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Characterising Sobolev inequalities by controlled coarse homology and applications for hyperbolic spaces

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Abstract. We give a Sobolev inequality characterisation for the vanishing of the fundamental class in the controlled coarse homology of Nowak and Špakula for quasiconvex uniform spaces that support a local weak $(1,1)$-Poincaré inequality. Among the applications, we consider visual Gromov hyperbolic spaces.

1. Introduction

In this article, a *metric measure space* (X, d, μ) is a metric space (X, d) with Borel regular outer measure μ such that $\mu(X) > 0$ and $\mu(B) < \infty$ for any open ball $B \subseteq X$. In what follows, we call a function $\varrho\colon [0,\infty) \to [0,\infty)$ a *control function* if it is non-decreasing, $\varrho(0) = 1$, and it satisfies

$$(\varrho_1) \qquad \varrho(\varepsilon + t) \le L(\varepsilon)\, \varrho(t) \quad \text{and}$$

$$(\varrho_2) \qquad \varrho(\varepsilon t) \le M(\varepsilon)\, \varrho(t)$$

for some functions $L, M\colon (0,\infty) \to (0,\infty)$ whenever $t > 0$ and $\varepsilon > 0$. The space (X, d, μ) satisfies the *global ϱ-weighted* $(1,1)$*-Sobolev inequality* $(S^{\varrho}_{1,1})$ if for the given control function ϱ there exist $o \in X$ and a constant $C > 0$ such that

$$\int_X |u|\, d\mu \le C \int_X |\nabla u|\, \varrho(d(o,\cdot))\, d\mu$$

for every $u \in N^{1,1}(X, d, \mu)$ with bounded support. The space $N^{1,1}(X, d, \mu)$ is the *Newton–Sobolev space* of equivalence classes of integrable functions $u\colon X \to [-\infty, \infty]$ with integrable upper gradient and $|\nabla u|\colon X \to [0,\infty]$ is its minimal 1-weak upper gradient; for details, see Section 7.1 in [10]. If $(S^{\varrho}_{1,1})$ holds for $\varrho \equiv 1$ we say that (X, d, μ) satisfies $(S_{1,1})$.

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In this article, we study the connection between $(S^{\varrho}_{1,1})$ on (X,d,μ) and the ϱ-isoperimetry of (X,d). Previously, ϱ-isoperimetry for metric spaces has been studied in [7], [16], [18], and the inequality $(S_{1,1})$ for Riemannian manifolds in [4], [5], [6] and for metric measure spaces in [8], [17]. Our main result is a metric measure version of Theorem 4.2 in [16] by Nowak and Špakula, and says the following.

Theorem A. *If (X,d,μ) is a quasiconvex uniform space that supports a local weak $(1,1)$-Poincaré inequality then it satisfies $(S^{\varrho}_{1,1})$ if and only if $0=[\Gamma]\in H^{\varrho}_0(\Gamma)$ for any quasi-lattice $\Gamma\subseteq X$.*

For the proof of Theorem A, see Theorem 12, noting the role of Lemma 2 in its proof. For the terminology related to controlled coarse homology and its connection to ϱ-isoperimetry, see Section 2. A subset $\Gamma\subseteq X$ in (X,d) is called *(C)-cobounded* if $N_C(\Gamma):=\{x\in X : d(x,\Gamma)<C\}=X$ for some constant $C>0$ and *uniformly locally finite* if there exists a function $N:(0,\infty)\to\mathbb{N}$ such that the cardinality $\#(\Gamma\cap B(x,r))\le N(r)$ for every $0<r<\infty$ and all $x\in X$ where $B(x,r)=\{y\in X : d(x,y)<r\}$. A *quasi-lattice* in (X,d) is a subset $\Gamma\subseteq X$ that is cobounded and uniformly locally finite. The space (X,d) is *(Q)-Quasiconvex* if there exists a constant $Q\ge 1$ such that for all $x,y\in X$ there is a path γ from x to y of length $\ell(\gamma)\le Q\,d(x,y)$. A Borel regular outer measure μ on (X,d) is *uniform* if there exist non-decreasing functions $f,g:(0,\infty)\to(0,\infty)$ such that $f(r)\le\mu(B(x,r))\le g(r)$ for every $0<r<\infty$ and all $x\in X$. We say that (X,d,μ) is uniform if μ is uniform. Given $1\le p<\infty$, we say that (X,d,μ) supports a *local weak $(1,p)$-Poincaré inequality (up to scale R_P)* if there exist constants $C_P>0$, $R_P>0$, and $\tau\ge 1$ such that for all $x\in X$ and all $0<r\le R_P$,

$$\fint_{B(x,r)}|u-u_{B(x,r)}|\,d\mu\le C_P\,r\left(\fint_{B(x,\tau r)}g_u^p\,d\mu\right)^{1/p}$$

whenever $u:X\to\mathbb{R}$ is an integrable function in $B(x,\tau r)$ and $g_u:X\to[0,\infty]$ is its minimal p-weak upper gradient; this is the local version of Proposition 8.1.3 in [10]. Here as usual, we write

$$f_A=\fint_A f\,d\mu=\frac{1}{\mu(A)}\int_A f\,d\mu$$

if $A\subseteq X$ is a μ-measurable set for which $0<\mu(A)<\infty$ and $f:X\to[-\infty,\infty]$ is integrable over A.

Contained in the proof of Theorem 12 is also the following implication that does not require a local weak $(1,1)$-Poincaré inequality.

Theorem B. *If (X,d,μ) is a quasiconvex uniform space satisfying $(S^{\varrho}_{1,1})$, then $0=[\Gamma]\in H^{\varrho}_0(\Gamma)$ for any quasi-lattice $\Gamma\subseteq X$.*

We now list some immediate applications to motivate Theorem A.

Corollary C. *Let (X,d,μ) and (X',d',μ') be quasiconvex uniform spaces that support a local weak $(1,1)$-Poincaré inequality. If (X,d) and (X',d') are quasi-isometric then (X,d,μ) satisfies $(S^{\varrho}_{1,1})$ if and only if (X',d',μ') satisfies $(S^{\varrho}_{1,1})$.*

A map $f\colon X \to X'$ between (X,d) and (X',d') is called a (λ,μ)-*quasi-isometric embedding* if there exist constants $\lambda \geq 1$ and $\mu \geq 0$ such that

$$\lambda^{-1}\, d(x,x') - \mu \leq d'(f(x),f(x')) \leq \lambda\, d(x,x') + \mu$$

for all $x, x' \in X$. If $f\colon X \to X'$ is a (λ,μ)-quasi-isometric embedding and $f(X) \subseteq X'$ is μ-cobounded we say that f is a (λ,μ)-*quasi-isometry* and that (X,d) and (X',d') are *quasi-isometric*. Corollary C now follows from Theorem A as the controlled coarse homology groups and the vanishing of the fundamental class are quasi-isometry invariants; see Corollary 2.3 in [16] and Lemma 2. A metric space (X,d) with a quasi-lattice $\Gamma \subseteq X$ is *amenable* if for all $\varepsilon > 0$ and all $r > 0$ there exists a non-empty finite subset $F \subseteq \Gamma$ such that

$$\frac{\#\partial_r F}{\#F} < \varepsilon,$$

where $\partial_r F = \{x \in \Gamma\colon d(x,\Gamma) < r \text{ and } d(x,\Gamma\setminus F) < r\}$. It is well known that amenability is independent of the choice of quasi-lattice, in fact it is a quasi-isometry invariant; see Corollary 2.2 in [1]. If (X,d) is not amenable we say that (X,d) is *non-amenable*. As observed in Theorem 3.1 in [1], a space (X,d) with a quasi-lattice $\Gamma \subseteq X$ is non-amenable if and only if $0 = [\Gamma] \in H_0^1(\Gamma)$ where $H_0^1(\Gamma)$ is the zeroth controlled coarse homology group of Γ for $\varrho \equiv 1$. With this in mind, we give a new characterisation of non-amenability.

Corollary D. *If (X,d,μ) is a quasiconvex uniform space that supports a local weak $(1,1)$-Poincaré inequality then (X,d) is non-amenable if and only if (X,d,μ) satisfies $(S_{1,1})$.*

Corollary D follows directly from Theorem A by Theorem 3.1 and Proposition 2.3 in [1]. Note in particular the similarity between Corollary D and Theorem 7.1 in [6]; related to this, see Example 5.8 in [17]. Corollary D has applications for visual Gromov hyperbolic metric measure spaces: recall that (X,d) is *Gromov hyperbolic* if it satisfies for some $\delta \in [0,\infty)$ the Gromov product inequality

$$(x|z)_w \geq \min\{(x|y)_w, (y|z)_w\} - \delta$$

for all $x,y,z,w \in X$; and (μ)-*visual* if there exist $o \in X$ and a constant $\mu \geq 0$ such that every point in X is contained in the image of some $(1,\mu)$-quasi-isometric embedding $\gamma\colon [0,\infty) \to X$ where $\gamma(0) = o$. Noting that a uniform space (X,d,μ) is uniformly coarsely proper, see Remark 3, Corollary D together with Theorem B in [15] thus implies the following.

Theorem E. *If (X,d,μ) is a quasiconvex uniform visual Gromov hyperbolic space that supports a local weak $(1,1)$-Poincaré inequality and its Gromov boundary consists of finitely many uniformly coarsely connected components each containing at least two points then (X,d,μ) satisfies $(S_{1,1})$.*

A *uniformly coarsely connected component*, say in (Z,d), is any set of the form $C(z,Z) = \bigcup\{A\colon z \in A \subseteq Z,\ A \text{ uniformly coarsely connected}\}$ where A is *uniformly coarsely connected* if for every $\varepsilon > 0$ and every $x,y \in A$ there exists a finite sequence of points $x = x_0, \dots, x_n = y$ in A such that $d(x_i, x_{i+1}) \le \varepsilon$ for all $0 \le i \le n-1$.

Theorem E has the following application to the Dirichlet problem at infinity and generalises Corollary 1.1 in [2]; see also [12].

Theorem F. *Suppose (X, d, μ) is a locally compact quasiconvex uniform visual Gromov hyperbolic space defined using the Gromov product that supports a local weak $(1,1)$-Poincaré inequality and with Gromov boundary ∂X consisting of finitely many uniformly coarsely connected components each containing at least two points. Then if $f\colon \partial X \to \mathbb{R}$ is a bounded continuous function there exists a continuous function $u\colon X^* \to \mathbb{R}$ on the Gromov closure X^* of X which is p-harmonic for $p > 1$ in X and $u|\partial X = f$.*

Proof. By Theorem E, the space (X, d, μ) satisfies $(S_{1,1})$ and hence the corresponding (p,p)-Sobolev inequality for $1 \le p < \infty$; see Example 8 in [12]. By Hölder's inequality, the space (X, d, μ) supports a local weak $(1,p)$-inequality for $1 \le p < \infty$ as well. Together with Remark 4 we conclude that (X, d, μ) satisfies all the assumptions of Theorem 1.1 in [12], from which the claim then follows. □

We finish with an example illustrating the case when $\varrho \not\equiv 1$. Write $f \preceq g$ for two non-decreasing functions $f, g\colon [0,\infty) \to [0,\infty)$ for which there exist constants $\lambda > 0$, $\mu > 0$, and $c \ge 0$ such that $f(r) \le \lambda g(\mu r + c)$ for all $r \ge 0$. Also, write $f \precnsim g$ if $f \preceq g$ but $g \not\preceq f$.

Example G. *The first real Heisenberg group $(\mathcal{H}_1(\mathbb{R}), d_{\mathcal{H}}, \mu)$ with Heisenberg metric satisfies $(S^{\varrho}_{1,1})$ for $\varrho(t) = t+1$ but not $(S^{\xi}_{1,1})$ for any other control function $\xi \precnsim \varrho$.*

Proof. As the first integer Heisenberg group $\mathcal{H}_1(\mathbb{Z})$ is a uniform lattice in $\mathcal{H}_1(\mathbb{R})$, there exists a quasi-isometry

$$f\colon (\mathcal{H}_1(\mathbb{Z}), d_S) \to (\mathcal{H}_1(\mathbb{R}), d_H),$$

where d_S is the word metric; see Definition 4.B.1 and Proposition 5.C.3 in [3]. In particular, $H^{\varrho}_0(\mathcal{H}_1(\mathbb{Z})) \cong H^{\varrho}_0(\mathcal{H}_1(\mathbb{R}))$ are isomorphic. As the group $\mathcal{H}_1(\mathbb{Z})$ is infinite polycyclic, $0 = [\mathcal{H}_1(\mathbb{Z})] \in H^{\varrho}_0(\mathcal{H}_1(\mathbb{Z}))$ if and only if $\varrho(t) = t+1$; see Corollary 5.5 in [16]. In particular, $0 \ne [\mathcal{H}_1(\mathbb{Z})] \in H^{\xi}_0(\mathcal{H}_1(\mathbb{Z}))$ for $\xi(t) \precnsim t+1$. The claim now follows from Theorem A. □

Similar arguments hold for Carnot groups.

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2. Tools of controlled coarse homology

Below we recall some terminology and results from [16], and prove a small lemma for the quasi-isometry invariance of the vanishing of the fundamental class. We say that (X, d) is *uniformly coarsely proper* if it has a quasi-lattice $\Gamma \subseteq X$.

Remark 1. A space (X, d) is uniformly coarsely proper if and only if there exist a constant $r_b > 0$ and $N\colon (0, \infty) \times (0, \infty) \to \mathbb{N}$ such that for all $R > r > r_b$ every open ball of radius R in X can be covered by $N(R, r)$ open balls of radius r in X; see Section 3.D.b in [3].

Let (X, d, o) be a pointed uniformly coarsely proper space, let $q \in \mathbb{N}$, and let $(X^{q+1}, d, \bar{o})$ be the pointed Cartesian product with basepoint $\bar{o} = (o, \ldots, o)$ and metric

$$d(\bar{x}, \bar{y}) = \max_{0 \le i \le q} d(x_i, y_i),$$

where $\bar{x} = (x_0, \ldots, x_q) \in X^{q+1}$ and $\bar{y} = (y_0, \ldots, y_q) \in X^{q+1}$. Given a quasi-lattice $\Gamma \subseteq X$ where $o \in \Gamma$ and a control function ϱ, we denote by $C_q^\varrho(\Gamma)$ the space of functions $c\colon \Gamma^{q+1} \to \mathbb{R}$ for which

(a) there exists a constant $K(c) \ge 0$ such that $|c(\bar{x})| \le K(c)\varrho(d(\bar{x}, \bar{o}))$ for all $\bar{x} \in \Gamma^{q+1}$;

(b) c is alternating, meaning that $c(x_{\sigma(0)}, \ldots, x_{\sigma(q)}) = \operatorname{sign}(\sigma) c(x_0, \ldots, x_q)$ for all $(x_0, \ldots, x_q) \in \Gamma^{q+1}$ and all permutations $\sigma\colon \{0, \ldots, q\} \to \{0, \ldots, q\}$;

(c) there exists a constant $P(c) \ge 0$ such that if $\max_{i \ne j} d(x_i, x_j) > P(c)$ then $c(x_0, \ldots, x_q) = 0$.

Note that $K(c)$ and $P(c)$ are allowed to depend on c. Note also that $C_q^\varrho(\Gamma)$ is an $\mathbb{R}$-module that does not depend on the choice of basepoint by (ϱ_1). A function $c \in C_q^\varrho(\Gamma)$ is called a *controlled coarse q-chain* and we write

$$c = \sum_{(x_0, \ldots, x_q) \in \Gamma^{q+1}} c(x_0, \ldots, x_q)\, [x_0, \ldots, x_q],$$

where $[x_0, \ldots, x_q] \in C_q^\varrho(\Gamma)$ is the characteristic function $\chi_{(x_0, \ldots, x_q)}$ of the point $(x_0, \ldots, x_q)$. The *controlled coarse homology* $H_*^\varrho(\Gamma)$ is the homology of the chain complex

$$\cdots \xrightarrow{\partial_3} C_2^\varrho(\Gamma) \xrightarrow{\partial_2} C_1^\varrho(\Gamma) \xrightarrow{\partial_1} C_0^\varrho(\Gamma) \xrightarrow{\partial_0} 0,$$

where the boundary homomorphism $\partial_q\colon C_q^\varrho(\Gamma) \to C_{q-1}^\varrho(\Gamma)$ is given by

$$\partial_q([x_0, \ldots, x_q]) = \sum_{i=0}^{q} (-1)^i\, [x_0, \ldots, \hat{x}_i, \ldots, x_q]$$

for each abstract q-cell $[x_0, \ldots, x_q]$ and extended linearly to $C^q(\Gamma)$ for $q \in \mathbb{N} \setminus \{0\}$; as usual, $[x_0, \ldots, \hat{x}_i, \ldots, x_q]$ denotes the abstract $(q-1)$-cell obtained from $[x_0, \ldots, x_q]$ by omitting its ith coordinate. In particular, $\partial_{q-1} \circ \partial_q = 0$ and $\partial_q c \in C_{q-1}^\varrho(\Gamma)$ by (ϱ_1).

The *q-dimensional controlled coarse homology group of* Γ is

$$H_q^\varrho(\Gamma) = \ker \partial_q / \operatorname{im} \partial_{q+1}.$$

An important role is played by the homology class $[\Gamma] \in H_0^\varrho(\Gamma)$ of the characteristic function of $\Gamma \subseteq X$

$$\chi_\Gamma = \sum_{x \in \Gamma} [x] \in C_0^\varrho(\Gamma),$$

called the *fundamental class*. Its vanishing characterises the ϱ-isoperimetry of the space as we now explain. In what follows we use the notation $|(x,y)| = d(\bar{o},(x,y))$.

Lemma 4.1 and Theorem 4.2 in [16]. *Let Γ be a quasi-lattice and $o \in \Gamma$. If there exists a constant $0 < C \leq 1$ such that $d(x,y) \geq C$ whenever $x, y \in \Gamma$ are distinct and for all $x, y \in \Gamma$ there is a sequence $x = x_0, \ldots, x_n = y$ in Γ such that $n \leq d(x,y)$ and $d(x_i, x_{i+1}) \leq 1$ for every $0 \leq i \leq n-1$, then the following are equivalent:*

(1) $0 = [\Gamma] \in H_0^\varrho(\Gamma)$;

(2) *there exists a constant $C' > 0$ such that*

$$\sum_{x \in \Gamma} |v(x)| \leq C' \Big(\sum_{x \in \Gamma} \sum_{\{y \in \Gamma \colon d(x,y) \leq 1\}} |v(x) - v(y)| \, \varrho\left(|(x,y)|\right) \Big)$$

for every $v \colon \Gamma \to \mathbb{R}$ with finite support;

(3) *there exists a constant $C'' > 0$ such that for every finite subset $F \subseteq \Gamma$*

$$\#F \leq C'' \sum_{x \in \partial F} \varrho(d(o,x)),$$

where $\partial F = \{x \in \Gamma \colon d(x,F) = 1 \,\mathrm{or}\, d(x, \Gamma \setminus F) = 1\}$.

We will also need the following.

Lemma 2.3 in [16]. *If $\Gamma \subseteq X$ is a quasi-lattice and $c = \sum_{x \in \Gamma} c(x)[x] \in C_0^\varrho(\Gamma)$ such that $\inf_{x \in \Gamma} c(x) > 0$ and $0 = [c] \in H_0^\varrho(\Gamma)$ then $0 = [\Gamma] \in H_0^\varrho(\Gamma)$.*

This leads us to the following central observation below: if $[\Gamma] = 0$ for some quasi-lattice $\Gamma \subseteq X$, then $[\Gamma'] = 0$ for every quasi-lattice $\Gamma' \subseteq X$.

Lemma 2. *Let $f \colon \Gamma \to \Gamma'$ be a quasi-isometry between quasi-lattices. Then $0 = [\Gamma] \in H_0^\varrho(\Gamma)$ if and only if $0 = [\Gamma'] \in H_0^\varrho(\Gamma')$.*

Proof. The quasi-isometry $f \colon \Gamma \to \Gamma'$ induces a chain map $f_q \colon C_q^\varrho(\Gamma) \to C_q^\varrho(\Gamma')$ extending the map $[x_0, \ldots, x_q] \mapsto [f(x_0), \ldots, f(x_q)]$ linearly to $C_q^\varrho(\Gamma)$. By (ϱ_1) and (ϱ_2), the function f_q is well-defined. In particular,

$$f_0\Big(\sum_{x \in \Gamma} [x]\Big) = \sum_{x \in \Gamma} [f(x)] = \sum_{y \in f(\Gamma)} c(y)[y] = c' \in C_0^\varrho(\Gamma'),$$

where $c(y) = \#f^{-1}(y) \geq 1$ for $y \in f(\Gamma)$.

Since $f(\Gamma) \subseteq \Gamma'$ is a quasi-lattice and $0 = [\Gamma]$ implies that $0 = [c'] \in H_0^{\varrho}(\Gamma')$ there exists for every $y \in f(\Gamma)$ a controlled coarse 1-chain

$$t_y = \sum_{i=0}^{\infty} [x_i, x_{i+1}] \in C_1^{\varrho}(f(\Gamma)),$$

where $x_0 = y$, so that

$$t = \sum_{y \in f(\Gamma)} t_y \in C_1^{\varrho}(f(\Gamma))$$

by the proof of Lemma 2.3 in [16]; see also Lemma 2.4 in [1]. By coboundedness, fix $C > 0$ such that $N_C(f(\Gamma)) = \Gamma'$. To begin, let $y_1 \in f(\Gamma)$ and let

$$t_{w,y_1} = [w, y_1] + t_{y_1} \in C_1^{\varrho}(\Gamma')$$

for each $w \in B(y_1, C) \setminus \{y_1\}$. Since Γ' is uniformly locally finite, there is at most $\#(B(y_1, C) \cap \Gamma') \leq N(C)$ chains t_{w,y_1}. Next, let $y_2 \in f(\Gamma) \setminus \{y_1\}$ and let

$$t_{w,y_2} = [w, y_2] + t_{y_2} \in C_1^{\varrho}(\Gamma')$$

for each $w \in (B(y_2, C) \setminus \{y_2\}) \setminus B(y_1, C)$. Again, there is at most $N(C)$ chains t_{w,y_2}. Continuing in the obvious way, we obtain a controlled coarse 1-chain

$$t' = \sum_{i=1}^{\infty} t_{w,y_i} + \sum_{y \in f(\Gamma)} t_y \in C_1^{\varrho}(\Gamma')$$

whose boundary is $\partial_1 t' = \sum_{y \in \Gamma'} [y]$. In other words, $0 = [\Gamma'] \in H_0^{\varrho}(\Gamma')$ as claimed. □

For uniformly finite homology, corresponding to $\varrho \equiv 1$, Lemma 2 follows directly by Proposition 2.3 in [1].

3. Discretisation and smoothing

A metric measure space (X, d, μ) is a $(DV)_{\text{loc}}$ *space* if it has the $(DV)_{\text{loc}}$ *property* saying that there exists a function $C\colon (0, \infty) \to (0, \infty)$ such that

$$0 < \mu(B(x, 2r)) \leq C(r)\mu(B(x, r)) < \infty$$

for every $0 < r < \infty$ and all $x \in X$; see [6]. This implies that the space is separable by Lemma 3.3.30 in [10]. Examples of $(DV)_{\text{loc}}$ spaces includes locally compact groups acting by measure preserving isometries on metric measure spaces by Example 5.4 in [17], and uniform spaces for $C(r) = g(2r)/f(r)$.

3.1. From discrete to smooth

A *maximal* ε*-net* in (X, d) is a ε-cobounded subset $N(X, \varepsilon) \subseteq X$ such that $d(x, y) \geq \varepsilon$ if $x, y \in N(X, \varepsilon)$ are distinct. We also write $q \sim p$ if $p, q \in N(X, \varepsilon)$ and $0 < d(p, q) \leq 3\varepsilon$, saying that q *is a neighbour of* p.

By Zorn's lemma, there exists for all $\varepsilon > 0$ and all $o \in X \neq \emptyset$ a maximal ε-net $N(X, \varepsilon) \ni o$.

Remark 3. A $(DV)_{\mathrm{loc}}$ space (X, d, μ) is uniformly coarsely proper. In particular any $N(X, \varepsilon)$ is a quasi-lattice.

Proof. Adapt the argument for doubling spaces in Section 4.1 in [10] □

Remark 4. A quasiconvex uniform space (X, d, μ) has at most exponential volume growth.

Proof. Fix a quasi-lattice $N(X, \varepsilon)$ and let $k \in \mathbb{N} \setminus \{0\}$. Since $N(X, \varepsilon)$ is uniformly locally finite any open ball $B(x, 2k\varepsilon) \subseteq X$ can be covered by $N(3\varepsilon)^k$ balls of radius ε. Since (X, d, μ) is uniform,

$$\mu(B(x, 2k\varepsilon)) \leq g(\varepsilon) N(3\varepsilon)^k$$

for every $k \in \mathbb{N} \setminus \{0\}$, from which the claim then follows. □

Lemma 5. *Let (X, d, μ) be an unbounded quasiconvex $(DV)_{\mathrm{loc}}$ space that supports a local weak $(1,1)$-Poincaré inequality up to scale R_P. Then, given a quasi-lattice $N(X, \varepsilon) \ni o$ where $\mu(\{o\}) = 0$ and $0 < \varepsilon \leq R_P/4$ and a control function ϱ, there exists a constant $C > 0$ for which*

$$\sum_{p \in N(X,\varepsilon)} \sum_{q \sim p} |u_{B(p,4\varepsilon)} - u_{B(q,4\varepsilon)}| \, \varrho(d(o,p)) \, \mu(B(p,\varepsilon)) \leq C \int_X |\nabla u(x)| \, \varrho(d(o,x)) \, d\mu(x)$$

for every $u \in N^{1,1}(X, d, \mu)$.

For $\varrho \equiv 1$, this is known for complete Riemannian manifolds of bounded geometry; see Lemma 33 in [11], as well as [14]. For the lemma at hand, the point to note is that using (ϱ_1) this can be weighted by ϱ.

Proof of Lemma 5. Let $p \in N(X, \varepsilon)$ and $x \in B(p, 8\tau\varepsilon)$ where $\tau \geq 1$. Now $d(o,p) \leq d(o,x) + d(x,p) \leq d(o,x) + 8\tau\varepsilon$, and since ϱ is non-decreasing,

$$(\star) \quad \begin{aligned} \varrho(d(o,p)) \int_{B(p,8\tau\varepsilon)} |\nabla u(x)| \, d\mu(x) &\leq \int_{B(p,8\tau\varepsilon)} |\nabla u(x)| \, \varrho(d(o,x) + 8\tau\varepsilon) \, d\mu(x) \\ &= \int_{B(p,8\tau\varepsilon)\setminus\{o\}} |\nabla u(x)| \, \varrho(d(o,x) + 8\tau\varepsilon) \, d\mu(x) \\ &\leq L(8\tau\varepsilon) \int_{B(p,8\tau\varepsilon)} |\nabla u(x)| \, \varrho(d(o,x)) \, d\mu(x), \end{aligned}$$

by (ϱ_1). The claim now follows by estimating ($\star$) from below using the local weak $(1,1)$-Poincaré inequality. First, choose $q \sim p$ noting that the space is quasiconvex and unbounded. Now $B(p, 4\tau\varepsilon) \cup B(q, 4\tau\varepsilon) \subseteq B(p, 8\tau\varepsilon)$ and

$$\int_{B(p,8\varepsilon\tau)} |\nabla u(x)| \, d\mu(x) \geq \frac{1}{2} \int_{B(p,4\tau\varepsilon)} |\nabla u(x)| \, d\mu(x) + \frac{1}{2} \int_{B(q,4\tau\varepsilon)} |\nabla u(x)| \, d\mu(x).$$

By the local weak $(1,1)$-Poincaré inequality,

$$\fint_{B(p,4\tau\varepsilon)} |\nabla u(x)|\, d\mu(x) \geq \frac{1}{4\varepsilon C_P} \fint_{B(p,4\varepsilon)} |u(x) - u_{B(p,4\varepsilon)}|\, d\mu(x),$$

and since $\mu(B(p,4\tau\varepsilon)) \geq \mu(B(p,4\varepsilon))$,

$$\int_{B(p,4\tau\varepsilon)} |\nabla u(x)|\, d\mu(x) \geq C \int_{B(p,4\varepsilon)} |u(x) - u_{B(p,4\varepsilon)}|\, d\mu(x)$$

for some constant $C > 0$. Hence,

$$\begin{aligned}
\int_{B(p,8\tau\varepsilon)} |\nabla u(x)|\, d\mu(x) &\geq \frac{1}{2} \int_{B(p,4\tau\varepsilon)} |\nabla u(x)|\, d\mu(x) + \frac{1}{2} \int_{B(q,4\tau\varepsilon)} |\nabla u(x)|\, d\mu(x) \\
&\geq \frac{C}{2} \int_{B(p,4\varepsilon)} |u(x) - u_{B(p,4\varepsilon)}|\, d\mu(x) + \frac{C}{2} \int_{B(q,4\varepsilon)} |u(x) - u_{B(q,4\varepsilon)}|\, d\mu(x) \\
&\geq \frac{C}{2} \int_{B(p,4\varepsilon)\cap B(q,4\varepsilon)} \left(|u(x) - u_{B(p,4\varepsilon)}| + |u(x) - u_{B(q,4\varepsilon)}|\right)\, d\mu(x) \\
&\geq \frac{C}{2} |u_{B(p,4\varepsilon)} - u_{B(q,4\varepsilon)}| \int_{B(p,\varepsilon)} d\mu(x) \\
&= \frac{C}{2} |u_{B(p,4\varepsilon)} - u_{B(q,4\varepsilon)}| \mu(B(p,\varepsilon)),
\end{aligned}$$

since $B(p,\varepsilon) \subseteq B(p,4\varepsilon) \cap B(q,4\varepsilon)$. Using this to estimate $(\star)$ from below gives

$$\begin{aligned}
\int_{B(p,8\tau\varepsilon)} |\nabla u(x)|\, \varrho(d(o,x))\, d\mu(x) &\geq \frac{\varrho(d(o,p))}{L(8\tau\varepsilon)} \int_{B(p,8\tau\varepsilon)} |\nabla u(x)|\, d\mu(x) \\
&\geq \frac{C\varrho(d(o,p))}{2L(8\tau\varepsilon)} |u_{B(p,4\varepsilon)} - u_{B(q,4\varepsilon)}| \mu(B(p,\varepsilon)).
\end{aligned}$$

Since $N(X,\varepsilon)$ is uniformly locally finite the number of neighbours $q \sim p$ is uniformly bounded and so

$$\begin{aligned}
&\int_{B(p,8\tau\varepsilon)} |\nabla u(x)| \varrho(d(o,x))\, d\mu(x) \\
&\qquad \geq C' \varrho(d(o,p)) \sum_{q\sim p} |u_{B(p,4\varepsilon)} - u_{B(q,4\varepsilon)}| \mu(B(p,\varepsilon))
\end{aligned}$$

for some constant $C' > 0$ independent of u. Similarly, every $x \in X$ belongs to a uniformly bounded number of open balls of radius $8\tau\varepsilon$ having a center in $N(X,\varepsilon)$, and altogether,

$$\begin{aligned}
&\sum_{p\in N(X,\varepsilon)} \sum_{q\sim p} |u_{B(p,4\varepsilon)} - u_{B(q,4\varepsilon)}|\, \varrho(d(o,p)) \mu(B(p,\varepsilon)) \\
&\leq C'^{-1} \sum_{p\in N(X,\varepsilon)} \int_{B(p,7\tau\varepsilon)} |\nabla u(x)|\, \varrho(d(o,x))\, d\mu(x) \leq C'' \int_X |\nabla u(x)|\, \varrho(d(o,x))\, d\mu(x)
\end{aligned}$$

for some constant $C'' > 0$ independent of u. The claim now follows. □

Proposition 6. *Let (X,d,μ) be a quasiconvex $(DV)_{\mathrm{loc}}$ space that supports a local weak $(1,1)$-Poincaré inequality up to scale R_P and $N(X,\varepsilon)\ni o$ a quasi-lattice where $\mu(\{o\})=0$ and $0<\varepsilon\le R_P/4$. If there exists a control function ϱ and a constant $C>0$ such that*

$$\sum_{p\in N(X,\varepsilon)}|v(p)|\mu(B(p,\varepsilon))\le C\sum_{p\in N(X,\varepsilon)}\sum_{q\sim p}|v(p)-v(q)|\,\varrho(|(p,q)|)\,\mu(B(p,\varepsilon))$$

for every $v\colon N(X,\varepsilon)\to\mathbb{R}$ with finite support, then (X,d,μ) satisfies $(S^{\varrho}_{1,1})$.

Proof. Let $u\colon X\to[0,\infty)$ be a representative in $N^{1,1}(X,d,\mu)$ having bounded support. Now,

$$u_{B(\cdot,4\varepsilon)}\colon N(X,\varepsilon)\to[0,\infty)$$

is finitely supported, and since $|(p,q)|=d(\bar{o},(p,q))\le 2d(o,p)+3\varepsilon$, we have

$$\begin{aligned}\sum_{p\in N(X,\varepsilon)}&u_{B(p,4\varepsilon)}\mu(B(p,\varepsilon))\\&\le C\sum_{p\in N(X,\varepsilon)}\sum_{q\sim p}|u_{B(p,4\varepsilon)}-u_{B(q,4\varepsilon)}|\,\varrho(|(p,q)|)\,\mu(B(p,\varepsilon))\\&\le C\sum_{p\in N(X,\varepsilon)\setminus\{o\}}\sum_{q\sim p}|u_{B(p,4\varepsilon)}-u_{B(q,4\varepsilon)}|\,\varrho(2d(o,p)+3\varepsilon)\,\mu(B(p,\varepsilon))\\&\quad+C\sum_{q\sim o}|u_{B(o,4\varepsilon)}-u_{B(q,4\varepsilon)}|\,\varrho(3\varepsilon)\,\mu(B(o,\varepsilon)).\end{aligned}$$

The first sum on the right-hand side contains every neighbour of o. To estimate the second sum observe that $\varrho(3\varepsilon)\le\varrho(2d(o,p)+3\varepsilon)$ for every $p\in N(X,\varepsilon)$, and when $p\sim o$ we have $B(o,\varepsilon)\subseteq B(o,4\varepsilon)\subseteq B(p,8\varepsilon)$, which gives $\mu(B(o,\varepsilon))\le C(4\varepsilon)C(2\varepsilon)C(\varepsilon)\,\mu(B(p,\varepsilon))$ using the $(DV)_{\mathrm{loc}}$ property. Put together, this gives the estimate

$$\begin{aligned}\sum_{p\in N(X,\varepsilon)}&u_{B(p,4\varepsilon)}\mu(B(p,\varepsilon))\\\le 2CC(4\varepsilon)C(2\varepsilon)C(\varepsilon)&\sum_{p\in N(X,\varepsilon)\setminus\{o\}}\sum_{q\sim p}|u_{B(p,4\varepsilon)}-u_{B(q,4\varepsilon)}|\,\varrho(2d(o,p)+3\varepsilon)\,\mu(B(p,\varepsilon)).\end{aligned}$$

Using both (ϱ_1) and (ϱ_2), this gives

$$\begin{aligned}\sum_{p\in N(X,\varepsilon)}&u_{B(p,4\varepsilon)}\mu(B(p,\varepsilon))\\&\le C'\sum_{p\in N(X,\varepsilon)\setminus\{o\}}\sum_{q\sim p}|u_{B(p,4\varepsilon)}-u_{B(q,4\varepsilon)}|\,\varrho(d(o,p))\,\mu(B(p,\varepsilon))\\&\le C'\sum_{p\in N(X,\varepsilon)}\sum_{q\sim p}|u_{B(p,4\varepsilon)}-u_{B(q,4\varepsilon)}|\,\varrho(d(o,p))\,\mu(B(p,\varepsilon))\end{aligned}$$

for some constant $C' > 0$ independent of u. By Lemma 5,

$$\sum_{p\in N(X,\varepsilon)}\sum_{q\sim p}|u_{B(p,4\varepsilon)} - u_{B(q,4\varepsilon)}|\varrho(d(o,p))\,\mu(B(p,\varepsilon)) \leq C'\int_X |\nabla u(x)|\,\varrho(d(o,x))\,d\mu(x),$$

so

$$\sum_{p\in N(X,\varepsilon)} u_{B(p,4\varepsilon)}\,\mu(B(p,\varepsilon)) \leq C''\int_X |\nabla u(x)|\,\varrho(d(o,x))\,d\mu(x)$$

for some constant $C'' > 0$ independent of u. On the other hand, by the $(DV)_{\mathrm{loc}}$ property,

$$\int_X u(x)\,d\mu(x) \leq \sum_{p\in N(X,\varepsilon)}\int_{B(p,4\varepsilon)} u(x)\,d\mu(x) = \sum_{p\in N(X,\varepsilon)} u_{4B(p,4\varepsilon)}\,\mu(B(p,4\varepsilon)) \leq C(2\varepsilon)C(\varepsilon)\sum_{p\in N(X,\varepsilon)} u_{4B(p,\varepsilon)}\,\mu(B(p,\varepsilon)),$$

from which the claim follows for $u\colon X\to[0,\infty)$ in $N(X,d,\mu)$ having bounded support. The claim for any $u\in N^{1,1}(X,d,\mu)$ having bounded support follows by replacing u with $|u|$ and noticing that $|\nabla|u||\leq|\nabla u|$. □

3.2. From smooth to discrete

To begin, we recall the notion of Lipschitz partition of unity associated to $N(X,\varepsilon)$ and that of Lipschitz extensions from Section 1.12 in [9].

Definition 7. A *Lipschitz partition of unity* associated to $N(X,\varepsilon)$ is a locally finite family $\{\varphi_p\colon p\in N(X,\varepsilon)\}$ of L-Lipschitz functions $\varphi_p\colon X\to[0,1]$ such that

$$\sum_{p\in N(X,\varepsilon)}\varphi_p(x) = 1$$

for every $x\in X$ and $\varphi_p|(X\setminus B(p,2\varepsilon))\equiv 0$.

The following lemma is a modification of that given in Section 1.12 in [9]; the proofs are essentially identical.

Lemma 8. *If (X,d) is a quasiconvex uniformly coarsely proper space and $N(X,\varepsilon)$ a quasi-lattice where $0<\varepsilon\leq 2$, then the family $\{\varphi_p\colon p\in N(X,\varepsilon)\}$, where*

$$\varphi_p(x) = \frac{\psi_p(x)}{\psi(x)},\quad \psi_p(x) = \min\left\{1,\frac{2}{\varepsilon}\operatorname{dist}(x,X\setminus B(p,3\varepsilon/2))\right\},\ \text{and}$$
$$\psi(x) = \sum_{p\in N(X,\varepsilon)}\psi_p(x),$$

gives a Lipschitz partition of unity associated to $N(X,\varepsilon)$.

Definition 9. Let (X,d) be a quasiconvex uniformly coarsely proper space and let $N(X,\varepsilon)$ be a quasi-lattice, where $0<\varepsilon\leq 2$. Given $v\colon N(X,\varepsilon)\to\mathbb{R}$, its *locally Lipschitz extension* $\overline{v}\colon X\to\mathbb{R}$ associated to $\{\varphi_p\colon p\in N(X,\varepsilon)\}$ is defined by

$$\overline{v}(x)=\sum_{p\in N(X,\varepsilon)} v(p)\,\varphi_p(x),$$

where $\{\varphi_p\colon p\in N(X,\varepsilon)\}$ is as in Lemma 8.

The *pointwise upper Lipschitz constant at* $x\in X$ of $v\colon X\to\mathbb{R}$ from (X,d) is given by

$$\operatorname{Lip} v(x)=\limsup_{r\to 0}\sup_{y\in B(x,r)}\frac{|v(x)-v(y)|}{r}.$$

Note that $\operatorname{Lip}\overline{v}\colon X\to[0,\infty]$ is an upper gradient of the locally Lipschitz extension $\bar{v}\colon X\to\mathbb{R}$ of $v\colon N(X,\varepsilon)\to\mathbb{R}$; see Lemma 6.2.6 in [10]. We are now ready to prove the following.

Lemma 10. *Let (X,d,μ) be a quasiconvex $(DV)_{\mathrm{loc}}$ space and $N(X,\varepsilon)\ni o$ a quasi-lattice where $\mu(\{o\})=0$ and $0<\varepsilon\leq 2$ and ϱ a control function. Then there exists a constant $C>0$ such that*

$$\int_X \operatorname{Lip}\overline{v}(x)\varrho(d(o,x))\,d\mu(x)\leq C\sum_{p\in N(X,\varepsilon)}\sum_{q\sim p}|v(p)-v(q)|\,\varrho(d(o,p))\,\mu(B(p,\varepsilon))$$

for every $v\colon N(X,\varepsilon)\to\mathbb{R}$.

Proof. Let $v\colon N(X,\varepsilon)\to\mathbb{R}$ be any function and let $\overline{v}\colon X\to\mathbb{R}$ be its locally Lipschitz extension; see Definition 9. Arguing as in Lemma 3.2 in [12], there exists a constant $C>0$ such that for any $p\in N(X,\varepsilon)$ and $x,y\in B(p,\varepsilon)$,

$$\frac{|\overline{v}(x)-\overline{v}(y)|}{d(x,y)}\leq C\sum_{q\in B(p,3\varepsilon)\cap N(X,\varepsilon)}|v(q)-v(p)|.$$

In particular,

$$\operatorname{Lip}\overline{v}(x)=\limsup_{r\to 0}\sup_{y\in B(x,r)}\frac{|\overline{v}(x)-\overline{v}(y)|}{r}\leq C\sum_{q\in B(p,3\varepsilon)\cap N(X,\varepsilon)}|v(q)-v(p)|.$$

Thus,

$$\begin{aligned}\int_X \operatorname{Lip}\overline{v}(x)\varrho(d(o,x))\,d\mu(x)&\leq\sum_{p\in N(X,\varepsilon)}\int_{B(p,\varepsilon)}\operatorname{Lip}\overline{v}(x)\varrho(d(o,x))\,d\mu(x)\\&\leq C\sum_{p\in N(X,\varepsilon)}\sum_{q\in B(p,3\varepsilon)\cap N(X,\varepsilon)}|v(q)-v(p)|\int_{B(p,\varepsilon)}\varrho(d(o,x))\,d\mu(x).\end{aligned}$$

The claim now follows by an application of (ϱ_1). Indeed, if $x \in B(p,\varepsilon)$, then $d(o,x) \le d(x,p) + d(p,o) \le \varepsilon + d(o,p)$, and we have $\varrho(d(o,x)) \le L(\varepsilon)\varrho(d(o,p))$ whenever $p \neq o$. Hence,

$$\int_X \operatorname{Lip} \overline{v}(x)\varrho(d(o,x))\, d\mu(x) \le CL(\varepsilon) \sum_{p\in N(X,\varepsilon)} \sum_{q\sim p} |v(q)-v(p)|\, \varrho(d(o,p))\, \mu(B(p,\varepsilon))$$

as claimed. ∎

At this point, we have the following version of Theorem 4.2 in [16] for quasi-convex $(DV)_{\text{loc}}$ spaces.

Theorem 11. *If (X,d,μ) is a quasiconvex $(DV)_{\text{loc}}$ space that supports a local weak $(1,1)$-Poincaré inequality up to scale R_P, then the following are equivalent:*

(1) *(X,d,μ) satisfies $(S^{\varrho}_{1,1})$;*

(2) *For any $N(X,\varepsilon) \ni o$ where $\mu(\{o\}) = 0$ and $0 < \varepsilon \le \min\{2, R_P/4\}$, there exists a constant $C > 0$ such that*

$$\sum_{p\in N(X,\varepsilon)} |v(p)|\mu(B(p,\varepsilon)) \le C \sum_{p\in N(X,\varepsilon)} \sum_{q\sim p} |v(p)-v(q)|\, \varrho(|(p,q)|)\, \mu(B(p,\varepsilon))$$

for every $v\colon N(X,\varepsilon) \to \mathbb{R}$ with finite support.

Proof. By Proposition 6 it follows that (2) implies (1). To prove that (1) implies (2), let $v\colon N(X,\varepsilon) \to [0,\infty)$ be finitely supported and let $\overline{v}\colon X \to [0,\infty)$ be its locally Lipschitz extension:

$$\overline{v}(x) = \sum_{p\in N(X,\varepsilon)} v(p)\, \varphi_p(x) = \sum_{p\in N(X,\varepsilon)} v(p) \frac{\psi_p(x)}{\psi(x)},$$

now with bounded support. Since $\overline{v}$ is locally Lipschitz, $\operatorname{Lip}\overline{v}$ is an upper gradient of $\overline{v}$. In particular, $\overline{v}$ has a minimal 1-weak upper gradient $|\nabla\overline{v}|$; see Theorem 6.3.20 in [10]. Thus, by $(S^{\varrho}_{1,1})$,

$$\int_X \overline{v}(x)\, d\mu(x) \le C \int_X |\nabla\overline{v}|\, \varrho(d(o,x))\, d\mu(x) \le C \int_X \operatorname{Lip}\overline{v}(x)\varrho(d(o,x))\, d\mu(x).$$

By Lemma 10,

$$\int_X \operatorname{Lip}\overline{v}(x)\varrho(d(o,x))\, d\mu(x) \le C' \sum_{p\in N(X,\varepsilon)} \sum_{q\sim p} |v(p)-v(q)|\, \varrho(|(p,q)|)\, \mu(B(p,\varepsilon)).$$

Since ψ in the Lipschitz partition of unity is uniformly bounded, there exists a constant $C'' > 0$ for which $\psi(x) \le C''$ for all $x \in X$ and

$$\begin{aligned}\int_X \overline{v}(x)\, d\mu(x) &= \int_X \sum_{p\in N(X,\varepsilon)} v(p)\varphi_p(x)\, d\mu(x) = \int_X \sum_{p\in N(X,\varepsilon)} v(p)\frac{\psi_p(x)}{\psi(x)}\, d\mu(x)\\ &\ge \frac{1}{C''}\int_X \sum_{p\in N(X,\varepsilon)} v(p)\psi_p(x)\, d\mu(x) \ge \frac{1}{C''} \sum_{p\in N(X,\varepsilon)} v(p)\, \mu(B(p,\varepsilon))\end{aligned}$$

as $\psi_p|B(p,\varepsilon) \equiv 1$; and altogether, for some constant $C''' > 0$ independent of v,

$$\sum_{p\in N(X,\varepsilon)} v(p)\,\mu(B(p,\varepsilon)) \le C''' \sum_{p\in N(X,\varepsilon)} \sum_{q\sim p} |v(p)-v(q)|\,\varrho(|(p,q)|)\,\mu(B(p,\varepsilon))$$

for every $v\colon N(X,\varepsilon) \to [0,\infty)$ with finite support. The general claim for any $v\colon N(X;\varepsilon) \to \mathbb{R}$ with finite support now follows observing that the claim holds for $|v|$ by the previous, and by the triangle inequality for v. □

3.3. Connecting H_0^ϱ to $(S_{1,1}^\varrho)$

We now prove that the vanishing of a fundamental class in H_0^ϱ is characterised by $(S_{1,1}^\varrho)$ for quasiconvex uniform spaces that support a local weak $(1,1)$-Poincaré inequality.

Theorem 12. *Let (X,d,μ) be a quasiconvex uniform space that supports a local weak $(1,1)$-Poincaré inequality up to scale R_P, let $N(X,\varepsilon) \ni o$ be a quasi-lattice, where $\mu(\{o\}) = 0$ and $0 < \varepsilon \le \min\{2, R_P/4\}$, and let ϱ be a control function. Then the following are equivalent:*

(1) *(X,d,μ) satisfies $(S_{1,1}^\varrho)$;*

(2) *there exists a constant $C > 0$ such that for every $v\colon N(X,\varepsilon) \to \mathbb{R}$ with finite support,*

$$\sum_{p\in N(X,\varepsilon)} |v(p)|\mu(B(p,\varepsilon)) \le C \sum_{p\in N(X,\varepsilon)} \sum_{q\sim p} |v(p)-v(q)|\,\varrho(|(p,q)|)\,\mu(B(p,\varepsilon));$$

(3) *there exists a constant $C' > 0$ such that for every $v\colon N(X,\varepsilon) \to \mathbb{R}$ with finite support,*

$$\sum_{p\in N(X,\varepsilon)} |v(p)| \le C' \sum_{p\in N(X,\varepsilon)} \sum_{q\sim p} |v(p)-v(q)|\,\varrho(|(p,q)|);$$

(4) *$0 = [\Gamma] \in H_0^\varrho(\Gamma)$ for any quasi-lattice $\Gamma \subseteq X$.*

Proof. By Theorem 11, parts (1) and (2) are equivalent. Since μ is uniform, $0 < f(\varepsilon) \le \mu(B(p,\varepsilon)) \le g(\varepsilon) < \infty$ for all $p \in N(X,\varepsilon)$, and so (2) and (3) are equivalent. Hence, it remains to prove that (3) and (4) are equivalent, and we first show that (3) implies (4). First, approximate (X,d) by the space obtained from equipping $N(X,\varepsilon)$ with the edge path length $\delta\colon N(X,\varepsilon) \times N(X,\varepsilon) \to \mathbb{N} \cup \{\infty\}$ given by

$\delta(x,y) = 0$ if $x = y$,

$\delta(x,y) = k$ if the shortest 3ε-path from x to y is of length k,

$\delta(x,y) = \infty$ if there is no 3ε-path from x to y,

where a 3ε-path from x to y of length k is any finite sequence of points $x = x_0,\dots,x_k = y$ in $N(X,\varepsilon)$ where $0 < d(x_i,x_{i+1}) \le 3\varepsilon$ for $0 \le i \le k-1$. Since (X,d)

is uniformly coarsely proper and quasiconvex, the edge path length is a metric on $N(X,\varepsilon)$ and $(N(X,\varepsilon),\delta)$ is quasi-isometric to (X,d) (see Proposition 3.D.15 in [3]), and

$$\text{(QI)}\qquad \frac{1}{3\varepsilon}\, d(q,p) \le \delta(p,q) \le \frac{Q}{\varepsilon}\, d(p,q) + 1$$

for all $p,q \in N(X,\varepsilon)$ adapting Lemma 2.5 in [13] for geodesic spaces to Q-quasiconvex spaces. Thus $\varrho(d(\bar{o},(p,q))) \le 3\varepsilon\delta(\bar{o},(p,q))$ by (QI), and by (ϱ_2) we see that $(N(X,\varepsilon),\delta)$ satisfies

$$\sum_{x\in N(X,\varepsilon)} |v(x)| \le C_2 M(3\varepsilon)\Bigg(\sum_{x\in N(X,\varepsilon)}\ \sum_{\{y\in N(X,\varepsilon)\colon\, \delta(y,x)=1\}} |v(x)-v(y)|\, \varrho\left(|(x,y)|\right)\Bigg)$$

for every $v\colon N(X,\varepsilon)\to\mathbb{R}$ with finite support. Equivalently, $0 = [N(X,\varepsilon)] \in H_0^{\varrho}(N(X,\varepsilon))$ where $H_0^{\varrho}(N(X;\varepsilon))$ is defined using the metric δ; see Lemma 4.1 and Theorem 4.2 in [16]. Since $\mathrm{id}\colon (N(X,\varepsilon),\delta)\to(N(X,\varepsilon),d)$ is a quasi-isometry, we conclude that $0 = [(N(X,\varepsilon))] \in H_0^{\varrho}(N(X,\varepsilon))$, where $H_0^{\varrho}(N(X,\varepsilon))$ is defined using the metric d, and hence that $0 = [\Gamma] \in H_0^{\varrho}(\Gamma)$ for any quasi-lattice $\Gamma \subseteq X$ by Lemma 2. It remains to prove that (4) implies (3). By assumption, $0 = [\Gamma] \in H_0^{\varrho}(\Gamma)$ for any quasi-lattice $\Gamma \subseteq X$; in particular for $N(X,\varepsilon) \subseteq X$. Since $\mathrm{id}\colon (N(X,\varepsilon),d)\to(N(X,\varepsilon),\delta)$ is a quasi-isometry, $0 = [(N(X,\varepsilon))] \in H_0^{\varrho}(N(X,\varepsilon))$, where $H_0^{\varrho}(N(X,\varepsilon))$ is defined defined using the metric δ, equivalently, for some constant $C>0$,

$$\sum_{x\in N(X,\varepsilon)} |v(x)| \le C\Bigg(\sum_{x\in N(X,\varepsilon)}\ \sum_{\{y\in N(X,\varepsilon)\colon\, \delta(y,x)=1\}} |v(x)-v(y)|\, \varrho\left(|(x,y)|\right)\Bigg)$$

for every $v\colon N(X,\varepsilon)\to\mathbb{R}$ with finite support. Applying (QI), (ϱ_1), and (ϱ_2) respectively, we conclude that $\varrho(\delta(\bar{o},(p,q))) \le L(1)M(Q/\varepsilon)\varrho(d(\bar{o},(p,q)))$. Using this to estimating the above inequality from above gives (3). □

Theorem A summarises this by stating the equivalence between (1) and (4) above. Theorem B follows from the observation that the local weak (1, 1)-Poincaré inequality is not needed to prove that (1) implies (2) in Theorem 11.

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