



The Loewner equation and Lipschitz graphs

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Abstract. The proofs of continuity of Loewner traces in the stochastic and in the deterministic settings employ different techniques. In the former setting of the Schramm–Loewner evolution SLE, Hölder continuity of the conformal maps is shown by estimating the derivatives, whereas the latter setting uses the theory of quasiconformal maps. In this note, we adopt the former method to the deterministic setting and obtain a new and elementary proof that Hölder-1/2 driving functions with norm less than 4 generate simple arcs. We also give a sufficient condition for driving functions to generate curves that are graphs of Lipschitz functions.

1. Introduction and results

The chordal Loewner differential equation

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t}, \quad g_0(z) = z,$$

which will be briefly reviewed in Section 2, provides a one-to-one correspondence between certain decreasing families of simply connected subdomains $\mathbb{H} \setminus K_t$ of the upper half plane, and real-valued continuous functions λ_t . Initially developed as a tool to study extremal problems in complex analysis [7], it has become an important tool in probability theory, based on Oded Schramm’s insight [11] that Brownian motion arises naturally as the driving function λ_t in various settings of random sets K_t .

While it is not hard to see that for simple curves $\gamma \subset \mathbb{H} \cup \{0\}$ with $\gamma_0 = 0$, the family $K_t = \gamma[0, t]$ yields continuous functions λ_t (see Section 2 for a precise statement), the converse is not true in general: There are continuous functions λ_t for which the associated hulls K_t are not locally connected, and hence not of the form $\gamma[0, t]$ for some continuous function γ .

In [8] and [6], the continuity problem was treated by viewing the hulls K_t as the result of “conformally welding” two intervals of the real line, and by applying the theory of quasiconformal maps to the welding problem. The main result is:

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Theorem 1.1 ([8], [6]). *If the driving function λ has Hölder-1/2 norm less than 4, then the chordal Loewner equation generates a simple curve γ . Moreover, γ is a quasiconformal arc that meets the real line non-tangentially.*

The constant 4 in the theorem is sharp in the sense that for each $c \geq 4$, there exists a driving function whose Hölder-1/2 norm is c and the Loewner equation generates a curve that meets the real line other than at the base, see examples in [3] and [6].

In the stochastic setting of the Schramm–Loewner evolution SLE_κ where the driving term is $\lambda_t = \sqrt{\kappa}B_t$ and B_t is a standard one-dimensional Brownian motion, the conformal welding approach leads to interesting and difficult problems, see [12] and [1] for related deep results. In [10], the almost sure continuity of the SLE_κ hulls was proved by different means: Based on estimates of the expectation $E[|(g_t^{-1})'(z)|^p]$ for suitable exponents p , it was shown that $\lim_{y \rightarrow 0} g_t^{-1}(\lambda_t + iy)$ exists, is continuous in t , and that this implies continuity of the hulls.

In this note, we will adopt the second method to the deterministic setting, and obtain a new and elementary proof of Theorem 1.1. The key observation is the following (see Theorem 3.1 below): under the “upward” flow (2.3), the point $z_t = f_t(i) - \lambda_t$ will never leave the cone $\{|x| < cy\}$, where c depends on the Hölder-1/2 norm only. Combined with the integral representation of $\log|f'_t(z)|$ ((2.8) below), this easily gives an estimate for $|(g_t^{-1})'(\lambda_t + iy)|$ that, when integrated, implies existence and continuity of $\lim_{y \rightarrow 0} g_t^{-1}(\lambda_t + iy)$. Standard conformal mapping techniques (particularly the Gehring–Hayman inequality) then yield the additional information that the trace is a quasiconformal arc approaching the real line non-tangentially. When applied to the integral representation (2.9) of $\arg f'_t(z)$, our approach also gives a sufficient condition for the driving functions to generate the graph of a Lipschitz function:

Theorem 1.2. *There exists a constant $C_0 > 0$ such that, for every continuous function λ satisfying*

$$(1.1) \quad \int_0^t \frac{N_{s,t}^\lambda}{(t-s)^{3/2}} ds \leq C_0 \quad \text{for all } 0 < t < T,$$

where $N_{s,t}^\lambda = \sup\{|\lambda_r - \lambda_s| : s \leq r \leq t\}$, the Loewner equation generates a graph of a Lipschitz function.

Theorem 1.2 is sharp, as the example $\lambda_t = c\sqrt{1-t}$ ($0 \leq t \leq 1$) shows (see [3]). In this example, the trace is asymptotic to a logarithmic spiral at the tip, hence it is not the graph of any function, and the integral in Theorem 1.2 diverges. On the other hand, the straight line of angle $\pi\alpha$ has driving function $\lambda_t = c\sqrt{t}$, where $c = 2(1-2\alpha)/\sqrt{\alpha(1-\alpha)}$, so that every Hölder-1/2 norm can arise from a simple curve. Notice that for $\lambda_t = c\sqrt{t}$, the integral has the finite value $c\sqrt{2} \sinh^{-1}(1)$.

The organization of the paper is as follows. In Section 2 we briefly review basic facts and definitions around the Loewner equation. The key points lie in Section 3, where we state and prove Theorem 3.1 regarding the trajectories $z_t = f_t(it) - \lambda_t$ of the backward Loewner flow when the Hölder-1/2 norm is less than 4. Finally, Theorems 1.1 and 1.2 will be proved in Section 4.

2. Preliminaries

Let $\gamma: [0, T] \rightarrow \mathbb{H} \cup \{0\}$ be a simple curve in the upper half-plane \mathbb{H} except that $\gamma_0 \in \mathbb{R}$. For each $t \in [0, T]$, by the Riemann mapping theorem there exists a unique conformal map $g_t: \mathbb{H} \setminus \gamma[0, t] \rightarrow \mathbb{H}$ satisfying the *hydrodynamic normalization*:

$$g_t(z) = z + \frac{c_t}{z} + \dots \quad \text{when } z \rightarrow \infty.$$

It can be shown that c_t is a nonnegative, strictly increasing, continuous function and $c_0 = 0$ [4]. Hence one can reparametrize γ so that $c_t = 2t$. Then for each $z \in \mathbb{H}$, the function $t \mapsto g_t(z)$ satisfies the (*downward*) *chordal Loewner equation*:

$$(2.1) \quad \partial_t g_t(z) = \frac{2}{g_t(z) - \lambda_t}, \quad g_0(z) = z,$$

where λ is a continuous, real-valued function and $g_t(\gamma_t) = \lambda_t$, see Chapter 4 in [4].

Conversely, if one starts with a continuous function $\lambda_t: [0, T] \rightarrow \mathbb{R}$, one can consider the initial value problem for each $z \in \mathbb{H}$:

$$\partial_t g(t, z) = \frac{2}{g(t, z) - \lambda_t}, \quad g(0, z) = z.$$

For each $z \in \mathbb{H}$ there is a maximal interval for which a solution $g(t, z)$ exists. Let $T_z = \sup\{s \in [0, T] : g(t, z) \text{ exists on } [0, s]\}$. It is easy to see that, if $T_z < T$, then

$$\lim_{t \rightarrow T_z} g(t, z) = \lambda_{T_z}.$$

Let $H_t = \{z \in \mathbb{H} : T_z > t\}$ and $g_t(z) = g(t, z)$. Then one can show that the set H_t is a simply connected subdomain of \mathbb{H} , and $g_t(z)$ is the unique conformal map from H_t onto \mathbb{H} with the following normalization near infinity:

$$g_t(z) = z + \frac{2t}{z} + O\left(\frac{1}{z^2}\right).$$

The *driving function* λ of the *Loewner chain* (g_t) is said to *generate a curve* if there exists a curve γ such that H_t is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$ for each $t \geq 0$. By Theorem 4.1 in [10], this is equivalent to the existence and continuity in $t > 0$ of

$$(2.2) \quad \gamma_t := \lim_{y \rightarrow 0^+} g_t^{-1}(\lambda_t + iy).$$

By Proposition 2.19 of [5] and Proposition 3.11 of [2], a very useful and simple criterion for this existence and continuity is the convergence to zero of

$$v(t, \varepsilon) := \int_0^\varepsilon |(g_t^{-1})'(\lambda_t + iy)| dy$$

as $\varepsilon \rightarrow 0$, uniformly in $t \in [0, T]$.

Rather than directly working with the Loewner equation (2.1), it is often easier to work with the *upward Loewner equation*:

$$(2.3) \quad \partial_t f_t(z) = \frac{-2}{f_t(z) - \xi_t}, \quad f_0(z) = z,$$

for $z \in \mathbb{H}$ and real-valued continuous function ξ_t . Since the imaginary part of $f_t(z)$ is strictly increasing, the solution exists for all time $t \geq 0$. If $(g_t)_{0 \leq t \leq T}$ is the solution to (2.1) with driving function λ and if $(f_t)_{0 \leq t \leq T}$ is the solution to (2.3) with $\xi_t = \lambda_{T-t}$, then

$$f_T(z) = g_T^{-1}(z).$$

We will frequently use the following two simple properties of the Loewner equation, regarding the translation and concatenation of driving functions:

First, if $a \in \mathbb{R}$ and $\tilde{\xi}_t = \xi_t + a$, then the Loewner chain (\tilde{f}_t) corresponding to $\tilde{\xi}$ is given by

$$\tilde{f}_t(z) = f_t(z - a) + a.$$

Second, let $(f_t^{(1)})_{0 \leq t \leq T_1}$ (respectively $(f_t^{(2)})_{0 \leq t \leq T_2}$) be the solution to (2.3) with the driving function $\xi^{(1)}$ defined on $[0, T_1]$ (respectively $\xi^{(2)}$ defined on $[0, T_2]$). Suppose $\xi_{T_1}^{(1)} = \xi_0^{(2)}$, and define the *concatenation* of $\xi^{(1)}$ and $\xi^{(2)}$ by

$$(2.4) \quad \xi_t = \begin{cases} \xi_t^{(1)}, & t \in [0, T_1], \\ \xi_{t-T_1}^{(2)}, & t \in [T_1, T_1 + T_2]. \end{cases}$$

Then the (upward) Loewner solution corresponding to ξ is given by

$$(2.5) \quad f_t = \begin{cases} f_t^{(1)}, & t \in [0, T_1], \\ f_{t-T_1}^{(2)} \circ f_{T_1}^{(1)}, & t \in [T_1, T_1 + T_2]. \end{cases}$$

The following notation will be used throughout the rest of the paper: if $z \in \mathbb{H}$ and $f_t(z)$ is the solution to (2.3), we define $x_t := x_t(z, \xi)$ and $y_t := y_t(z, \xi)$ by

$$x_t + iy_t := z_t := f_t(z) - \xi_t.$$

It follows that

$$(2.6) \quad \partial_t(x_t + \xi_t) = \frac{-2x_t}{x_t^2 + y_t^2}$$

and

$$(2.7) \quad \partial_t y_t = \frac{2y_t}{x_t^2 + y_t^2}.$$

The following expressions for $|f'_t(z)|$ and $\arg f'_t(z)$ in terms of x_t and y_t will be used to prove Theorems 1.1 and 1.2. Since

$$f'_t(z) = e^{\log f'_t(z)} = e^{\int_0^t \partial_s \log f'_s(z) ds}$$

and

$$\partial_s \log f'_s(z) = \frac{\partial_s f'_s(z)}{f'_s(z)} = \frac{2}{(f_s(z) - \xi(s))^2},$$

we have

$$(2.8) \quad |f'_t(z)| = \exp\left(2 \int_0^t \frac{x_s^2 - y_s^2}{(x_s^2 + y_s^2)^2} ds\right) = \exp\left(\int_0^t \frac{x_s^2 - y_s^2}{x_s^2 + y_s^2} \cdot \frac{2ds}{x_s^2 + y_s^2}\right)$$

and

$$(2.9) \quad \arg f'_t(z) = -4 \int_0^t \frac{x_s y_s}{(x_s^2 + y_s^2)^2} ds.$$

Finally, we will frequently use the following simple estimate for the oscillation of x_t for general driving functions.

Lemma 2.1. *Let ξ be an arbitrary continuous function.*

- a) *If $x_s \geq 0$ for all $0 \leq s \leq t$, then $x_t \leq x_0 + \xi_0 - \xi_t$.*
- b) *In general, $|x_t| \leq |x_0| + M_{0,t}^\xi$, where $M_{0,t}^\xi = \sup\{|\xi_r - \xi_t| : r \in [0, t]\}$.*

Proof. Since $x_s \geq 0$, the sum $x_t + \xi_t$ is nonincreasing by (2.6), and part a) follows. To prove b), by symmetry we may assume that $x_0 \geq 0$, and we may also assume $|x_t| > x_0$, else b) is trivial. Let $S = \sup\{0 \leq s < t : |x_s| \leq x_0\}$ so that $|x_S| = x_0$.

If $x_t \geq 0$ then $x_t > x_0$ and $x_0 = x_S < x_s$ for $S < s \leq t$. Applying a) with z replaced by $x_S + iy_S$ and ξ replaced by $\xi(\cdot + S)$ we get

$$x_t \leq x_S + \xi_S - \xi_t = x_0 + \xi_S - \xi_t.$$

If $x_t < 0$ then $x_t < -x_0$ and $x_s < x_S = -x_0$ for $S < s \leq t$. Now replacing z by $-(x_S + iy_S)$ and ξ by $-\xi_{+S}$, the claim follows again from a). □

3. Staying in a fixed cone

In this section, we restrict our attention to the upward Loewner equation (2.3) with driving function ξ whose Hölder-1/2 norm satisfies

$$\sigma := \|\xi\|_{1/2} = \sup_{s \neq t} \frac{|\xi_t - \xi_s|}{|t - s|^{1/2}} < 4.$$

Denote A_c the cone $\{x + iy : |x| \leq cy\}$ and $A_c(v) = v + A_c$ for $v \in \mathbb{R}$. The main result of this section is:

Theorem 3.1. *There is a constant c_σ such that, if $z_0 = iy$, then $z_t = f_t(z_0 + \xi_0) - \xi_t$ stays in the cone A_{c_σ} for all t . Moreover,*

$$(3.1) \quad \sqrt{\frac{4t}{1 + c_\sigma^2} + y^2} \leq y_t \leq \sqrt{4t + y^2}$$

for all $t \geq 0$, and $c_\sigma \leq \sigma/\sqrt{4 - \sigma^2}$ for $\sigma < 2$.

This theorem easily implies the Hölder continuity of f_t , Corollary 3.5 below. The intuition behind the proof of Theorem 3.1 is as follows. To first order, $\Delta z_t = \frac{-2}{z_t} \Delta t - \Delta \xi$. Therefore, the larger x_t/y_t , the stronger $\frac{2}{z_t} \Delta t$ pushes towards the middle of the cone, and dominates $\Delta \xi$ if the Hölder-1/2 norm is small.

We will first show that an upper bound on the growth rate of x_t implies a lower bound on y_t that is comparable to the optimal upper bound $y_t \leq \sqrt{4t + y_0^2}$.

Lemma 3.2. *If $|x_t| < M\sqrt{t}$ for all $t \geq Cy_0^2$ with some $M < 2$ and $C > 0$, then*

$$(3.2) \quad y_t^2 \geq Lt$$

for all $t \geq 0$, where $L = \min(1/C, 4 - M^2) > 0$.

Proof. Since $L \leq 1/C$, we have $Lt \leq t/C \leq y_0^2 < y_t^2$ for $0 < t \leq t_0 := Cy_0^2$, where the last inequality comes from the fact that y_t is increasing in t which can be seen from the equation (2.7). If (3.2) were not true, there would be a minimal $s > t_0$ such that $y_s^2 = Ls$ and $y_t^2 \geq Lt$ on $[0, s]$. It follows from (2.7) that

$$\partial_t y_t^2 = \frac{4y_t^2}{x_t^2 + y_t^2} \geq \frac{4Lt}{M^2t + Lt} = \frac{4L}{M^2 + L} \geq L$$

for all $t_0 \leq t \leq s$, which implies $y_s^2 - y_{t_0}^2 \geq L(s - t_0)$. This contradicts the fact

$$y_s^2 - y_{t_0}^2 = Ls - y_{t_0}^2 < L(s - t_0). \quad \square$$

If $\sigma < 2$, then the assumption $|x_t| < M\sqrt{t}$ of the Lemma is satisfied with $M = \sigma$ by Lemma 2.1 and arbitrarily small C , and Theorem 3.1 follows easily. The reader who is only interested in a short proof of Theorem 1.1 for small Hölder-1/2 norm may thus skip ahead to Corollary 3.5. To deal with the case $2 \leq \sigma < 4$, we will show that the trivial bound $|x_t| \leq \sigma\sqrt{t}$ can be improved to an estimate $|x_t| < M\sqrt{t}$ for some $M < 2$ and t large enough, if we assume that z_t stays outside a cone. As a first step, we will show:

Lemma 3.3. *Let K and M be finite positive constants. If*

$$Ky_t \leq x_t \leq M\sqrt{t} \quad \text{for all } t \in [t_0, T],$$

then

$$x_t \leq \left(\sigma - \frac{4K^2}{K^2 + 1} \frac{1}{M} \right) \sqrt{t} + C \quad \text{for all } t \in [t_0, T],$$

where $C = (M + 4K^2/(M(K^2 + 1)))\sqrt{t_0}$.

Proof. It follows from the differential equation (2.6) for $x_t + \xi_t$ that

$$x_t + \xi_t - x_{t_0} - \xi_{t_0} = \int_{t_0}^t \frac{-2x_s}{x_s^2 + y_s^2} ds = \int_{t_0}^t \frac{-2(\frac{x_s}{y_s})^2}{(\frac{x_s}{y_s})^2 + 1} \frac{1}{x_s} ds \leq \frac{-2K^2}{K^2 + 1} \int_{t_0}^t \frac{1}{x_s} ds,$$

so that

$$\begin{aligned} x_t &\leq x_{t_0} - \frac{2K^2}{K^2 + 1} \int_{t_0}^t \frac{1}{x_s} ds + \sigma\sqrt{t} \leq M\sqrt{t_0} - \frac{2K^2}{K^2 + 1} \int_{t_0}^t \frac{1}{M\sqrt{s}} ds + \sigma\sqrt{t} \\ &= \left(\sigma - \frac{4K^2}{(K^2 + 1)M} \right) \sqrt{t} + \left(M + \frac{4K^2}{(K^2 + 1)M} \right) \sqrt{t_0}. \end{aligned} \quad \square$$

Lemma 3.4. *For every $\sigma < 4$ and $\sigma' > \sigma/2$ there are $K > 0$ and $C > 0$ such that, if $x_0 = Ky_0$ and if $x_t \geq Ky_t$ for all $t \geq 0$, then $|x_t| \leq \sigma'\sqrt{t}$ for all $t \geq Cy_0^2$.*

Proof. Let $M_0 = \sigma$, and $K = \sigma/\sqrt{16 - \sigma^2}$. Recursively define

$$M_{n+1} = \sigma - \frac{4K^2}{(K^2 + 1)M_n}$$

and notice that $M_n \rightarrow \sigma/2$ as $n \rightarrow \infty$. Hence there is N such that $M_N < \sigma'$. Because $x_t \leq x_0 + \sigma\sqrt{t}$, for every $M'_0 > \sigma$ there is C_0 such that $x_t \leq M'_0\sqrt{t}$ for all $t \in [C_0y_0^2, T]$. It follows from Lemma 3.3 that for every $M'_1 > M_1$ there is C_1 such that $x_t \leq M'_1\sqrt{t}$ for all $t \in [C_1y_0^2, T]$. Similarly, by continuity and N applications of Lemma 3.3, for every $M'_N > M_N$ there is C_N such that $x_t \leq M'_N\sqrt{t}$ for all $t \in [C_Ny_0^2, T]$. The lemma follows by choosing $M'_N = \sigma'$ and setting $C = C_N$. \square

We are now ready to give the:

Proof of Theorem 3.1. If $\sigma < 2$, we simply apply Lemma 3.2 with arbitrarily small C and find that

$$\frac{|x_t|}{y_t} \leq \frac{\sigma\sqrt{t}}{L\sqrt{t}} = \frac{\sigma}{\sqrt{4 - \sigma^2}}$$

for all t so that we can take $c_\sigma = \sigma/\sqrt{4 - \sigma^2}$. In general, fix $\sigma' \in (\sigma/2, 2)$ and let K and C be the constants of Lemma 3.4. Since $x_0 = 0$, the points z_t are in the cone A_K for all small t (see Figure 1). If for some t , the point z_t is outside A_K , then we can find an interval $[t_1, t_2]$ containing t so that

$$|x_{t_1}| = Ky_{t_1} \text{ and } |x_s| \geq Ky_s \text{ for all } t_1 \leq s \leq t_2,$$

and without loss of generality we may assume $x_s > 0$ on $[t_1, t_2]$. Replacing ξ by $\tilde{\xi}(\cdot) = \xi(\cdot + t_1) - \xi(t_1)$ on $[0, t_2 - t_1]$, we are now in the situation where $x_0 = Ky_0$ and $x_t \geq Ky_t$ for all $t \in [0, T]$ (where $T = t_2 - t_1$). By Lemma 3.4, we can apply Lemma 3.2 and obtain

$$\frac{x_t}{y_t} \leq \frac{\sigma'\sqrt{t}}{L\sqrt{t}} = \sigma' \max(C, 1/(4 - \sigma'^2))$$

for $t \geq Cy_0^2$, whereas

$$\frac{x_t}{y_t} \leq \frac{x_0 + \sigma\sqrt{t}}{y_0} \leq K + \sigma\sqrt{C}$$

for $t \leq Cy_0^2$. It follows that z_t never leaves the cone A_c where $c = \max(\sigma'C, \sigma'/(4 - \sigma'^2), K + \sigma\sqrt{C})$.

Finally, the estimate for y_t follows from $|x_t| \leq cy_t$ and

$$\partial_t y_t^2 = \frac{4y_t^2}{x_t^2 + y_t^2}. \quad \square$$

A simple consequence of Theorem 3.1 is the Hölder continuity in bounded subsets of the upper half plane of the solutions f_t to the upward Loewner equation (2.3) with driving functions satisfying $\sigma = \|\xi\|_{1/2} < 4$:

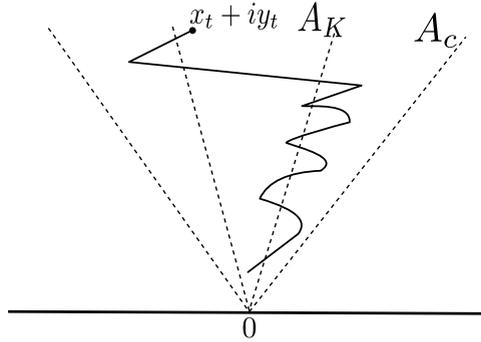


FIGURE 1: A trajectory of $x_t + iy_t$. It never leaves the cone A_c once outside A_K .

Corollary 3.5. *If $\sigma = \|\xi\|_{1/2} < 4$, then*

$$|f'_t(\xi_0 + iy)| \leq (4t + y^2)^{(1-\alpha)/2} y^{\alpha-1}$$

for every $y > 0$ and $t \in [0, T]$, where α is a constant in $(0, 1]$ depending on σ only.

Proof. By (2.7) and (2.8), Theorem 3.1 implies that

$$|f'_t(\xi_0 + iy)| \leq \exp\left(\int_0^t \frac{c^2 - 1}{c^2 + 1} \frac{2ds}{x_s^2 + y_s^2}\right) = \left(\frac{yt}{y}\right)^{\frac{c^2-1}{c^2+1}} \leq (4t + y^2)^{(1-\alpha)/2} y^{\alpha-1},$$

where $c = c_\sigma$ and $\alpha = \min\{1 - \frac{c^2-1}{c^2+1}, 1\} \in (0, 1]$. □

Remark 3.6. The proof of Theorem 3.1 can easily be modified to give the following statement: for every $0 < c_1 < c_2$, there is σ_0 such that, if $z_0 \in A_{c_1}$ and $\sigma \leq \sigma_0$, then $z_t \in A_{c_2}$ for all t . Then (3.1) holds with c_σ replaced by c_2 .

Corollary 3.7. *There is a constant σ_0 such that the following is true: if $\|\xi\|_{1/2} \leq \sigma_0$, if $0 \leq c \leq 1$ and z is in the cone $A_c(\xi_0)$, and if*

$$\int_0^T \frac{M_{0,s}^\xi}{s^{3/2}} ds < \infty,$$

then

$$|\arg f'_T(z)| \leq 8c + 4 \int_0^T \frac{M_{0,s}^\xi}{s^{3/2}} ds.$$

Proof. Let σ_0 be the constant from Remark 3.6 with $c_1 = 1$ and $c_2 = \sqrt{3}$. Then if $z_0 \in A_c$ and $c \leq 1$, we have $z_t \in A_{c_2}$ and $y_t \geq \sqrt{y_0^2 + t}$ for all t by (3.1). By (2.9) and Lemma 2.1,

$$\begin{aligned} |\arg f'_T(z)| &\leq 4 \int_0^T \frac{|x_s|}{y_s^3} ds \leq 4 \int_0^T \frac{cy_0 + M_{0,s}^\xi}{(y_0^2 + s)^{3/2}} ds \\ &= 8cy_0 \left(\frac{1}{y_0} - \frac{1}{\sqrt{y_0^2 + T}}\right) + 4 \int_0^T \frac{M_{0,s}^\xi}{s^{3/2}} ds \leq 8c + 4 \int_0^T \frac{M_{0,s}^\xi}{s^{3/2}} ds. \quad \square \end{aligned}$$

4. The proofs of Theorems 1.1 and 1.2

Throughout this section, we maintain our notation $\sigma = \|\lambda\|_{1/2}$, and denote by $\alpha = \alpha_\sigma$ the constant of Corollary 3.5. As explained in Section 2, in order to show that the Loewner equation generates a curve it suffices to show that

$$v(t, \varepsilon) := \int_0^\varepsilon |(g_t^{-1})'(\lambda_t + iy)| dy$$

goes to zero as $\varepsilon \rightarrow 0$, uniformly in $t \in [0, T]$. In our setting, this follows easily from Corollary 3.5.

Lemma 4.1. *Suppose that $\lambda: [0, T] \rightarrow \mathbb{R}$ is Hölder continuous with $\sigma < 4$ and (g_t) is the solution to (2.1). Then for every $\varepsilon > 0$ and $0 \leq t \leq T$,*

$$\int_0^\varepsilon |(g_t^{-1})'(\lambda_t + iy)| dy \leq \frac{(4t + \varepsilon^2)^{(1-\alpha)/2}}{\alpha} \varepsilon^\alpha.$$

Proof. Fix $0 \leq t \leq T$ and $\varepsilon > 0$. Let $\xi_s = \lambda_{t-s}$ for $0 \leq s \leq t$. Let $(f_s)_{0 \leq s \leq t}$ be the solution to (2.3) with the driving function ξ , so that $g_t^{-1} = f_t$. Hence by Corollary 3.5,

$$(4.1) \quad |(g_t^{-1})'(\lambda_t + iy)| = |f_t'(\xi_0 + iy)| \leq (4t + y^2)^{(1-\alpha)/2} y^{\alpha-1},$$

and the lemma follows by integration. □

Remark 4.2. By Proposition 3.9 of [2], we get a quantitative estimate for the modulus of continuity of the trace $\gamma_t := \lim_{y \rightarrow 0^+} g_t^{-1}(\lambda_t + iy)$, namely γ is Hölder continuous with exponent $\alpha/2$.

To complete the proof of Theorem 1.1, it only remains to show that γ is a simple curve and satisfies the Ahlfors geometric characterization of quasiconformal arcs

$$(4.2) \quad |\gamma_t - \gamma_s| \leq M |\gamma_t - \gamma_r|$$

for some constant $M = M_\gamma$ and all $0 \leq r \leq s \leq t \leq T$. The key idea is to use the Gehring–Hayman inequality, see page 72 in [9], which says that among all curves in a simply connected plane domain with two fixed end points, the hyperbolic geodesic minimizes the euclidean length, up to a universal multiplicative constant.

Lemma 4.3. *If $\sigma < 4$, then γ is a simple curve that stays inside the cone $A_{c_\sigma}(\lambda_0)$ and satisfies (4.2).*

Proof. Again consider the upward Loewner equation (2.3) with the driving function $\xi(s) = \lambda(t - s)$ for $s \in [0, t]$, for fixed $t \in [0, T]$. It follows from Theorem 3.1 that, for $z = \xi_0 + i\varepsilon = \lambda_t + i\varepsilon$,

$$|x_t| \leq c_\sigma y_t$$

and

$$\sqrt{\frac{4}{1 + c_\sigma^2} t + \varepsilon^2} \leq y_t^2 \leq \sqrt{4t + \varepsilon^2}.$$

Since $\gamma_t = \lim_{\varepsilon \rightarrow 0^+} (g_t^{-1})(\lambda_t + i\varepsilon) = \lim_{\varepsilon \rightarrow 0^+} (x_t + iy_t + \xi_t)$, it follows that

$$(4.3) \quad \frac{2\sqrt{t}}{\sqrt{1+c_\sigma^2}} \leq \text{Im } \gamma_t \leq 2\sqrt{t},$$

and

$$(4.4) \quad |\text{Re } \gamma_t - \lambda_0| \leq c_\sigma \text{Im } \gamma_t \leq 2c_\sigma\sqrt{t}.$$

This implies that the curve γ is contained in the cone $A_{c_\sigma}(\lambda_0)$ and meets the real line non-tangentially. It also implies that $\gamma(0, T) \cap \mathbb{R} = \emptyset$, which easily implies that γ is simple (Lemma 4.34 in [4]): just notice that, if $\gamma_t = \gamma_{t'}$ for some $t < t'$, then $g_t(\gamma(t, T))$ intersects the real line at λ_t , but that the curve $g_t(\gamma[t, T])$ has driving function $\tilde{\lambda}_t = \lambda_{t+t'}$ so that $g_t(\gamma(t, T)) \cap \mathbb{R} = \emptyset$ by the above.

To prove (4.2), fix $0 \leq r \leq s \leq t \leq T$, denote $\gamma_r, \gamma_s, \gamma_t$ by u, v, w , and their images under g_r by u', v', w' . We may assume that the line segment (u, w) is contained in $H_r = \mathbb{H} \setminus \gamma[0, r]$ (else replace u by the point \hat{u} that is closest to w on $(u, w) \cap \gamma[0, r]$, and replace r by $\hat{r} = \gamma^{-1}(\hat{u})$). By (4.3),

$$\text{Im } v' \leq 2\sqrt{s-r} \leq 2\sqrt{t-r} \leq \sqrt{1+c_\sigma^2} \text{Im } w',$$

so that the hyperbolic geodesic $\text{geo}_{\mathbb{H}}(u', v')$ from u' to v' in \mathbb{H} is within bounded hyperbolic distance from $\text{geo}_{\mathbb{H}}(u', w')$. In particular, there is a point $z' = g_r(z)$ on $\text{geo}_{\mathbb{H}}(u', w')$ of bounded hyperbolic distance from v' (where all bounds depend on c_σ only). Denoting ℓ the euclidean length, it follows from the Koebe distortion theorem that

$$|v - w| \leq |v - z| + |z - w| \leq C \text{dist}(z, \partial H_r) + \ell(\text{geo}_{H_r}(z, w)) \leq C \ell(\text{geo}_{H_r}(u, w)).$$

Since the line segment (u, w) is contained in H_r , the Gehring-Hayman inequality implies $\ell(\text{geo}_{H_r}(u, w)) \leq C'|u - w|$ and (4.2) follows. This finishes the proof of Lemma 4.3 and of Theorem 1.1. \square

Proof of Theorem 1.2. Since we did not assume *a priori* that λ generates a curve, we first observe that

$$(4.5) \quad \|\lambda\|_{1/2} \leq 3C_0.$$

Indeed, since $|\lambda_{t_2} - \lambda_{t_1}| \leq 2N_{s,t_2}^\lambda$ for $0 < s \leq t_1 < t_2 \leq T$, it is not hard to see that λ has a finite Hölder-1/2 norm on every interval $[t_1, t_2]$ inside $(0, T]$. Next, for $t_1 \leq s \leq r \leq t \leq t_2$ we have $|\lambda_t - \lambda_s| \leq N_{r,t}^\lambda + \|\lambda\|_{1/2} \sqrt{r-s}$. Integrating both sides of this inequality from s to t with respect to r , dividing by $(t-s)^{3/2}$ and estimating the integral involving N^λ by C_0 , (4.5) easily follows by choosing s and t appropriately.

If $C_0 < 4/3$, Theorem 1.1 applies and λ generates a curve γ . We will show that, if C_0 is small enough, then for every pair of points $\gamma_{t_1}, \gamma_{t_2}$ on γ with $0 \leq t_1 < t_2 \leq T$ we have

$$(4.6) \quad \left| \arg(\gamma_{t_2} - \gamma_{t_1}) - \frac{\pi}{2} \right| \leq C < \frac{\pi}{2},$$

where C depends on C_0 and $\sigma = \|\lambda\|_{1/2}$ only, and where $\arg(z) \in (-\pi, \pi]$ is the principal argument of z . This implies that γ grows vertically and is the graph of a Lipschitz function.

Let $\tilde{\gamma}$ be the image of γ under the map g_{t_1} . This is the curve generated by the driving function $\tilde{\lambda}_t = \lambda_{t+t_1}$, $t \in [0, T - t_1]$, see the figure below.

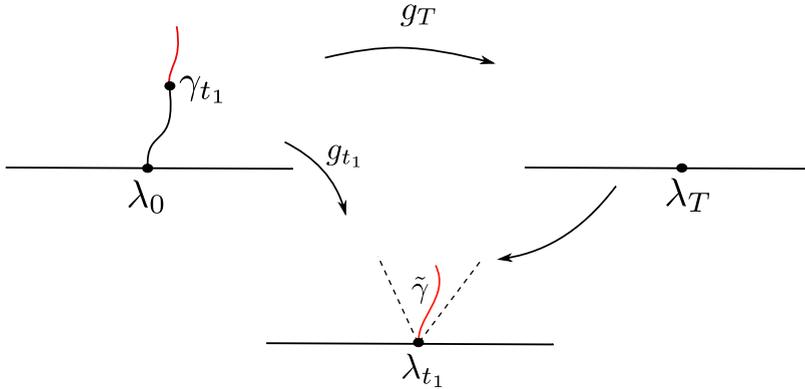


FIGURE 2: The curve $\tilde{\gamma}$ is inside the cone $A_c(\lambda_{t_1})$.

By Lemma 4.3 the curve $\tilde{\gamma}$ is in the cone $A_c(\lambda_{t_1})$, where $c = c_\sigma$ is defined in Section 3. With $w = g_{t_1}(\gamma_{t_2}) - \lambda_{t_1}$ we therefore have

$$\begin{aligned} \left| \arg(\gamma_{t_2} - \gamma_{t_1}) - \frac{\pi}{2} \right| &= \left| \arg\left(w \int_0^1 (g_{t_1}^{-1})'(\lambda_{t_1} + sw) ds\right) - \frac{\pi}{2} \right| \\ &\leq \arctan c + \sup_{z \in A_c(\lambda_{t_1})} |\arg(g_{t_1}^{-1})'(z)|. \end{aligned}$$

Applying Corollary 3.7 to the driving function $\xi_t = \lambda_{t_1-t}$ with $t \in [0, t_1]$, assuming C_0 is small enough such that $3C_0 < \sigma_0$ and $c_{3C_0} \leq 1$, we get

$$\sup_{z \in A_c(\lambda_{t_1})} |\arg(g_{t_1}^{-1})'(z)| \leq 8c + 4 \int_0^{t_1} \frac{M_{0,s}^\xi}{s^{3/2}} ds = 8c + 4 \int_0^{t_1} \frac{N_{s,t_1}^\lambda}{(t_1 - s)^{3/2}} ds.$$

Thus

$$\left| \arg(\gamma_{t_2} - \gamma_{t_1}) - \frac{\pi}{2} \right| \leq \arctan c + 8c + 4C_0.$$

If $C_0 \rightarrow 0$, then $\sigma \rightarrow 0$ by (4.5) and therefore $c \rightarrow 0$ by Theorem 3.1. Thus (4.6) follows if C_0 is sufficiently small, and the theorem is proved. \square

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