c > 0$ and a decomposition

$$T_x M = E^s(x) \oplus E^u(x)$$

for each $x \in \Lambda$, such that

$$d_x f E^s(x) = E^s(f(x)), \quad d_x f E^u(x) = E^u(f(x)),$$

$$\|d_x f^n v\| \le c\tau^n \|v\| \quad \text{whenever} \quad v \in E^s(x),$$

and

$$\|d_x f^{-n} v\| \le c\tau^n \|v\| \quad \text{whenever} \quad v \in E^u(x)$$

for every $x \in \Lambda$ and $n \in \mathbb{N}$. We say that Λ is *locally maximal* if there exists an open set $U \supset \Lambda$ such that

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U).$$

Given continuous functions $\varphi, \psi\colon \Lambda \to \mathbb{R}$ with $\psi > 0$, we consider again the sets X_I in (2.1) for $X = \Lambda$. The following result is an immediate consequence of Theorem 3.1.

Theorem 3.6. *If Λ is a locally maximal hyperbolic set and $\varphi, \psi\colon \Lambda \to \mathbb{R}$ are continuous functions with* $\inf \psi > 0$, *then*

$$h(f|X_I) = \inf_{\alpha\in I} H(\alpha)$$

for any closed interval $I \subset \mathcal{L}$.

One can also formulate analogous results for the irregular sets obtained from local entropies, pointwise dimensions and Lyapunov exponents.

4. Proof of Theorem 3.1

We separate the proof into several steps.

Step 1. Preliminaries. We first introduce a quantity that can be described as a variation of the lower capacity topological pressure (see [15]). We also relate it to the function $H(\alpha)$.

Given $\alpha \in \mathcal{L}$, $\delta > 0$ and $n \in \mathbb{N}$, let

$$P(\alpha, \delta, n) = \Big\{x \in X : \Big|\sum_{i=0}^{n-1} \varphi(f^i(x)) - \alpha \sum_{i=0}^{n-1} \psi(f^i(x))\Big| < n\delta\Big\}.$$

We define

$$\Lambda(\alpha) = \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N(\alpha, \delta, n, \varepsilon), \tag{4.1}$$

where $N(\alpha, \delta, n, \varepsilon)$ is the least number of d_n-balls of radius ε that are needed to cover the set $P(\alpha, \delta, n)$. Moreover, we recall that a set $E \subset X$ is said to be (n, ε)-*separated* if $d_n(x, y) > \varepsilon$ for any $x, y \in E$ with $x \neq y$. Let $M(\alpha, \delta, n, \varepsilon)$ be the cardinality of a maximal (n, ε)-separated set in $P(\alpha, \delta, n)$. A simple argument shows that

$$N(\alpha, \delta, n, \varepsilon) \leq M(\alpha, \delta, n, \varepsilon) \leq N(\alpha, \delta, n, \varepsilon/2)$$

for $n \in \mathbb{N}$ and $\varepsilon, \delta > 0$. Therefore,

$$\Lambda(\alpha) = \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log M(\alpha, \delta, n, \varepsilon). \tag{4.2}$$

Moreover, proceeding in a similar manner, for example, to that in the proof of Theorem 3.4 in [4] one can show that

$$\begin{aligned} \Lambda(\alpha) &= \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N(\alpha, \delta, n, \varepsilon) \\ &= \lim_{\varepsilon\to 0} \lim_{\delta\to 0} \limsup_{n\to\infty} \frac{1}{n} \log M(\alpha, \delta, n, \varepsilon). \end{aligned} \tag{4.3}$$

We note that for an expansive map f the limits in (4.1), (4.2) and (4.3) when $\varepsilon \to 0$ are not needed provided that ε is smaller than the constant ζ in the notion of expansivity.

Lemma 4.1. $\Lambda(\alpha) \geq H(\alpha)$ *for* $\alpha \in \mathcal{L}$.

Proof. We need the following result of Katok in [12] that gives a characterization of the metric entropy. Let ν be an ergodic f-invariant probability measure on X and given $\varepsilon, \delta > 0$, let $N^\nu(\delta, \varepsilon, n)$ be the least number of d_n-balls of radius ε covering each set of measure at least $1 - \delta$.

Lemma 4.2. *For each* $\delta \in (0, 1)$, *we have*

$$h_\nu(f) = \lim_{\varepsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N^\nu(\delta, \varepsilon, n) = \lim_{\varepsilon\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N^\nu(\delta, \varepsilon, n).$$

Since f is expansive, for $\varepsilon < \zeta$ (with ζ as in the notion of expansivity), we have

$$h_\nu(f) = \lim_{n\to\infty} \frac{1}{n} \log N^\nu(\delta, \varepsilon, n).$$

Given $\alpha \in \mathcal{L}$ and $\eta > 0$, there exist $\varepsilon \in (0, \zeta/2)$ and sequences $\delta_k \searrow 0$ and $a_k \searrow 0$ with $a_k \le 1/2$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log M(\alpha, c\,\delta_k, n, \varepsilon) < \Lambda(\alpha) + \eta$$

and

$$\operatorname{var}(\varphi, a_k\,\varepsilon) := \sup\bigl\{|\varphi(x) - \varphi(y)| : |x - y| < a_k\varepsilon\bigr\} < \delta_k$$

for $k \in \mathbb{N}$, where

$$c = \frac{6(1+\alpha)}{\inf \psi}. \tag{4.4}$$

Now take $\mu \in \mathcal{M}$ such that

$$\frac{\int_X \varphi\, d\mu}{\int_X \psi\, d\mu} = \alpha. \tag{4.5}$$

By a result of Young in [23], there exist measures $\nu_k \in \mathcal{M}$ such that:

1. $\nu_k = \sum_{i=1}^{j(k)} \lambda_{k,i}\nu_{k,i}$, for some numbers $\lambda_{k,i} > 0$ with $\sum_{i=1}^{j(k)} \lambda_{k,i} = 1$, where $\nu_{k,i}$ is an ergodic measure for $k \in \mathbb{N}$ and $i = 1, \ldots, j(k)$;
2. $h_{\nu_k}(f) \ge h_\mu(f) - \eta$;
3. $\bigl|\int_X \varphi\, d\mu - \int_X \varphi\, d\nu_k\bigr| < \delta_k$ and $\bigl|\int_X \psi\, d\mu - \int_X \psi\, d\nu_k\bigr| < \delta_k$.

Since the measures $\nu_{k,i}$ are ergodic, there exists a sequence $\ell_k \nearrow +\infty$ such that the set $A_{k,i}$ formed by the points $x \in X$ such that

$$\Bigl|\sum_{p=0}^{n-1} \varphi(f^p(x)) - n\int_X \varphi\, d\nu_{k,i}\Bigr| < n\delta_k \quad \text{and} \quad \Bigl|\sum_{p=0}^{n-1} \psi(f^p(x)) - n\int_X \psi\, d\nu_{k,i}\Bigr| < n\delta_k$$

for $n > \ell_k$ has measure $\nu_{k,i}(A_{k,i}) > 1 - \eta$ for $k \in \mathbb{N}$ and $i = 1, \ldots, j(k)$.

Now we take a sequence $n_k \nearrow +\infty$ and $\delta > 0$ such that for each $k \in \mathbb{N}$:

1) $[\lambda_{k,i}\, n_k] \ge \ell_k$ for $i = 1, \ldots, j(k)$, and

$$\frac{j(k)\, m(a_k\,\varepsilon)\, \max\{1, \|\varphi\|\}}{\min_i \lambda_{k,i}\, n_k - 1} < \delta_k, \tag{4.6}$$

where $[\cdot]$ is the integer part and $\|\varphi\| = \sup|\varphi|$, with m as in (2.2);

2) for $n \ge n_k(1 - \delta_k)$,

$$\frac{1}{n} \log M(\alpha, c\,\delta_k, n, \varepsilon) < \Lambda(\alpha) + 2\eta \tag{4.7}$$

and, for $i = 1, \ldots, j(k)$,

$$N^{\nu_{k,i}}(\delta, 2\varepsilon, [\lambda_{k,i}\, n_k]) \ge \exp\bigl([\lambda_{k,i}\, n_k]\,(h_{\nu_{k,i}}(f) - \eta)\bigr). \tag{4.8}$$

Let $S_{k,i} \subset A_{k,i}$ be a maximal $([\lambda_{k,i}\, n_k], 2\varepsilon)$-separated set. By (4.8), its cardinality is at least $\exp([\lambda_{k,i}\, n_k](h_{\nu_{k,i}}(f) - \eta))$. On the other hand, for each

$$z = (x_{k,1}, \ldots, x_{k,j(k)}) \in S_{k,1} \times \cdots \times S_{k,j(k)},$$

by the specification property there exists $y = y(z) \in X$ such that

$$d_{[\lambda_{k,i}\, n_k]}(x_{k,i}, f^{a_i}(y)) < a_k\, \varepsilon,$$

where

$$a_i = (i-1)([\lambda_{k,i}\, n_k] + m(a_k\, \varepsilon)),$$

for $k \in \mathbb{N}$ and $i = 1, \ldots, j(k)$. Let

$$R_k = \left\{ y(z) : z \in S_{k,1} \times \cdots \times S_{k,j(k)} \right\} \quad \text{and} \quad \hat{n}_k = \sum_{i=1}^{j(k)} [\lambda_{k,i}\, n_k] + (j(k)-1) m(a_k\, \varepsilon).$$

By (4.6), we have

$$\hat{n}_k \geq \sum_{i=1}^{j(k)} (\lambda_{k,i}\, n_k - 1) = n_k - j(k) \geq n_k - n_k \delta_k \tag{4.9}$$

and

$$\hat{n}_k \leq \sum_{i=1}^{j(k)} \lambda_{k,i}\, n_k + j(k)\, m(a_k \varepsilon) \leq n_k + n_k \delta_k.$$

Hence,

$$1 - \delta_k \leq \frac{\hat{n}_k}{n_k} \leq 1 + \delta_k \tag{4.10}$$

for $k \in \mathbb{N}$. Given distinct vectors

$$(x_{k,1}, \ldots, x_{k,j(k)}), (\overline{x}_{k,1}, \ldots, \overline{x}_{k,j(k)}) \in S_{k,1} \times \cdots \times S_{k,j(k)},$$

say with $x_{k,s} \neq \overline{x}_{k,s}$, we have

$$\begin{aligned} d_{\hat{n}_k}(y, \overline{y}) &\geq d_{[\lambda_{k,s} n_k]}(f^{a_s}(y), f^{a_s}(\overline{y})) \\ &\geq d_{[\lambda_{k,s} n_k]}(x_{k,s}, \overline{x}_{k,s}) - d_{[\lambda_{k,s} n_k]}(x_{k,s}, f^{a_s}(y)) - d_{[\lambda_{k,s} n_k]}(\overline{x}_{k,s}, f^{a_s}(\overline{y})) \\ &\geq 2\varepsilon - a_k \varepsilon - a_k\, \varepsilon \geq \varepsilon, \end{aligned}$$

since $a_k \leq 1/2$. Therefore, R_k is an $(\hat{n}_k, \varepsilon)$-separated set.

Finally, we show that

$$\left| \frac{\sum_{p=0}^{\hat{n}_k - 1} \varphi(f^p(y))}{\sum_{p=0}^{\hat{n}_k - 1} \psi(f^p(y))} - \alpha \right| < c\, \delta_k \tag{4.11}$$

for $k \in \mathbb{N}$ and $y \in R_k$, with c as in (4.4). In order to prove (4.11), it is sufficient to show that

$$\left| \sum_{p=0}^{\hat{n}_k - 1} \varphi(f^p(y)) - \hat{n}_k \int_X \varphi\, d\mu \right| < 6\, \hat{n}_k\, \delta_k \tag{4.12}$$

and

$$\left|\sum_{p=0}^{\hat{n}_k-1}\psi(f^p(y))-\hat{n}_k\int_X\psi\,d\mu\right|<6\,\hat{n}_k\,\delta_k \tag{4.13}$$

for $y\in R_k$ with k sufficiently large. We only prove inequality (4.12) since the proof of (4.13) is identical. Then

$$\begin{aligned}
&\left|\frac{\sum_{p=0}^{\hat{n}_k-1}\varphi(f^p(y))}{\sum_{p=0}^{\hat{n}_k-1}\psi(f^p(y))}-\alpha\right|\\
&\quad\le\frac{\left|\sum_{p=0}^{\hat{n}_k-1}\varphi(f^p(y))-\hat{n}_k\int_X\varphi\,d\mu\right|+\left|\hat{n}_k\int_X\varphi\,d\mu-\alpha\sum_{p=0}^{\hat{n}_k-1}\psi(f^p(y))\right|}{\left|\sum_{p=0}^{\hat{n}_k-1}\psi(f^p(y))\right|}\\
&\quad=\frac{\left|\sum_{p=0}^{\hat{n}_k-1}\varphi(f^p(y))-\hat{n}_k\int_X\varphi\,d\mu\right|+\left|\hat{n}_k\int_X\psi\,d\mu-\sum_{p=0}^{\hat{n}_k-1}\psi(f^p(y))\right|\alpha}{\left|\sum_{p=0}^{\hat{n}_k-1}\psi(f^p(y))\right|}\\
&\quad\le\frac{6\,\hat{n}_k\,\delta_k+6\,\hat{n}_k\,\delta_k\,\alpha}{\hat{n}_k\inf\psi}=c\,\delta_k.
\end{aligned}$$

First we observe that

$$\begin{aligned}
0&\le\frac{-1+m(a_k\,\varepsilon)}{n_k}\le\frac{\lambda_{k,i}(\lambda_{k,j}n_k-1+m(a_k\varepsilon))-\lambda_{k,j}\lambda_{k,i}\,n_k}{\lambda_{k,i}\,n_k}\\
&\le\frac{\lambda_{k,i}([\lambda_{k,j}n_k]+m(a_k\,\varepsilon))}{[\lambda_{k,i}\,n_k]}-\lambda_{k,j}\\
&\le\frac{\lambda_{k,i}\lambda_{k,j}n_k+\lambda_{k,j}m(a_k\,\varepsilon)-\lambda_{k,j}\lambda_{k,i}\,n_k+\lambda_{k,j}}{\min_i\lambda_{k,i}\,n_k-1}\le\frac{m(a_k\,\varepsilon)+1}{\min_i\lambda_{k,i}\,n_k-1}
\end{aligned}$$

for $i,j\in\{1,2,\dots,j(k)\}$, and it follows from (4.6) that

$$0\le\frac{\lambda_{k,i}\left(\sum_{j=1}^{j(k)}[\lambda_{k,j}n_k]+j(k)\,m(a_k\,\varepsilon)\right)}{[\lambda_{k,i}\,n_k]}-1<\frac{\delta_k}{\|\varphi\|}$$

(when $\|\varphi\|=0$ there is nothing to prove). Therefore,

$$0\le\lambda_{k,i}\,\hat{n}_k+\lambda_{k,i}\,m(a_k\,\varepsilon)-[\lambda_{k,i}\,n_k]<[\lambda_{k,i}\,n_k]\,\frac{\delta_k}{\|\varphi\|}$$

for $k\in\mathbb{N}$ and $i\in\{1,2,\dots,j(k)\}$. Since

$$\lambda_{k,i}\,m(a_k\,\varepsilon)\le m(a_k\,\varepsilon)\le\frac{[\lambda_{k,i}\,n_k]}{\|\varphi\|}\cdot\frac{j(k)\,m(a_k\,\varepsilon)\,\|\varphi\|}{[\lambda_{k,i}\,n_k]}\le[\lambda_{k,i}\,n_k]\,\frac{\delta_k}{\|\varphi\|},$$

it also follows from (4.6) that

$$\left|\lambda_{k,i}\,\hat{n}_k-[\lambda_{k,i}\,n_k]\right|<2\,[\lambda_{k,i}\,n_k]\,\frac{\delta_k}{\|\varphi\|}\le2\,\hat{n}_k\,\frac{\delta_k}{\|\varphi\|}.$$

Now we prove (4.12). Observe that

$$
\begin{aligned}
&\Big|\sum_{p=0}^{\hat n_k-1}\varphi(f^p(y))-\hat n_k\int_X\varphi\,d\mu\Big|\\
&<\Big|\sum_{p=0}^{\hat n_k-1}\varphi(f^p(y))-\hat n_k\int_X\varphi\,d\nu_k\Big|+\hat n_k\,\delta_k\\
&\le\sum_{i=1}^{j(k)}\Big|\sum_{p=0}^{[\lambda_{k,i}\,n_k]-1}\varphi(f^{p+a_i}(y))-\lambda_{k,i}\hat n_k\int_X\varphi\,d\nu_{k,i}\Big|+\hat n_k\,\delta_k+j(k)\,m(a_k\,\varepsilon)\,\|\varphi\|\\
&\le\sum_{i=1}^{j(k)}\Big|\sum_{p=0}^{[\lambda_{k,i}\,n_k]-1}\varphi(f^{p+a_i}(y))-[\lambda_{k,i}\,n_k]\int_X\varphi\,d\nu_{k,i}\Big|\\
&\quad+\hat n_k\,\delta_k+2\,\hat n_k\,\delta_k+j(k)\,m(a_k\,\varepsilon)\,\|\varphi\|\\
&\le\sum_{i=1}^{j(k)}\Big|\sum_{p=0}^{[\lambda_{k,i}\,n_k]-1}\varphi(f^{p+a_i}(y))-\sum_{p=0}^{[\lambda_{k,i}\,n_k]-1}\varphi(f^p(x_{k,i}))\Big|\\
&\quad+\sum_{i=1}^{j(k)}\Big|\sum_{p=0}^{[\lambda_{k,i}\,n_k]-1}\varphi(f^p(x_{k,i}))-[\lambda_{k,i}\,n_k]\int_X\varphi\,d\nu_{k,i}\Big|\\
&\quad+\hat n_k\,\delta_k+2\,\hat n_k\,\delta_k+j(k)\,m(a_k\,\varepsilon)\,\|\varphi\|\\
&\le\sum_{i=1}^{j(k)}\big([\lambda_{k,i}\,n_k]\,\mathrm{var}(\varphi,a_k\,\varepsilon)+[\lambda_{k,i}\,n_k]\delta_k\big)+2\hat n_k\,\delta_k+j(k)\,m(a_k\,\varepsilon)\,\|\varphi\|\\
&\le\hat n_k\left(\mathrm{var}(\varphi,a_k\,\varepsilon)+2\delta_k+2\delta_k+\frac{j(k)\,m(a_k\,\varepsilon)\,\|\varphi\|}{\hat n_k}\right)\\
&<\hat n_k\,(\delta_k+2\delta_k+2\delta_k+\delta_k)=6\hat n_k\,\delta_k.
\end{aligned}
$$

This establishes (4.12) which together with (4.13) yields inequality (4.11).

It follows from (4.11) that $R_k\subset P(\alpha,c\,\delta_k,\hat n_k)$ for each k. Therefore, the cardinality of a maximal $(\hat n_k,\varepsilon)$-separated set in $P(\alpha,c\,\delta_k,\hat n_k)$ is at least

$$
\begin{aligned}
&\#S_{k,1}\times\cdots\times\#S_{k,j(k)}\\
&\qquad\ge\exp\big([\lambda_{k,1}n_k](h_{\nu_{k,1}}(f)-\eta)+\cdots+[\lambda_{k,j(k)}n_k](h_{\nu_{k,j(k)}}(f)-\eta)\big)\\
&\qquad\ge\exp\Big(\sum_{i=1}^{j(k)}(\lambda_{k,i}\,n_k-1)\big(h_{\nu_{k,i}}(f)-\eta\big)\Big)\\
&\qquad=\exp\Big(n_k\sum_{i=1}^{j(k)}\lambda_{k,i}\big(h_{\nu_{k,i}}(f)-\eta\big)-\sum_{i=1}^{j(k)}\big(h_{\nu_{k,i}}(f)-\eta\big)\Big)\\
&\qquad\ge\exp\big(n_k(h_{\nu_k}(f)-\eta)-j(k)(h(f)-\eta)\big)\\
&\qquad\ge\exp\big(n_k(h_{\nu_k}(f)-\eta)-n_k\delta_k(h(f)-\eta)\big)\\
&\qquad\ge\exp\big(n_k(h_{\mu}(f)-2\eta)-n_k\delta_k(h(f)-\eta)\big)
\end{aligned}
$$

(in view of (4.6), we have $j(k) \le n_k\delta_k$). Finally, it follows from (4.7), (4.9) and (4.10) that

$$\begin{aligned}\Lambda(\alpha)+2\eta &\ge \limsup_{k\to\infty}\frac{1}{\hat n_k}\log M(\alpha, c\,\delta_k, \hat n_k, \varepsilon)\\ &\ge \limsup_{k\to\infty}\frac{1}{n_k}\log M(\alpha, c\,\delta_k, \hat n_k, \varepsilon) \ge h_\mu(f)-2\eta.\end{aligned}$$

Since η is arbitrary this implies that $\Lambda(\alpha) \ge H(\alpha)$ (see (4.5)) and the proof of the lemma is complete. □

Step 2. Construction of a set of Moran type. Following [9] and [20] we describe a construction of a set of Moran type. This construction will be used later on to approximate the set X_I in (2.1).

Take $\varepsilon > 0$. For each $k \in \mathbb{N}$ let $m_k = m(2^{-k}\varepsilon)$, where m is the integer in the notion of specification. Moreover, let W_k be a sequence of finite sets in X and let n_k, N_k be sequences of positive integers. We assume that

$$d_{n_k}(x,y) \ge 8\varepsilon \quad \text{for } x,y \in W_k,\ x \ne y.$$

Given $x_1,\dots,x_{N_k} \in W_k$, by the specification property one can take some point $y = y(x_1,\dots,x_{N_k}) \in X$ such that

$$d_{n_k}(x_j, f^{a_j}(y)) < \frac{\varepsilon}{2^k}, \quad j=1,\dots,N_k,$$

where $a_j = (j-1)(n_k+m_k)$. Now let $D_1 = W_1$ and

$$D_k = \{y(x_1,\dots,x_{N_k}) : x_1,\dots,x_{N_k} \in W_k\}$$

for $k \ge 2$. We define recursively sets L_k and integers ℓ_k as follows. Let

$$L_1 = D_1 = W_1, \quad \ell_1 = n_1$$

and

$$\ell_{k+1} = N_1 n_1 + \sum_{i=2}^{k+1} N_i(n_i+m_i). \tag{4.14}$$

Given $x \in L_k$ and $y \in D_{k+1}$, by the specification property one can take some point $z = z(x,y)$ such that

$$d_{\ell_k}(x,z) < \frac{\varepsilon}{2^{k+1}} \quad \text{and} \quad d_{t_{k+1}}(y, f^{\ell_k+m_{k+1}}(z)) < \frac{\varepsilon}{2^{k+1}},$$

where

$$t_{k+1} = (N_{k+1}-1)\,m_{k+1} + N_{k+1}\,n_{k+1}.$$

Finally, let

$$L_{k+1} = \{z(x,y) : x \in L_k, y \in D_{k+1}\}.$$

The set of Moran type is defined by

$$F = \bigcap_{k=1}^{\infty}\bigcup_{x\in L_k}\overline{B_{\ell_k}(x, \varepsilon/2^{k-1})}. \tag{4.15}$$

The following result gives a lower bound for the topological entropy on F.

Lemma 4.3 (see [9] and [20]). *For each $n \in \mathbb{N}$, let k and $0 \le p < N_{k+1}$ be the unique nonnegative integers such that*

$$\ell_k + p(m_{k+1} + n_{k+1}) < n \le \ell_k + (p+1)(m_{k+1} + n_{k+1}).$$

Then

$$h(f|F) \ge \liminf_{n\to\infty} \frac{1}{n}(N_1 \log M_1 + \cdots + N_k \log M_k + p \log M_{k+1}),$$

where $M_k = \#W_k$.

Step 3. Construction of a specific set F. Let $I \in \mathcal{L}$ be a closed interval and write

$$\beta = \inf_{\alpha\in I} \Lambda(\alpha). \tag{4.16}$$

We construct a specific set $F \subset X$ such that

$$F \subset X_I \tag{4.17}$$

and

$$h(f|F) \ge \beta. \tag{4.18}$$

Given $k \in \mathbb{N}$, take $\alpha_{k,1}, \dots, \alpha_{k,q_k} \in I$ with q_k increasing such that

$$I \subset \bigcup_{i=1}^{q_k} B\Big(\alpha_{k,i}, \frac{1}{k}\Big), \quad |\alpha_{k,i+1} - \alpha_{k,i}| < \frac{1}{k} \text{ for all } i, \quad |\alpha_{k,q_k} - \alpha_{k+1,1}| < \frac{1}{k}. \tag{4.19}$$

Since f is expansive, for $\varepsilon \in (0, \zeta)$ we have

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \frac{1}{n} \log M(\alpha, \delta, n, 8\varepsilon) = \Lambda(\alpha).$$

Given $\gamma > 0$, we consider sequences $\{\delta_{k,i}\}_{k\in\mathbb{N}, i=1,\dots,q_k}$ and $\{n_{k,i}\}_{k\in\mathbb{N}, i=1,\dots,q_k}$ satisfying

$$\delta_{1,1} > \delta_{1,2} > \cdots > \delta_{1,q_1} > \delta_{2,1} > \delta_{2,2} > \cdots > \delta_{2,q_2} > \cdots$$

and

$$n_{1,1} < n_{1,2} < \cdots < n_{1,q_1} < n_{2,1} < n_{2,2} < \cdots < n_{2,q_2} < \cdots$$

such that $n_{k,i} \ge 2^{m_k}$ and

$$M_{k,i} := M(\alpha_{k,i}, \delta_{k,i}, n_{k,i}, 8\varepsilon) > \exp(n_{k,i}(\Lambda(\alpha_{k,i}) - \gamma/2)) \tag{4.20}$$

for $k \in \mathbb{N}$ and $i = 1, \dots, q_k$. Moreover, let

$$W_{k,i} = \{x_j^{k,i} : j = 1, \dots, M_{k,i}\}$$

be a maximal $(n_{k,i}, 8\varepsilon)$-separated set in $P(\alpha_{k,i}, \delta_{k,i}, n_{k,i})$.

Finally, let $\{N_{k,i}\}_{k\in\mathbb{N}, i=1,\dots,q_k}$ be a sequence of positive integers such that:

1) $N_{1,1} = 1$ and $N_{1,i} \geq 2^{n_{1,i+1}+m_1}$ for $2 \leq i \leq q_1 - 1$;
2) $N_{k,i} \geq 2^{n_{k,i+1}+m_k}$ for $k \geq 2, 1 \leq i \leq q_k - 1$, and $N_{k,q_k} \geq 2^{n_{k+1,1}+m_{k+1}}$ for $k \geq 1$;
3) for $k \geq 1$, $1 \leq i \leq q_k - 1$,
$$N_{k,i+1} \geq 2^{N_{1,1}n_{1,1}+N_{1,2}(n_{1,2}+m_1)+\cdots+N_{k,i}(n_{k,i}+m_k)},$$
and, for $k \geq 1$,
$$N_{k+1,1} \geq 2^{N_{1,1}n_{1,1}+N_{1,2}(n_{1,2}+m_1)+\cdots+N_{k,q_k}(n_{k,q_k}+m_k)}.$$

We consider from now on the set F in (4.15) obtained from the sequences
$$\begin{aligned}(n_1, n_2, n_3, \ldots) &= (n_{1,1}, n_{1,2}, \ldots, n_{1,q_1}, n_{2,1}, n_{2,2}, \ldots),\\ (W_1, W_2, W_3, \ldots) &= (W_{1,1}, W_{1,2}, \ldots, W_{1,q_1}, W_{2,1}, W_{2,2}, \ldots),\\ (N_1, N_2, N_3, \ldots) &= (N_{1,1}, N_{1,2}, \ldots, N_{1,q_1}, N_{2,1}, N_{2,2}, \ldots).\end{aligned}$$

We first establish inequality (4.18). Let $\ell_{k,i}$ be the integer obtained as in (4.14) but now indexed as the numbers $n_{k,i}$, that is,
$$\begin{aligned}\ell_{k,i_k} = N_{1,1}n_{1,1} &+ \sum_{i=2}^{q_1} N_{1,i}(n_{1,i} + m_1)\\ &+ \sum_{i=2}^{k-1}\sum_{j=1}^{q_i} N_{i,j}(n_{i,j} + m_i) + \sum_{j=1}^{i_k} N_{k,j}(n_{k,j} + m_k).\end{aligned}$$
For each $n \geq \ell_{1,1}$, there exist either k, i_k and p with $i_k \in \{1, \ldots, q_k - 1\}$, $0 \leq p \leq N_{k,i_k+1} - 1$ such that
$$\ell_{k,i_k} + p(n_{k,i_k+1} + m_k) < n \leq \ell_{k,i_k} + (p+1)(n_{k,i_k+1} + m_k) \tag{4.21}$$
or k and p with $0 \leq p \leq N_{k+1,1} - 1$ such that
$$\ell_{k,q_k} + p(n_{k+1,1} + m_{k+1}) < n \leq \ell_{k,q_k} + (p+1)(n_{k+1,1} + m_{k+1}). \tag{4.22}$$

We consider only the first case (the proof is analogous in the second case). By (4.16), (4.20) and the choice of the integers $N_{k,i}$ together with (4.21) and (4.22), we obtain
$$\begin{aligned}&\frac{1}{n}(N_{1,1} \log M_{1,1} + \cdots + N_{k,i_k} \log M_{k,i_k} + p \log M_{k,i_k+1})\\ &\qquad \geq \frac{N_{1,1}n_{1,1} + N_{1,2}n_{1,2} + \cdots + N_{k,i_k}n_{k,i_k} + pn_{k,i_k+1}}{n}(\beta - \gamma/2)\\ &\qquad \geq \frac{\ell_{k,i_k} + p(n_{k,i_k+1} + m_k)}{n}(\beta - \gamma) \geq \Big(1 - \frac{n_{k,i_k+1} + m_k}{n}\Big)(\beta - \gamma)\\ &\qquad \geq \Big(1 - \frac{n_{k,i_k+1} + m_k}{N_{k,i_k}}\Big)(\beta - \gamma)\end{aligned}$$
for any sufficiently large k. Since γ is arbitrary, inequality (4.18) follows now readily from Lemma 4.3.

Now we turn to the proof of (4.17). We must show that, for each $x \in F$,

$$I \subset A(x), \tag{4.23}$$

and

$$A(x) \subset I. \tag{4.24}$$

Step 4. Proof of inclusion (4.23). Given $\alpha \in I$, take $i_k \in \{2, \ldots, q_k - 1\}$ such that $\alpha \in B(\alpha_{k,i_k}, 1/k)$. Again, for simplicity of the notation, without loss of generality we assume that $i_k \notin \{1, q_k\}$. Let

$$R_{k,i} = \max_{z \in L_{k,i}} \left| \sum_{p=0}^{\ell_{k,i}-1} \varphi(f^p(z)) - \sum_{p=0}^{\ell_{k,i}-1} \psi(f^p(z))\alpha_{k,i} \right|.$$

Lemma 4.4. *We have*

$$\lim_{k\to\infty} \frac{R_{k,i_k}}{\ell_{k,i_k}} = 0.$$

Proof. Given $y \in D_{k,i_k}$, there exist $x_1^{k,i_k}, \ldots, x_{N_{k,i_k}}^{k,i_k} \in W_{k,i_k}$ such that

$$d_{n_{k,i_k}}(x_j^{k,i_k}, f^{a_j}(y)) < \frac{\varepsilon}{2^k},$$

where $a_j = (j-1)(n_{k,i_k} + m_k)$ for $j = 1, \ldots, N_{k,i_k}$. Therefore,

$$\left| \sum_{p=0}^{n_{k,i_k}-1} \varphi(f^p(x_j^{k,i_k})) - \sum_{p=0}^{n_{k,i_k}-1} \varphi(f^{a_j+p}(y)) \right| \le n_{k,i_k}\, b_k, \tag{4.25}$$

where

$$b_k = \max\left\{ \operatorname{var}\left(\varphi, \frac{\varepsilon}{2^k}\right), \operatorname{var}\left(\psi, \frac{\varepsilon}{2^k}\right) \right\}.$$

Now we consider the decomposition

$$\begin{aligned} [0, t_{k,i_k} - 1] = &\bigcup_{j=1}^{N_{k,i_k}} [a_j, a_j + n_{k,i_k} - 1] \\ &\cup \bigcup_{j=1}^{N_{k,i_k}-1} [a_j + n_{k,i_k}, a_j + n_{k,i_k} + m_k - 1], \end{aligned} \tag{4.26}$$

where

$$t_{k,i_k} = (N_{k,i_k} - 1)m_k + N_{k,i_k} n_{k,i_k}.$$

On each interval $[a_j, a_j + n_{k,i_k} - 1]$ we have the estimate in (4.25). On the other hand, on each interval in the second union in (4.26), since $|\alpha_{k,i_k}| \le \eta$ for some constant $\eta > 0$ depending only on φ and ψ, we have

$$\begin{aligned} &\left| \sum_{p=0}^{m_k-1} \varphi(f^{a_j+n_{k,i_k}+p}(y)) - \sum_{p=0}^{m_k-1} \psi(f^{a_j+n_{k,i_k}+p}(y))\alpha_{k,i_k} \right| \\ &\qquad\qquad \le m_k(\|\varphi\| + \|\psi\| \cdot |\alpha_{k,i_k}|) \le m_k\, \eta', \end{aligned}$$

where

$$\eta' = \|\varphi\| + \|\psi\|\,\eta.$$

Therefore,

$$\begin{aligned}
&\Big|\sum_{p=0}^{t_{k,i_k}-1}\varphi(f^p(y)) - \sum_{p=0}^{t_{k,i_k}-1}\psi(f^p(y))\alpha_{k,i_k}\Big| \\
&\le \sum_{j=1}^{N_{k,i_k}}\Big|\sum_{p=a_j}^{a_j+n_{k,i_k}-1}\varphi(f^p(y)) - \sum_{p=a_j}^{a_j+n_{k,i_k}-1}\psi(f^p(y))\alpha_{k,i_k}\Big| \\
&\quad + \sum_{j=1}^{N_{k,i_k}-1}\Big|\sum_{p=a_j+n_{k,i_k}}^{a_j+n_{k,i_k}+m_k-1}\varphi(f^p(y)) - \sum_{p=a_j+n_{k,i_k}}^{a_j+n_{k,i_k}+m_k-1}\psi(f^p(y))\alpha_{k,i_k}\Big| \\
&\le \sum_{j=1}^{N_{k,i_k}}\Big|\sum_{p=0}^{n_{k,i_k}-1}\varphi(f^{p+a_j}(y)) - \sum_{p=0}^{n_{k,i_k}-1}\varphi(f^p(x_j^{k,i_k}))\Big| \\
&\quad + \sum_{j=1}^{N_{k,i_k}}\Big|\sum_{p=0}^{n_{k,i_k}-1}\varphi(f^p(x_j^{k,i_k})) - \sum_{p=0}^{n_{k,i_k}-1}\psi(f^{p+a_j}(y))\alpha_{k,i_k}\Big| + (N_{k,i_k}-1)m_k\eta' \\
&\le N_{k,i_k}n_{k,i_k}b_k + (N_{k,i_k}-1)m_k\eta' \\
&\quad + \sum_{j=1}^{N_{k,i_k}}\Big|\sum_{p=0}^{n_{k,i_k}-1}\varphi(f^p(x_j^{k,i_k})) - \sum_{p=0}^{n_{k,i_k}-1}\psi(f^{p+a_j}(y))\alpha_{k,i_k}\Big|.
\end{aligned} \tag{4.27}$$

Since $x_j^{k,i_k} \in W_{k,i_k} \subset P(\alpha_{k,i_k}, \delta_{k,i_k}, n_{k,i_k})$, we have

$$\Big|\sum_{p=0}^{n_{k,i_k}-1}\varphi(f^p(x_j^{k,i_k})) - \sum_{p=0}^{n_{k,i_k}-1}\psi(f^p(x_j^{k,i_k}))\alpha_{k,i_k}\Big| \le n_{k,i_k}\delta_{k,i_k}$$

and thus,

$$\begin{aligned}
&\sum_{j=1}^{N_{k,i_k}}\Big|\sum_{p=0}^{n_{k,i_k}-1}\varphi(f^p(x_j^{k,i_k})) - \sum_{p=0}^{n_{k,i_k}-1}\psi(f^{p+a_j}(y))\alpha_{k,i_k}\Big| \\
&\le \sum_{j=1}^{N_{k,i_k}}\Big|\sum_{p=0}^{n_{k,i_k}-1}\varphi(f^p(x_j^{k,i_k})) - \sum_{p=0}^{n_{k,i_k}-1}\psi(f^p(x_j^{k,i_k}))\alpha_{k,i_k}\Big| \\
&\quad + \sum_{j=1}^{N_{k,i_k}}\Big|\sum_{p=0}^{n_{k,i_k}-1}\psi(f^p(x_j^{k,i_k})) - \sum_{p=0}^{n_{k,i_k}-1}\psi(f^{p+a_j}(y))\Big| \cdot |\alpha_{k,i_k}| \\
&\le N_{k,i_k}n_{k,i_k}\delta_{k,i_k} + N_{k,i_k}n_{k,i_k}b_k\eta.
\end{aligned} \tag{4.28}$$

Finally, it follows from (4.27) and (4.28) that

$$\left|\sum_{p=0}^{t_{k,i_k}-1}\varphi(f^p(y))-\sum_{p=0}^{t_{k,i_k}-1}\psi(f^p(y))\alpha_{k,i_k}\right| \le N_{k,i_k}n_{k,i_k}(b_k+\delta_{k,i_k}+b_k\eta)+(N_{k,i_k}-1)\,m_k\,\eta'. \tag{4.29}$$

On the other hand, by the definition of L_{k,i_k}, for each $z\in L_{k,i_k}$ there exist $x\in L_{k,i_k-1}$ and $y\in D_{k,i_k}$ such that

$$d_{\ell_{k,i_k-1}}(x,z)<\frac{\varepsilon}{2^k} \quad\text{and}\quad d_{t_{k,i_k}}(y,f^{\ell_{k,i_k-1}+m_k}(z))<\frac{\varepsilon}{2^k}. \tag{4.30}$$

Therefore,

$$\left|\sum_{p=0}^{\ell_{k,i_k}-1}\varphi(f^p(z))-\sum_{p=0}^{\ell_{k,i_k}-1}\varphi(f^p(z))\alpha_{k,i_k}\right|\le S_1(k)+S_2(k)+S_3(k),$$

where

$$S_1(k)=\left|\sum_{p=0}^{\ell_{k,i_k-1}-1}\varphi(f^p(z))-\sum_{p=0}^{\ell_{k,i_k-1}-1}\psi(f^p(z))\alpha_{k,i_k}\right|,$$

$$S_2(k)=\left|\sum_{p=\ell_{k,i_k-1}}^{\ell_{k,i_k-1}+m_k-1}\varphi(f^p(z))-\sum_{p=\ell_{k,i_k-1}}^{\ell_{k,i_k-1}+m_k-1}\psi(f^p(z))\alpha_{k,i_k}\right|,$$

$$S_3(k)=\left|\sum_{p=\ell_{k,i_k-1}+m_k}^{\ell_{k,i_k}-1}\varphi(f^p(z))-\sum_{p=\ell_{k,i_k-1}+m_k}^{\ell_{k,i_k}-1}\psi(f^p(z))\alpha_{k,i_k}\right|.$$

Clearly,

$$S_1(k)\le \ell_{k,i_k-1}\eta' \quad\text{and}\quad S_2(k)\le m_k\eta'.$$

Moreover, it follows from (4.29) and (4.30) that

$$\begin{aligned}
S_3(k)&\le\left|\sum_{p=\ell_{k,i_k-1}+m_k}^{\ell_{k,i_k}-1}\varphi(f^p(z))-\sum_{p=\ell_{k,i_k-1}+m_k}^{\ell_{k,i_k}-1}\psi(f^p(z))\alpha_{k,i_k}\right|\\
&\le\left|\sum_{p=0}^{t_{k,i_k}-1}\varphi(f^{\ell_{k,i_k-1}+m_k+p}(z))-\sum_{p=0}^{t_{k,i_k}-1}\varphi(f^p(y))\right|\\
&\quad+\left|\sum_{p=0}^{t_{k,i_k}-1}\varphi(f^p(y))-\sum_{p=0}^{t_{k,i_k}-1}\psi(f^p(y))\alpha_{k,i_k}\right|\\
&\quad+\left|\sum_{p=0}^{t_{k,i_k}-1}\psi(f^p(y))-\sum_{p=0}^{t_{k,i_k}-1}\psi(f^{\ell_{k,i_k-1}+m_k+p}(z))\right|\cdot|\alpha_{k,i_k}|\\
&\le t_{k,i_k}b_k+N_{k,i_k}n_{k,i_k}(b_k+\delta_{k,i_k}+b_k\eta)+(N_{k,i_k}-1)m_k\eta'+t_{k,i_k}b_k\eta.
\end{aligned}$$

Therefore,

$$R_{k,i_k} \le (\ell_{k,i_k-1} + N_{k,i_k} m_k)\eta' \\ + N_{k,i_k} n_{k,i_k}(b_k + \delta_{k,i_k} + b_k\eta) + t_{k,i_k}(b_k + b_k\eta).$$

By the choice of the integers $N_{k,i}$, we have $\ell_{k,i_k} \ge 2^{\ell_{k,i_k-1}}$ and hence,

$$\frac{R_{k,i_k}}{\ell_{k,i_k}} \to 0 \quad \text{when} \quad k \to \infty.$$

This completes the proof of the lemma. □

We proceed with the proof of inclusion (4.23). Since $x \in F$, there exists $z \in L_{k,i_k+1}$ such that

$$d_{\ell_{k,i_k+1}}(x,z) < \frac{\varepsilon}{2^k}. \tag{4.31}$$

Moreover, since $z \in L_{k,i_k+1}$, there exist $\overline{x} \in L_{k,i_k}$ and $y \in D_{k,i_k+1}$ such that

$$d_{\ell_{k,i_k}}(\overline{x},z) < \frac{\varepsilon}{2^k} \quad \text{and} \quad d_{t_{k,i_k+1}}(y, f^{\ell_{k,i_k}+m_k}(z)) < \frac{\varepsilon}{2^k}.$$

Therefore,

$$d_{\ell_{k,i_k}}(\overline{x},x) < \frac{\varepsilon}{2^{k-1}} \quad \text{and} \quad d_{t_{k,i_k+1}}(y, f^{\ell_{k,i_k}+m_k}(x)) < \frac{\varepsilon}{2^{k-1}}. \tag{4.32}$$

It follows from (4.32) and Lemma 4.4 that

$$\begin{aligned}
&\left|\sum_{p=0}^{\ell_{k,i_k}-1} \varphi(f^p(x)) - \sum_{p=0}^{\ell_{k,i_k}-1} \psi(f^p(x))\alpha_{k,i_k}\right| \\
&\le \left|\sum_{p=0}^{\ell_{k,i_k}-1} \varphi(f^p(x)) - \sum_{p=0}^{\ell_{k,i_k}-1} \varphi(f^p(\overline{x}))\right| + \left|\sum_{p=0}^{\ell_{k,i_k}-1} \varphi(f^p(\overline{x})) - \sum_{p=0}^{\ell_{k,i_k}-1} \psi(f^p(x))\alpha_{k,i_k}\right| \\
&\le \ell_{k,i_k} b_{k-1} + \left|\sum_{p=0}^{\ell_{k,i_k}-1} \varphi(f^p(\overline{x})) - \sum_{p=0}^{\ell_{k,i_k}-1} \psi(f^p(\overline{x}))\alpha_{k,i_k}\right| \\
&\quad + \left|\sum_{p=0}^{\ell_{k,i_k}-1} \psi(f^p(\overline{x})) - \sum_{p=0}^{\ell_{k,i_k}-1} \psi(f^p(x))\right| \cdot |\alpha_{k,i_k}| \\
&\le \ell_{k,i_k} b_{k-1} + R_{k,i_k} + \ell_{k,i_k} b_{k-1}\eta. \qquad (4.33)
\end{aligned}$$

Finally, it follows from Lemma 4.4 and (4.33) that

$$\left|\frac{\sum_{p=0}^{\ell_{k,i_k}-1} \varphi(f^p(x))}{\sum_{p=0}^{\ell_{k,i_k}-1} \psi(f^p(x))} - \alpha_{k,i_k}\right| \to 0$$

when $k \to \infty$. Therefore,

$$\left|\frac{\sum_{p=0}^{\ell_{k,i_k}-1}\varphi(f^p(x))}{\sum_{p=0}^{\ell_{k,i_k}-1}\psi(f^p(x))}-\alpha\right| \le \left|\frac{\sum_{p=0}^{\ell_{k,i_k}-1}\varphi(f^p(x))}{\sum_{p=0}^{\ell_{k,i_k}-1}\psi(f^p(x))}-\alpha_{k,i_k}\right| + |\alpha_{k,i_k}-\alpha|$$
$$\le \left|\frac{\sum_{p=0}^{\ell_{k,i_k}-1}\varphi(f^p(x))}{\sum_{p=0}^{\ell_{k,i_k}-1}\psi(f^p(x))}-\alpha_{k,i_k}\right| + \frac{1}{k} \to 0$$

when $k \to \infty$, which implies that $\alpha \in A(x)$ and inclusion (4.23) holds.

Step 5. Proof of inclusion (4.24). Take $x \in F$ and $n > \ell_{1,1}$. There exist k, i_k and j with $i_k \in \{1, \dots, q_k\}$ (again, without loss of generality we assume that $i_k \ne q_k$) and $0 \le j \le N_{k,i_k+1} - 1$ such that

$$\ell_{k,i_k} + j(n_{k,i_k+1} + m_k) < n \le \ell_{k,i_k} + (j+1)(n_{k,i_k+1} + m_k). \tag{4.34}$$

Lemma 4.5. *$|S(x,n) - \alpha_{k,i_k}| \to 0$ when $k \to \infty$.*

Proof. As in (4.31), since $x \in F$, there exists $z \in L_{k,i_k+1}$ such that

$$d_{\ell_{k,i_k+1}}(x,z) < \frac{\varepsilon}{2^k}.$$

Moreover, since $z \in L_{k,i_k+1}$, there exist $\overline{x} \in L_{k,i_k}$ and $y \in D_{k,i_k+1}$ such that

$$d_{\ell_{k,i_k}}(\overline{x},z) < \frac{\varepsilon}{2^k} \quad \text{and} \quad d_{t_{k,i_k+1}}(y, f^{\ell_{k,i_k}+m_k}(z)) < \frac{\varepsilon}{2^k}.$$

Therefore,

$$d_{\ell_{k,i_k}}(\overline{x},x) < \frac{\varepsilon}{2^{k-1}} \quad \text{and} \quad d_{t_{k,i_k+1}}(y, f^{\ell_{k,i_k}+m_k}(x)) < \frac{\varepsilon}{2^{k-1}}.$$

When $j > 0$ in (4.34), there exist $x_1^{k,i_k+1}, \dots, x_j^{k,i_k+1} \in W_{k,i_k+1}$ such that

$$d_{n_{k,i_k+1}}(x_r^{k,i_k+1}, f^{a_r}(y)) < \frac{\varepsilon}{2^k},$$

where $a_r = (n_{k,i_k+1} + m_k)(r-1)$, $r = 1, \dots, j$, and hence,

$$d_{n_{k,i_k+1}}(x_r^{k,i_k+1}, f^{\ell_{k,i_k}+m_k+a_r}(x)) < \frac{\varepsilon}{2^{k-2}}. \tag{4.35}$$

Now we write

$$[0,n-1] = [0,\ell_{k,i_k}-1]$$
$$\cup \bigcup_{r=1}^{j}\left[\ell_{k,i_k}+(r-1)(m_k+n_{k,i_k+1}), \ell_{k,i_k}+r(m_k+n_{k,i_k+1})-1\right]$$
$$\cup \left[\ell_{k,i_k}+j(m_k+n_{k,i_k+1}), n-1\right].$$

On the interval $[0, \ell_{k,i_k} - 1]$ we have the estimate in (4.33). On each interval $[b_r, b_r + (m_k + n_{k,i_k+1}) - 1]$, where

$$b_r = \ell_{k,i_k} + (r-1)(m_k + n_{k,i_k+1}),$$

we have

$$\begin{aligned}
\Bigg|&\sum_{p=b_r}^{b_r+(m_k+n_{k,i_k+1})-1} \varphi(f^p(x)) - \sum_{p=b_r}^{b_r+(m_k+n_{k,i_k+1})-1} \psi(f^p(x))\alpha_{k,i_k}\Bigg| \\
&\le \Bigg|\sum_{p=b_r}^{b_r+m_k-1} \varphi(f^p(x)) - \sum_{p=b_r}^{b_r+m_k-1} \psi(f^p(x))\alpha_{k,i_k}\Bigg| \\
&\quad + \Bigg|\sum_{p=0}^{n_{k,i_k+1}-1} \varphi(f^{b_r+m_k+p}(x)) - \sum_{p=0}^{n_{k,i_k+1}-1} \psi(f^{b_r+m_k+p}(x))\alpha_{k,i_k}\Bigg| \\
&\le m_k\eta' + \Bigg|\sum_{p=0}^{n_{k,i_k+1}-1} \varphi(f^{b_r+m_k+p}(x)) - \sum_{p=0}^{n_{k,i_k+1}-1} \psi(f^{b_r+m_k+p}(x))\alpha_{k,i_k}\Bigg|.
\end{aligned}$$

Since $x_r^{k,i_k+1} \in W_{k,i_k+1} \subset P(\alpha_{k,i_k+1}, \delta_{k,i_k+1}, n_{k,i_k+1})$, it follows from (4.35) that

$$\begin{aligned}
\Bigg|&\sum_{p=0}^{n_{k,i_k+1}-1} \varphi(f^{b_r+m_k+p}(x)) - \sum_{p=0}^{n_{k,i_k+1}-1} \psi(f^{b_r+m_k+p}(x))\alpha_{k,i_k}\Bigg| \\
&\le \Bigg|\sum_{p=0}^{n_{k,i_k+1}-1} \varphi(f^{b_r+m_k+p}(x)) - \sum_{p=0}^{n_{k,i_k+1}-1} \varphi(f^p(x_r^{k,i_k+1}))\Bigg| \\
&\quad + \Bigg|\sum_{p=0}^{n_{k,i_k+1}-1} \varphi(f^p(x_r^{k,i_k+1})) - \sum_{p=0}^{n_{k,i_k+1}-1} \psi(f^p(x_r^{k,i_k+1}))\alpha_{k,i_k+1}\Bigg| \\
&\quad + \Bigg|\sum_{p=0}^{n_{k,i_k+1}-1} \psi(f^p(x_r^{k,i_k+1})) - \sum_{p=0}^{n_{k,i_k+1}-1} \psi(f^{b_r+m_k+p}(x))\Bigg| \cdot |\alpha_{k,i_k}| \\
&\quad + \sum_{p=0}^{n_{k,i_k+1}-1} \psi(f^p(x_r^{k,i_k+1}))|\alpha_{k,i_k} - \alpha_{k,i_k+1}| \\
&\le n_{k,i_k+1}\Big(b_{k-2} + \delta_{k,i_k+1} + b_{k-2}\eta' + \frac{\|\psi\|}{k}\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\Bigg|&\sum_{p=b_r}^{b_r+(m_k+n_{k,i_k+1})-1} \varphi(f^p(x)) - \sum_{p=b_r}^{b_r+(m_k+n_{k,i_k+1})-1} \psi(f^p(x))\alpha_{k,i_k}\Bigg| \\
&\le m_k\eta' + n_{k,i_k+1}\Big(b_{k-2} + \delta_{k,i_k+1} + b_{k-2}\eta + \frac{\|\psi\|}{k}\Big).
\end{aligned} \tag{4.36}$$

Finally, on the interval $[\ell_{k,i_k}+j(m_k+n_{k,i_k+1}), n-1]$ we have

$$\tag{4.37} \begin{aligned} \Big| \sum_{p=\ell_{k,i_k}+j(m_k+n_{k,i_k+1})}^{n-1} \varphi(f^p(x)) - \sum_{p=\ell_{k,i_k}+j(m_k+n_{k,i_k+1})}^{n-1} \psi(f^p(x))\alpha_{k,i_k} \Big| \\ \le (n-\ell_{k,i_k}-j(m_k+n_{k,i_k+1}))\eta' \le (n_{k,i_k+1}+m_k)\eta'. \end{aligned}$$

By (4.33), (4.36) and (4.37), we obtain

$$\begin{aligned} &\Big| \sum_{p=0}^{n-1} \varphi(f^p(x)) - \sum_{p=0}^{n-1} \psi(f^p(x))\alpha_{k,i_k} \Big| \\ &\quad \le \ell_{k,i_k}(b_{k-1}+b_{k-1}\eta) + R_{k,i_k} + (n_{k,i_k+1}+(j+1)m_k)\eta' \\ &\quad\quad + jn_{k,i_k+1}\Big(b_{k-2}+\delta_{k,i_k+1}+b_{k-2}\eta+\frac{\|\psi\|}{k}\Big) \end{aligned}$$

and hence,

$$\begin{aligned} \left| \frac{\sum_{p=0}^{n-1}\varphi(f^p(x))}{\sum_{p=0}^{n-1}\psi(f^p(x))} - \alpha_{k,i_k} \right| &< b_{k-1}+b_{k-1}\eta+\frac{R_{k,i_k}}{\ell_{k,i_k}}+\frac{n_{k,i_k+1}+(j+1)m_k}{N_{k,i_k}}\eta' \\ &\quad + b_{k-2}+\delta_{k,i_k+1}+b_{k-2}\eta+\frac{\|\psi\|}{k}. \end{aligned}$$

By Lemma 4.4 and the choice of the integers $N_{k,i}$, the right-hand side tends to zero when $k\to\infty$. This completes the proof of the lemma. □

Now we use Lemma 4.5 to prove inclusion (4.24). Fix $x \in F$. For $n \in \mathbb{N}$, by (4.19) and Lemma 4.5, we have

$$\operatorname{dist}(A(x), I) \le \left| \frac{\sum_{p=0}^{n-1}\varphi(f^p(x))}{\sum_{p=0}^{n-1}\psi(f^p(x))} - \alpha_{k,i_k} \right| + \operatorname{dist}(\alpha_{k,i_k}, I) \to 0$$

when $n\to\infty$. Since I is closed, this implies that $A(x)\subset I$.

Step 6. Conclusion of the proof. It follows from (4.17) and (4.18) that

$$\tag{4.38} h(f|X_I) \ge \inf_{\alpha\in I} \Lambda(\alpha).$$

In particular, taking $I=\{\alpha\}$ with $\alpha\in\mathcal{L}$, we obtain

$$\tag{4.39} \Lambda(\alpha) \le h(f|X(\alpha)).$$

On the other hand, Thompson showed in [21] that

$$\tag{4.40} h(f|X(\alpha)) \le H(\alpha).$$

It follows from (4.39) and (4.40) together with Lemma 4.1 that

$$\tag{4.41} h(f|X(\alpha)) = H(\alpha) = \Lambda(\alpha)$$

for $\alpha \in \mathcal{L}$. In particular, by (4.38), we obtain

$$h(f|X_I) \geq \inf_{\alpha \in I} H(\alpha). \tag{4.42}$$

Now we establish an auxiliary result.

Lemma 4.6. *$h(f|X_\alpha) \leq \Lambda(\alpha)$ for each $\alpha \in \mathcal{L}$.*

Proof. For each $\delta > 0$ we have

$$X_\alpha \subset \bigcap_{p=1}^{\infty} \bigcup_{k \geq p} P(\alpha, \delta, k) = \bigcap_{p=1}^{\infty} G_p,$$

where

$$G_p = \bigcup_{k \geq p} P(\alpha, \delta, k).$$

Let

$$h(f|Z, \varepsilon) = \inf\big\{s \geq 0 : m(Z, s, \varepsilon) = 0\big\}.$$

We will show that $h(f|G_p, \varepsilon) \leq \Lambda(\alpha)$ for every $p \in \mathbb{N}$ and all sufficiently small $\varepsilon > 0$. Therefore, $h(f|X_\alpha, \varepsilon) \leq \Lambda(\alpha)$ and

$$h(f|X_\alpha) = \lim_{\varepsilon \to 0} h(f|Z, \varepsilon) \leq \Lambda(\alpha).$$

For each $k \in \mathbb{N}$ the set $P_k := P(\alpha, \delta, k)$ can be covered by a number $N(\alpha, \delta, k, \varepsilon)$ of d_k-balls of radius ε. Hence, for each $s \geq 0$, we have

$$m(P_k, s, \varepsilon) \leq N(\alpha, \delta, k, \varepsilon) \exp(-ks). \tag{4.43}$$

Now let $s > \Lambda(\alpha)$ and $\gamma = (s - \Lambda(\alpha))/2 > 0$. By (4.3), for all sufficiently small $\varepsilon > 0$ and $\delta > 0$ (possibly depending on ε), we have

$$N(\alpha, \delta, k, \varepsilon) \leq \exp(k(\Lambda(\alpha) + \gamma))$$

for $k \in \mathbb{N}$. Hence, it follows from (4.43) that

$$m(P_k, s, \varepsilon) \leq \exp(-k\gamma).$$

Therefore, $m(P_k, s, \varepsilon) = 0$, which implies that $h(f|P_k, \varepsilon) \leq s$. Taking the union over k we obtain $h(f|G_p, \varepsilon) \leq s$ and letting $s \to \Lambda(\alpha)$ yields the desired result. □

Finally, for $\alpha \in \mathcal{L}$ we define

$$X_\alpha = \{x \in X : \alpha \in A(x)\}.$$

Since $X_I \subset X_\alpha$ for every $\alpha \in I$, it follows immediately from (4.41) and Lemma 4.6 that

$$h(f|X_I) \leq \inf_{\alpha \in I} H(\alpha).$$

Together with (4.42) this yields statement 2 of the theorem.

For statement 1, note that if a_n is a bounded sequence such that $a_{n+1}-a_n \to 0$ when $n \to \infty$, then its set of accumulation points A is a bounded closed interval. In particular, since

$$\begin{aligned}
&\big|S(x,n+1)-S(x,n)\big| \\
&\qquad \le \frac{1}{n(n+1)(\inf\psi)^2}\left|\varphi(f^n(x))\sum_{i=0}^{n-1}\psi(f^i(x))-\psi(f^n(x))\sum_{i=0}^{n-1}\varphi(f^i(x))\right| \\
&\qquad \le \frac{2\,\|\varphi\|\cdot\|\psi\|}{(n+1)(\inf\psi)^2}\to 0
\end{aligned}$$

when $n \to \infty$, we obtain statement 1. This completes the proof.

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Received February 1, 2016.

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L. B. and C. V. were supported by FCT/Portugal through UID/MAT/04459/2013. J. L. was supported by the National Natural Science Foundation of China (11671189) and the Natural Science Foundation of Fujian Province (2017J01403).