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Uniqueness for discrete Schrödinger evolutions

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Abstract. We prove that if a solution of the discrete time-dependent Schrödinger equation with bounded potential decays fast at two distinct times then the solution is trivial. For the free Schrödinger operator, as well as for operators with compactly supported time-independent potentials, a sharp analog of the Hardy uncertainty principle is obtained, using an argument based on the theory of entire functions. Logarithmic convexity of weighted norms is employed in the case of general bounded potentials.

1. Introduction

In the present work we study the discrete Schrödinger evolution

$$\partial_t u = i(\Delta_d u + V u), \tag{1.1}$$

where $u\colon \mathbb{R}_+ \times \mathbb{Z} \to \mathbb{C}$, the potential $V = V(t,n)$ is a bounded function, and Δ_d is the discrete Laplacian: given a function $f\colon \mathbb{Z} \to \mathbb{C}$,

$$\Delta_d f(n) := f(n+1) + f(n-1) - 2f(n).$$

We refer the reader to the surveys [4] and [12] for insight into the growing interest in discrete variants of Schrödinger equations. Our main objective is to prove that a non-trivial solution of equation (1.1) cannot decay fast at two distinct moments in time. This statement can be viewed as a manifestation of the uncertainty principle, which limits the localization of a quantum state at two different moments in time, depending on the distance between the moments.

The usual formulation of the uncertainty principle is that a function and its Fourier transform cannot both be arbitrarily well localized. In the continuous setting, for the formulation of the uncertainty principle due to Hardy, the localization is measured in terms of the decay rate at infinity:

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if $f \in L^2(\mathbb{R})$ is such that f and its Fourier transform $\hat{f}$ satisfy

$$|f(x)| \le C \exp(-\pi|x|^2), \quad |\hat{f}(\xi)| \le C \exp(-\pi|\xi|^2), \quad x, \xi \in \mathbb{R},$$

for some constant $C > 0$, then there is a constant A such that $f(x) = Ae^{-\pi|x|^2}$.

Hardy's uncertainty principle can also be given a dynamical interpretation in terms of solutions of the free Schrödinger equation [20], [14], [17]. It is equivalent to the following statement:

($\star$) *if $u(t,x)$ is a solution of the free Schrödinger equation $\partial_t u = i\Delta u$ and $|u(0,x)| + |u(1,x)| \le C \exp(-x^2/4)$, then $u(0,x) = A \exp(-(1+i)x^2/4)$.*

The reason that these two statements are equivalent is that the free Schrödinger equation can be explicitly solved via the Fourier transform, from which the two formulations of the Hardy uncertainty principle are easily seen to be the same.

In a remarkable series of papers ([13], [14], [15] [16]), L. Escauriaza, C. E. Kenig, G. Ponce and L. Vega, and also in collaboration with M. Cowling [9], have extended the uniqueness statement ($\star$) to solutions of Schrödinger equations with potentials, as well as to solutions of some nonlinear Schrödinger equations; we refer here also to the initial work [21]. The main tools for obtaining these uniqueness results are logarithmic convexity estimates and Carleman type inequalities. Further results for covariant Schrödinger evolutions were obtained in [2] and [6]. Concerning the discrete setting, we note that a discrete dynamical interpretation of the Heisenberg uncertainty principle was given in [17].

We study our problem using two essentially different approaches. In the simpler case of the free equation or one with a compactly supported time-independent potential, we apply the machinery of complex analysis to obtain what can be considered as the discrete analog of Hardy's uncertainty principle. This machinery provides us with precise results and also hints towards the answer in the general case. The other approach is based on logarithmic convexity and Carleman type estimates. It allows us to study equations with general bounded potentials.

Our results bear similarities to the continuous case, but at the same time there are fundamental differences. In particular the critical rate of decay is different for the continuous and the discrete cases. This fact is related to different behavior of the heat kernels: for the continuous case the standard heat kernel is $k(1,0,x) = (4\pi)^{-1/2} \exp(-x^2/4)$, while for the discrete case the heat kernel is $K(1,0,n) = e^{-1}|I_n(1)| \asymp e^{-1}(n!2^n)^{-1}$, where I_n are the modified Bessel functions, $I_n(z) = (-i)^n J_n(iz)$. Computations of the discrete heat kernel for the lattice and asymptotics connecting the two cases can be found in [7], [8]. It is also worth mentioning that discrete heat kernels appeared as weights in logarithmic convexity results for discrete harmonic functions in recent work by G. Lippner and D. Mangoubi [23].

To complete this introduction, we describe the main results in greater details. First, in Section 2 we consider the model cases and apply the theory of entire functions to prove that if $u(t,n)$ solves the free equation

$$\partial_t u = i\Delta_d u$$

and satisfies

$$|u(0,n)| + |u(1,n)| \le C \frac{1}{\sqrt{|n|}} \Big(\frac{e}{2|n|}\Big)^{|n|}, \quad n \in \mathbb{Z} \setminus \{0\}, \tag{1.2}$$

then $u(t,n) = Ai^{-n}e^{-2it}J_n(1-2t)$, where J_n is the Bessel function. This result is sharp: both $|J_n(-1)|$ and $|J_n(1)|$ decay precisely as the right-hand side in (1.2) as $|n| \to \infty$.

Further, we prove that if the potential is compactly supported and u is a strong solution of (1.1) satisfying the one-sided estimates

$$\text{at times } t = 0 \text{ and } t = 1, \quad |u(t,n)| \le C \left(\frac{e}{(2+\epsilon)n}\right)^n, \quad n > 0,$$

for some $\epsilon > 0$, then $u \equiv 0$. Note that in the continuous setting, one-sided Hardy uncertainty principles have previously appeared in works of F. Nazarov [25] and B. Demange [11]. The corresponding results for the continuous Schrödinger evolution can be also found in the recent survey [16].

In the second part of the paper, we use the real-variable approach, following ideas of [14]. The main step is to construct a weight function $\psi(t,n)$ which provides logarithmic convexity of the weighted ℓ^2 norms $\|\psi(t,\cdot)u(t,\cdot)\|_{\ell^2(\mathbb{Z})}$, where $u \in C^1([0,\infty),\ell^2)$ is a strong solution of (1.1). We use the general formalism developed in [14] and the main difficulty is to find the correct weight and prove the convexity; computations and estimates of the main terms differ from the continuous case. The general line of reasoning has its roots in celebrated results of T. Carleman and S. Agmon; the technique of the Carleman estimates goes back to [5], and the convexity principles for elliptic operators were described in [1]. This method allows us to consider general bounded potentials V, at the cost of having to assume stronger decay of $u(0,n)$ and $u(1,n)$ in both directions $n \to \pm\infty$. The main result, Theorem 4.3, says that if

$$\big\|(1+|n|)^{\gamma(1+|n|)}\, u(0,n)\big\|_{\ell^2(\mathbb{Z})}, \ \big\|(1+|n|)^{\gamma(1+|n|)}\, u(1,n)\big\|_{\ell^2(\mathbb{Z})} < \infty$$

for some $\gamma > (3+\sqrt{3})/2$, then $u \equiv 0$. We do not expect this result to be sharp, but it does provide a universal decay condition which implies uniqueness of solutions of Schrödinger equations with general bounded potentials.

In both parts, we consider for the sake of simplicity only the one-dimensional lattice, but we remark that our results generalize to higher dimension. Moreover, we expect that other higher-dimensional uncertainty principles, not following directly from our techniques, hold for discrete evolutions. For instance, it would be interesting to find a discrete counterpart to Beurling's uncertainty principle, as well as "non-radial" discrete versions of Hardy's uncertainty principle (see [3], [11] for results in the continuous setting). As an example of an immediate extension of our results, we note here that the complex analytic techniques in the first part of the present work yield the following result about the free Schrödinger evolution on the two dimensional lattice $\mathbb{Z}^2$ with the standard lattice-Laplacian:

If $u \in C^1([0,1], l^2(\mathbb{Z}^2))$ and $i\partial_t u = \Delta_d u$, and if

$$\sum_{m\in\mathbb{Z}} |u(t,m,n)|^2 \le \Big(\frac{e}{(2+\epsilon)n}\Big)^{2n}, \quad n>0,\ t\in\{0,1\},$$

then $u \equiv 0$.

Our results can be applied to non-linear equations similar to the continuous case. For example, if u is a bounded solution of the discrete non-linear Schrödinger equation

$$\partial_t u = i(\Delta_d u + c|u|^2 u),$$

the latter can be viewed as a linear Schrödinger equation with an (unknown) real bounded potential. Our main result now states that u cannot be sharply localized at two distinct moments in time. Note, however, that this cannot be interpreted as a uniqueness theorem since the equation is nonlinear.

The paper is organized as follows: in Section 2 we discuss preliminaries on entire functions and obtain sharp results for the free evolution and operators with compactly supported potentials. Section 3 contains a precursory energy estimate for solutions of (1.1), this estimate justifies the validity of our further computations. Section 4 splits into several subsections discussing and proving the logarithmic convexity results we require; the main result (Theorem 4.3) is proved in the final subsection.

While preparing this manuscript for journal publication, we learned that some of our results were independently, and with a different method, obtained by Fernández-Bertolin [18], [19], who also proved a number of interesting convexity estimates. We would also thank A. Fernández-Bertolin for pointing out for us how to drop the assumption that the potential is real-valued.

Notation. We use the symbol C to denote various constants. Unless otherwise indicated, their value may change from line to line. We mention also that all $\|\cdot\|_2$-norms are to be understood as the ℓ^2-norm in the discrete variable n.

2. A uniqueness result for Schrödinger operators with compactly supported potentials

In this section, we use methods from complex analysis. For the reader's convenience, we begin by briefly outlining some definitions and facts on entire functions of exponential type that we need. Details can be found in [22] (see in particular Lectures 8 and 9). Recall that an entire function f is said to be of exponential type if, for some $k > 0$,

$$|f(z)| \le C\exp(k|z|). \tag{2.1}$$

In this case the type of an entire function f is defined by

$$\sigma = \limsup_{r\to\infty} \frac{\log\max\{|f(re^{i\phi})|;\phi\in[0,2\pi]\}}{r} < \infty. \tag{2.2}$$

In particular, an entire function f is of zero exponential type if for any $k > 0$ there exists $C = C(k)$ such that (2.1) holds.

Let $f(z)$ be an entire function of exponential type, $f(z) = \sum_{n=0}^{\infty} c_n z^n$. Then the type of f can be expressed in terms of its Taylor coefficients as

$$\limsup_{n\to\infty} n\,|c_n|^{1/n} = e\sigma. \tag{2.3}$$

The growth of a function f of exponential type along different directions is described by the indicator function

$$h_f(\varphi) = \limsup_{r\to\infty} \frac{\log|f(re^{i\varphi})|}{r}.$$

This function is the support function of some convex compact set $I_f \subset \mathbb{C}$ which is called the indicator diagram of f. In particular,

$$h_f(\varphi) + h_f(\pi+\varphi) \geq 0. \tag{2.4}$$

For example the indicator function of e^{az} for $a \in \mathbb{C}$ is $h(\varphi) = \Re(ae^{i\varphi})$, and its indicator diagram consists of a single point, $\bar{a}$.

Clearly, $h_{fg}(\varphi) \leq h_f(\varphi) + h_g(\varphi)$, implying that

$$I_{fg} \subset I_f + I_g := \{z = z_1 + z_2 : z_1 \in I_f, z_2 \in I_g\}.$$

Recall that the Bessel functions J_n can be defined by

$$\exp(x(z - z^{-1})/2) = \sum_{n=-\infty}^{\infty} J_n(x)\, z^n, \quad z \neq 0.$$

For fixed x, the asymptotic of $J_n(x)$ is

$$|J_n(x)| \sim \frac{1}{\sqrt{|n|}}\Big(\frac{ex}{2|n|}\Big)^{|n|} \quad \text{as } |n| \to \infty.$$

Our first observation is the following discrete analog of the classical Hardy uncertainty principle.

Proposition 2.1. *Let $u \in C^1([0,1], \ell^2)$ satisfy the discrete free Schrödinger equation $\partial_t u = i\Delta_d u$, and suppose that*

$$|u(0,n)|,\, |u(1,n)| \leq C\frac{1}{\sqrt{|n|}}\Big(\frac{e}{2|n|}\Big)^{|n|}, \quad n \in \mathbb{Z}\setminus\{0\}, \tag{2.5}$$

for some $C > 0$. Then $u(t,n) = Ai^{-n}e^{-2it}J_n(1-2t)$ for all $n \in \mathbb{Z}$ and $0 \leq t \leq 1$, for some constant A.

Proof. Consider the discrete Fourier transforms of $u(t,\cdot)$,

$$\Phi(t,\theta) = \sum_{k=-\infty}^{\infty} u(t,k)\,\theta^k \ \in L^2(\mathbb{T}).$$

We have $\partial_t \Phi(t,\theta) = i(\theta+\theta^{-1}-2)\,\Phi(t,\theta)$. Thus

$$\Phi(t,\theta) = e^{i(\theta+\theta^{-1}-2)t}\,\Phi(0,\theta),$$

and in particular,

$$\Phi(1,\theta) = e^{i(\theta+\theta^{-1}-2)}\,\Phi(0,\theta). \tag{2.6}$$

It follows from (2.5) that the functions $\theta \mapsto \Phi(s,\theta)$, for $s=0$ and $s=1$, admit holomorphic extensions to $\mathbb{C}\setminus\{0\}$:

$$\Phi(s,\theta) = \sum_{k<0} u(k,s)\,\theta^k + \sum_{k\ge 0} u(k,s)\,\theta^k = \Phi^-(s,\theta) + \Phi^+(s,\theta), \quad s\in\{0,1\}. \tag{2.7}$$

Furthermore, (2.5) implies that $\Phi^+(s,\theta)$ and $\Phi^-(s,1/\theta)$, $s=0$ and $s=1$, are entire functions of exponential type whose respective indicator diagrams I_s^+ and I_s^- are contained in the disk of radius $1/2$ centered at zero. Actually one can say more:

$$|\Phi^+(s,\theta)|,\ |\Phi^-(s,1/\theta)| \le Ce^{|\theta|/2}, \quad s\in\{0,1\}. \tag{2.8}$$

This follows from the fact that the right-hand side of (2.5) is asymptotically equivalent to the coefficients in the Taylor expansion of $\exp(z/2)$. On the other hand, it follows from (2.6) that $I_1^\pm \subset I_0^\pm + i$. Thus $I_0^\pm = \{-i/2\}$ and $I_1^\pm = \{i/2\}$.

Now let

$$g(z) = g^+(z) + g^-(z) = e^{i(z+z^{-1})/2}\,\Phi(0,z) = e^{2i}\,e^{-i(z+z^{-1})/2}\,\Phi(1,z), \tag{2.9}$$

where, as before, $g^\pm$ are the parts of the Laurent series of g with respectively non-negative and negative powers. It follows that the indicator diagrams $I^\pm$ of the entire functions $g^+(z)$ and $g^-(1/z)$ coincide with $\{0\}$, so $g^+(z)$ and $g^-(1/z)$ are entire functions of type zero.

The relations (2.8) and (2.9) now yield that $g^+(iy)$ and $g^-(1/iy)$ are bounded for $y\in\mathbb{R}\setminus\{0\}$ and by the Phragmén–Lindelöf principle (see [22], Lecture 6), g^+ and g^-, and hence g, are constants. Finally $\Phi(0,z) = A\exp(-i(z+z^{-1})/2)$, yielding the required expression for $u(t,n)$. □

Corollary 2.2. *Let u be as in Proposition 2.1. If in addition*

$$|u(0,n)|\Big(\frac{2|n|}{e}\Big)^{|n|}\sqrt{|n|} = o(1)$$

as $n\to+\infty$ or $n\to-\infty$, then $u\equiv 0$.

Assuming only slightly stronger decay, one can apply similar techniques in order to obtain a uniqueness result for solutions of discrete Schrödinger equations with compactly supported time-independent potentials. In this case it suffices to demand that the solution decays just in one direction. Recall that by *strong solutions* of (1.1) we mean functions in $C^1([0,1],\ell^2)$ which solve this equation.

Theorem 2.3. *Let $u(t,n)$, $t>0$, $n\in\mathbb{Z}$ be a strong solution of (1.1), where the potential V does not depend on time and also $V(n)\neq 0$ just for a finite number of n's. If, for some $\varepsilon>0$,*

$$|u(t,n)|\leq C\Big(\frac{e}{(2+\varepsilon)n}\Big)^n,\quad n>0,\ t\in\{0,1\}, \tag{2.10}$$

then $u\equiv 0$.

Proof. We may assume that $V_n=0$ for $n>N$ and for $n<0$. Consider the bounded operator $H=\Delta_d+V\colon \ell^2\to\ell^2$. The solution $u(t,n)$ is then defined by

$$u(\cdot,t)=e^{iHt}u(\cdot,0),$$

and hence belongs to ℓ^2 for all $t>0$.

The absolutely continuous spectrum of $H\colon\ell^2\to\ell^2$ is the segment $[-4,0]$, each point with multiplicity 2. This spectrum is parametrized naturally by the unit circle $\mathbb{T}$: each $\lambda\in(-4,0)$ can be written in two ways as $\lambda=\lambda(\theta):=\theta+\theta^{-1}-2$ for some $\theta\in\mathbb{T}$. We denote by $e^{\pm}(\theta)=e^{\pm}(\theta,n)$ the corresponding Jost solutions of the spectral problem

$$Hx=\lambda(\theta)x, \tag{2.11}$$

i.e., the solutions of (2.11) satisfying $e^{+}(\theta,n)=\theta^n$ for $n>N$ and $e^{-}(\theta,n)=\theta^n$ for $n<0$. We refer the reader to [26] and [24] for the precise definition and detailed discussions of Jost solutions, and note that (2.11) implies

$$e^{\pm}(\theta,n+1)=(\theta+\theta^{-1}-2+V_n)e^{\pm}(\theta,n)-e^{\pm}(\theta,n-1).$$

Therefore $e^{\pm}(\theta,n)$ are polynomials of θ and θ^{-1} and more precisely, for $0\leq n\leq N$ the functions $e^{\pm}(\theta,n)$ are linear combinations of θ^j, $j\in\{-N,-N+1,\dots,2N\}$.

Except for $\theta=\pm1$, each of the pairs $\{e^{+}(\theta,\cdot),e^{+}(\theta^{-1},\cdot)\}$, $\{e^{-}(\theta,\cdot),e^{-}(\theta^{-1},\cdot)\}$ is a fundamental system of solutions of (2.11). Hence we have the representations

$$\begin{aligned}e^{+}(\theta,\cdot)&=a^{-}(\theta)e^{-}(\theta,\cdot)+b^{-}(\theta)e^{-}(\theta^{-1},\cdot),\\ e^{-}(\theta,\cdot)&=a^{+}(\theta)e^{+}(\theta,\cdot)+b^{+}(\theta)e^{+}(\theta^{-1},\cdot).\end{aligned}$$

It can be easily verified, see e.g. [24], that $a^{\pm}$ and $b^{\pm}$ are rational functions of θ, with no poles on $\mathbb{T}$. In particular,

$$\lim_{|\theta|\to+\infty}\frac{\log|a^{+}(\theta)|}{|\theta|}=\lim_{|\theta|\to+\infty}\frac{\log|b^{+}(\theta)|}{|\theta|}=0. \tag{2.12}$$

For $\theta\in\mathbb{T}$, let $\Phi(t,\theta)=\sum_{-\infty}^{\infty}u(t,n)e^{-}(\theta,n)$ and note that $\Phi(t,\cdot)\in L^2(\mathbb{T})$. Assume towards a contradiction that $u\neq 0$ thus $\Phi\neq 0$. Decompose $\Phi(t,\cdot)$ on $\mathbb{T}$ into four functions:

$$\begin{aligned}\Phi(t,\theta)&=\sum_{-\infty}^{N}u(t,n)e^{-}(\theta,n)+\sum_{N+1}^{\infty}u(t,n)\big(a^{+}(\theta)e^{+}(\theta,n)+b^{+}(\theta)e^{+}(\theta^{-1},n)\big)\\ &=\sum_{-\infty}^{-1}u(t,n)\,\theta^n+b^{+}(\theta)\sum_{N+1}^{\infty}u(t,n)\,\theta^{-n}+\sum_{0}^{N}u(t,n)e^{-}(\theta,n)+a^{+}(\theta)\sum_{N+1}^{\infty}u(t,n)\,\theta^n.\end{aligned}$$

First, as $u(t,\cdot)\in\ell^2$, $\Phi_1(t,\theta):=\sum_{-\infty}^{-1}u(t,n)\,\theta^n$ and $\tilde\Phi_2(t,\theta):=\sum_{N+1}^{\infty}u(t,n)\,\theta^{-n}$ extend to holomorphic functions on $\mathbb{D}^c:=\{\theta\in\mathbb{C},|\theta|>1\}$ and, for every $\alpha\in[0,2\pi]$,

$$\limsup_{r\to\infty}\frac{\log|\Phi_1(t,re^{i\alpha})|}{r}=\limsup_{r\to\infty}\frac{\log|\tilde\Phi_2(t,re^{i\alpha})|}{r}=0.$$

Further, $\Phi_2(t,\theta):=b^+(\theta)\tilde\Phi_2(t,\theta)$ extends to a holomorphic function on $\mathbb{D}^c\setminus N_b$, where N_b is the set of poles of b^+, and, from (2.12), for every $\alpha\in[0,2\pi]$,

$$\limsup_{r\to\infty}\frac{\log|\Phi_2(t,re^{i\alpha})|}{r}=0.$$

Next, $\Phi_3(t,\theta):=\sum_0^N u(t,n)e^-(\theta,n)$ is a polynomial in θ,θ^{-1} and therefore extends to a holomorphic function on $\mathbb{C}\setminus\{0\}$ and, for every $\alpha\in[0,2\pi]$,

$$\limsup_{r\to\infty}\frac{\log|\Phi_3(t,re^{i\alpha})|}{r}=0.$$

It remains to estimate $\Phi_4(t,\theta):=a^+(\theta)\tilde\Phi_4(t,\theta)$, where $\tilde\Phi_4(t,\theta):=\sum_{N+1}^{\infty}u(t,n)\,\theta^n$. From (2.10) we get that, at times $t=0$ and $t=1$, $\tilde\Phi_4(t,\cdot)$ is an entire function of exponential type at most $(2+\varepsilon)^{-1}$. In particular, for $t\in\{0,1\}$, $\Phi_4(t,\cdot)$ extends to a holomorphic function on $\mathbb{C}\setminus N_a$ where N_a is the set of poles of a^+. Further, for each $\alpha\in[0,2\pi]$ we have

$$\limsup_{r\to\infty}\frac{\log|\tilde\Phi_4(t,re^{i\alpha})|}{r}\le\frac{1}{2+\varepsilon},\qquad t\in\{0,1\}.$$

By (2.4), we also have

$$\limsup_{r\to\infty}\frac{\log|\tilde\Phi_4(t,re^{i\alpha})|}{r}\ge-\frac{1}{2+\varepsilon},\qquad t\in\{0,1\}.$$

From (2.12) we deduce that

$$-\frac{1}{2+\varepsilon}\le\limsup_{r\to\infty}\frac{\log|\Phi_4(t,re^{i\alpha})|}{r}\le\frac{1}{2+\varepsilon},\qquad t\in\{0,1\}.$$

Finally, grouping all estimates on $\Phi_1,\dots,\Phi_4$, we obtain that $\Phi=\Phi_1+\cdots+\Phi_4$ satisfies

$$-\frac{1}{2+\varepsilon}\le\limsup_{r\to\infty}\frac{\log|\Phi(t,re^{i\alpha})|}{r}\le\frac{1}{2+\varepsilon},\quad \alpha\in[0,2\pi],\ t\in\{0,1\}. \tag{2.13}$$

In order to obtain a contradiction, note that

$$-i\frac{\partial\Phi(t,\theta)}{\partial t}=\sum_{n=-\infty}^{\infty}(Hu)(t,n)e^-(n,\theta)=\sum_{n=-\infty}^{\infty}u(t,n)(He^-)(n,\theta)=(\theta+\theta^{-1}-2)\Phi(t,\theta).$$

Hence $\Phi(t,\theta) = e^{it(\theta+\theta^{-1}-2)}\Phi(0,\theta)$, $\theta \in \mathbb{T}$, in particular,

$$\Phi(1,\theta) = e^{i(\theta+\theta^{-1}-2)}\Phi(0,\theta),$$

and this relation extends to $\theta \in \mathbb{D}^c \setminus (0 \cup N_a \cup N_b)$. But then

$$\frac{1}{2+\varepsilon} > \limsup_{y\to+\infty} \frac{\log|\Phi(1,iy)|}{y} = 1 + \limsup_{y\to+\infty} \frac{\log|\Phi(0,iy)|}{y} > 1 - \frac{1}{2+\varepsilon},$$

which leads to a contradiction. □

It would be of interest to extend this result to the case of potentials with fast decay, not necessarily compactly supported. The technique of the Jost solution is available for fast decaying potentials, see [10], [26].

3. First energy estimate

In the remaining of the paper we will follow the ideas of [14]. We prove that a solution of the discrete Schrödinger equation which decays sufficiently fast along both half-axes at two different moments of time is trivial.

We begin with an energy estimate for solutions of a non-homogeneous initial value problem and show that if the initial data is well-concentrated, the energy cannot spread out too fast.

Given $\alpha > 0$ and $t \geq 0$, denote $\psi_\alpha(t) = \{\psi_\alpha(t,n)\}_{n\in\mathbb{Z}} = \{(1+|n|)^{\alpha|n|/(1+t)}\}_{n\in\mathbb{Z}}$.

Proposition 3.1. *Let $V = V_1 + iV_2$, with $V_1, V_2 \colon [0,T]\times\mathbb{Z} \to \mathbb{R}$ and V_2 bounded and $F\colon [0,T]\times\mathbb{Z}\to\mathbb{C}$ bounded. Let $u\colon [0,T]\times\mathbb{Z}\to\mathbb{C}$, $u \in \mathrm{C}^1\big([0,T],\ell^2(\mathbb{Z})\big)$, be a strong solution of the equation*

$$\partial_t u(t,n) = i(\Delta_d u(t,n) + V(t,n)u + F(t,n)). \tag{3.1}$$

Assume that $\{\psi_\alpha(0,n)u(0,n)\} \in \ell^2(\mathbb{Z})$ for some $\alpha \in (0,1]$. Then, for $T > 0$,

$$\|\psi_\alpha(T,n)u(T,n)\|_2^2 \leq e^{CT}\Big(\|\psi_\alpha(0,n)u(0,n)\|_2^2 + \int_0^T \|\psi_\alpha(s,n)F(s,n)\|_2^2\,ds\Big). \tag{3.2}$$

Proof. Consider $f(t,n) = \psi_\alpha(t,n)u(t,n)$ and let $H(t) = \|f(t,n)\|_2^2$. We fix α till the end of the proof and write $\psi = \psi_\alpha$.

We will perform several formal computations, assuming that $H(t)$ is finite for all $t \in [0,T]$, and then justify these computations at the end of the proof.

Define

$$\kappa(n,t) = \log\psi(t,n) = \frac{\alpha}{1+t}|n|\log(1+|n|).$$

Then

$$\partial_t f = i\psi\,\Delta_d(\psi^{-1}f) + iVf + \partial_t\kappa f + i\psi F,$$

which we rewrite as $\partial_t f = \mathcal{S}f + \mathcal{A}f + iVf + i\psi F$, where $\mathcal{S}$ and $\mathcal{A}$ are symmetric and anti-symmetric operators, respectively. Explicitly,

$$\mathcal{S}f = \frac{i}{2}\left(\psi\Delta_d(\psi^{-1}f) - \psi^{-1}\Delta_d(\psi f)\right) + \partial_t\kappa f,$$
$$\mathcal{A}f = \frac{i}{2}\left(\psi\Delta_d(\psi^{-1}f) + \psi^{-1}\Delta_d(\psi f)\right).$$

Denote

$$a_n = \frac{\psi_{n+1}}{\psi_n} - \frac{\psi_n}{\psi_{n+1}}, \quad b_n = \frac{\psi_{n+1}}{\psi_n} + \frac{\psi_n}{\psi_{n+1}}. \tag{3.3}$$

In what follows we will use the notation $a_n = a(t,n)$, and similarly for ψ_n, etc.

We then rewrite

$$(\mathcal{S}f)_n = -\frac{i}{2}(a_n f_{n+1} - a_{n-1}f_{n-1}) + (\partial_t\kappa)_n f_n, \tag{3.4}$$

$$(\mathcal{A}f)_n = \frac{i}{2}(b_n f_{n+1} + b_{n-1}f_{n-1}) - 2if_n. \tag{3.5}$$

We want to control the growth of $H(t)$. Clearly, $\partial_t H(t) = 2\Re\langle\partial_t f, f\rangle$ and thus

$$\begin{aligned}\partial_t H(t) &= 2\langle\mathcal{S}f, f\rangle - 2\Im\langle Vf, f\rangle - 2\Im\langle\psi F, f\rangle \\ &= 2\Im\sum_n a_n f_{n+1}\,\overline{f}_n + 2\langle\partial_t\kappa f, f\rangle - 2\langle V_2 f, f\rangle - 2\Im\langle\psi F, f\rangle.\end{aligned}$$

This implies

$$\partial_t H(t) \le 2\|\psi F\|_2\,\|f\|_2 + \|V_2\|_\infty\,\|f\|_2^2 + \sum_n (2\partial_t\kappa_n + |a_n| + |a_{n-1}|)\,|f_n|^2.$$

Our aim is to prove that for all $n \in \mathbb{Z}$,

$$2\partial_t\kappa_n + |a_n| + |a_{n-1}| \le 2C, \tag{3.6}$$

where C is a constant. We have

$$\partial_t\kappa_n = -\frac{\alpha}{(1+t)^2}\,|n|\log(|n|+1).$$

Further, $|a_n| \le e^\alpha(|n|+1)^\alpha$. Hence, as $\alpha \le 1$ we obtain (3.6), for $t \in [0,1]$.

Therefore $\partial_t\|f\|_2 \le C\|f\|_2 + \|\psi F\|_2$ and (3.2) follows.

In order to justify these computations, we truncate the weight function ψ to an interval $[-N, N]$:

$$\psi_N(n,t) = \begin{cases}(|n|+1)^{(1+t)^{-1}\alpha|n|}, & |n| \le N,\\ (|N|+1)^{(1+t)^{-1}\alpha|N|}, & |n| > N.\end{cases}$$

Since the solution u is in ℓ^2, the relevant norms weighted by ψ_N are guaranteed to be finite and by running the above argument we obtain (3.6) and (3.2) for the weight ψ_N, this time rigorously. The desired inequality follows by passing to the limit as $N \to \infty$. □

Corollary 3.2. *Let $u\colon [0,1]\times\mathbb{Z}\to\mathbb{C}$ be a strong solution of the Schrödinger equation*

$$\partial_t u(t,n) = i\,(\Delta_d u(t,n) + V(t,n)u),$$

where $V = V_1 + iV_2$ is as above. Further suppose that

$$\sum_{n>0} n^{2\alpha n}\,|u(0,n)|^2 < \infty$$

for some $\alpha \le 1$. Then for each $t\in[0,1]$ we have

$$\sum_{n>0} n^{\alpha n}|u(t,n)|^2 < \infty.$$

Proof. Define $\tilde u(t,n) = 0$ for $n<0$ and $\tilde u(t,n) = u(t,n)$ for $n\ge 0$. Then $\tilde u$ satisfies (3.1) with $F(t,n)$ bounded and vanishing for $n\notin\{-1,0\}$. If we apply Proposition 3.1 to $\tilde u$ we obtain the required estimate □

4. Logarithmic convexity of weighted ℓ^2-norms

4.1. Preliminary discussion

From now on we fix $\gamma_0>0$ and suppose that $V\colon [0,T]\times\mathbb{Z}\to\mathbb{C}$ is bounded. Further, we assume that u is a strong solution of

$$\partial_t u = i(\Delta_d u + Vu)$$

such that $\|(1+|n|)^{\gamma_0(1+|n|)}u(0,n)\|_2$ and $\|(1+|n|)^{\gamma_0(1+|n|)}u(1,n)\|_2$ are finite.

Following the ideas of [14], we are looking for a weight

$$\psi(t,n) = \exp(\kappa(t,n)) \tag{4.1}$$

to give us a logarithmically convex function $e^{-Ct(1-t)}H(t)$, where

$$H(t) = \|\psi(t,n)\,u(t,n)\|_2^2$$

and C depends on V and ψ.

We will first use such a convexity argument to show that for any $0<\gamma<\gamma_0$ and any $t\in[0,1]$,

$$\|(1+|n|)^{\gamma(1+|n|)}\,u(t,n)\|_2 < \infty. \tag{4.2}$$

This also implies that

$$\left\|(C_0+|n|+R_0\,t(1-t))^{\gamma(C_0+|n|+R_0t(1-t))}\,u(t,n)\right\|_2 < +\infty \tag{4.3}$$

for any $C_0, R_0>0$ and $t\in[0,1]$, and we then set out to prove the logarithmic convexity in t of this latter norm.

In both steps we consider weights of the form (4.1), with

$$\kappa(t,n) = \gamma(|n|+R(t))\ln^b\big(|n|+R(t)\big)$$

where either $1/2<b<1$ and $R(t)=1$, or $b=1$ and $R(t) = C_0 + R_0 t(1-t)$. As before we set $f(t,n) = \psi(t,n)u(t,n)$.

We will first assume that $b < 1$, prove estimates independent of b, and let $b \to 1$ to establish (4.2). This will allow us to justify the computations involved in the second step, when $b = 1$ and we prove the convexity of (4.3).

4.2. Formal computations

We collect here a number of formal identities which we need in the sequel. The first identities are the same as in the continuous case, found for example in [16], others are specific to the discrete case. We use the notation established in the proof of Proposition 3.1.

We already know that $\partial_t H(t) = 2\langle \mathcal{S}f, f\rangle + 2\Re\langle iVf, f\rangle$, and thus

$$\begin{aligned}\partial_t(\partial_t H(t) - 2\Re\langle iVf, f\rangle) &= 2\langle \mathcal{S}_t f, f\rangle + 4\Re\langle \mathcal{S}f, f_t\rangle \\ &= 2\langle \mathcal{S}_t f, f\rangle + 4\|\mathcal{S}f\|^2 + 2\langle [\mathcal{S},\mathcal{A}]f, f\rangle + 4\Re\langle \mathcal{S}f, iVf\rangle \\ &= 2\langle \mathcal{S}_t f, f\rangle + 2\langle [\mathcal{S},\mathcal{A}]f, f\rangle + 4\Re\langle \mathcal{S}f + iVf, \mathcal{S}f\rangle \\ &= 2\langle \mathcal{S}_t f, f\rangle + 2\langle [\mathcal{S},\mathcal{A}]f, f\rangle + \|2\mathcal{S}f + iVf\|^2 - \|Vf\|^2.\end{aligned}$$

Therefore we obtain that

$$\begin{aligned}&\|f\|^2\partial_t\left(\partial_t \log H(t) - 2\|f\|^{-2}\Re\langle iVf, f\rangle\right) \\ &\quad= 2(\langle \mathcal{S}_t f, f\rangle + \langle [\mathcal{S},\mathcal{A}]f, f\rangle) - \|Vf\|^2 \\ &\qquad+ \left(\|2\mathcal{S}f + iVf\|^2\|f\|^2 - 4\langle \mathcal{S}f, f\rangle\Re\langle \mathcal{S}f + iVf, f\rangle\right)\|f\|^{-2} \\ &\quad= 2(\langle \mathcal{S}_t f, f\rangle + \langle [\mathcal{S},\mathcal{A}]f, f\rangle) - \|Vf\|^2 \\ &\qquad+ \left(\|2\mathcal{S}f + iVf\|^2\|f\|^2 - |\Re\langle 2\mathcal{S}f + iVf, f\rangle|^2 + |\Re\langle iVf, f\rangle|^2\right)\|f\|^{-2} \\ &\quad\geq 2(\langle \mathcal{S}_t f, f\rangle + \langle [\mathcal{S},\mathcal{A}]f, f\rangle) - \|Vf\|^2.\end{aligned}$$

Our aim is to show that

$$\partial_t^2\left(\log H(t) + G(t)\right) \geq -2C \tag{4.4}$$

for some $C \geq 0$, where G satisfies $\partial_t G(t) = -2\|f\|^{-2}\Re\langle iVf, f\rangle$ and $|G(t)| \leq 2\|\Im V\|_\infty$ on $[0,1]$. Inequality (4.4) implies the log-convexity of $\exp(-Ct(1-t) + G(t))H(t)$.

The last term $-\|Vf\|^2$ is bounded below by $-C\|f\|^2$ since V is bounded. It suffices to establish an estimate of the first two terms of the form

$$\langle \mathcal{S}_t f, f\rangle + \langle [\mathcal{S},\mathcal{A}]f, f\rangle > -C\|f\|^2. \tag{4.5}$$

We refer now to (3.4) and (3.5). It follows that

$$(\mathcal{S}_t f)_n = -\frac{i}{2}\left(a'_n f_{n+1} - a'_{n-1} f_{n-1}\right) + \kappa''_n f_n,$$

and finally,

$$(2\mathcal{S}_t f + 2[\mathcal{S},\mathcal{A}]f)_n = \nu_{n+1} f_{n+2} + \lambda_n f_{n+1} + \mu_n f_n + \overline{\lambda_{n-1}} f_{n-1} + \nu_{n-1} f_{n-2},$$

where

$$\nu_{n+1} = \frac{1}{2}(a_n b_{n+1} - a_{n+1} b_n), \quad \lambda_n = -ib_n(\kappa'_{n+1} - \kappa'_n) - ia'_n,$$
$$\mu_n = a_n b_n - a_{n-1} b_{n-1} + 2\kappa''_n,$$

and, as before, the coefficients a_n and b_n are defined in (3.3).

Clearly $\psi'_n = \kappa'_n \psi_n$, implying that $a'_n = (\kappa'_{n+1} - \kappa'_n) b_n$ and

$$\lambda_n = -2ib_n(\kappa'_{n+1} - \kappa'_n).$$

4.3. Estimates with an auxiliary weight

Proposition 4.1. *Let $\gamma > 0$. Assume that u is a strong solution of*

$$\partial_t u = i(\Delta_d u + V u)$$

where the potential V is a bounded function. Assume that

$$\left\|(1+|n|)^{\gamma(1+|n|)} u(t,n)\right\|_2 < +\infty, \quad t \in \{0,1\}. \tag{4.6}$$

Then, for all $t \in [0,1]$, $\left\|(1+|n|)^{\gamma(1+|n|)} u(t,n)\right\|_2 < +\infty$.

Proof. Consider the weight function

$$\psi(n) = e^{\kappa_b(n)}, \quad \kappa_b(n) = \gamma(1+|n|)\ln^b\big(1+|n|\big),$$

where $1/2 < b < 1$. Note that the hypotheses (4.6) combined with Proposition 3.1 show that $H_b(t) = \|\exp(\kappa_b(n))u(t,n)\|_2^2$ is finite for all t, allowing us to justify the computations of the preceding section for this choice of weight. We will show that $H(t) = H_b(t)$ satisfies (4.4) with some C independent of b, whence

$$\begin{aligned}\|\exp(\kappa_b(n))u(t,n)\|_2^2 &\le G_0\, e^{\frac{C}{2}t(1-t)} H_b(0)^{1-t} H_b(1)^t \\ &\le G_0\, e^{\frac{C}{2}t(1-t)} \left\|(1+|n|)^{\gamma(1+|n|)} u(0,n)\right\|_2^{2(1-t)} \left\|(1+|n|)^{\gamma(1+|n|)} u(1,n)\right\|_2^{2t}.\end{aligned}$$

Letting $b \to 1$ and applying the monotone convergence theorem then concludes the proof.

We refer to computations in the previous section. In the current setting $S_t = 0$ and $\lambda_n = 0$ so relation (4.5) reduces to

$$\langle 2[\mathcal{S},\mathcal{A}]f, f\rangle \ge -C\|f\|^2. \tag{4.7}$$

We have

$$\langle 2[\mathcal{S},\mathcal{A}]f, f\rangle = \sum_n \mu_n\, |f_n|^2 + 2\Re \sum_n \nu_{n+1}\, f_{n+2}\, \overline{f_n},$$

where

$$\mu_n = a_n b_n - a_{n-1} b_{n-1} = \frac{\psi_{n+1}^2}{\psi_n^2} - \frac{\psi_n^2}{\psi_{n-1}^2} - \frac{\psi_n^2}{\psi_{n+1}^2} + \frac{\psi_{n-1}^2}{\psi_n^2},$$

and

$$\nu_{n+1} = \frac{1}{2}(a_n b_{n+1} - a_{n+1} b_n) = -\frac{\psi_n \psi_{n+2}}{\psi_{n+1}^2} + \frac{\psi_{n+1}^2}{\psi_{n+2}\, \psi_n},$$

where the coefficients a_n and b_n are defined in (3.3). By appealing to the second derivative of $x \mapsto (1+x)\ln^b(1+x)$ it is easy to verify that $\kappa_b(n+2)+\kappa_b(n)-2\kappa_b(n+1)$ is always non-negative and uniformly bounded from above. Thus ν_{n+1} is uniformly bounded and $\mu_n \geq 0$. This implies (4.7). □

4.4. Convexity estimate

In this subsection we consider the weight function given by

$$\psi(t,n)=e^{\kappa(t,n)}, \quad \text{where } \kappa(t,n)=\gamma(|n|+R(t))\ln(|n|+R(t)),$$

and $R(t)=C_0+R_0t(1-t)$, $R_0>0$, C_0 being large enough. As before we define $H(t)=\|u(t,n)\psi(t,n)\|_2^2$.

Lemma 4.2. *For every $\gamma>(3+\sqrt{3})/2$, there exists $C(\gamma)$ such that for $C_0>C(\gamma)$ and $R(t)=C_0+R_0t(1-t)$ we have*

$$\partial_t^2(\log H(t)+G(t)) \geq -\frac{4\gamma}{2\gamma-3}R_0\log R_0 - C_1R_0 - C_2,$$

where C_1 and C_2 depend on γ and $\|V\|_\infty$ only, and $\|G\|_\infty \leq 2\|\Im V\|_\infty$.

Proof. For $n \geq 0$ we have

$$\frac{\psi(t,n+1)}{\psi(t,n)}=(n+1+R(t))^\gamma\Big(1+\frac{1}{n+R(t)}\Big)^{\gamma(n+R(t))},$$

and $\psi_n=\psi_{-n}$ for $n<0$. Hence $a_n=-a_{-n-1}$ and $b_n=b_{-n+1}$ for $n<0$, which in turn implies that $\mu_n=\mu_{-n}$ and $\lambda_n=-\lambda_{-n-1}$ when $n<0$. We have also $\mu_0=2a_0b_0+2\kappa_0''$.

As before, we get

$$|\nu_{n+1}|=\Big|\frac{\psi_{n+1}^2}{\psi_n\psi_{n+2}}-\frac{\psi_n\psi_{n+2}}{\psi_{n+1}^2}\Big| \leq C_3,$$

where C_3 depends on γ only.

Let $\phi(M)=\gamma M\ln M$ and $M=M(t,n)=|n|+R(t)$. In this notation we have, for $n\neq 0$,

$$\mu_n \geq \exp(2\phi(M+1)-2\phi(M))-\exp(2\phi(M)-2\phi(M-1))-C_4+2\kappa_n'',$$

where C_4 is a constant that depends only on γ. The derivatives of κ_n are

$$\kappa_n'(t)=R'(t)\phi'(|n|+R(t)), \quad \kappa_n''(t)=-2R_0\phi'(|n|+R(t))+(R'(t))^2\phi''(|n|+R(t)).$$

Then, by the Taylor expansions, we obtain that, for each $\epsilon>0$ and $C_0=C_0(\epsilon)$ large enough,

$$\mu_n \geq 2\gamma e^{2\gamma}M^{2\gamma-1}+\gamma e^{2\gamma}\Big(\frac{(\gamma-1)^2}{3}-\epsilon\Big)M^{2\gamma-3} + 2A^2\gamma M^{-1}-4R_0\gamma(1+\ln M)-C_4,$$

where $A=|R'(t)|$ and $n\neq 0$. Further,

$$\mu_0 \geq (2-\epsilon)M^{2\gamma}e^{2\gamma}+2A^2\gamma M^{-1}-4R_0\gamma(1+\ln M)-C_4.$$

We introduce the notation

$$\sigma_n = 2\gamma\, e^{2\gamma}\, M^{2\gamma-1} + \gamma\, e^{2\gamma}\Big(\frac{(\gamma-1)^2}{3} - 2\epsilon\Big) M^{2\gamma-3} + 2A^2\,\gamma\, M^{-1},$$

and

$$\rho_n = \epsilon\,\gamma\, e^{2\gamma} M^{2\gamma-3} - 4R_0\,\gamma\,(1+\ln M),$$

so that $\mu_n \geq \sigma_n + \rho_n - C_4$ for all n. Note that by the inequality of arithmetic and geometric means we have

$$\sigma_n^2 \geq 8A^2\,\gamma^2\, e^{2\gamma}\Big(2M^{2\gamma-2} + \Big(\frac{(\gamma-1)^2}{3} - 2\epsilon\Big) M^{2\gamma-4}\Big).$$

For $n \geq 0$ we have also

$$|\kappa'_{n+1} - \kappa'_n| = |R'(t)|(\phi'(M+1) - \phi'(M)) = A\,\gamma\,\ln(1+M^{-1}).$$

Hence, for sufficiently large C_0,

$$|\lambda_n| = 2|(\kappa'_{n+1} - \kappa'_n)|\,|b_n| \leq 2A\,\gamma\, e^{\gamma} M^{\gamma-1} + A\,\gamma\, e^{\gamma}(\gamma-1)\, M^{\gamma-2} + A\,\gamma\, e^{\gamma}\Big(\frac{3\gamma^2 - 10\gamma + 8}{12} + \epsilon\Big) M^{\gamma-3}, \quad n \geq 0.$$

To estimate $\partial_t^2(\log H(t) + G(t))$ (where $\partial_t G(t) = -2\Re\langle iVf, f\rangle$), we note that

$$\begin{aligned}\langle 2\mathcal{S}_t f + 2[\mathcal{S},\mathcal{A}]f, f\rangle &= \sum_n \mu_n |f_n|^2 + 2\Re\sum_n \nu_{n+1} f_{n+2}\overline{f_n} + 2\Re\sum_n \lambda_n f_{n+1}\overline{f_n}\\ &\geq \sum_n \sigma_n |f_n|^2 + 2\Re\sum_n \lambda_n f_{n+1}\overline{f_n} + \sum_n \rho_n |f_n|^2 - (C_3 + C_4)\sum_n |f_n|^2.\end{aligned}$$

Let us start with the first two terms. If we show that, for any $x, y \geq 0$,

$$\sigma_n x^2 + \sigma_{n+1}\, y^2 \geq 4\,|\lambda_n|\, xy, \tag{4.8}$$

then the summation of these inequalities with $x = f_n$, $y = f_{n+1}$ yields

$$\sum_n \sigma_n\,|f_n|^2 + 2\Re\sum_n \lambda_n\, f_{n+1}\,\overline{f_n} \geq 0.$$

To show (4.8) we have to check that

$$\sigma_n \sigma_{n+1} \geq 4|\lambda_n|^2, \quad n \geq 0. \tag{4.9}$$

Actually we show (4.8) only for $n \geq 0$. The relations for negative integers given in the beginning of the proof then imply the inequality for all n.

Using the estimates above, we have

$$\begin{aligned}\sigma_n^2\,\sigma_{n+1}^2 \geq{}& 64\, A^4\,\gamma^4\, e^{4\gamma}\,\big(4M^{4\gamma-4} + 8(\gamma-1)M^{4\gamma-5}\big)\\ &+ 64\, A^4\,\gamma^4\, e^{4\gamma}\Big(4(\gamma-1)(2\gamma-3) + 4\Big(\frac{(\gamma-1)^2}{3} - 2\epsilon\Big)\Big) M^{4\gamma-6},\end{aligned}$$

while

$$16|\lambda_n|^4 \le 64\, A^4\, \gamma^4\, e^{4\gamma}\left(4M^{4\gamma-4} + 8(\gamma-1)M^{4\gamma-5}\right) + 64\, A^4\, \gamma^4\, e^{4\gamma}\Big(6(\gamma-1)^2 + 8\Big(\frac{3\gamma^2-10\gamma+8}{12} + \epsilon\Big)\Big)M^{4\gamma-6}.$$

Inequality (4.9) hence follows for sufficiently small ϵ when

$$2(\gamma-1)(2\gamma-3) + \frac{2(\gamma-1)^2}{3} > 3(\gamma-1)^2 + \frac{3\gamma^2-10\gamma+8}{3}.$$

The last inequality is equivalent to $2\gamma^2-6\gamma+3>0$, which holds for $\gamma > (3+\sqrt{3})/2$.

Finally, by minimizing in M one obtains that, for $\gamma > 3/2$,

$$\rho_n \ge \min_{M>0}\{\epsilon\,\gamma\, e^{2\gamma} M^{2\gamma-3} - 4R_0\gamma(1+\ln M)\} \ge -\frac{4\gamma}{2\gamma-3}R_0\ln R_0 - C_1R_0,$$

where C_1 depends on γ and ϵ. The conclusion of the lemma follows. □

4.5. Concluding arguments

Using the weight $\psi(n,t,R_0)$ from the last section and Lemma 4.2, we obtain that $H_{R_0}(t)\exp(-d(R_0,\gamma)t(1-t)+G(t))$ is logarithmically convex, where $d(R_0,\gamma) = \frac{2\gamma}{2\gamma-3}R_0\ln R_0 + \frac{C_1}{2}R_0 + \frac{C_2}{2}$ and $\|G\|_\infty \le 2\|\Im Vf\|_\infty$. Hence, for $t=1/2$ we obtain

$$H_{R_0}(1/2) \le \exp\Big(\frac{\gamma}{2(2\gamma-3)}R_0\ln R_0 + \frac{C_1}{8}R_0 + C_3\Big)H_{R_0}(0)^{1/2}H_{R_0}(1)^{1/2}.$$

But since $R(0)=R(1)=C_0$ we see that $H(0)$ and $H(1)$ do not depend on the choice of R_0. We obtain that

$$|u(1/2,n)|^2\exp(2\gamma(|n|+C_0+R_0/4)\ln(|n|+C_0+R_0/4)) \le D\exp\Big(\frac{\gamma}{2(2\gamma-3)}R_0\ln R_0 + \frac{C_1}{8}R_0\Big),$$

where D is a constant independent of n and R_0. However, this last inequality is clearly impossible for large R_0 when $\gamma > 2$, unless $u(1/2,\cdot)\equiv 0$, which of course implies that $u\equiv 0$. Our work of this section can thus be summarized as follows.

Theorem 4.3. *Assume that $\gamma > (3+\sqrt{3})/2$ and that $V(t,n)$ is a bounded function. If u is a strong solution of $\partial_t u = i(\Delta_d u + Vu)$ such that*

$$\big\|(1+|n|)^{\gamma(1+|n|)}u(0,n)\big\|_2,\ \big\|(1+|n|)^{\gamma(1+|n|)}u(1,n)\big\|_2 < +\infty,$$

then $u\equiv 0$.

Remark. This result is most likely not sharp. The authors expect that a milder decay condition (with $\gamma = 1+\epsilon$) and even just one-sided decay should imply uniqueness as in the case of free Schrödinger evolution.

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