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L^p-bounds on spectral clusters associated to polygonal domains

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Abstract. We look at the L^p bounds on eigenfunctions for polygonal domains (or more generally Euclidean surfaces with conic singularities) by analysis of the wave operator on the flat Euclidean cone $C(\mathbb{S}^1_\rho) \overset{\text{def}}{=} \mathbb{R}_+ \times (\mathbb{R}/2\pi\rho\mathbb{Z})$ of radius $\rho > 0$ equipped with the metric $\mathrm{h}(r,\theta) = \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2$. Using explicit oscillatory integrals and relying on the fundamental solution to the wave equation in geometric regions related to flat wave propagation and diffraction by the cone point, we can prove spectral cluster estimates equivalent to those in works on smooth Riemannian manifolds.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a compact polygonal domain in the plane, that is, a compact, connected region in $\mathbb{R}^2$ whose boundary, $\partial\Omega$, is piecewise linear. Note, we place no restrictions here on the polygon in terms of convexity or rationality. Suppose $\{\phi_j\}$, $\phi_j : \Omega \longrightarrow \mathbb{C}$ is an orthonormal $L^2(\Omega)$ eigenbasis for the (positive) Laplacian operator on Ω with either Dirichlet or Neumann boundary conditions on $\partial\Omega$,

$$\Delta\phi_j = \lambda_j^2\,\phi_j, \quad 0 \le \lambda_0 < \lambda_1 \le \cdots \le \lambda_j \le \lambda_{j+1} \le \cdots, \quad \|\phi_j\|_{L^2(\Omega)} = 1. \tag{1.1}$$

We study L^p boundedness properties of the ϕ_j's depending upon their frequency, which can be achieved by proving estimates on clusters of eigenfunctions. There is a rich history of spectral cluster estimates on smooth, closed Riemannian manifolds, classically going back to the work of Avakumovič, Levitan, and Hörmander and more recently in the work of Sogge [21], with many further extensions to manifolds with boundary such as [12], [19], [20], [2]. Other extensions to metrics of less regularity can be found in for instance [18], [14], [1]. However, the estimates in the present work appear to be the first on domains with corners or conic singularities except for rectangles. See for instance the recent work of Bourgain–Demeter [5], where restriction estimates on general tori are studied.

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Indeed, L^p bounds on the eigenfunctions can be viewed via the Stein–Tomas restriction theorem as a version of the adjoint restriction estimate on the sphere. The authors have previously treated the analogs of adjoint restriction estimates for polygonal domains in cases of the parabola in [10], [3] and the cone in [4] by proving Strichartz estimates for the Schrödinger equation and wave equation respectively in the setting polygonal domains. Arguably, the sphere presents unique challenges since Strichartz bounds for the Schrödinger and wave equations rely only on fixed time bounds for the corresponding kernel, whereas the spectral cluster bounds typically require integrating/averaging the wave kernel and estimating the contributions of the jumps in the transition from geometric to diffracted wave fronts.

Remark 1.1. The Neumann Laplacian on Ω is taken to be the Friedrichs extension of the Laplace operator acting on smooth functions which vanish in a neighborhood of the vertices and whose normal derivative is zero on the rest of the boundary. The Dirichlet Laplacian is taken to be the typical Friedrichs extension of the Laplace operator acting on smooth functions which are compactly supported in the interior of Ω.

The spectral projection operator Π_λ is defined for any $\lambda \geq 0$ such that

$$\Pi_\lambda f = \sum_{\lambda_j \in [\lambda, \lambda+1]} \langle f, \phi_j \rangle \phi_j. \tag{1.2}$$

We refer to functions in the range of Π_λ as "spectral clusters". Then, the desired spectral cluster estimates are stated as the following theorem.

Theorem 1.2. *For any polygonal domain Ω in $\mathbb{R}^2$, $f \in L^2(\Omega)$, we have*

$$\|\Pi_\lambda f\|_{L^q(\Omega)} \leq C\lambda^{\delta(q)} \|f\|_{L^2(\Omega)}, \quad \delta(q) = \begin{cases} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) & \text{for } 2 \leq q \leq 6, \\ 2\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{2} & \text{for } 6 \leq q \leq \infty \end{cases} \tag{1.3}$$

for C independent of $\lambda \geq 1$. Consequently, given any L^2 normalized eigenfunction $\Delta\phi_\lambda = \lambda^2 \phi_\lambda$, we have

$$\|\phi_\lambda\|_{L^q(\Omega)} \leq C\, \lambda^{\delta(q)}.$$

As in [3], [4], we will in reality establish Theorem 1.2 for a Euclidean surface with conical singularities (ESCS). When the cone angle is a rational multiple of π, this has a special type of orbifold structure. An ESCS is a Riemannian surface (X, g) that can be covered by a finite number of coordinate charts, each of which is isometric to a subset of $\mathbb{R}^2$ or $C(\mathbb{S}^1_\rho)$. Let $C(\mathbb{S}^1_\rho)$ denote the Euclidean cone of radius $\rho > 0$, defined as the product manifold $C(\mathbb{S}^1_\rho) = \mathbb{R}_+ \times \left(\mathbb{R}/2\pi\rho\mathbb{Z}\right)$, equipped with the metric $\mathrm{g}(r, \theta) = \mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2$. This is an incomplete manifold which is locally isometric to $\mathbb{R}^2$ away from the cone points and hence flat. For a more precise definition, see [3]. Even though the manifolds we consider have conic singularities, the power $\delta(q)$ that appears in Theorem 1.2 is the same as that in Sogge's original estimates for spectral clusters on C^∞ manifolds [21]. The same work shows that this is the sharp exponent for spectral clusters on any Riemannian manifold, though this exponent may not be optimal for individual eigenfunctions.

Any compact planar polygonal domain Ω can be doubled across its boundary to produce a compact ESCS. In this procedure, a vertex of Ω of angle α gives rise to a conic point of X with cone angle 2α. We then take the Laplace–Beltrami operator Δ_{g} on X to be the Friedrichs extension of the Laplacian on $\mathcal{C}_c^\infty(X_0)$, where X_0 is X less the singular points. To see this clearly, let us recall the procedure outlined in Section 2 of [3]. Begin with two copies Ω and $\sigma\Omega$ of the polygonal domain, where σ is a reflection of the plane. An ESCS X is then obtained by taking the formal union $\Omega \cup \sigma\Omega$, where two corresponding sides are identified pointwise. Taking polar coordinates near each vertex of the polygon, it can be seen that the flat metric g extends smoothly across the sides. In particular, a vertex in Ω of angle α gives rise to a conic point of X locally isometric to $C(\mathbb{S}^1_\rho)$ with $\rho = \alpha/\pi$. Such a doubling procedure produces a conic point of angle 2α.

The reflection σ of Ω gives rise to an involution of X commuting with the Laplace–Beltrami operator. We thus have a decomposition into two operators acting on functions which are either odd (even) with respect to σ, which are equivalent to the Laplace operator on Ω with Dirichlet (Neumann) boundary conditions respectively. For us, the key observation is that for any eigenfunction φ_j of the Dirichlet, resp. Neumann, Laplace operator on Ω, we can construct an eigenfunction of the Laplace operator on X by taking φ_j in Ω and $-\varphi_j \circ \sigma$, resp. $\varphi_j \circ \sigma$, in $\sigma\Omega$. As a consequence, the spectrum over X can be seen to extend that for Ω. See the previous works of the authors [3], Section 2, for a thorough description of ESCSs and Δ_{g}, as well as [4], Section 2, for a general treatment of Cheeger's functional calculus on cones.

Theorem 1.2 then follows from the equivalent statement for ESCSs.

Theorem 1.3. *For X any compact ESCS, $f \in L^2(X)$, we have*

$$\|\Pi_\lambda f\|_{L^q(X)} \le C\lambda^{\delta(q)}\|f\|_{L^2(X)}, \quad \delta(q) = \begin{cases} \frac{1}{2}\left(\frac{1}{2} - \frac{1}{q}\right) & \text{for } 2 \le q \le 6, \\ 2\left(\frac{1}{2} - \frac{1}{q}\right) - \frac{1}{2} & \text{for } 6 \le q \le \infty \end{cases} \tag{1.4}$$

for C independent of λ.

1.1. Obtaining spectral cluster estimates

As is well understood and explored below (see also the result from [22]), Theorem 1.2 can be related to forming an oscillatory integral which integrates the wave kernel in time on the Euclidean cone. In order to pursue such estimates, we will consider the fundamental solution of the wave equation on the Euclidean cone,

$$\begin{cases} \left(D_t^2 - \Delta_{\mathrm{g}}\right) u(t,r,\theta) = F\,, \\ u(0,r,\theta) = f(r,\theta)\,, \\ \partial_t u(0,r,\theta) = g(r,\theta)\,, \end{cases} \tag{1.5}$$

for $u\colon \mathbb{R} \times C(\mathbb{S}^1_\rho) \to \mathbb{R}$. A pioneering work regarding the fundamental solution to the wave equation on manifolds with conic singularities is that of Cheeger and

Taylor [7], [8] who studied the propagation of singularities for solutions amongst other properties. Further progress on the regularity of the fundamental solution was made by Melrose and Wunsch in [16]. Let us recall, from Section 4 in [8] and Equations (3.14)–(3.16) in [4], that the wave fundamental solution kernel for $\sin(t\sqrt{\Delta_g})/\sqrt{\Delta_g}$ on the cone can be written as a decomposition of a geometric component,

$$(1.6)\quad K^{\mathrm{geom}}(t,r_1,\theta_1;r_2,\theta_2) = \sum_{-\pi\le(\theta_1-\theta_2)+j\cdot 2\pi\rho\le\pi} \frac{1}{(t^2-r_1^2-r_2^2+2r_1r_2\cos((\theta_1-\theta_2)+j\cdot 2\pi\rho))_+^{1/2}},$$

and a diffracted component,

$$(1.7)\quad K^{\mathrm{diff}}(t,r_1,\theta_1;r_2,\theta_2) = -\frac{\mathbf{1}_{(0,t)}(r_1+r_2)}{4\pi^2\rho(2r_1r_2)^{1/2}} \times \int_0^\beta (\alpha-\cosh s)^{-1/2}\left[\frac{\sin\varphi_1}{\cosh(s/\rho)-\cos\varphi_1}+\frac{\sin\varphi_2}{\cosh(s/\rho)-\cos\varphi_2}\right]ds,$$

where we have used the abbreviations

$$\alpha = \frac{t^2-r_1^2-r_2^2}{2r_1r_2} = \frac{t^2-(r_1+r_2)^2}{2r_1r_2}+1,\quad \beta=\cosh^{-1}(\alpha),$$

$$\varphi_1 = \frac{\pi+\theta_1-\theta_2}{\rho},\quad \varphi_2 = \frac{\pi-(\theta_1-\theta_2)}{\rho}.$$

Remark 1.4. As shown in [6], [20], [18], spectral cluster estimates are equivalent to proving a dispersive estimate that holds on the representative geometry of each coordinate patch of the domain Ω. Namely, using Fourier analysis in the t-variable, spectral cluster estimates can be related to dispersive estimates for a solution to the wave equation on an ESCS X,

$$(1.8)\quad \begin{cases} \left(D_t^2-\Delta_{\mathrm{g}}\right)u(t,x)=F\,, \\ u(0,x)=f(x)\,, \\ \partial_t u(0,x)=g(x)\,. \end{cases}$$

To be more precise, Theorem 1.3 on a Riemannian manifold, M, is equivalent to the dispersive-type estimate

$$(1.9)\quad \|u\|_{L^q_x(M;L^2_t[-T,T])} \le C\big(\|(f,g)\|_{H^{\delta(q)}\times H^{\delta(q)-1}} + \|F\|_{L^1_t([-T,T];H^{\delta(q)-1})}\big)$$

for u a solution to (1.8), see [18], [6]. Note that these estimates are typically associated with a measure of decay away from the light cone and hence differ in form from the standard Strichartz estimates which capture dispersive decay. See [4] for more on Strichartz estimates in this setting as well. In addition, L^p regularity for wave operators on product cones and their applications to L^p bounds for spectral multipliers have been studied in [17].

The proof of Theorem 1.3 will follow once we derive proper representations of the spectral projection operators as oscillatory integrals. One proof of (1.3) on $\mathbb{R}^2$ begins by first observing (cf. p. 130, 137 in [22]) that one may replace Π_λ by $\chi(\sqrt{\Delta_g} - \lambda)$ with $\chi \in \mathcal{S}(\mathbb{R})$ even and real-valued, with $\chi > 0$ in a neighborhood of 0, and $\operatorname{supp}(\widehat{\chi}) \subset \{|t| \in (\delta, 2\delta)\}$ for some $\delta > 0$. Note, here we are considering the wave operator on $\mathbb{R}^2$ but similar approaches work on more general manifolds. It can then be seen that the Schwartz kernel of $\chi(\sqrt{\Delta} - \lambda)$ is a convolution kernel, which as a function of z is of the form

$$\lambda^{1/2} \sum_{\pm} e^{\pm i\lambda|z|}\, a_{\lambda,\pm}(|z|) + R_\lambda(z)\,, \tag{1.10}$$

where $a_{\lambda,\pm}(\cdot)$ is compactly supported in $(\delta/2, 4\delta)$ and $R_\lambda(z)$ satisfies much better bounds than is needed: $|\partial_z^\alpha R_\lambda(z)| \lesssim_{N,\alpha} \lambda^{-N}$. The phase function $|x-y|$ is a Carleson–Sjölin phase, so the desired $L^2(\mathbb{R}^2) \to L^6(\mathbb{R}^2)$ bounds then follow from oscillatory integral estimates in [13]. For a generalization of this result to higher dimensions, see for instance Stein's variable coefficient generalization of the Stein–Tomas restriction theorem (see e.g. Corollary 2.2.3 in [22]).

The easiest way to see (1.10) is to write the Schwartz kernel as a Fourier integral in polar coordinates

$$\int_0^\infty \Big(\int_0^{2\pi} e^{irz\cdot\theta} d\theta \Big) \chi(r-\lambda)\, r\, dr.$$

Stationary phase shows that

$$\int_0^{2\pi} e^{irz\cdot\theta} d\theta = |rz|^{-1/2} \sum_{\pm} e^{\pm ir|z|}\, a_\pm(r|z|)\,,$$

where $a_\pm$ are smooth and bounded. When $|z| \in (\delta/2, 4\delta)$, (2.4) follows by using that the fact that χ is Schwartz allows one to essentially replace r by λ. Seeing the rapid decay in λ when $|z| \notin (\delta/2, 4\delta)$ takes some extra work. In short, one has to replace χ by its Fourier transform, but we will see it by a different method below in Section 2.

Such a representation of the fundamental solution generally allows one to establish the L^6 bounds we desire. In the case of the geometric wave, we will observe that the leading order fundamental solution representation has the correct form of a Carleson–Sjölin phase, and the result holds from standard arguments. The diffracted component presents a different challenge in that the phase function is not of the desired form, thus we need a modified argument to get the correct decay.

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2. Spectral cluster estimates on polygonal domains

2.1. Treatment of the geometric term

Let X be an ESCS of dimension 2. We are interested in establishing the bound

$$\|\Pi_\lambda\|_{L^2(X)\to L^p(X)} \lesssim \lambda^{\max(1/4-1/(2p),1/2-2/p)}, \tag{2.1}$$

where Π_λ projects onto eigenspaces corresponding to frequencies λ_j satisfying $\lambda_j \in [\lambda, \lambda+1]$. Note that this is a discrete analog of the Fourier multiplier determined by the symbol $\mathbf{1}_{[\lambda,\lambda+1]}(\xi)$ on $\mathbb{R}^2$. As noted above, (cf. [22], p. 130, 137) that it suffices to prove this replacing Π_λ by $\chi(\sqrt{\Delta_g}-\lambda)$, where $\chi \in \mathcal{S}(\mathbb{R})$, is even and real valued and $\chi > 0$ in a neighborhood of 0 with $\operatorname{supp}(\widehat{\chi}) \subset \{|t| \in (\delta, 2\delta)\}$ for some $\delta > 0$. Hence

$$\begin{aligned}\chi(\sqrt{\Delta_g}-\lambda) &= \frac{1}{2\pi}\int e^{it(\sqrt{\Delta_g}-\lambda)}\,\widehat{\chi}(t)\,dt\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-it\lambda}\cos(t\sqrt{\Delta_g})\,\widehat{\chi}(t)\,dt + \tilde{\chi}(\sqrt{\Delta_g}+\lambda),\end{aligned}$$

where $\tilde{\chi}$ is some other Schwartz class function. Since the spectrum of $\sqrt{\Delta_g}$ is positive, $\tilde{\chi}(\sqrt{\Delta_g}+\lambda)$ is a rapidly decaying function of an elliptic operator, and hence $\|\tilde{\chi}(\sqrt{\Delta_g}+\lambda)\|_{L^2(X)\to L^p(X)} = O(\lambda^{-N})$ for any $N > 0$. Consequently it suffices to restrict attention to the operator valued integral here.

Integration by parts yields

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-it\lambda}\cos(t\sqrt{\Delta_g})\,\widehat{\chi}(t)\,dt \\ &\qquad = i\lambda\int_{-\infty}^{\infty} e^{-it\lambda}\frac{\sin(t\sqrt{\Delta_g})}{\sqrt{\Delta_g}}\,\widehat{\chi}(t)\,dt - \int_{-\infty}^{\infty} e^{-it\lambda}\frac{\sin(t\sqrt{\Delta_g})}{\sqrt{\Delta_g}}\,\widehat{\chi}'(t)\,dt.\end{aligned} \tag{2.2}$$

By the Sobolev embedding theorem, the operator defined second term here satisfies stronger $L^2(X) \to L^p(X)$ bounds than needed, so we may also neglect its contribution. We further note that since $\widehat{\chi}$ is even, the first term on the right can be rewritten as

$$\lambda\int_{-\infty}^{\infty}\sin(t\lambda)\,\frac{\sin(t\sqrt{\Delta_g})}{\sqrt{\Delta_g}}\,\widehat{\chi}(t)\,dt = 2\lambda\int_0^{\infty}\sin(t\lambda)\,\frac{\sin(t\sqrt{\Delta_g})}{\sqrt{\Delta_g}}\,\widehat{\chi}(t)\,dt. \tag{2.3}$$

By finite speed of propagation, the Schwartz kernel of this operator thus vanishes when the distance between the two points on X is larger than 2δ. See (1.16) in [4], or (3.41) in [7], for a complete definition of this notion of distance on the cone. Consequently, it suffices to prove $L^2 \to L^p$ bounds for data supported in a chart where X can be identified with a flat cone, $C(\mathbb{S}^1_\rho)$. Moreover, using the fact that the wave kernels respect periodicity, by a doubling argument, if the bounds hold when the radius is ρ, then they also hold when the radius is $\rho/2$. We may thus assume that $\rho > 1$ (recalling that the $\rho = 1$ follows simply by identification with $\mathbb{R}^2$, see the treatment below).

We finally remark that it suffices to establish $p = \infty$ and $p = 6$ bounds on the operator in (2.3) as the remaining bounds will follow from interpolation.

2.1.1. The Schwartz kernel of (2.3) on $\mathbb{R}^2$. We begin by computing the Schwartz kernel of the operator in (2.3) when $X = \mathbb{R}^2$ and $\Delta_g = \Delta$ is the standard Laplacian on $\mathbb{R}^2$. While this can be accomplished by employing the methods in [22], we include an alternative presentation as it can be used to help treat the "geometric" contribution below. In particular, we will only use that the fundamental solution of the wave equation is of the form $\frac{1}{2\pi}(t^2 - |z|^2)_+^{-1/2}$. It will be seen that the Schwartz kernel for the integral in (2.3) is a convolution kernel, which as a function of z, is of the form

$$\mathrm{Re}\left(\lambda^{1/2} e^{i\lambda|z|} a_\lambda(|z|)\right) + R_\lambda(z), \tag{2.4}$$

where $a_\lambda(\cdot)$ is compactly supported and smooth in $(\delta, 2\delta)$ and $R_\lambda(z)$ satisfies stronger $L^2(X) \to L^p(X)$ bounds. In particular, $R_\lambda(z)$ is $O(\lambda^{-N})$ for any N. The bounds when $p = \infty$ are then immediate. Moreover, the phase function $|x-y|$ is a Carleson–Sjölin phase, so the desired $L^2(X) \to L^6(X)$ bounds then follow from Hörmander [13] as stated in the introduction.

The kernel of the operator in (2.3) is a convolution kernel, and neglecting harmless constants, this as a function of z is given by:

$$\begin{aligned}
\lambda \int_{|z|}^{\infty} & \sin(t\lambda)(t^2 - |z|^2)_+^{-1/2} \widehat{\chi}(t)\, dt \\
&= \lambda \int_{-\infty}^{\infty} \chi(\tau) \int_{|z|}^{\infty} \sin(t\lambda)\cos(t\tau)(t^2 - |z|^2)_+^{-1/2}\, dt\, d\tau \\
&= \frac{\lambda}{2} \int_{-\infty}^{\infty} \chi(\tau) \int_{|z|}^{\infty} \left(\sin(t(\lambda+\tau)) + \sin(t(\lambda-\tau))\right)(t^2 - |z|^2)_+^{-1/2}\, dt\, d\tau \\
&= \lambda \int_{-\infty}^{\infty} \chi(\tau) \int_{|z|}^{\infty} \sin(t(\lambda-\tau))(t^2 - |z|^2)_+^{-1/2}\, dt\, d\tau. \qquad (2.5)
\end{aligned}$$

Here the second equality follows from trigonometric identities and the first and third equalities use that $\chi(\tau)$ is even. Now observe that after a change of variables $t = s|z|$ and the identities of p. 170 in [24], we have

$$\frac{2}{\pi}\int_{|z|}^{\infty} \sin(t\zeta)(t^2 - |z|^2)_+^{-1/2}\, dt = \frac{2}{\pi}\int_1^{\infty} \sin(s|z|\zeta)(s^2-1)_+^{-1/2}\, ds = \mathrm{sgn}(\zeta) J_0(|z\zeta|) \tag{2.6}$$

where J_0 is the Bessel function of order 0. Neglecting harmless constants once again, we are now led to consider

$$\lambda \int_{-\infty}^{\infty} \chi(\tau)\, \mathrm{sgn}(\lambda-\tau)\, J_0(|z||\lambda-\tau|)\, d\tau. \tag{2.7}$$

Let ψ be a smooth, even bump function such that $\mathrm{supp}(\psi) \subset (-1/2, 1/2)$ and $\mathrm{supp}(1-\psi) \subset (-1/4, 1/4)^c$, and observe that

$$\lambda \int_{-\infty}^{\infty} \chi(\tau)(1-\psi)(\lambda^{-1}\tau)\ \mathrm{sgn}(\lambda-\tau)\, J_0(|z||\lambda-\tau|)\, d\tau = O(\lambda^{-N})$$

for any $N > 0$ given that $\chi(\tau)$ is a Schwartz class function and J_0 is bounded. Note that this relation is uniform in z. Consequently, since $\lambda - \tau \geq \lambda/2$ when $\lambda^{-1}\tau \in \operatorname{supp}(\psi)$, we may restrict attention to the contribution of

$$\lambda \int_{-\infty}^{\infty} \chi(\tau)\, \psi(\lambda^{-1}\tau)\, J_0(|z|(\lambda - \tau))\, d\tau. \tag{2.8}$$

Typical stationary phase arguments imply that for $\zeta \in (0, \infty)$,

$$J_0(\zeta) = \operatorname{Re}\left(e^{i\zeta}\, b(\zeta)\right),$$

where the k-th derivative of b satisfies

$$|b^{(k)}(\zeta)| \lesssim_k (1+\zeta)^{-k-1/2}, \quad k \geq 0.$$

We thus rewrite (2.8) as

$$\operatorname{Re}\left(\lambda e^{i\lambda|z|} \int_{-\infty}^{\infty} e^{-i\tau|z|}\, \chi(\tau)\, \tilde{\psi}(\lambda^{-1}\tau, \lambda|z|)\, d\tau\right), \tag{2.9}$$

where for $\tau \in \mathbb{R}$ and $w \geq 0$,

$$\tilde{\psi}(\tau, w) \stackrel{\text{def}}{=} b(w(1-\tau))\psi(\tau).$$

Recalling that $\operatorname{supp}(\psi) \subset (-1/2, 1/2)$, we have

$$|\partial_\tau^j \partial_w^k \tilde{\psi}(\tau, w)| \lesssim_{j,k} (1+w)^{-1/2-k} \quad j, k \geq 0.$$

Consequently, the Fourier integral in (2.9) is

$$\int_{-\infty}^{\infty} \widehat{\chi}(s)\, \widehat{\tilde{\psi}}(\lambda(|z| - s), \lambda|z|)\, \lambda\, ds, \tag{2.10}$$

which is seen to be smooth in $|z|$ with derivatives which are $O(\lambda^{-1/2})$ and is $O(\lambda^{-N})$ for any $N > 0$ when $|z| \notin (\delta, 2\delta)$ (since $\operatorname{supp}(\widehat{\chi}) \subset (\delta, 2\delta)$). This establishes (2.4) and in turn (2.1).

2.1.2. The geometric contribution on a flat cone. Let us recall from (1.6) that the "geometric" contribution to $\sin(t\sqrt{\Delta_g})/\sqrt{\Delta_g}$ on $C(\mathbb{S}^1_\rho)$ when $\rho > 1$ has a Schwartz kernel of the form (neglecting harmless constants as before)

$$\left(t^2 - G^2(r_1, r_2; \theta_1 - \theta_2)\right)_+^{-1/2}, \quad \text{where } G(r_1, r_2; \theta) \stackrel{\text{def}}{=} (r_1^2 + r_2^2 - 2r_1 r_2 \cos\theta)^{1/2},$$

and supported in $|(\theta_2 - \theta_1) \,(\operatorname{mod} 2\pi\rho)| \leq \pi$. Consequently, given two points (r_1, θ_1), (r_2, θ_2) such that $|\theta_1 - \theta_2| \leq \pi\rho$, the previous subsection shows that the leading order contribution of this term in (2.3) gives rise to the (real part of the) kernel

$$K(r_1, r_2; \theta_1 - \theta_2) \stackrel{\text{def}}{=} \mathbf{1}_{[-\pi,\pi]}(\theta_1 - \theta_2)\, e^{i\lambda G(r_1, r_2; \theta_1 - \theta_2)}\, a_\lambda(G(r_1, r_2; \theta_1 - \theta_2)),$$

and we recall that $\mathrm{supp}(a_\lambda) \subset (\delta, 2\delta)$. Note that the factor of $\lambda^{1/2}$ is not included here and we will thus show that this integral operator contributes to a gain of $\lambda^{-2/p}$ in the $L^2 \to L^p$ estimates for $p = 6, \infty$. For

$$\mathrm{supp}(g) \subset \{(r,\theta) \in C(\mathbb{S}^1_\rho) : r \in (0, 4\delta)\},$$

we have the straightforward L^∞ bound

$$\sup_{(r_2,\theta_2)} \left| \int K(r_1, r_2; \theta_2 - \theta_1)\, g(r_1, \theta_1)\, r_1\, dr_1\, d\theta_1 \right| \lesssim \|g\|_{L^2(r_1 dr_1 d\theta_1)}.$$

Note that without loss of generality, we can always assume localization of g this throughout the proof of our theorem on cones due to the localization of $\hat{\chi}$.

Consequently, we are left to show $L^2 \to L^6$ bounds on the operator determined by K. Due to the sharp cutoff to $|\theta| < \pi$, there is a jump to contend with and the estimates are not a trivial consequence of the standard theory. It suffices to further assume that g is supported in a small arc of length ε where ε is sufficiently small, but otherwise uniform. In particular, we assume that $\varepsilon < \min(\pi(\rho-1)/2, \pi/4)$. We then take coordinates such that

$$\mathrm{supp}(g) \subset \{(r_1,\theta_1) \in C(\mathbb{S}^1_\rho) : r_1 \in (0, 4\delta), \theta_1 \in (0, \varepsilon)\}.$$

Taking coordinates (r_2, θ_2) such that $\theta_2 \in (-\pi\rho, \pi\rho]$, we now have that

$$\left\| \mathbf{1}_{(-\pi\rho,\pi]}(\theta_2) \int K(r_1, r_2; \theta_2 - \theta_1)\, g(r_1, \theta_1)\, r_1\, dr_1\, d\theta_1 \right\|_{L^6(r_2 dr_2 d\theta_2)} \lesssim \lambda^{-1/3}\, \|g\|_{L^2(r_1 dr_1 d\theta_1)}.$$

Indeed, given our assumptions on $\mathrm{supp}(g)$, K vanishes for $\theta_2 < -\pi$. Hence the characteristic function $\mathbf{1}_{(-\pi\rho,\pi]}(\theta_2)$ ensures that the integral operator identifies with the operator determined by (2.4) on $\mathbb{R}^2$, at which point the bound follows from the standard theory of Carleson–Sjölin oscillatory integral operators referenced above.

We are left to show that

$$\left\| \mathbf{1}_{(\pi,\pi+\varepsilon)}(\theta_2) \int K(r_1, r_2; \theta_2 - \theta_1)\, g(r_1, \theta_1)\, r_1\, dr_1\, d\theta_1 \right\|_{L^6(r_2 dr_2 d\theta_2)} \lesssim \lambda^{-1/3}\, \|g\|_{L^2(r_1 dr_1 d\theta_1)}.$$

Let $\eta \in C_c^\infty(\mathbb{R})$ be such that $\mathrm{supp}(\eta) \subset (\pi - 2\varepsilon, \pi + 2\varepsilon)$ and $\eta(\theta) = 1$ for $\theta \in [\pi - \varepsilon, \pi + \varepsilon]$. Given the support hypothesis on the data g, we may replace $K(r_1, r_2; \theta_2 - \theta_1)$ by $K(r_1, r_2; \theta_2 - \theta_1)\eta(\theta_2 - \theta_1)$. Moreover, by applying the inequalities of Minkowski and Hölder in the r_1 variable, it suffices to show that with $r_1 \in (0, 4\delta)$ fixed,

$$\left\| \int K(r_1, r_2; \theta_2 - \theta_1)\, \eta(\theta_2 - \theta_1) f(\theta_1) d\theta_1 \right\|_{L^6(r_2 dr_2 d\theta_2)} \lesssim \lambda^{-1/3} r_1^{-1/2} \|f\|_{L^2(d\theta_1)}. \tag{2.11}$$

To show (2.11), we let T^{λ,r_1} denote the oscillatory integral operator defined by the left hand side of the inequality. Consider the mapping defined by

$$(T_{r_2}^{\lambda,r_1} f)(\theta_2) = \int K(r_1, r_2; \theta_2 - \theta_1)\, \eta(\theta_2 - \theta_1)\, f(\theta_1)\, d\theta_1$$

so that for a function $F \in L^{6/5}(\tilde{r}_2 d\tilde{r}_2 d\theta_1)$, we have

$$\left(T^{\lambda,r_1} \circ (T^{\lambda,r_1})^*(F)\right)(r_2, \theta_2) = \int_{-\infty}^{\infty} \left(T_{r_2}^{\lambda,r_1} \circ (T_{\tilde{r}_2}^{\lambda,r_1})^*(F(\tilde{r}_2, \cdot))\right)(\theta_2)\, \tilde{r}_2\, d\tilde{r}_2.$$

A standard duality argument implies that (2.11) will follow from

$$\left\|\left(T^{\lambda,r_1} \circ (T^{\lambda,r_1})^*(F)\right)\right\|_{L^6(r_2 dr_2 d\theta_2)} \lesssim \lambda^{-2/3}\, r_1^{-1} \|F\|_{L^{6/5}(\tilde{r}_2 d\tilde{r}_2 d\theta_1)}\,,$$

which in turn follows from interpolating the bounds

$$\left\|\left(T_{r_2}^{\lambda,r_1} \circ (T_{\tilde{r}_2}^{\lambda,r_1})^*(f)\right)\right\|_{L^\infty(d\theta_2)} \lesssim (\lambda|r_2 - \tilde{r}_2|)^{-1/2} r_1^{-1} \|f\|_{L^1(d\theta_1)}\,, \tag{2.12}$$

$$\left\|\left(T_{r_2}^{\lambda,r_1} \circ (T_{\tilde{r}_2}^{\lambda,r_1})^*(f)\right)\right\|_{L^2(d\theta_2)} \lesssim (\lambda r_1 r_2)^{-1/2} (\lambda r_1 \tilde{r}_2)^{-1/2} \|f\|_{L^2(d\theta_1)}\,, \tag{2.13}$$

followed by the Hardy–Littlewood–Sobolev fractional integration inequality in r_2.

To see (2.12), we first observe that the Schwartz kernel of $T_{r_2}^{\lambda,r_1} \circ (T_{\tilde{r}_2}^{\lambda,r_1})^*$ is

$$\int \mathbf{1}_{[\pi-2\varepsilon,\pi]}(\theta_1 - \theta) \mathbf{1}_{[\pi-2\varepsilon,\pi]}(\theta_2 - \theta) e^{i\lambda \Psi_{r_1}(r_2, \tilde{r}_2, \theta_1, \theta_2, \theta)} A_{\lambda,r_1}(r_2, \tilde{r}_2, \theta_1, \theta_2, \theta)\, d\theta, \tag{2.14}$$

where

$$\Psi_{r_1}(r_2, \tilde{r}_2, \theta_1, \theta_2, \theta) = G(r_1, r_2, \theta_2 - \theta) - G(r_1, \tilde{r}_2, \theta_1 - \theta)$$

and

$$A_{\lambda,r_1}(r_2, \tilde{r}_2, \theta_1, \theta_2, \theta) = a_\lambda(G(r_1, r_2, \theta_2 - \theta)) \overline{a_\lambda}(G(r_1, \tilde{r}_2, \theta_1 - \theta)) \eta(\theta_2 - \theta) \eta(\theta_1 - \theta).$$

Note that given the compact support of η, including the left endpoint $\pi - 2\varepsilon$ in the characteristic functions in (2.14) is redundant, but is kept for emphasis. The limits of integration in the integral can thus be taken as $\max(\theta_1 - \pi, \theta_2 - \pi)$ and $\min(\theta_1 - \pi + 2\varepsilon, \theta_2 - \pi + 2\varepsilon)$. The bound (2.12) will follow by applying standard oscillatory integral estimates to (2.14). We may assume that $r_1^2 |r_2 - \tilde{r}_2| \geq \lambda^{-1}$ below as the other case is trivial. The crucial lower bound is thus

$$|\partial_\theta \Psi_{r_1}(r_2, \tilde{r}_2, \theta_1, \theta_2, \theta)| + \left|\partial_\theta^2 \Psi_{r_1}(r_2, \tilde{r}_2, \theta_1, \theta_2, \theta)\right| \gtrsim |r_2 - \tilde{r}_2|\, r_1^2. \tag{2.15}$$

Once this is established, either the phase lacks critical points or we can appeal to the stationary phase estimates in [23], §VIII.1.2. In either case, we have that (2.14) is $O((\lambda|r_2 - \tilde{r}_2|)^{-1/2} r_1^{-1})$. Note that this version of stationary phase is uniform regardless of the location of any critical points in the domain of integration.

To see the lower bound (2.15), first observe that by taking ε sufficiently small, we may assume that if $A_{\lambda,r_1}(r_2, \tilde{r}_2, \theta_1, \theta_2, \theta) \neq 0$, then $(r_2 + r_1), (\tilde{r}_2 + r_1) \geq \delta/4$. Next, by using that

$$\frac{1}{G(r_1, r_2, \theta)} = \frac{1}{r_1 + r_2} + \frac{r_1 r_2 (\cos\theta + 1)}{(r_1 + r_2)^3}$$

we see that

$$\partial_\theta G(r_1,r_2,\theta) = \frac{r_1r_2}{r_1+r_2}\sin(\theta) + O((r_1r_2)^2(\theta-\pi)^3), \tag{2.16}$$

$$\partial_\theta^2 G(r_1,r_2,\theta) = \frac{r_1r_2}{r_1+r_2}\cos(\theta) + O((r_1r_2)^2(\theta-\pi)^2). \tag{2.17}$$

Hence up to acceptable error, $\partial_\theta^2\Psi_{r_1}(r_2,\tilde r_2,\theta_1,\theta_2,\theta)$ is

$$\frac{r_1^2(r_2-\tilde r_2)}{(r_1+r_2)(r_1+\tilde r_2)}\cos(\theta-\theta_2) + \frac{r_1\tilde r_2}{r_1+\tilde r_2}\left(\cos(\theta-\theta_2)-\cos(\theta-\theta_1)\right).$$

The first term has the proper lower bound we desire and the estimate will hold provided the second term is lower order. The second term is $O(r_1\tilde r_2\varepsilon|\theta_1-\theta_2|)$, so this shows (2.15) provided

$$r_1\tilde r_2\,|\theta_1-\theta_2| \le C\,\varepsilon^{-1}\,r_1^2\,|r_2-\tilde r_2|$$

for some constant C which can be taken as absolute as long as ε is sufficiently small. We are thus left to see that if

$$r_1\tilde r_2\,|\theta_1-\theta_2| > C\,r_1^2\,|r_2-\tilde r_2|,$$

then the first derivative term in (2.15) satisfies the lower bound. But (2.16) shows that up to acceptable error, $\partial_\theta\Psi_{r_1}(r_2,\tilde r_2,\theta_1,\theta_2,\theta)$ is

$$\frac{r_1^2(r_2-\tilde r_2)}{(r_1+r_2)(r_1+\tilde r_2)}\sin(\theta-\theta_2) + \frac{r_1\tilde r_2}{r_1+\tilde r_2}\left(\sin(\theta-\theta_2)-\sin(\theta-\theta_1)\right).$$

This time we may take ε small so that the absolute value of the second term is bounded below by $|\theta_1-\theta_2|(r_1\tilde r_2)/(4\delta)$. Since the first term is $O(\varepsilon r_1^2|r_2-\tilde r_2|)$, the inequality now follows by taking ε sufficiently small.

The bound (2.13) will follow from

$$\|T_{r_2}^{\lambda,r_1}\|_{L^2(d\theta_1)\to L^2(d\theta_2)} \lesssim (\lambda r_1 r_2)^{-1/2}.$$

The assumption on the data g and the supports of K, η in $\theta_2-\theta_1$ mean that we may treat the θ_i as variables in $\mathbb{R}$. Since $T_{r_2}^{\lambda,r_1}$ is a convolution kernel in θ it thus suffices to show that the corresponding Fourier multiplier satisfies

$$\left|\int_{\pi-2\varepsilon}^{\pi} e^{i\zeta+i\lambda G(r_1,r_2,\theta)}a_\lambda\left(G(r_1,r_2,\theta)\right)\eta(\theta)\,d\theta\right| \lesssim (\lambda r_1 r_2)^{-1/2}.$$

But this follows from (2.17) and the same version of stationary phase used above, once we recall that we may assume that $r_1+r_2\ge\delta/4$.

2.2. The diffractive contribution

Recall from (1.7), the contribution of the diffractive term is supported where $t > r_1+r_2$ and is a sum of terms of the form

$$-\frac{1}{4\pi^2\rho}\int_0^{\cosh^{-1}\left(\frac{t^2-r_1^2-r_2^2}{2r_1r_2}\right)} \left(t^2-r_1^2-r_2^2-2r_1r_2\cosh(s)\right)^{-1/2}\frac{\sin\varphi}{\cosh(s/\rho)-\cos\varphi}\,ds,$$

where $\varphi = (\pi \pm (\theta_1 - \theta_2))/\rho$. Recalling the left hand side of (2.3) and that $\mathrm{supp}(\widehat{\chi}) \subset (\delta, 2\delta)$, this has a nontrivial contribution to the Schwartz kernel only when $r_1 + r_2 < 2\delta$. In this case, reasoning as in (2.5) the contribution is

$$\begin{aligned}
&-\frac{\lambda}{4\pi^2\rho}\int_{-\infty}^{\infty}\int_{r_1+r_2}^{\infty}\int_0^{\cosh^{-1}\left(\frac{t^2-r_1^2-r_2^2}{2r_1r_2}\right)} \chi(\tau)\left(\sin(t(\lambda+\tau))+\sin(t(\lambda-\tau))\right)\\
&\qquad\times\left(t^2-r_1^2-r_2^2-2r_1r_2\cosh(s)\right)^{-1/2}\frac{\sin\varphi}{\cosh(s/\rho)-\cos\varphi}\,ds\,dt\,d\tau\\
&\quad= -\frac{\lambda}{2\pi^2\rho}\int_{-\infty}^{\infty}\int_{r_1+r_2}^{\infty}\int_0^{\cosh^{-1}\left(\frac{t^2-r_1^2-r_2^2}{2r_1r_2}\right)} \chi(\tau)\sin(t(\lambda-\tau))\\
&\qquad\times\left(t^2-r_1^2-r_2^2-2r_1r_2\cosh(s)\right)^{-1/2}\frac{\sin\varphi}{\cosh(s/\rho)-\cos\varphi}\,ds\,dt\,d\tau,
\end{aligned}\tag{2.18}$$

where the second expression follows from the fact that τ is even. Switching the order of integration in t, s yields

$$\begin{aligned}
&-\frac{\lambda}{2\pi^2\rho}\int_{-\infty}^{\infty}\chi(\tau)\int_0^{\infty}\frac{\sin\varphi}{\cosh(s/\rho)-\cos\varphi}\\
&\qquad\times\left(\int_{(r_1^2+r_2^2+2r_1r_2\cosh s)^{1/2}}^{\infty}\frac{\sin(t(\lambda-\tau))}{(t^2-r_1^2-r_2^2-2r_1r_2\cosh s)^{1/2}}\,dt\right)ds\,d\tau.
\end{aligned}$$

Neglecting harmless constants, (2.6) shows that this is

$$\lambda\int_{-\infty}^{\infty}\chi(\tau)\int_0^{\infty}\frac{\mathrm{sgn}(\lambda-\tau)\sin\varphi}{\cosh(s/\rho)-\cos\varphi}\,J_0\left((r_1^2+r_2^2+2r_1r_2\cosh s)^{1/2}|\lambda-\tau|\right)ds\,d\tau.$$

Proceeding as above, we use the smooth, even bump function ψ satisfying $\mathrm{supp}(\psi) \subset (-1/2, 1/2)$ and $\mathrm{supp}(1-\psi) \subset (-1/4, 1/4)^c$. As before, it suffices to restrict attention to the integral

$$\begin{aligned}
\lambda\int_{-\infty}^{\infty}\chi(\tau)\psi(\tau/\lambda)\int_0^{\infty}&\frac{\sin\varphi}{\cosh(s/\rho)-\cos\varphi}\\
&\times J_0\left((r_1^2+r_2^2+2r_1r_2\cosh s)^{1/2}(\lambda-\tau)\right)ds\,d\tau
\end{aligned}\tag{2.19}$$

as the error is $O(\lambda^{-N})$ uniformly in φ by the same argument as above. We now follow the same approach as in (2.10) and in particular we use the function $\tilde{\psi}$ from that same discussion. First set

$$D(r_1, r_2, s) \stackrel{\text{def}}{=} (r_1^2 + r_2^2 + 2r_1r_2\cosh s)^{1/2}.$$

Similarly to (2.10), we define

$$a_\lambda(\zeta) = \int \widehat{\chi}(s)\,\widehat{\tilde{\psi}}(\lambda(\zeta - s); \lambda\zeta)\,\lambda\,ds, \tag{2.20}$$

so that a_λ is rapidly decreasing outside a λ^{-1} neighborhood of $\operatorname{supp}(\chi)$. Note that in contrast to the previous sections, there is a small difference in the definition of a_λ here in that we do not assume that it is compactly supported in $(\delta, 2\delta)$. As in the geometric case, from (2.19) it now suffices to consider the real part of

$$K(r_1,r_2,\theta) \stackrel{\text{def}}{=} \int_0^\infty e^{i\lambda D(r_1,r_2,s)} \frac{\sin(\theta/\rho)}{\cosh(s/\rho)-\cos(\theta/\rho)}\, a_\lambda\left(D(r_1,r_2,s)\right) ds \tag{2.21}$$

as we may take translations in θ to remove the term π in the definition of φ. As before, we remove the factor of $\lambda^{1/2}$ in the kernel and instead we will prove the following bound.

Lemma 2.1. *For $p = 6$ and $p = \infty$ and $K(r_1, r_2, \theta_2 - \theta_1)$ defined in (2.21), we obtain the gain*

$$\left\| \iint K(r_1,r_2,\theta_2-\theta_1) g(r_1,\theta_1) r_1 dr_1 d\theta_1 \right\|_{L^p(r_2 d\theta_2 dr_2)} \lesssim \lambda^{-2/p} \|g\|_{L^2(r_1 d\theta_1 dr_1)}. \tag{2.22}$$

We begin by observing that if $s > 0$, $2 < p \le \infty$, and $q > 1$ satisfies $1/q = 1/2 + 1/p$ then

$$\left(\int_{\mathbb{S}^1_\rho} \left| \frac{\sin(\theta/\rho)}{\cosh(s/\rho)-\cos(\theta/\rho)} \right|^q d\theta \right)^{1/q} \lesssim \begin{cases} s^{1/q-1} & s<1, \\ e^{-s/\rho} & s \ge 1. \end{cases} \tag{2.23}$$

Note that as a consequence, for each s, the integrand in (2.21) is in $L^q(r_i dr_i d\theta)$ when $i = 1, 2$ and hence by Young's inequality this maps $L^2 \to L^p$ with operator norm integrable in s. In particular, by Minkowski's integral inequality this already yields (2.22) when $p = \infty$. For the remainder of the section, we thus assume that $p = 6$. Moreover, since $s^{-1/3}$ presents a locally integrable singularity, this gives the estimate

$$\begin{aligned} &\left\| \mathbf{1}_{(0,1/\lambda)}(r_2) \iint K(r_1,r_2,\theta_2-\theta_1) g(r_1,\theta_1) r_1\, dr_1 d\theta_1 \right\|_{L^6(r_2 d\theta_2 dr_2)} \\ &\lesssim \left(\int_0^{\lambda^{-1}} \left| \int_0^{2\delta} \|g(r_1,\cdot)\|_{L^2(d\theta_1)}\, r_1\, dr_1 \right|^6 r_2\, dr_2 \right)^{1/6} \lesssim \lambda^{-1/3} \|g\|_{L^2(r_1 d\theta_1 dr_1)}, \end{aligned} \tag{2.24}$$

showing that we may further localize K to $r_2 \ge \lambda^{-1}$. The same argument shows that the same restricted kernel maps $L^{6/5} \to L^2$ with an even stronger gain. Consequently, by duality and symmetry of K in r_1, r_2 we may also restrict attention to $r_1 \ge \lambda^{-1}$.

2.2.1. The case $r_1 + r_2 \approx \delta$. Here we obtain estimates on the contribution of

$$\mathbf{1}_{(\lambda^{-1},\infty)}(r_1)\, \mathbf{1}_{(\lambda^{-1},\infty)}(r_2)\, \mathbf{1}_{(\delta/2,2\delta)}(r_1+r_2)\, K(r_1,r_2,\theta_2-\theta_1)$$

to (2.22). Here we also note that it suffices to replace K by $e^{-i\lambda(r_1+r_2)}K$ in (2.22) as these modulations to g and its image under the integral operator do not change their L^p norms. Define the following integral kernel H which will serve as a sufficiently accurate approximation to $e^{-i\lambda(r_1+r_2)}K(r_1,r_2,\theta_2-\theta_1)$:

$$H(r_1,r_2,\theta) \stackrel{\text{def}}{=} 2\rho(r_1+r_2)^{1/2}a_\lambda(r_1+r_2)\int_0^\infty e^{i\lambda r_1 r_2 s^2}\frac{\theta}{s^2+\theta^2}\,ds. \tag{2.25}$$

Strictly speaking, we should be multiplying H by characteristic functions which localize us to $r_1, r_2 \geq \lambda^{-1}$ and $r_1+r_2 \in (\delta/2, 2\delta)$ but we suppress these for both H and K to avoid cluttering the notation.

Lemma 2.2. *Suppose $r_1, r_2 \geq \lambda^{-1}$ and $r_1+r_2 \in (\delta/2, 2\delta)$ as above. Then the difference*

$$\tilde{K}(r_1,r_2,\theta) = e^{-i\lambda(r_1+r_2)}K(r_1,r_2,\theta) - H(r_1,r_2,\theta)$$

satisfies

$$|\tilde{K}(r_1,r_2,\theta)| \lesssim (\lambda r_1 r_2)^{-1/2}. \tag{2.26}$$

Given the lemma, by applying Young's inequality in θ along with the inequalities of Minkowski and Hölder, we have

$$\begin{aligned}\Big\|\iint \tilde{K}(r_1,r_2,\theta_2-\theta_1)\,g(r_1,\theta_1)r_1\,dr_1\,d\theta_1\Big\|_{L^6(r_2 d\theta_2 dr_2)} \\ \lesssim \lambda^{-1/2}\Big(\int_{\lambda^{-1}}^1\Big|\int_{\lambda^{-1}}^1 \|g(r_1,\cdot)\|_{L^2(d\theta_1)}\,r_1^{1/2}dr_1\Big|^6 r_2^{-2}dr_2\Big)^{1/6} \\ \lesssim \lambda^{-1/3}\,\|g\|_{L^2(r_1 dr_1 d\theta_1)}.\end{aligned} \tag{2.27}$$

Hence it suffices to replace $e^{-i\lambda(r_1+r_2)}K$ by H below.

The lemma makes use of the stationary phase estimates in [23], §VIII.1.2, or [9], §2.9, which imply if $a \in C^1([0,1])$ and $\phi(s)$ has a single nondegenerate critical point at $s=0$, then

$$\int_0^1 e^{i\mu\phi(s)}a(s)\,ds = O(\mu^{-1/2}). \tag{2.28}$$

Proof. We will see that $\tilde{K}$ is a sum of terms, each of which is $O((\lambda r_1 r_2)^{-1/2})$. Note that while the domain of integration in (2.21) is over $[0,\infty)$, we can include the contribution of the integral over $[1,\infty)$ in $\tilde{K}$ provided δ is sufficiently small as the phase function over this interval lacks any critical points.

We first observe that the difference between the integral defining K, now restricted to $s \in [0,1]$, and the integral

$$\int_0^1 e^{i\lambda(r_1^2+r_2^2+2r_1r_2\cosh s)^{1/2}-i\lambda(r_1+r_2)}\frac{2\rho\theta}{s^2+\theta^2}a_\lambda(r_1+r_2)\,ds \tag{2.29}$$

can be included in $\tilde{K}$. Observe that for $s \in [0,1]$,

$$\Big|\partial_s^j\Big(\frac{\sin(\theta/\rho)}{\cosh(s/\rho)-\cos(\theta/\rho)} - \frac{2\rho\theta}{s^2+\theta^2}\Big)\Big| \lesssim \theta^{1-j}, \quad j=0,1.$$

Since (2.28) only requires that the amplitude is C^1, replacing $\sin(\theta/\rho)/(\cosh(s/\rho)-\cos(\theta/\rho))$ by this difference in (2.21) yields a term which is $O((\lambda r_1 r_2)^{-1/2})$ and hence this can indeed be included in $\tilde{K}$. Moreover, a Taylor expansion shows that

$$(2.30)\qquad \left(r_1^2+r_2^2+2r_1r_2\cosh s\right)^{1/2}-(r_1+r_2)=\frac{r_1r_2s^2}{r_1+r_2}\left(1+O(r_1r_2s^2)\right).$$

Hence

$$\left|\partial_s^j\left(a_\lambda((r_1^2+r_2^2+2r_1r_2\cosh s)^{1/2})-a_\lambda(r_1+r_2)\right)\right|\lesssim s^{2-j},\quad j=0,1,$$

and applying (2.28) a second time completes the proof of the claim that error introduced by replacing the integral in K by (2.29) is acceptable.

We now make a change of variables in (2.29), defining $\tilde{s}$ as a function of s by

$$r_1r_2\tilde{s}^2=\left(r_1^2+r_2^2+2r_1r_2\cosh s\right)^{1/2}-(r_1+r_2).$$

Therefore by (2.30), we have $\frac{d\tilde{s}}{ds}=\frac{1}{(r_1+r_2)^{1/2}}+O(r_1r_2s^2)$ and hence since $s=O(\tilde{s})$,

$$\frac{ds}{d\tilde{s}}=(r_1+r_2)^{1/2}+O(r_1r_2\tilde{s}^2)$$

Applying stationary phase as before, the contribution of the second term on the right here can be included into $\tilde{K}$. If the domain of integration of the integral in (2.25) were over $[0,1]$, this would conclude the proof. Since this is not the case, we simply observe that

$$\int_1^\infty e^{i\lambda r_1r_2s^2}\frac{\theta}{s^2+\theta^2}\,ds=O((\lambda r_1r_2)^{-1}),$$

as the phase function here lacks critical points. □

Returning to H in (2.25), we redefine ψ to be an even bump function supported in a small interval about the origin of size much less than ρ. Define $\widetilde{H}$ as the kernel obtained by multiplying the expression on the left-hand side of (2.25) by $\psi(\theta)$. Applying stationary phase, we have that

$$2\rho(1-\psi)(\theta)(r_1+r_2)^{1/2}a_\lambda(r_1+r_2)\int_0^\infty e^{i\lambda r_1r_2s^2}\frac{\theta}{s^2+\theta^2}\,ds=O\big((\lambda r_1r_2)^{-1/2}\big).$$

Given (2.27), it now suffices to show (2.22) with K replaced by $\widetilde{H}$.

Next, we recall that the convolution kernel defined for $\theta\in\mathbb{R}$ by

$$Q_s(\theta)\stackrel{\text{def}}{=}\pi\,\frac{\theta}{s^2+\theta^2}$$

is known as the *conjugate Poisson kernel*, see for example [11], p. 265. Its action is equivalent to the Fourier multiplier with symbol

$$\widehat{Q_s}(\xi)=-i\,\mathrm{sgn}(\xi)e^{-s|\xi|}$$

and hence if we let $H(r_1, r_2, \theta)$ be the expression in (2.25) but with $\theta \in \mathbb{R}$ (as opposed to $\theta \in \mathbb{S}^1_\rho$), we have that its partial Fourier transform in θ satisfies

$$\begin{aligned} \widehat{H}(r_1, r_2, \xi) &= \frac{2\rho}{i\pi} \operatorname{sgn}(\xi)(r_1 + r_2)^{1/2} a_\lambda(r_1 + r_2) \int_0^\infty e^{i\lambda r_1 r_2 s^2} e^{-s|\xi|}\, ds \\ &= \frac{2\rho}{i\pi}(r_1 + r_2)^{1/2} \operatorname{sgn}(\xi) a_\lambda(r_1 + r_2)\Big(\sum_{k=0}^\infty \frac{(-1)^k |\xi|^k}{k!} \int_0^\infty e^{i\lambda r_1 r_2 s^2} s^k\, ds\Big). \end{aligned} \tag{2.31}$$

Lemma 2.3. *The multipliers* $\widehat{H}(r_1, r_2, \xi)$, $\widehat{\widetilde{H}}(r_1, r_2, \xi)$ *satisfy the bounds*

$$|\widehat{\widetilde{H}}(r_1, r_2, \xi)|, |\widehat{H}(r_1, r_2, \xi)| \lesssim \begin{cases} (\lambda r_1 r_2)^{-1/2} & |\xi| \lesssim (\lambda r_1 r_2)^{1/2}, \\ 1/|\xi| & |\xi| \gg (\lambda r_1 r_2)^{1/2}. \end{cases} \tag{2.32}$$

Proof. First note that since $\widehat{\psi}$ is a Schwartz class function rapidly decreasing on the unit scale, the bound on $\widehat{\widetilde{H}} = \widehat{\psi} * \widehat{H}$ follows from the one on $\widehat{H}$. We now consider the identity (cf. [9], p. 54)

$$\int_0^\infty e^{i\lambda r_1 r_2 s^2} s^k\, ds = \frac{1}{2}\Gamma\Big(\frac{k+1}{2}\Big) e^{\frac{i\pi}{4}(k+1)} (\lambda r_1 r_2)^{-\frac{k+1}{2}}.$$

The power series in parentheses in (2.31) thus takes the form

$$\frac{1}{2}\sum_{k=0}^\infty \frac{(-1)^k |\xi|^k}{k!} \Gamma\Big(\frac{k+1}{2}\Big) e^{\frac{i\pi}{4}(k+1)} (\lambda r_1 r_2)^{-\frac{k+1}{2}}. \tag{2.33}$$

The power series

$$F(z) = \sum_{k=0}^\infty \frac{(-1)^k \Gamma\left(\frac{k+1}{2}\right)}{k!} z^k = \sum_{l=0}^\infty \frac{\Gamma(l + 1/2)}{(2l)!} z^{2l} + \sum_{l=1}^\infty \frac{\Gamma(l)}{(2l-1)!} z^{2l-1},$$

is seen to converge uniformly on compact sets and satisfies

$$\widehat{H}(r_1, r_2, \xi) = 2\rho (r_1 + r_2)^{1/2} \operatorname{sgn}(\xi) a_\lambda(r_1 + r_2) (\lambda r_1 r_2)^{-1/2} e^{-\frac{i\pi}{4}} F\big((\lambda r_1 r_2)^{-1/2} e^{\frac{i\pi}{4}} |\xi|\big).$$

The desired bound (2.32) will then follow from

$$|F(z)| \lesssim \begin{cases} 1 & |z| \le 1\,, \\ 1/|z| & |z| \ge 1\,, \end{cases} \tag{2.34}$$

for $z \in \mathbb{C}$ such that $\arg(z) = \pi/4$. The bound for $|z| \le 1$ is immediate from the uniform convergence noted above.

To analyze the behavior of $F(z)$ when $|z|$ is large, we split the series into even and odd terms as above. When $k = 2l$ is even, the duplication formula

$$\Gamma(z)\,\Gamma(z + 1/2) = 2^{1-2z}\sqrt{\pi}\;\Gamma(2z)$$

gives

$$\Gamma\Big(\frac{k+1}{2}\Big) = \Gamma\big(l+1/2\big) = \frac{(2l)!\,\sqrt{\pi}}{4^l\,l!}.$$

The contribution of the even terms to $F(z)$ is therefore easy to describe as

$$\sum_{l=0}^{\infty} \frac{\sqrt{\pi}}{4^l\,l!}\,z^{2l} = \sqrt{\pi}\,\exp(z^2/4). \tag{2.35}$$

When $k=2l-1$ is odd, we use Pochhammer notation $(1/2)_l=(1/2)(3/2)\cdots(l-1/2)$ to write $(2l-1)! = 2^{2l-1}\,(1/2)_l\,(l-1)!$. Hence

$$\frac{\Gamma\left(\frac{k+1}{2}\right)}{k!} = \frac{(l-1)!}{(2l-1)!} = \frac{2}{4^l(1/2)_l} = \frac{2l!}{4^l(1/2)_l\,l!}.$$

The odd terms thus yield a series that can be expressed in terms of a Kummer function $\Phi(\alpha,\gamma;w) = \sum_{l=0}^{\infty} \frac{(\alpha)_l}{(\gamma)_l l!} w^l$ (cf. [15], §9.9, though note that several other authors denote this as $M(\alpha,\gamma;w)$):

$$-\frac{2}{z}\sum_{l=1}^{\infty} \frac{(1)_l}{(1/2)_l l!}\Big(\frac{z^2}{4}\Big)^l = -\frac{2}{z}\Big(\Phi\Big(1,1/2;\frac{z^2}{4}\Big)-1\Big).$$

Using the asymptotics of $\Phi(\alpha,\gamma;w)$ for large $|w|$ and $\arg w = \pi/2$ from [15], (9.12.7), we have

$$\Phi\Big(1,1/2;w\Big) = \Gamma(1/2)\left(\frac{e^w w^{1/2}}{\Gamma(1)} + O(w^{-1/2})\right).$$

Hence, since $\Gamma(1/2)=\sqrt{\pi}$,

$$-\frac{2}{z}\left(\Phi\big(1,1/2;z^2/4\big)-1\right) = -\sqrt{\pi}\exp(z^2/4) + 2/z + O(z^{-2}).$$

The bound (2.34) now follows from the cancellation between (2.35) and the first term in the asymptotic expansion here. □

Theorem 2.4. *The operator determined by the integral kernel $\widetilde{H}$ maps $L^2 \to L^6$ with operator norm bounded by $\lambda^{-1/3}$:*

$$\Big\| \iint \widetilde{H}(r_1,r_2,\theta_2-\theta_1)g(r_1,\theta_1)\,d\theta_1 r_1 dr_1\Big\|_{L^6(d\theta_2 r_2 dr_2)} \lesssim \lambda^{-1/3}\,\|g\|_{L^2(d\theta_1 r_1 dr_1)}. \tag{2.36}$$

Proof. By Minkowski's inequality, (2.36) is reduced to

$$\int \big\|\widetilde{H}(r_1,r_2,\cdot) * g(r_1,\cdot)\big\|_{L^6(d\theta_2 r_2 dr_2)} r_1 dr_1 \lesssim \lambda^{-1/3}\,\|g\|_{L^2(d\theta_1 r_1 dr_1)}.$$

In particular, if we can show that for $f \in L^2(d\theta_1)$ and $r_1 \in (0,\delta)$ fixed, we have

$$\big\|\widetilde{H}(r_1,r_2,\cdot) * f\big\|_{L^6(d\theta_2 r_2 dr_2)} \lesssim \lambda^{-1/3}\,r_1^{-1/3}\,\|f\|_{L^2(d\theta_1)}, \tag{2.37}$$

then by Hölder's inequality, (2.36) will follow.

Now let $\{\beta_\ell\}_{\ell=0}^\infty$ be a sequence of smooth Littlewood–Paley cutoffs satisfying, for $\xi \in \mathbb{R}$,

$$\sum_{\ell=0}^{\infty} \beta_\ell(\xi) = 1 \quad \text{and} \quad \beta_\ell(\xi) = \beta(2^{1-\ell}\xi) \ \text{ for } \ell \geq 1,$$

with $\operatorname{supp}(\beta) \subset \{|\xi| \in (1/2, 2)\}$ and $\operatorname{supp}(\beta_0) \subset (-2, 2)$. Now define $\widetilde{H}_\ell$ by

$$\widehat{\widetilde{H}}_\ell(r_1, r_2, \xi) \stackrel{\text{def}}{=} \beta_\ell(\xi)\widehat{\widetilde{H}}(r_1, r_2, \xi).$$

By the Sobolev embedding/Young's inequality, we have that

$$\|H_\ell(r_1, r_2, \cdot) * f\|_{L^6(d\theta_2)} \lesssim \begin{cases} (\lambda r_1 r_2)^{-1/2} 2^{\ell/3} \|\beta_\ell f\|_{L^2(d\theta_1)} & 2^\ell \leq (\lambda r_1 r_2)^{1/2}, \\ 2^{-2\ell/3} \|\beta_\ell f\|_{L^2(d\theta_1)} & 2^\ell > (\lambda r_1 r_2)^{1/2}. \end{cases} \tag{2.38}$$

The classical Littlewood–Paley square function bound and Minkowski's inequality imply that the left-hand side of (2.37) is dominated by

$$\Big(\int \Big(\sum_{\ell=0}^{\infty} \|H_\ell(r_1, r_2, \cdot) * f\|^2_{L^6(d\theta_2)}\Big)^3 r_2\, dr_2\Big)^{1/6}.$$

Applying (2.38), this in turn is bounded by

$$(\lambda r_1)^{-1/2}\Big(\int_{\lambda^{-1}}^{1} \Big(\sum_{2^\ell \leq (\lambda r_1 r_2)^{1/2}} 2^{2\ell/3} \|\beta_\ell f\|^2_{L^2(d\theta_1)}\Big)^3 r_2^{-2}\, dr_2\Big)^{1/3 \cdot 1/2} + \Big(\int_{\lambda^{-1}}^{1} \Big(\sum_{2^\ell > (\lambda r_1 r_2)^{1/2}} 2^{-4\ell/3} \|\beta_\ell f\|^2_{L^2(d\theta_1)}\Big)^3 r_2\, dr_2\Big)^{1/3 \cdot 1/2}.$$

To bound the first expression here we use Minkowski's inequality to get that

$$\begin{aligned} (\lambda r_1)^{-1}\Big(\int \Big(\sum_{2^\ell \leq (\lambda r_1 r_2)^{1/2}} 2^{2\ell/3}\, \|\beta_\ell f\|^2_{L^2(d\theta_1)}\, r_2^{-2/3}\Big)^3 dr_2\Big)^{1/3} &\lesssim (\lambda r_1)^{-1} \sum_{\ell=0}^{\infty} 2^{2\ell/3}\, \|\beta_\ell f\|^2_{L^2(d\theta_1)} \Big(\int_{2^{2\ell}(\lambda r_1)^{-1} \leq r_2} r_2^{-2}\, dr_2\Big)^{1/3} \\ &\lesssim (\lambda r_1)^{-2/3} \sum_{\ell=0}^{\infty} \|\beta_\ell f\|^2_{L^2(d\theta_1)} \approx (\lambda r_1)^{-2/3} \|f\|^2_{L^2(d\theta_1)}. \end{aligned}$$

and after taking square roots, the contribution of this expression is bounded above by the right-hand side of (2.37). For the second expression, we use Minkowski's inequality again:

$$\Big(\int_{\lambda^{-1}}^{1}\Big(\sum_{2^\ell>(\lambda r_1 r_2)^{1/2}} 2^{-4\ell/3}\,\|\beta_\ell f\|^2_{L^2(d\theta_1)}\,r_2^{1/3}\Big)^3 dr_2\Big)^{1/3}$$
$$\lesssim \sum_{\mu} 2^{-4\ell/3}\,\|\beta_\ell f\|^2_{L^2(d\theta_1)}\Big(\int_{r_2\le 2^{2\ell}(\lambda r_1)^{-1}} r_2\,dr_2\Big)^{1/3}$$
$$\lesssim (\lambda r_1)^{-2/3}\sum_{\ell=0}^{\infty}\|\beta_\ell f\|^2_{L^2(d\theta_1)} \approx (\lambda r_1)^{-2/3}\|f\|^2_{L^2(d\theta_1)}.$$

The desired bound (2.37) now follows as before after taking square roots. □

2.2.2. The case $r_1 + r_2 \le \delta/2$. We now consider the contribution of

$$\mathbf{1}_{(\lambda^{-1},\infty)}(r_1)\,\mathbf{1}_{(\lambda^{-1},\infty)}(r_2)\,\mathbf{1}_{(0,\delta/2)}(r_1+r_2)\,K(r_1,r_2,\theta_2-\theta_1)$$

to (2.22). As before, we will assume that r_1, r_2 evaluate to one along the characteristic functions here throughout this subsection to avoid cluttering notation. Recall that the amplitude a_λ in (2.20) decays rapidly outside a λ^{-1} neighborhood of $\operatorname{supp}(\widehat{\chi}) \subset (\delta, 2\delta)$. Hence for any $N > 0$, $a_\lambda(D(r_1,r_2,s)) = O(\lambda^{-N})$ if $D(r_1,r_2,s) \notin (\delta,2\delta)$. But $D(r_1,r_2,s) \in (\delta,2\delta)$ implies that

$$2r_1r_2(\cosh s - 1) \ge \delta^2 - (r_1+r_2)^2 \ge 3\delta^2/4,$$

which means there exists a sufficiently small constant c_0 such that $s \ge c_0(r_1r_2)^{-1/2}$. Since we are assuming $r_i \ge \lambda^{-1}$, $i = 1,2$, we can recall (2.23) to see that the integral operator with kernel

$$\int_0^{c_0(r_1r_2)^{-1/2}} e^{i\lambda D(r_1,r_2,s)}\frac{\sin(\theta/\rho)}{\cosh(s/\rho)-\cos\varphi}\,a_\lambda\left(D(r_1,r_2,s)\right)\,ds$$

maps $L^2 \to L^6$ with a gain of $O(\lambda^{-N})$ for any $N > 0$ using the decay in $a_\lambda(D)$ in this region.

It thus suffices to consider the contribution of

$$\int_{c_0(r_1r_2)^{-1/2}}^{\infty} e^{i\lambda D(r_1,r_2,s)}\frac{\sin(\theta/\rho)}{\cosh(s/\rho)-\cos\varphi}\,a_\lambda\left(D(r_1,r_2,s)\right)\,ds.$$

Since

$$\partial_s D(r_1,r_2,s) = \frac{r_1r_2\sinh s}{D(r_1,r_2,s)}$$

the phase function has no critical points in $[c_0(r_1r_2)^{-1/2},\infty)$. Using that $D\times a_\lambda(D)$ is bounded, the integral is $O(\lambda^{-1}(r_1r_2)^{-1/2})$, and by the argument in (2.27), this yields a kernel which maps $L^2 \to L^6$ with a gain of $\lambda^{-5/6} = \lambda^{-1/3-1/2}$.

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