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On zeros of analytic functions satisfying non-radial growth conditions

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Abstract. Extending the results of Borichev–Golinskii–Kupin (2009), we obtain refined Blaschke-type necessary conditions on the zero distribution of analytic functions on the unit disk and on the complex plane with a cut along the positive semi-axis satisfying some non-radial growth restrictions.

To Peter Yuditskii on occasion of his 60-th anniversary

1. Introduction and main results

The study of relations between the zero distribution of an analytic function and its growth is likely to be one of the most basic problems of complex analysis. We have no intention to review a vast literature on it, but just give several references related to the points of our interest. Perhaps, the first results in this direction were obtained in the second half of 19th century by Hadamard, Borel, Weierstrass and others, see Levin [22], Chapter 1, for a modern presentation. These results completely described the behavior of zeros of an entire function of finite type. Later, Blaschke [2], Nevanlinna [23] and Smirnov [27] described the zero sets of functions from the Hardy spaces $H^p(\mathbb{D})$, $p > 0$, or, more generally, the Nevanlinna class $\mathcal{N}(\mathbb{D})$. Here, as usual, $\mathbb{D} = \{|z| < 1\}$. Namely, for $f \in \mathcal{N}(\mathbb{D})$, $f \not\equiv 0$, one has

$$(1.1) \qquad \sum_{\zeta\in Z(f)} (1 - |\zeta|) \le \sup_{0\le r<1} \frac{1}{2\pi} \int_0^{2\pi} \log^+ |f(re^{i\theta})|\, d\theta - \log|f(0)|,$$

where $Z(f)$ stands for the zero set of f counting multiplicities. Hence, a discrete subset $Z(f)$ of the unit disk is a zero set of a function from $H^p(\mathbb{D})$ (or $\mathcal{N}(\mathbb{D})$) if and only if the series at the LHS of (1.1) converges. This condition is usually called the "Blaschke condition" after [2].

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Let $\mathcal{A}(\mathbb{D})$ be the set of analytic functions on the unit disk. An argument similar to the proof of (1.1) shows that if $f \in \mathcal{A}(\mathbb{D})$, $|f(0)| = 1$, satisfies the growth condition

$$\log|f(z)| \le \frac{K}{(1-|z|)^p},$$

where $p \ge 1$, then for any $\varepsilon > 0$,

$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\varepsilon} \le C_0 \cdot K, \tag{1.2}$$

where the constant $C_0 = C_0(p, \varepsilon)$ depends on p and ε, see, e.g., Golubev [16].

Of course, the study of the zero distribution of analytic functions from other classes is much more involved; see, for instance, papers of Korenblum [18], [19] on the zero distribution for functions from spaces $A^{-p}(\mathbb{D}), A^{-\infty}(\mathbb{D})$. Interesting results on zeros of functions from some Bergman-type spaces are given in Seip [26].

The above mentioned spaces of analytic functions are defined with the help of radial (i.e., invariant with respect to rotations of the unit disk) growth conditions. However, it turns out that one often needs to deal with classes of analytic functions subject to *non-radial* growth relations. These classes appear, in particular, if one wants to study the distribution of the discrete spectrum of non-self-adjoint perturbations for certain self-adjoint or unitary operators.

The study of such classes was initiated in [3], and the main result therein looks as follows, see [3], Theorem 0.2. Given a finite set $F = \{\xi_k\}_{k=1}^m$ on the unit circle $\mathbb{T} = \{|z| = 1\}$, let $d(z, F) = \min_k |z - \xi_k|$ denote the Euclidian distance between a point $z \in \mathbb{D}$ and F. In what follows, $a_+ := \max(a, 0)$, $a_- := \max(-a, 0)$, and K is a positive constant.

Theorem A. Let $f \in \mathcal{A}(\mathbb{D})$, $|f(0)| = 1$, satisfy the growth condition

$$\log|f(z)| \le \frac{K}{(1-|z|)^p \; d^q(z,F)}, \quad z \in \mathbb{D}, \quad p, q \ge 0.$$

Then, for each $\varepsilon > 0$ there is a positive number $C_1 = C_1(F, p, q, \varepsilon)$ such that the following Blaschke-type condition holds:

$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\varepsilon} \; d^{(q-1+\varepsilon)_+}(\zeta, F) \le C_1 \cdot K. \tag{1.3}$$

Moreover, in the case $p = 0$ the term $(1-|\zeta|)^{p+1+\varepsilon}$ can be replaced by $(1-|\zeta|)$.

Theorem A effectively applies to the study of the discrete spectrum of complex perturbations of certain self-adjoint operators of mathematical physics in Demuth–Hansmann–Katriel [5], [6], Golinskii–Kupin [13], [14], [15], Dubuisson [7], [8], and Sambou [25]. We also mention recent interesting papers by Cuenin–Laptev–Tretter [4], Frank–Sabin [12], Frank [11], and Laptev–Safronov [20] in this connection. For some extensions of this result to the case of arbitrary closed sets F and subharmonic on $\mathbb{D}$ functions f, and applications in perturbation theory see Favorov–Golinskii [9], [10].

Let us go over to the main results of the present paper which extend Theorem A. Let $E = \{\zeta_j\}_{j=1}^n$ and $F = \{\xi_k\}_{k=1}^m$ be two disjoint finite sets of distinct points on the unit circle $\mathbb{T}$.

Theorem 1.1. *Let $f \in \mathcal{A}(\mathbb{D})$, $|f(0)| = 1$, satisfy the growth condition*

$$\log|f(z)| \le \frac{K}{(1-|z|)^p}\, \frac{d^r(z,E)}{d^q(z,F)}, \qquad z \in \mathbb{D}, \quad p,q,r \ge 0. \tag{1.4}$$

Then for every $\varepsilon > 0$, there is a positive number $C_2 = C_2(E, F, p, q, r, \varepsilon)$ such that the following Blaschke-type condition holds:

$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\varepsilon}\, \frac{d^{(q-1+\varepsilon)_+}(\zeta,F)}{d^{\min(p,r)}(\zeta,E)} \le C_2 \cdot K. \tag{1.5}$$

Of course, Theorem A is exactly Theorem 1.1 with $r = 0$.

An obvious inequality for an arbitrary finite set $B = \{\beta_j\}_{j=1}^n \subset \mathbb{T}$,

$$c(B) \prod_{j=1}^n |z-\beta_j| \le d(z,B) \le C(B) \prod_{j=1}^n |z-\beta_j|,$$

along with Theorem 0.3 from [3] prompt a more general statement.

Theorem 1.2. *Let $f \in \mathcal{A}(\mathbb{D})$, $|f(0)| = 1$, satisfy the growth condition*

$$\log|f(z)| \le \frac{K}{(1-|z|)^p}\, \frac{\prod_{j=1}^n |z-\zeta_j|^{r_j}}{\prod_{k=1}^m |z-\xi_k|^{q_k}}, \qquad z \in \mathbb{D}, \quad p, q_k, r_j \ge 0. \tag{1.6}$$

Then for every $\varepsilon > 0$, there is a positive number $C_3 = C_3(E, F, p, \{q_k\}, \{r_j\}, \varepsilon)$ such that the following Blaschke-type condition holds:

$$\sum_{\zeta \in Z(f)} (1-|\zeta|)^{p+1+\varepsilon}\, \frac{\prod_{k=1}^m |\zeta-\xi_k|^{(q_k-1+\varepsilon)_+}}{\prod_{j=1}^n |\zeta-\zeta_j|^{\min(p,r_j)}} \le C_3 \cdot K. \tag{1.7}$$

Once again, in the case $p = 0$ the factor $(1-|\zeta|)^{1+\varepsilon}$ in (1.5) and (1.7) can be replaced by $(1-|\zeta|)$.

Remark. An observation due to Hansmann–Katriel [17] applies in our setting. It turns out that the stronger assumption

$$\log|f(z)| \le \frac{K|z|^\gamma}{(1-|z|)^p}\, \frac{\prod_{j=1}^n |z-\zeta_j|^{r_j}}{\prod_{k=1}^m |z-\xi_k|^{q_k}}, \qquad z \in \mathbb{D}, \quad p, q_k, r_j, \gamma \ge 0,$$

implies the stronger conclusion

$$\sum_{\zeta \in Z(f)} \frac{(1-|\zeta|)^{p+1+\varepsilon}}{|\zeta|^{(\gamma-\varepsilon)_+}}\, \frac{\prod_{k=1}^m |\zeta-\xi_k|^{(q_k-1+\varepsilon)_+}}{\prod_{j=1}^n |\zeta-\zeta_j|^{\min(p,r_j)}} \le C(E, F, p, \{q_k\}, \{r_j\}, \varepsilon) \cdot K.$$

The result of Theorem 1.1 can be extended in another direction involving arbitrary closed subsets F of the unit circle. A key ingredient in such extensions is the following quantitative characteristic of F known as the *Ahern–Clark type* [1]:

$$\alpha(F) := \sup\big\{\alpha\in\mathbb{R} : |\{t\in\mathbb{T}:\ d(t,F)<x\}| = O(x^\alpha),\ x\to +0\big\}.$$

Here $|A|$ denotes the Lebesgue measure of a measurable set $A\subset\mathbb{T}$.

Theorem 1.3. *Let $E=\{\zeta_j\}_{j=1}^n$ be a finite subset of $\mathbb{T}$, $F\subset\mathbb{T}$ be an arbitrary closed set, and $E\cap F=\emptyset$. Let $f\in\mathcal{A}(\mathbb{D})$, $|f(0)|=1$, satisfy the growth condition (1.4). Then for every $\varepsilon>0$, there is a positive number $C_4=C_4(E,F,p,q,r,\varepsilon)$ such that the following Blaschke-type condition holds:*

$$\sum_{\zeta\in Z(f)}(1-|\zeta|)^{p+1+\varepsilon}\,\frac{d^{(q-\alpha(F)+\varepsilon)_+}(\zeta,F)}{d^{\min(p,r)}(\zeta,E)}\le C_4\cdot K. \tag{1.8}$$

Clearly, Theorem 1.1 is a special case of the latter result, since $\alpha(F)=1$ for finite sets F.

As we will see later in Section 4, inequalities (1.5), (1.7) are in some sense "local" with respect to the singular points $\{\zeta_j\}_{j=1}^n$ and $\{\xi_k\}_{k=1}^m$ on the unit circle, so we can restrict ourselves to the case $n=m=1$ and $E=\{\zeta_0\}$, $F=\{\xi_0\}$. The following "one-point" version of the main result will be crucial in the sequel.

Theorem 1.4. *Let $\zeta_0,\xi_0\in\mathbb{T}$, $\zeta_0\ne\xi_0$, and let $f\in\mathcal{A}(\mathbb{D})$, $|f(0)|=1$, satisfy the growth condition*

$$\log|f(z)|\le\frac{K}{(1-|z|)^p}\,\frac{|z-\zeta_0|^r}{|z-\xi_0|^q},\qquad z\in\mathbb{D},\qquad p,q,r\ge 0. \tag{1.9}$$

Then for every $\varepsilon>0$, there is a positive number $C_5=C_5(\zeta_0,\xi_0,p,q,r,\varepsilon)$ such that the following inequality holds:

$$\sum_{\zeta\in Z(f)}(1-|\zeta|)^{p+1+\varepsilon}\,\frac{|\zeta-\xi_0|^{(q-1+\varepsilon)_+}}{|\zeta-\zeta_0|^{\min(p,r)}}\le C_5\cdot K. \tag{1.10}$$

The paper is organized in a straightforward manner. The preliminaries are given in Section 2. In Sections 3 and 4 we prove Theorem 1.4 and then deduce the general statements in Theorems 1.2 and 1.3 from this one-point version. Some further results (the analogs for the upper half-plane and the plane with a cut) are given in Section 5.

To keep the notation reasonably simple and consistent, we usually number the constants C_k appearing in the formulations of theorems, propositions, etc. The constants C arising in the proofs are generic, i.e., the same symbol does not necessarily denote the same constant in different occurrences.

2. Conformal mappings, Pommerenke lemma and Stolz angles

We start with some general preliminaries from complex analysis.

The known distortion inequalities [24], Corollary 1.4, play a key role in what follows.

Lemma 2.1. *Let Ω be a bounded, simply connected domain with the boundary $\partial\Omega$, and φ be a conformal mapping of Ω onto $\mathbb{D}$. Then*

$$\frac{1}{2}\, d(w,\partial\Omega)\cdot|\varphi'(w)| \le 1-|\varphi(w)| \le 4\, d(w,\partial\Omega)\cdot|\varphi'(w)|, \quad w\in\Omega. \tag{2.1}$$

This result will be applied in the following situation, wherein the bounds on derivatives can be specified. It is related to the Stolz angle with the vertex at $\zeta_0\in\mathbb{T}$, that is, a domain inside the unit disk of the form

$$\mathcal{S}_A(\zeta_0) := \Big\{ z\in\mathbb{D} : \frac{|z-\zeta_0|}{1-|z|} < A \Big\}, \quad A>1. \tag{2.2}$$

When $\zeta_0=1$, we use the abbreviation $\mathcal{S}_A := \mathcal{S}_A(1)$, see Figure 1. The interior angle of $\mathcal{S}_A$ at 1 equals 2ω, $0<\omega:=\arccos A^{-1}<\pi/2$. The Stolz angles $\{\mathcal{S}_A\}_{A>1}$ form an increasing family of sets which exhaust the unit disk as $A\to\infty$. The boundary of $\mathcal{S}_A$ is denoted by $\partial\mathcal{S}_A$.

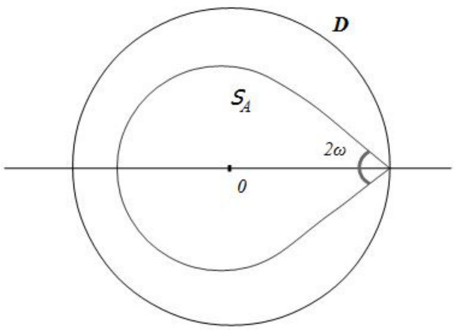

Figure 1. Stolz angle $\mathcal{S}_A=\mathcal{S}_A(1)$, $A>1$, $\omega=\arccos A^{-1}$.

Let φ_A denote the conformal mapping $\varphi_A:\mathcal{S}_A\to\mathbb{D}$, $\varphi_A(0)=0$, $\varphi_A(1)=1$. The following result provides a local uniform bound for its derivative φ_A'.

Lemma 2.2. *Let*

$$\alpha=\alpha_A := \frac{\pi}{2\omega} = \frac{\pi}{2\arccos A^{-1}}, \quad \alpha>1.$$

Then the following bounds hold uniformly for $A\ge 2$:

$$\frac{1}{16} < \frac{|\varphi_A'(z)|}{|z-1|^{\alpha-1}} < 48, \quad z\in\mathcal{S}_A^+ := \mathcal{S}_A \bigcap \Big\{ |z-1| < \frac{1}{16} \Big\}. \tag{2.3}$$

Proof. We just sketch the proof, which is rather standard. Let $\psi\colon \mathbb{D} \to \mathbb{C}_r := \{z : \operatorname{Re} z > 0\}$ be the linear-fractional mapping of $\mathbb{D}$ onto the right half-plane, $\psi(0) = 1$, $\psi(1) = \infty$. A crucial observation is that ψ maps $\mathcal{S}_A$ onto the interior H_i of the right branch of the hyperbola[1]

$$H : \frac{x^2}{\cos^2 \omega} - \frac{y^2}{\sin^2 \omega} = 1, \quad z = x + iy.$$

Set $\mathbb{C}_\pm = \{z : \pm \operatorname{Im} z > 0\}$, $H_\pm = H_i \cap \mathbb{C}_\pm$, and define $\phi_1(z) = z + \sqrt{z^2 - 1} = \exp(\operatorname{arch} z)$, $\phi_1\colon H_\pm \to A_\pm = \{re^{i\theta} : r > 1, \pm\theta \in (0, \omega)\}$, $\phi_2(z) = (z^{\pi/\omega} + z^{-\pi/\omega})/2$, $\phi_2\colon A_\pm \to \mathbb{C}_\pm$, $\phi_3(z) = \sqrt{1+z}$, $\phi_3 : \mathbb{C}_\pm \to \mathbb{C}_\pm \cap \mathbb{C}_r$. Since $\phi_3 \circ \phi_2 \circ \phi_1$ extends continuously to $H_i \cap \mathbb{R}$, we obtain a conformal map $\phi\colon H_i \to \mathbb{C}_r$, $\phi(1) = \sqrt{2}$, $\phi(\infty) = \infty$. Finally, if $\phi_4(z) = (z - \sqrt{2})/(z + \sqrt{2})$, then

$$\varphi_A = \phi_4 \circ \phi \circ \psi = \left(\frac{(1 \pm \sqrt{z})^\alpha - (1 \mp \sqrt{z})^\alpha}{(1 \pm \sqrt{z})^\alpha + (1 \mp \sqrt{z})^\alpha}\right)^2 = 1 - \frac{4(1-z)^\alpha}{((1 \pm \sqrt{z})^\alpha + (1 \mp \sqrt{z})^\alpha)^2}$$

(see also Lavrent'ev–Shabat [21], Section 2.3.36). Next,

$$|\varphi_A'(z)| = \frac{4\alpha|1-z|^{\alpha-1}}{\sqrt{|z|}} \cdot \frac{|(1 \pm \sqrt{z})^\alpha - (1 \mp \sqrt{z})^\alpha|}{|(1 \pm \sqrt{z})^\alpha + (1 \mp \sqrt{z})^\alpha|^3}. \tag{2.4}$$

Since $\alpha \le 3/2$ for $A \ge 2$, the elementary bounds

$$\begin{aligned} &\frac{1}{2} \le \sqrt{|z|} \le 1, \\ &1 \le |(1 \pm \sqrt{z})^\alpha + (1 \mp \sqrt{z})^\alpha| \le 4, \\ &1 \le |(1 \pm \sqrt{z})^\alpha - (1 \mp \sqrt{z})^\alpha| \le 4, \end{aligned}$$

valid for $z \in \mathcal{S}_A^+$, yield (2.3). □

The following simple relation between two Stolz angles is casted as a lemma for convenience only; its elementary proof is omitted.

Lemma 2.3. *Let $A < B$, so $\mathcal{S}_A \subset \mathcal{S}_B$. Then, for $z \in \mathcal{S}_A$,*

$$\frac{B-A}{B+1}(1 - |z|) \le d(z, \partial\mathcal{S}_B) < 1 - |z|. \tag{2.5}$$

For $0 < a < 1$, consider a nested family of domains (curvilinear quadrangles) $\{L_a\}$, see Figure 2,

$$L_{a_1} \subset L_{a_2} \subset \mathbb{D}, \quad 0 < a_1 < a_2 < 1. \tag{2.6}$$

We denote by $\eta = \eta_a$ the conformal mapping of L_a onto $\mathbb{D}$ with normalization $\eta(0) = 0$, $\eta(1) = 1$, and write η_j, $j = 1, 2$ for the domains L_{a_j}. Although there

[1] We thank D. Tulyakov for this remark.

is no explicit formula for η, it is easily seen from Theorem 3.9 in [24] that both η and η' have continuous extensions on the closure $\overline{L}_a$, and so

$$|\eta'(z)| \le c(a), \quad z \in L_a. \tag{2.7}$$

The relations below follow directly from (2.7) and Lemma 2.1. First,

$$1 - |\eta_2(z)| \le c_1(a_2)\,(1 - |z|), \quad z \in L_{a_2}, \tag{2.8}$$

holds with some positive constant $c_1(a_2)$. Next, since 1 is a regular point for η (η is analytic at some neighborhood of 1),

$$c_2(a_2) \le \frac{|1 - \eta_2(z)|}{|1 - z|} \le c_3(a_2), \quad z \in L_{a_2}, \tag{2.9}$$

holds with some positive constants $c_j(a_2)$, $j = 2, 3$. Finally, there are positive constants $c_j = c_j(a_1, a_2)$, $j = 4, 5$, such that

$$c_4(a_1, a_2) \le \frac{1 - |\eta_2(z)|}{1 - |z|} \le c_5(a_1, a_2), \quad z \in L_{a_1}. \tag{2.10}$$

We will exploit these relations later in Section 4.

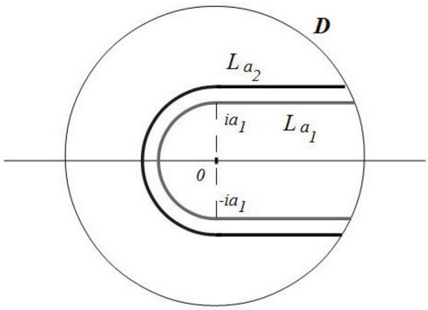

FIGURE 2. Domains $L_{a_1}, L_{a_2}, 0 < a_1 < a_2$.

3. Proof of Theorem 1.4 for $q = 0$

Without loss of generality we assume that $\zeta_0 = 1$. By (1.9),

$$\log|f(z)| \le \frac{2^r K}{(1 - |z|)^p}, \quad z \in \mathbb{D},$$

and, by (1.2), for each $\varepsilon > 0$,

$$\sum_{\zeta \in Z(f)} (1 - |\zeta|)^{p+1+\varepsilon} \le C(p, r, \varepsilon) \cdot K. \tag{3.1}$$

To clarify the local character of the problem, put

$$Z^+(f) := Z(f)\bigcap\Big\{|z-1| < \frac{1}{16}\Big\},\quad Z^-(f) := Z(f)\bigcap\Big\{|z-1| \ge \frac{1}{16}\Big\}, \tag{3.2}$$

so that with $s := \min(p, r)$ we have

$$\sum_{\zeta\in Z(f)} \frac{(1-|\zeta|)^{p+1+\varepsilon}}{|1-\zeta|^s} = \Big\{\sum_{\zeta\in Z^+(f)} + \sum_{\zeta\in Z^-(f)}\Big\} \frac{(1-|\zeta|)^{p+1+\varepsilon}}{|1-\zeta|^s} = \Sigma^+ + \Sigma^-.$$

The bound for Σ^- follows directly from (3.1) and the inequality $|1-\zeta| \ge 1/16$:

$$\Sigma^- = \sum_{\zeta\in Z^-(f)} \frac{(1-|\zeta|)^{p+1+\varepsilon}}{|1-\zeta|^s} \le C(p,r,\varepsilon)\cdot K, \quad p, r \ge 0. \tag{3.3}$$

Thus, the main problem is to prove (1.10) for Σ^+.

3.1. Case $p \le r$

Given a function $f \in \mathcal{A}(\mathbb{D})$ and a number $A \ge 2$, put (see (3.2) and (2.3))

$$Z_A(f) := Z^+(f) \cap \mathcal{S}_A = Z(f) \cap \mathcal{S}_A^+. \tag{3.4}$$

Step 1. Let $\psi_A = \varphi_A^{(-1)}$ be the conformal mapping from $\mathbb{D}$ onto $\mathcal{S}_A$, $\psi_A(0) = 0$. Set $f_A = f(\psi_A)$. Then $f_A \in \mathcal{A}(\mathbb{D})$, $|f_A(0)| = 1$ and by (1.9) we have

$$\log|f_A(w)| \le K\,\frac{|1-\psi_A(w)|^r}{(1-|\psi_A(w)|)^p} \le 2^{r-p}A^p\cdot K, \quad w \in \mathbb{D}.$$

The Poisson–Jensen formula implies

$$\sum_{w\in Z(f_A)} (1-|w|) \le 2^{r-p}A^p\cdot K. \tag{3.5}$$

However, $Z(f_A) = \varphi_A(Z(f)\cap\mathcal{S}_A)$, and so

$$\sum_{w\in Z(f_A)} (1-|w|) = \sum_{\zeta\in Z(f)\cap\mathcal{S}_A} (1-|\varphi_A(\zeta)|) \le 2^{r-p}A^p\cdot K. \tag{3.6}$$

By Lemma 2.1,

$$\sum_{\zeta\in Z(f)\cap\mathcal{S}_A} |\varphi_A'(\zeta)|\cdot d(\zeta,\partial\mathcal{S}_A) \le 2^{r-p+1}A^p\cdot K,$$

and, since $Z(f)\cap\mathcal{S}_A \supset Z_A(f) = Z(f)\cap\mathcal{S}_A^+$, it follows from Lemma 2.2 that, for $A \ge 2$,

$$\begin{aligned}\sum_{\zeta\in Z(f)\cap\mathcal{S}_A} |\varphi_A'(\zeta)|\cdot d(\zeta,\partial\mathcal{S}_A) &\ge \sum_{\zeta\in Z_A(f)} |\varphi_A'(\zeta)|\cdot d(\zeta,\partial\mathcal{S}_A)\\ &\ge \frac{1}{32}\sum_{\zeta\in Z_A(f)} |\zeta-1|^{\alpha-1}\cdot d(\zeta,\partial\mathcal{S}_A).\end{aligned}$$

Hence,

$$\sum_{\zeta\in Z_A(f)} |1-\zeta|^{\alpha-1}\cdot d(\zeta,\partial\mathcal{S}_A) \le 2^{r-p+6}A^p\cdot K, \quad \alpha=\frac{\pi}{2\arccos A^{-1}}\,. \tag{3.7}$$

Step 2. In what follows $A=A_k=2^k$, $k\in\mathbb{N}$, so the Stolz angles $\mathcal{S}_k:=\mathcal{S}_{A_k}$ (with a little abuse of notation) exhaust the unit disk, as $k\to\infty$. Relation (3.7) with $A=A_{k+1}$ takes the form

$$\sum_{\zeta\in Z_{k+1}} |1-\zeta|^{\alpha_{k+1}-1}\cdot d(\zeta,\partial\mathcal{S}_{k+1}) \le 2^{kp+r+6}K, \quad Z_k:=Z_{A_k}(f)=Z^+(f)\cap\mathcal{S}_k,$$

see (3.2), (3.4), or, since $Z_k\subset Z_{k+1}$,

$$\sum_{\zeta\in Z_k} |1-\zeta|^{\beta_{k+1}}\cdot d(\zeta,\partial\mathcal{S}_{k+1}) \le 2^{kp+r+6}K, \quad \beta_{k+1}:=\alpha_{k+1}-1. \tag{3.8}$$

To apply Lemma 2.3 with $A=2^k$, $B=2^{k+1}$, notice that

$$\frac{B-A}{B+1}=\frac{2^{k+1}-2^k}{2^{k+1}+1}\ge\frac{2}{5}\,,$$

so (3.8) entails

$$\sum_{\zeta\in Z_k} (1-|\zeta|)|1-\zeta|^{\beta_{k+1}} \le 5K\cdot 2^{kp+r+5}=C(r)\,2^{kp}\cdot K, \tag{3.9}$$

for $k\in\mathbb{N}$. It is convenient to deal with a chain of inequalities

$$\sum_{\zeta\in Z_k\setminus Z_{k-1}} (1-|\zeta|)|1-\zeta|^{\beta_{k+1}} \le C(r)\,2^{kp}\cdot K, \quad k\in\mathbb{N}, \quad Z_0:=\emptyset.$$

Take an arbitrary $0<\varepsilon<1/16$ and write

$$\frac{1}{2^{k(p+\varepsilon)}}\sum_{\zeta\in Z_k\setminus Z_{k-1}} (1-|\zeta|)|1-\zeta|^{\beta_{k+1}} \le C(r)\,2^{-\varepsilon k}\cdot K. \tag{3.10}$$

On the set $Z_k\setminus Z_{k-1}$ we have

$$2^{-k}<\frac{1-|\zeta|}{|1-\zeta|}\le 2^{-k+1}, \quad \Bigl(\frac{1-|\zeta|}{|1-\zeta|}\Bigr)^{p+\varepsilon}\le\frac{2^{p+\varepsilon}}{2^{k(p+\varepsilon)}}\,,$$

and so

$$\sum_{\zeta\in Z_k\setminus Z_{k-1}} \frac{(1-|\zeta|)^{p+\varepsilon+1}}{|1-\zeta|^{p+\varepsilon}}|1-\zeta|^{\beta_{k+1}} \le C(p,r)\,2^{-\varepsilon k}\cdot K. \tag{3.11}$$

Step 3. We have

$$\alpha_k=\frac{\pi}{2\arccos 2^{-k}}, \quad \beta_k=\alpha_k-1=\frac{\arcsin 2^{-k}}{\arccos 2^{-k}},$$

and as $x\le\arcsin x\le\pi x/2$ for $0\le x\le 1$, and $\arccos 1/2=\pi/3$, we see that

$$\frac{2}{\pi}\le 2^k\beta_k\le\frac{3}{2}\,. \tag{3.12}$$

By definition, $\beta_k \searrow 0$ as $k \to \infty$. Now, choose $k_0 = k_0(\varepsilon)$ from the relations

$$2^{-k_0-1} \le \varepsilon < 2^{-k_0}, \tag{3.13}$$

and hence

$$\mathcal{S}_{k_0} \subset \mathcal{S}_{1/\varepsilon} \subset \mathcal{S}_{k_0+1}, \quad Z_{k_0} \subset Z^+(f) \cap \mathcal{S}_{1/\varepsilon} \subset Z_{k_0+1}\,. \tag{3.14}$$

By (3.12) and (3.13), one has for $k \ge k_0 + 1$

$$\beta_{k+1} \le \beta_{k_0+2} \le \frac{3}{2}\, 2^{-k_0-2} < \frac{3}{4}\,\varepsilon.$$

Let $z \in Z^+(f)$. Since $|1-z| < 1/16$, we see that $|1-z|^{\beta_{k+1}} \ge |1-z|^{\varepsilon}$. Hence, (3.11) implies that

$$\sum_{\zeta \in Z_k \setminus Z_{k-1}} \frac{(1-|\zeta|)^{p+\varepsilon+1}}{|1-\zeta|^p} \le C(p,r)\, 2^{-\varepsilon k} \cdot K, \quad k \ge k_0 + 1. \tag{3.15}$$

Summation over k from $k = k_0 + 1$ to infinity gives

$$\sum_{\zeta \in Z^+(f) \setminus Z_{k_0}} \frac{(1-|\zeta|)^{p+\varepsilon+1}}{|1-\zeta|^p} \le C(p,r,\varepsilon) \cdot K. \tag{3.16}$$

Next, write

$$Z^+(f) = (Z^+(f) \cap \mathcal{S}_{1/\varepsilon}) \bigcup (Z^+(f) \cap \mathcal{S}^c_{1/\varepsilon}), \quad \mathcal{S}^c_{1/\varepsilon} := \mathbb{D} \setminus \mathcal{S}_{1/\varepsilon}.$$

By (3.14), $Z^+(f) \setminus Z_{k_0} \supset Z^+(f) \cap \mathcal{S}^c_{1/\varepsilon}$, so (3.16) provides

$$\sum_{\zeta \in Z^+(f) \cap \mathcal{S}^c_{1/\varepsilon}} \frac{(1-|\zeta|)^{p+\varepsilon+1}}{|1-\zeta|^p} \le C(p,r,\varepsilon) \cdot K. \tag{3.17}$$

On the other hand, put $k = k_0 + 1$ in (3.9). By (3.13), $\beta_{k_0+2} < \varepsilon$, and (3.14) implies that

$$\sum_{\zeta \in Z^+(f) \cap \mathcal{S}_{1/\varepsilon}} (1-|\zeta|)\, |1-\zeta|^{\varepsilon} \le \sum_{\zeta \in Z_{k_0+1}} (1-|\zeta|)\, |1-\zeta|^{\varepsilon} \le C(r,\varepsilon) \cdot K. \tag{3.18}$$

The sum of (3.17) and (3.18) gives

$$\sum_{\zeta \in Z^+(f) \cap \mathcal{S}^c_{1/\varepsilon}} \frac{(1-|\zeta|)^{p+\varepsilon+1}}{|1-\zeta|^p} + \sum_{\zeta \in Z^+(f) \cap \mathcal{S}_{1/\varepsilon}} (1-|\zeta|)\, |1-\zeta|^{\varepsilon} \le C(p,r,\varepsilon) \cdot K, \tag{3.19}$$

and it remains to note again that the inequality $|1-z| \ge 1 - |z|$ for $z \in \mathbb{D}$ implies

$$(1-|z|)\, |1-z|^{\varepsilon} \ge \frac{(1-|z|)^{p+\varepsilon+1}}{|1-z|^p}\,.$$

Finally,

$$\Sigma^+ = \sum_{\zeta\in Z^+(f)} \frac{(1-|\zeta|)^{p+\varepsilon+1}}{|1-\zeta|^p} \le C(p,r,\varepsilon)\cdot K. \tag{3.20}$$

Note that now $p = s = \min(p,r)$. A combination of (3.20) and (3.3) completes the proof of Theorem 1.4 in the case $q = 0$, $p \le r$.

3.2. Case $p > r$

Let $f \in \mathcal{A}(\mathbb{D})$, $|f(0)| = 1$, satisfy

$$\log|f(z)| \le K\,\frac{|1-z|^r}{(1-|z|)^p}\,, \quad z\in\mathbb{D}, \tag{3.21}$$

with $0 \le r < p$. Recalling the notation $f_A = f(\psi_A)$ (see Section 3.1) we have

$$\log|f_A(w)| \le K\,\frac{|1-\psi_A(w)|^r}{\big(1-|\psi_A(w)|\big)^r}\cdot\frac{1}{\big(1-|\psi_A(w)|\big)^{p-r}} \le \frac{K\,A^r}{\big(1-|\psi_A(w)|\big)^{p-r}}\,.$$

By the Schwarz lemma, $|\psi_A(w)| \le |w|$, and so

$$\log|f_A(w)| \le \frac{K\,A^r}{(1-|w|)^{p-r}}\,. \tag{3.22}$$

As above in (3.1), we get for each $\varepsilon > 0$

$$\sum_{w\in Z(f_A)} (1-|w|)^\gamma \le C(p,r,\varepsilon)\,A^r\cdot K, \tag{3.23}$$

where $\gamma = \gamma(p,r,\varepsilon) := p-r+1+\varepsilon$. So we come to (3.5) with exponent γ instead of 1.

The rest is essentially the same as in the argument for the case $p \le r$. For instance, (3.7) becomes

$$\sum_{\zeta\in Z_A(f)} |1-\zeta|^{\gamma(\alpha-1)}\cdot d^\gamma(\zeta,\partial\mathcal{S}_A) \le C(p,r)\,A^r\cdot K, \tag{3.24}$$

and (3.11) turns into

$$\sum_{\zeta\in Z_k\setminus Z_{k-1}} \frac{(1-|\zeta|)^{p+1+2\varepsilon}}{|1-\zeta|^{r+\varepsilon}}\,|1-\zeta|^{\gamma\beta_{k+1}} \le C(p,r)\,2^{-\varepsilon k}\cdot K. \tag{3.25}$$

The choice of k_0 is somewhat different from (3.13):

$$2^{-k_0-1} \le \frac{\varepsilon}{p-r+2} < 2^{-k_0},$$

and again $\gamma\beta_{k+1} \le \varepsilon$ for $k \ge k_0+1$. Thereby we come to

$$\sum_{\zeta\in Z^+(f)\setminus Z_{k_0}} \frac{(1-|\zeta|)^{p+1+2\varepsilon}}{|1-\zeta|^r} \le C(p,r,\varepsilon)\cdot K; \tag{3.26}$$

compare this inequality to (3.16). Finally,

$$\sum_{\zeta\in Z^+(f)} \frac{(1-|\zeta|)^{p+1+2\varepsilon}}{|1-\zeta|^r} \le C(p,r,\varepsilon)\cdot K. \tag{3.27}$$

A combination of (3.27) and (3.3) completes the proof of Theorem 1.4 for $q=0$.

4. Proofs of Theorem 1.4 with $q>0$, Theorems 1.2 and 1.3

We proceed with a local version of the result obtained in Section 3, see also Favorov–Golinskii [10].

Proposition 4.1. *Given the quadrangle L_{a_2} on Figure 2, let $g\in\mathcal{A}(L_{a_2})$ satisfy*

$$\log|g(w)| \le K\,\frac{|1-w|^r}{(1-|w|)^p}\,, \quad w\in L_{a_2}, \quad p,r\ge 0. \tag{4.1}$$

Then for every $\varepsilon>0$ and every $0<a_1<a_2$ there exists a positive constant $C=C(p,r,\varepsilon;a_1,a_2)$ such that

$$\sum_{\zeta\in Z(g)\cap L_{a_1}} \frac{(1-|\zeta|)^{p+1+\varepsilon}}{|\zeta-1|^s} \le C\cdot K, \quad s=\min(p,r). \tag{4.2}$$

Proof. Recall that η_2 stands for the normalized conformal map from L_{a_2} onto $\mathbb{D}$. Put $f:=g\circ\eta_2^{-1}$, so

$$\log|f(z)| \le K\,\frac{|1-\eta_2^{-1}(z)|^r}{(1-|\eta_2^{-1}(z)|)^p}\,, \quad z\in\mathbb{D}.$$

In view of (2.8), (2.9) we have

$$\log|f(z)| \le CK\,\frac{|1-z|^r}{(1-|z|)^p}\,, \quad z\in\mathbb{D}, \quad p,r\ge 0.$$

By the result obtained in Section 3, for every $\varepsilon>0$,

$$\sum_{v\in Z(f)} \frac{(1-|v|)^{p+1+\varepsilon}}{|1-v|^s} = \sum_{\zeta\in Z(g)} \frac{(1-|\eta_2(\zeta)|)^{p+1+\varepsilon}}{|1-\eta_2(\zeta)|^s} \le C(p,r,\varepsilon;a_2)\cdot K, \quad s=\min(p,r),$$

and moreover, for $0<a_1<a_2$,

$$\sum_{\zeta\in Z(g)\cap L_{a_1}} \frac{(1-|\eta_2(\zeta)|)^{p+1+\varepsilon}}{|1-\eta_2(\zeta)|^s} \le C(p,r,\varepsilon;a_1,a_2)\cdot K.$$

The result now follows from (2.9) and (2.10). □

Proof of Theorem 1.4. Recall that, by convention, $\zeta_0 = 1$. To complete the proof of Theorem 1.4, we note that (1.9) implies (4.1) locally inside the domain L_a with $4a = |1-\xi_0|$ and with K replaced by $C(\zeta_0,\xi_0)\cdot K$. Put $\rho := (q-1+\varepsilon)_+$. By Proposition 4.1,

$$\sum_{\zeta\in Z(f)\cap L_{a/2}} (1-|\zeta|)^{p+1+\varepsilon}\frac{|\zeta-\xi_0|^\rho}{|\zeta-1|^s} \le 2^\rho \sum_{\zeta\in Z(f)\cap L_{a/2}} \frac{(1-|\zeta|)^{p+1+\varepsilon}}{|\zeta-1|^s} \le C(p,q,r,\xi_0,\varepsilon)\cdot K. \tag{4.3}$$

On the other hand, condition (1.9) implies the global bound

$$\log|f(z)| \le \frac{2^r K}{(1-|z|)^p\,|z-\xi_0|^q}, \quad z\in\mathbb{D},$$

and so

$$\begin{aligned}\sum_{\zeta\in Z(f)\setminus L_{a/2}} (1-|\zeta|)^{p+1+\varepsilon}\frac{|\zeta-\xi_0|^\rho}{|\zeta-1|^s} &\le C \sum_{\zeta\in Z(f)\setminus L_{a/2}} (1-|\zeta|)^{p+1+\varepsilon}|\zeta-\xi_0|^\rho \\ &\le C \sum_{\zeta\in Z(f)} (1-|\zeta|)^{p+1+\varepsilon}|\zeta-\xi_0|^\rho \le C(p,q,r,\xi_0,\varepsilon)\cdot K. \end{aligned}\tag{4.4}$$

The latter inequality follows from Theorem A. The combination of (4.3) and (4.4) completes the proof of Theorem 1.4. □

Proof of Theorem 1.2. We follow the line of reasoning of the above proof. In view of (1.6) one has the bound, which holds inside the turned quadrangle,

$$L_a(\zeta_i) = \zeta_i\, L_a, \quad a := \frac{1}{2}\min_{1\le j\le n} d(\zeta_j, E\backslash\{\zeta_j\}).$$

Precisely,

$$\log|f(z)| \le C\,K\frac{|z-\zeta_i|^{r_i}}{(1-|z|)^p}, \quad z\in L_a(\zeta_i), \quad i=1,2,\dots,n. \tag{4.5}$$

By Proposition 4.1, for $i=1,2,\dots,n$ and $s_i = \min(p,r_i)$

$$\sum_{\zeta\in Z(f)\cap L_{a/2}(\zeta_i)} \frac{(1-|\zeta|)^{p+1+\varepsilon}}{\prod_{j=1}^n |\zeta-\zeta_j|^{s_j}} \le C\cdot \sum_{\zeta\in Z(f)\cap L_{a/2}(\zeta_i)} \frac{(1-|\zeta|)^{p+1+\varepsilon}}{|\zeta-\zeta_i|^{s_i}} \le C\cdot K. \tag{4.6}$$

On the other hand, if we “ignore” the product in the numerator of (1.6), we get the global bound

$$\log|f(z)| \le \frac{K}{(1-|z|)^p \prod_{k=1}^m |z-\xi_k|^{q_k}}, \quad z\in\mathbb{D},$$

and Theorem 0.2 in [3] gives

$$\sum_{\zeta\in Z(f)}(1-|\zeta|)^{p+1+\varepsilon}\prod_{k=1}^{m}|\zeta-\xi_k|^{(q_k-1+\varepsilon)_+}\le C\cdot K. \tag{4.7}$$

As above, the combination of (4.6) and (4.7) yields (1.7), as claimed. □

Proof of Theorem 1.3. The argument is close to the one above. Within the domain $L_a(\zeta_i)$ with

$$a:=\frac{1}{2}\min_{1\le j\le n} d(\zeta_j, F\cup E\backslash\{\zeta_j\}),$$

the effect of the second factor in the denominator of (1.4) is negligible. Therefore, as above in (4.6), we have, with $s=\min(p,r)$,

$$\sum_{\zeta\in Z(f)\cap L_{a/2}(\zeta_i)}\frac{(1-|\zeta|)^{p+1+\varepsilon}}{d^s(\zeta,E)}\le\sum_{\zeta\in Z(f)\cap L_{a/2}(\zeta_i)}\frac{(1-|\zeta|)^{p+1+\varepsilon}}{|\zeta-\zeta_i|^s}\le C\cdot K. \tag{4.8}$$

The global bound now looks as

$$\log|f(z)|\le\frac{K}{(1-|z|)^p\ d^q(z,F)},\quad z\in\mathbb{D}. \tag{4.9}$$

The Blaschke-type condition for f in (4.9) with $p=0$ is a particular case of Theorem 3 [10]:

$$\sum_{\zeta\in Z(f)}(1-|\zeta|)\ d^\rho(\zeta,F)\le C\cdot K,\quad \rho:=(q-\alpha(F)+\varepsilon)_+. \tag{4.10}$$

There is a standard way to carry the later result over to the case $p>0$, see the proof of Theorem 0.2 in [3]. For the sake of completeness we outline the idea of this method.

Consider the sequence of functions

$$f_n(z):=f(\lambda_n z),\quad \lambda_n:=1-2^{-n},\quad n\in\mathbb{N}.$$

By (4.9) and elementary inequality $d(z,F)\le 2\,d(\lambda_n z,F)$ we have

$$\log|f_n(z)|\le\frac{2^{np+q}K}{d^q(z,F)},\quad z\in\mathbb{D}.$$

The latter is (4.9) with $p=0$, so, in view of (4.10),

$$\sum_{j:|\zeta_j(f)|\le\lambda_{n-1}}(1-|\zeta_j(f_n)|)\ d^\rho(\zeta_j(f_n),F)\le C\,2^{np}\cdot K, \tag{4.11}$$

where $\zeta_j(f)$, $\zeta_j(f_n)$ are the zeros of f and f_n, respectively, so $\zeta_j(f)=\lambda_n\zeta_j(f_n)$.

To obtain the lower bound of the left-hand side in (4.11), we note that $|\zeta_j(f)| \le \lambda_{n-1}$ implies that

$$1 - |\zeta_j(f_n)| = 1 - \frac{|\zeta_j(f)|}{\lambda_n} \ge \frac{1 - |\zeta_j(f)|}{2}, \quad d(\zeta_j(f_n), F) \ge \frac{1}{2}\, d(\zeta_j(f), F),$$

and hence

$$\sum_{\lambda_n < |\zeta_j(f)| \le \lambda_{n+1}} (1 - |\zeta_j(f)|)\ d^{\rho}(\zeta_j(f), F) \le C\, 2^{np} \cdot K.$$

Since now

$$1 - |\zeta_j(f)| < 1 - \lambda_n = 2^{-n}, \quad (1 - |\zeta_j(f)|)^{p+\varepsilon+1} < 2^{-n(p+\varepsilon)},$$

the summation over n leads to

$$\sum_{\zeta \in Z(f)} (1 - |\zeta|)^{p+1+\varepsilon}\ d^{(q-\alpha(F)+\varepsilon)_+}(\zeta, F) \le C \cdot K, \tag{4.12}$$

which is the Blaschke-type condition for the functions f in (4.9) with $p > 0$. Again, a combination of (4.8) and (4.12) gives (1.8), as claimed. □

5. Some further Blaschke-type conditions

5.1. Generalized Stolz domains

There is a seemingly more general form of the Blaschke-type condition (1.5) which states that, under assumption (1.4), for every $0 \le \tau' < \tau < \infty$ there is a positive constant $C = C(E, F, p, q, r, \tau, \tau')$ such that

$$\sum_{\zeta \in Z(f)} (1 - |\zeta|)^{p+1+\tau}\, \frac{d^{(q-1+\tau)_+}(\zeta, F)}{d^{\min(p,r)+\tau'}(\zeta, E)} \le C \cdot K. \tag{5.1}$$

In fact, it is a direct consequence of (1.5) with $\varepsilon = \tau - \tau'$, since

$$\frac{1 - |\zeta|}{d(\zeta, E)} \le 1, \quad \zeta \in \mathbb{D}.$$

However, it turns out that in some instances (5.1) holds with $\tau' = \tau$, see Corollary 5.3.

Recall the notation $\mathcal{S}_A(\zeta_0)$, $\zeta_0 \in \mathbb{T}, A > 0$ introduced in (2.2). In the proof of Theorem 1.4, we actually obtained a little stronger conclusion than the claimed one.

Proposition 5.1. *Let $f \in \mathcal{A}(\mathbb{D})$ be a function satisfying the assumptions of Theorem 1.4. Then for each $0 \le \tau' < \tau < \infty$ and $\varepsilon = \tau - \tau' > 0$ there is a positive* *number* $C_6 = C_6(\zeta_0, \xi_0, p, q, r, \tau, \tau')$ *such that the following condition holds:*

$$\sum_{\zeta\in Z(f)\cap\mathcal{S}^c_{1/\varepsilon}(\zeta_0)} \frac{(1-|\zeta|)^{p+\tau+1}|\zeta-\xi_0|^{(q-1+\tau)_+}}{|\zeta-\zeta_0|^{\min(p,r)+\tau'}} + \sum_{\zeta\in Z(f)\cap\mathcal{S}_{1/\varepsilon}(\zeta_0)} (1-|\zeta|)^{p+1+\varepsilon}|\zeta-\xi_0|^{(q-1+\varepsilon)_+} \le C_6\cdot K. \tag{5.2}$$

Obviously, inequality (5.2) reads as

$$\sum_{\zeta\in Z(f)\cap\mathcal{S}^c_{1/\varepsilon}(\zeta_0)} \frac{(1-|\zeta|)^{p+\tau+1}}{|\zeta-\zeta_0|^{\min(p,r)+\tau'}} + \sum_{\zeta\in Z(f)\cap\mathcal{S}_{1/\varepsilon}(\zeta_0)} (1-|\zeta|)^{p+1+\varepsilon} \le C_6\cdot K.$$

when $q=0$. Of course, the above remark also holds for Theorems 1.1 and 1.2.

To get sharper results we could replace summation along the Stolz angles by that along larger approach domains. For simplicity, we formulate here just the result for one point $\zeta_0 = 1$.

Theorem 5.2. *Let* $f\in\mathcal{A}(\mathbb{D})$, $|f(0)|=1$, *satisfy the growth condition*

$$\log|f(z)| \le \frac{K|1-z|^r}{(1-|z|)^p}, \quad z\in\mathbb{D},$$

where $0<p<r+1$. *Then for each* $\tau>0$ *there is a positive number* $C_7 = C_7(p,r,\tau)$ *such that*

$$\sum_{\zeta\in Z(f),\, \frac{1-|\zeta|}{|1-\zeta|}>|1-\zeta|^\beta} (1-|\zeta|) + \sum_{\zeta\in Z(f),\, \frac{1-|\zeta|}{|1-\zeta|}\le|1-\zeta|^\beta} \frac{(1-|\zeta|)^{p+1+\tau}}{|1-\zeta|^{\min(p,r)+1+\tau}} \le C_7\cdot K, \tag{5.3}$$

where

$$\beta = \begin{cases} 1/(p+\tau), & p\le r;\\ (r+1-p)/(p+\tau), & r<p<r+1.\end{cases}$$

Corollary 5.3. *Under the same conditions,*

$$\sum_{\zeta\in Z(f)} \frac{(1-|\zeta|)^{p+1+\tau}}{|1-\zeta|^{\min(p,r)+\tau}} \le C_8\cdot K.$$

Proof. We use that $(1-|\zeta|)/(|1-\zeta|)\le 1$ and that $1-|\zeta|\le 1$. Then

$$\sum_{\zeta\in Z(f),\, \frac{1-|\zeta|}{|1-\zeta|}>|1-\zeta|^\beta} \frac{(1-|\zeta|)^{p+1+\tau}}{|1-\zeta|^{\min(p,r)+\tau}} \le \sum_{\zeta\in Z(f),\, \frac{1-|\zeta|}{|1-\zeta|}>|1-\zeta|^\beta} (1-|\zeta|).$$

It remains to use (5.3). □

Proof of Theorem 5.2. Let

$$\psi(z) = \frac{1-z}{1+z}, \quad F(z) = f(\psi(z)).$$

Then F is analytic in the right half-plane $\mathbb{C}_r$ and

$$\log|f(z)| \le C'K \cdot \frac{|z|^r}{x^p}, \quad z \in \mathbb{C}_r,\ |z| < C, \tag{5.4}$$

where $z = x + iy$, $C > 1$ is arbitrary and C' depends on p, r, C.

Let $\lambda > 1$ be fixed later on. Consider the domain

$$\Omega_0 = \{x + iy : x > |y|^\lambda\}.$$

Let ϕ_0 be a conformal map of Ω_0 onto $\mathbb{C}_r$ such that $\phi_0(\Omega_0 \cap \mathbb{R}) = \mathbb{C}_r \cap \mathbb{R}$. To obtain good asymptotic information on ϕ_0 at 0, we use the results of Warschawski [28] (see also [24], Theorem 11.16). For some C and C' depending only on λ we obtain

$$0 < C < \frac{|\phi_0(z)|}{|z|} < C' < \infty, \quad z \in \Omega_0,\ |z| < 1. \tag{5.5}$$

Furthermore, the same results show that given $\gamma \in (0, 1]$ we have

$$0 < C < \frac{\operatorname{Re}\phi_0(x+iy)}{x} < C' < \infty, \quad x > \max(\gamma|y|, 2|y|^\lambda),\ |x+iy| < 1, \tag{5.6}$$

with C, C' depending only on γ and λ.

Next, we need a similar estimate for

$$2|y|^\lambda \le x < \gamma|y|, \quad |x+iy| < \frac{1}{4}. \tag{5.7}$$

For $y \in [-1/4, 1/4]$ we consider the points

$$\begin{aligned} A &= |y|^\lambda/2 + i(y - \gamma|y|), & B &= |y|^\lambda/2 + i(y + \gamma|y|),\\ A' &= 3|y|^\lambda/2 + i(y - \gamma|y|), & B' &= 3|y|^\lambda/2 + i(y + \gamma|y|),\\ A'' &= 2|y| + i(y - \gamma|y|), & B'' &= 2|y| + i(y + \gamma|y|) \end{aligned}$$

and the rectangles $ABB''A''$, $A'B'B''A''$, see Figure 3. From now on we fix $\gamma = \gamma(\lambda)$ as the maximal number in $(0, 1]$ such that

$$A'B'B''A'' \subset ABB''A'' \cap \overline{\Omega_0}$$

for all $y \in [-1/4, 1/4]$.

Fix $x + iy$ satisfying (5.7), the corresponding points A, B, A', B', A'', B'' and the rectangles $ABB''A''$, $A'B'B''A''$. Then

$$[A'', B''] \subset \{x + iy : x > \max(\gamma|y|, 2|y|^\lambda),\ |x+iy| < 1\}.$$

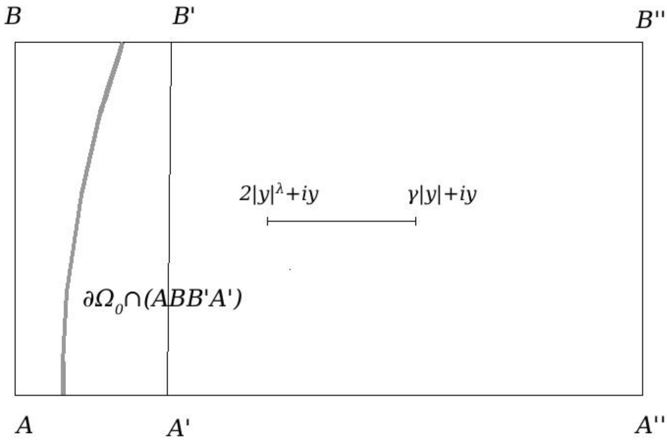

FIGURE 3. Rectangles $ABB''A''$, $A'B'B''A''$, and the part of the boundary $\partial\Omega_0 \cap (ABB'A')$.

Set $u = \operatorname{Re}\phi_0$. By (5.5) and (5.6) we have

$$\begin{aligned} 0 \le u(w) \le C''|y|, &\quad w \in ABB''A'' \cap \Omega_0, \\ C''|y| \le u(w), &\quad w \in [A'', B''], \end{aligned}$$

with C'''s depending only on λ.

Since $0 < x < |y|$,

$$\begin{aligned} d(x+iy, [A', B']) &\ge x/4, \\ d(x+iy, [A', A'']) &= d(x+iy, [B', B'']) = \gamma|y|, \\ d(x+iy, [A'', B'']) &\le 2|y|, \end{aligned}$$

an elementary estimate of harmonic measure shows that

$$\omega(x+iy, [A'', B''], A'B'B''A'') \ge C'' \cdot \frac{x}{|y|},$$

with C'' depending only on λ. Hence,

$$u(x+iy) \ge C \cdot x.$$

Since $0 < x < |y|$,

$$\begin{aligned} d(x+iy, [A, B]) &\le x, \\ d(x+iy, [A, A'']) &= d(x+iy, [B, B'']) = \gamma|y|, \\ d(x+iy, [A'', B'']) &\ge |y|, \end{aligned}$$

another elementary estimate of harmonic measure gives that

$$\omega(x+iy,\partial(ABB''A''\cap\Omega_0)\setminus\partial\Omega_0, ABB''A''\cap\Omega_0) \\ \le \omega(x+iy,\partial(ABB''A'')\setminus[A,B], ABB''A'') \le C''\cdot\frac{x}{|y|},$$

with C'' depending only on λ. Hence,

$$u(x+iy)\le C\cdot x.$$

As a result, we obtain

$$0<C<\frac{\operatorname{Re}\phi_0(x+iy)}{x}<C'<\infty,\quad x\ge 2|y|^\lambda,\ |x+iy|<1, \tag{5.8}$$

with C, C' depending only on λ.

Now, for $n\ge 1$ we define

$$\Omega_n=\{x+iy: x>2^{-n}|y|^\lambda\},$$

and $\phi_n:\Omega_n\mapsto\mathbb{C}_r$,

$$\phi_n(z)=2^{n/(\lambda-1)}\phi_0(2^{-n/(\lambda-1)}z).$$

By (5.5) and (5.8), for some C, C' and for $n\ge 1$, $z\in\Omega_n$, $|z|<1$ we have

$$0<C<\frac{|\phi_n(z)|}{|z|}<C'<\infty,$$
$$0<C<\frac{\operatorname{Re}\phi_n(x+iy)}{x}<C'<\infty.$$

Next, we define

$$G_n=F\circ\phi_n^{-1},\quad n\ge 1.$$

Then $|G_n(\phi_n(1))|=1$. Set $Q=\max(2|\phi_n(1)|,1)$. By (5.4) we have

$$\log|G_n(iy)|\le CK\cdot 2^{np}|y|^{r-\lambda p},\quad y\in[-Q,Q],$$
$$\log|G_n(e^{i\theta}Q)|\le CK\cdot 2^{np},\quad \theta\in[-\pi/2,\pi/2],$$

with C depending only on λ.

From now on we suppose that $r-\lambda p>-1$. By the Poisson–Jensen formula in the right half-disk $\{z\in\mathbb{C}_r: |z|<Q\}$ we obtain that

$$\sum_{G_n(x+iy)=0,\ |x+iy|<1/2} x\le CK\cdot 2^{np},\quad n\ge 1,$$

and hence,

$$\sum_{F(x+iy)=0,\ x>2^{1-n}|y|^\lambda,\ |x+iy|<C} x\le C'K\cdot 2^{np},\quad n\ge 1,$$

with C and C' depending only on λ, p, r. Theorem 1.4 implies that

$$\sum_{F(x+iy)=0,\, x>2^{1-n}|y|^\lambda,\, C\le|x+iy|<1} x \le C'K\cdot 2^{np}, \quad n\ge 1,$$

with C and C' depending only on λ, p, r. Hence,

$$\sum_{F(x+iy)=0,\, x>2^{1-n}|y|^\lambda,\, |x+iy|<1} x \le CK\cdot 2^{np}, \quad n\ge 1,$$

with C depending only on λ, p, r.

Let $\delta > 1$. Then

$$\sum_{F(x+iy)=0,\, x>|y|^\lambda,\, |x+iy|<1} x + \sum_{F(x+iy)=0,\, x\le|y|^\lambda,\, |x+iy|<1} \frac{x^{1+\delta p}}{|x+iy|^{\delta\lambda p}} \le CK,$$

with C depending only on λ, p, r, δ. If $p \le r$, then, given $\tau > 0$, we can choose $\delta = 1+\tau/p$, $\lambda = 1+1/(p+\tau)$ to get

$$\sum_{F(x+iy)=0,\, x>|y|^\lambda,\, |x+iy|<1} x + \sum_{F(x+iy)=0,\, x\le|y|^\lambda,\, |x+iy|<1} \frac{x^{p+1+\tau}}{|x+iy|^{p+1+\tau}} \le CK.$$

If $r < p < r+1$, then, given $\tau > 0$, we can choose $\delta = 1+\tau/p$, $\lambda = (r+1+\tau)/(p+\tau)$ to get

$$\sum_{F(x+iy)=0,\, x>|y|^\lambda,\, |x+iy|<1} x + \sum_{F(x+iy)=0,\, x\le|y|^\lambda,\, |x+iy|<1} \frac{x^{p+1+\tau}}{|x+iy|^{r+1+\tau}} \le CK.$$

Returning to the zeros of f and estimating those far from the point 1 as in the proof of Theorem 1.4 we obtain (5.3). □

5.2. Upper half-plane and plane with a cut

A version of Theorem 1.2 for the upper half-plane looks as follows. We use a convenient notation

$$\{u\}_{c,\varepsilon} := (u_- - 1+\varepsilon)_+ - \min(c, u_+), \quad c\ge 0, \quad \varepsilon > 0.$$

Recall that $u_+ = \max(u,0)$, $u_- = \max(-u,0)$, and so $u = u_+ - u_-$.

Theorem 5.4. *Let $X = \{x_j\}_{j=1}^n$ and $X' = \{x'_k\}_{k=1}^m$ be two disjoint finite sets of distinct points on the real line. Let $g \in \mathcal{A}(\mathbb{C}_+)$, $|g(i)| = 1$, satisfy the growth condition*

$$\log|g(w)| \le K\frac{(1+|w|)^{2b}}{(\operatorname{Im} w)^a}\,\frac{\prod_{j=1}^n |w-x_j|^{c_j}}{\prod_{k=1}^m |w-x'_k|^{d_k}}, \quad w\in\mathbb{C}_+, \tag{5.9}$$

and $a, b, c_j, d_k \geq 0$. *Denote*

$$l := 2a - 2b - \sum_{j=1}^{n} c_j + \sum_{k=1}^{m} d_k = l_+ - l_-.$$

Then for each $\varepsilon > 0$ *there exists a positive number* $C_9 = C_9(X, X', a, b, c_j, d_k, \varepsilon)$ *such that the following Blaschke-type condition holds:*

$$\sum_{\zeta \in Z(g)} \frac{(\operatorname{Im}\zeta)^{a+1+\varepsilon}}{(1+|\zeta|)^{l_1}} \frac{\prod_{k=1}^{m} |\zeta - x_k'|^{(d_k-1+\varepsilon)_+}}{\prod_{j=1}^{n} |\zeta - x_j|^{\min(a,c_j)}} \leq C_9 \cdot K, \tag{5.10}$$

where the parameter l_1 *is defined by the relation*

$$l_1 := 2(a+1+\varepsilon) + \{l\}_{a,\varepsilon} - \sum_{j=1}^{n} \min(a, c_j) + \sum_{k=1}^{m} (d_k - 1 + \varepsilon)_+.$$

Proof. Since the result follows directly from Theorem 1.2, we give only a sketch of the proof. Consider the standard conformal mappings

$$z = z(w) = \frac{w-i}{w+i} : \mathbb{C}_+ \to \mathbb{D}, \quad w = w(z) = i\,\frac{1+z}{1-z} : \mathbb{D} \to \mathbb{C}_+, \tag{5.11}$$

and the following elementary relations between the corresponding quantities in $\mathbb{C}_+$ and $\mathbb{D}$:

$$\frac{2}{1+|w|} \leq |1-z| \leq \frac{2\sqrt{2}}{1+|w|}, \quad \frac{2\operatorname{Im} w}{(1+|w|)^2} \leq 1-|z| \leq \frac{8\operatorname{Im} w}{(1+|w|)^2}.$$

We have

$$|w - x_j| = \frac{2|z-\zeta_j|}{|1-z||1-\zeta_j|}, \quad \frac{2|w-x_j|}{(1+|w|)|x_j+i|} \leq |z-\zeta_j| \leq \frac{2\sqrt{2}\,|w-x_j|}{(1+|w|)|x_j+i|}$$

with $\zeta_j = z(x_j)$. Similar inequalities hold for $|w - x_k'|$ and $|z - z(x_k')|$. Then, we map $\mathbb{C}_+$ onto $\mathbb{D}$ using $w(z)$ defined in (5.11), and rewrite inequality (5.9) in terms of $z \in \mathbb{D}$. To complete, we apply Theorem 1.2 and go back to $\mathbb{C}_+$ using $z(w)$ defined in (5.11). □

In view of applications to the spectral theory we give yet another version of Theorem 1.2 related to the domain $\mathbb{C}\backslash\mathbb{R}_+$.

Theorem 5.5. *Let* $T = \{t_j\}_{j=1}^{n}$ *and* $T' = \{t_k'\}_{k=1}^{m}$ *be two disjoint finite sets of distinct positive numbers. Let* $h \in \mathcal{A}(\mathbb{C}\backslash\mathbb{R}_+)$, $|h(-1)| = 1$, *satisfy the growth condition*

$$\log|h(\lambda)| \leq \frac{K}{|\lambda|^r} \frac{(1+|\lambda|)^b}{d^a(\lambda, \mathbb{R}_+)} \frac{\prod_{j=1}^{n} |\lambda - t_j|^{c_j}}{\prod_{k=1}^{m} |\lambda - t_k'|^{d_k}}, \quad \lambda \in \mathbb{C}\backslash\mathbb{R}_+,$$

and $a, b, c_j, d_k \geq 0$, $r \in \mathbb{R}$. *Denote*

$$s := 3a - 2b + 2r - 2\sum_{j=1}^{n} c_j + 2\sum_{k=1}^{m} d_k = s_+ - s_-.$$

Then for each $\varepsilon > 0$ *there is a positive number* C *which depends on all parameters involved such that the following inequality holds:*

$$\sum_{\zeta \in Z(h)} d^{a+1+\varepsilon}(\zeta, \mathbb{R}_+) \frac{|\zeta|^{s_1}}{(1+|\zeta|)^{s_2}} \cdot \frac{\prod_{k=1}^{m} |\zeta - t'_k|^{(d_k-1+\varepsilon)_+}}{\prod_{j=1}^{n} |\zeta - t_j|^{\min(a,c_j)}} \leq C \cdot K, \tag{5.12}$$

where the parameters s_1 *and* s_2 *are defined by the relations*

$$s_1 := \frac{\{-2r-a\}_{a,\varepsilon} - a - 1 - \varepsilon}{2},$$

$$s_2 := a + 1 + \varepsilon + \frac{\{-2r-a\}_{a,\varepsilon} + \{s\}_{a,\varepsilon}}{2} - \sum_{j=1}^{n} \min(a, c_j) + \sum_{k=1}^{m} (d_k - 1 + \varepsilon)_+ .$$

The result is a direct consequence of Theorem 5.4 applied to the function $g(w) := h(w^2)$, $w \in \mathbb{C}_+$, and the elementary inequalities

$$|w| \operatorname{Im} w \leq d(w^2, \mathbb{R}_+) \leq 2|w| \operatorname{Im} w, \quad w \in \mathbb{C}_+.$$

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