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Critical points of non-regular integral functionals

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Dedicated to Liliane Maia and Marco Degiovanni,
without and with (60) birthday.

Abstract. We prove the existence of a bounded positive critical point for a class of functionals such as

$$J(v) = \frac{1}{2}\int_\Omega [a(x) + b(x)|v|^\gamma]\, |\nabla v|^2 - \int_\Omega |v|^p$$

for Ω a bounded open set in $\mathbb{R}^N$, $N > 2$, $\gamma + 2 < p < 2N/(N-2)$, $\gamma > 0$, $\gamma \neq 1$ and $a(x)$, $b(x)$ measurable function satisfying $0 < \alpha \leq a(x) \leq \beta$, $0 \leq b(x) \leq \beta$ almost everywhere in Ω.

1. Introduction

The existence of critical points for integral functionals defined on the Sobolev space $W_0^{1,r}(\Omega)$ by

$$I(v) = \int_\Omega \mathcal{J}(x, v, Dv), \quad v \in W_0^{1,r}(\Omega),$$

is widely studied. Unfortunately, the differentiability of I can fail even for very simple examples of functionals defined through smooth functions $\mathcal{J}(x, s, \xi)$, so that the Ambrosetti–Rabinowitz theorem cannot be employed. In this paper, we carry on the study of critical points for multiple integrals of the calculus of variations by studying the functional $I\colon W_0^{1,2}(\Omega) \mapsto \mathbb{R} \cup \{+\infty\}$ defined by

$$I(v) = \frac{1}{2}\int_\Omega [a(x) + b(x)|v|^\gamma]\, |\nabla v|^2 - \frac{1}{p}\int_\Omega |v|^p, \tag{1.1}$$

where Ω is a bounded open set in $\mathbb{R}^N$, $N > 2$, γ and p are positive numbers such that

$$0 < \gamma < p - 2 < \frac{4}{N-2}, \tag{1.2}$$

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and a and b are measurable functions satisfying the following condition:

$$0 < \alpha \le a(x) \le \beta, \quad 0 \le b(x) \le \beta. \tag{1.3}$$

The interest in the study of this class of functionals has increased in the last decades also due to the relation with the so called "modified Schrödinger equations". This kind of models arises in different physical phenomena (see [14], [15] and the references therein). Many papers deal with functionals having

$$a(x) \equiv b(x) \equiv 1 \quad \text{and} \quad \gamma = 2. \tag{1.4}$$

The first existence results in the whole $\mathbb{R}^N$ appeared in [23], [18], where constrained minimization arguments are used in the case $\gamma = 2$. Then, taking advantage of (1.4), many authors tackled the problem via suitable changes of variable in order to recover a semi-linear equation to deal with. This kind of strategy has been frequently used in the whole $\mathbb{R}^N$ (see [1], [11], [19], [20] and the references therein). Studying the problem on bounded domains leads to the following quasi-linear variational problem:

$$\begin{cases} -\operatorname{div}(A(x,u)\nabla u) + \dfrac{1}{2}\,\partial_s A(x,u)|\nabla u|^2 = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

with $A(x,s)$ given by

$$A(x,s) = a(x) + b(x)|s|^\gamma. \tag{1.6}$$

Every solution of equation (1.5) can formally be seen as a critical point of the associated action functional I given by (1.1). Unfortunately, I is not well defined in the whole $W_0^{1,2}(\Omega)$ and it is not of class C^1, as $\langle I'(u), v\rangle$ can be computed only for $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and not for all $u \in W_0^{1,2}(\Omega)$.

In the 90's, two different approaches have been introduced to handle general functionals which are not differentiable. An abstract, general theory of critical point for lower semi-continuous functionals has been developed by Marco Degiovanni et al. in [10] and [12]; a version of the Ambrosetti–Rabinowitz theorem (see [2]) for non-differentiable functionals is proved, with a different approach, in [3]. By means of these abstract results it is possible to study the existence of critical points for non-differentiable functionals, and this has been done in the case in which the coefficient in the principal part is uniformly bounded with respect to s in [3], [10], [21], [24]. The case of unbounded coefficients $a(x,s)$ which are differentiable with respect to s (which includes our situation when $\gamma > 1$) has been considered (either exploiting [3] or [12]) in [4], [6], [9], [22] for $p < 2N/(N-2)$. On the other hand, when $b(x) \equiv 1$, it is possible to allow the exponent p to overcome the usual critical Sobolev exponent $2N/(N-2)$. Indeed, the new critical exponent depends on γ and for $a(x,s) = 1 + |s|^\gamma$ it is given by $(\gamma+2)N/(N-2)$. This effect has been seen in $\mathbb{R}^N$ in all the above quoted papers when (1.4) holds, and in bounded domains in [5] (see also [17]) for more general coefficients $a(x,s)$

uniformly bounded from below by $\alpha_0 + \beta_0|s|^\gamma$ for α_0, $\beta_0 \in \mathbb{R}^+$. Here we consider a coefficient $a(x,s)$ given by (1.6), so that it depends on the spatial variable and it is not possible to make a change of variable. Moreover, the coefficient $b(x)$ is only supposed to be greater or equal than zero, so that we cannot expect the exponent p to overcome the Sobolev critical exponent. That is the reason why we have assumed the bound from above on p in (1.2). Our existence result in the model case is stated as follows.

Theorem 1.1. *Assume* (1.2), (1.3). *Then there exists a function* $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $u > 0$ *in* Ω, *such that* $b(x)u^{\gamma-1}|\nabla u|^2 \in L^1(\Omega)$ *and* u *satisfies*

$$\int_\Omega [a(x) + b(x)u^\gamma]\, \nabla u \nabla v + \frac{1}{2}\int_\Omega b(x)u^{\gamma-1}|\nabla u|^2 v - \int_\Omega u^{p-1}v = 0,$$

for every $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Since we will look for positive critical points of I we will prove Theorem 1.1 for the functional $J\colon W_0^{1,2}(\Omega) \mapsto \mathbb{R} \cup \{+\infty\}$ given by

$$J(v) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega a(x,v)|\nabla v|^2 - \dfrac{1}{p}\int_\Omega (v^+)^p, & \text{if } J(v) < +\infty, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.7}$$

We will handle the case $\gamma > 1$ in the Appendix, where we give a new proof of the existence result. Analogously to [6], [9], our argument relies strongly on the knowledge of the boundedness of a weak limit of a "Palais–Smale" sequence, before knowing that it is actually a critical point of J.

Our main interest will however, be to face the case $\gamma \in (0,1)$. In this range of exponents, the situation is completely different since that principal part is still unbounded from above when "u is large", while the derivative is unbounded from above when "u is small". Indeed, in this case we have, at least formally,

$$\langle J'(u), v\rangle = \int_\Omega [a(x) + b(x)(u^+)^\gamma]\nabla u \nabla v + \frac{\gamma}{2}\int_\Omega \frac{b(x)|\nabla u|^2}{(u^+)^{1-\gamma}} v - \int_\Omega (u^+)^{p-1}v,$$

so that it is not enough to differentiate along direction $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, nor to know that $u \in L^\infty(\Omega)$, but it is crucial to show that u is strictly positive. This strong irregularity of J forces us to proceed, as done in [4] for $\gamma > 1$, by approximating J with functionals with a C^1 coefficient $a(x,s)$ in the principal part. We will be able to study the approximating functionals by using the existence result obtained for $\gamma > 1$, and then we will pass to the limit.

2. Small exponents: $\gamma < 1$

In this section we will prove our main result, that is Theorem 1.1 when $\gamma \in (0,1)$.

In order to stress this difference with the case $\gamma > 1$, we will denote γ as θ. In this context, the definition of a critical point of the functional J defined in (1.7) is clarified in the following definition.

Definition 2.1. A function u is a critical point of J if $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $u > 0$ in Ω, $b(x)|\nabla u|^2/u^{1-\theta} \in L^1(\Omega)$ and u satisfies

$$\int_\Omega \left[a(x) + b(x)u^\theta\right] \nabla u \nabla \varphi + \frac{\theta}{2} \int_\Omega b(x) \frac{|\nabla u|^2}{u^{1-\theta}} \varphi = \int_\Omega u^{p-1} \varphi, \tag{2.1}$$

for every $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Then proving Theorem 1.1 is equivalent to show the existence of a critical point of J under the following condition on θ:

$$\theta \in (0,1), \quad \theta + 2 < p < \frac{2N}{N-2}. \tag{2.2}$$

Let us be more precise about the ranges of exponents we will deal with.

Remark 2.2. In hypothesis (2.2) it is implicitly assumed that

$$\theta < \frac{2N}{N-2} - 2 \quad \Longleftrightarrow \quad \theta < \frac{4}{N-2};$$

this condition is always satisfied when $N \leq 6$, because $\theta < 1$. While for $N > 6$, our existence result does not hold for every $\theta \in (0,1)$ but only for $\theta \in (0, 4/(N-2))$.

When $p \in (2, \theta + 2)$ the situation is more delicate even if $b(x) \equiv 1$, as illustrated in Theorem 4.2 in [5] (see also [17]).

As already explained, we will proceed by approximating our functional J by a sequence of C^1 functionals to which it is possible to apply the result of the Appendix. Namely, for every $\delta > 0$, let us define the functional $J_\delta \colon W_0^{1,2}(\Omega) \mapsto \mathbb{R} \cup \{+\infty\}$ by

$$J_\delta(v) = \begin{cases} \dfrac{1}{2} \displaystyle\int_\Omega a_\delta(x,v)|\nabla v|^2 - \dfrac{1}{p} \displaystyle\int_\Omega (v^+)^p & \text{if } J_\delta(v) < +\infty, \\ +\infty & \text{otherwise,} \end{cases} \tag{2.3}$$

where the function $a_\delta(x,s) : \Omega \times \mathbb{R} \mapsto \mathbb{R}^+$ is defined by

$$a_\delta(x,s) = a(x) + b(x)\left[(\delta + s^+)^\theta + \frac{\delta\theta}{(1-\theta)(\delta + s^+)^{1-\theta}}\right]. \tag{2.4}$$

The function $a_\delta(x,s)$ is measurable with respect to $x \in \Omega$ and C^1 with respect to s and its partial derivative is given by

$$\partial_s a_\delta(x,s) = b(x) \frac{\theta s^+}{(\delta + s^+)^{2-\theta}}. \tag{2.5}$$

Taking into account hypotheses (1.3) and (2.2), one observes that $a_\delta(x,s)$ satisfies hypotheses (A.2) and (A.3) with $\beta(s)$ given by

$$\beta(s) := \beta\, \xi_\delta(s), \quad \text{with } \ \xi_\delta(s) = 1 + (\delta + s^+)^\theta + \delta^\theta \frac{\theta}{(1-\theta)}. \tag{2.6}$$

Concerning the derivative of $a_\delta(x,s)$ with respect to s, condition (A.4) is satisfied with $\eta(s)$ given by

$$\eta(s) := \beta\,\theta(\delta)^{\theta-2}s^+. \tag{2.7}$$

Moreover, notice that for every $s\in\mathbb{R}$ the assumptions (1.3), and (2.2) imply that $a_\delta(x,s)$ satisfies condition (A.5). Indeed, for for $s\le 0$, (A.5) is a direct consequence of (2.5) as

$$(p-2)a_\delta(x,s)-\partial_s a_\delta(x,s)s=(p-2)a_\delta(x,s)\ge(p-2)\alpha; \tag{2.8}$$

on the other hand, for $s>0$, (2.2) and (1.3) imply

$$(p-2)a_\delta(x,s)-\partial_s a_\delta(x,s)s$$
$$\ge(p-2)\alpha+\frac{b(x)}{(\delta+s)^{2-\theta}}\left\{(p-2)\Big[\ (\delta+s)^2+\frac{\delta\theta}{1-\theta}(\delta+s)\ \Big]-\theta s^2\right\}\ge(p-2)\alpha.$$

Finally, hypothesis (A.6) is also satisfied.

Remark 2.3. One may think that the natural approximating functional should have the coefficient

$$a_\delta(x,s)=a(x)+b(x)(\delta+s^+)^\theta.$$

But in this case $a_\delta(x,s)$ would not satisfy hypothesis (A.6).

2.1. Proof of Theorem 1.1

In order to prove Theorem 1.1, it is possible to use the existence results in [6], [9], and [22], or Theorem A.3, obtaining the existence of a positive, bounded, critical point u_δ of J_δ. Namely, u_δ satisfies

$$u_\delta\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega),\ u_\delta\ge 0,\ J_\delta(u_\delta)=c_\delta>0, \tag{2.9}$$

$$\int_\Omega a_\delta(x,u_\delta)\nabla u_\delta\nabla v+\int_\Omega\partial_s a_\delta(x,u_\delta)|\nabla u_\delta|^2v=\int_\Omega u_\delta^{p-1}v, \tag{2.10}$$

for every $v\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$.

In the next proposition we derive uniform bounds on c_δ and on the norm of u_δ in $W_0^{1,2}(\Omega)$ with respect to δ . Before starting the proof of Theorem 1.1, let us observe that, in the whole paper, we will denote the norm of $W_0^{1,2}(\Omega)$ with $\|\cdot\|$, while the norm in the Lebesgue space $L^p(\Omega)$ will be denoted with $\|\cdot\|_p$. Moreover, for every measurable set A, we will denote with $|A|$ the Lebesgue measure of A.

Proposition 2.4. *There exist three positive constant $\sigma_1<\sigma_2$ and L, such that*

$$0<\sigma_1\le c_\delta\le\sigma_2, \tag{2.11}$$

$$\|u_\delta\|\le L. \tag{2.12}$$

Proof. Hypothesis (1.3) and definition (2.4) imply that $J_\delta(v)\ge\frac{\alpha}{2}\|v\|^2-\frac{1}{p}\|v\|_p^p$, so that, from Sobolev embedding theorem one deduces that, for every v with norm $\|v\|=R$, it results

$$J_\delta(v)\ge R^2\left[\frac{\alpha}{2}-CR^{p-2}\right]=\frac{\alpha R^2}{4},\quad\text{with } R=\left(\frac{\alpha}{4C}\right)^{1/(p-2)}.$$

Then the left-hand side inequality in (2.11) is satisfied for $\sigma_1 = \alpha R^2/4$. In order to prove the right-hand side inequality we note that (A.8) implies that c_δ is smaller than the maximum of $J_\delta(t\,\varphi_1)$ for $t \in [0,T]$. Then, hypothesis (2.6) yields

$$c_\delta \le J_\delta(t\,\varphi_1) \le \frac{t^2}{2p}\left[p\beta\xi_\delta(\|\,\varphi_1\|_\infty)\|\,\varphi_1\|^2 - 2t^{p-2}\|\,\varphi_1\|_p^p\right]$$

so that

$$c_\delta \le \sigma_2 := \max_{[0,T]} g(t), \quad \text{with } g(t) := t^2\left[\beta\xi_\delta(\|\,\varphi_1\|_\infty)\|\,\varphi_1\|^2 - t^{p-2}\|\,\varphi_1\|_p^p\right].$$

To show (2.12), we compute $pJ_\delta(u_\delta) - \langle J'_\delta(u_\delta), u_\delta\rangle$ and use (2.8), (2.9) and (2.10) to obtain

$$2pc_\delta = \int_\Omega [(p-2)a_\delta(x,u_\delta) - \partial_s a_\delta(x,u_\delta)u_\delta]\,|\nabla u_\delta|^2 \ge (p-2)\alpha\|u_\delta\|^2.$$

Then (2.11) gives the conclusion. □

The next proposition is the key result in the compactness argument. It has also been employed in [6],[9] in the case of unbounded, regular coefficients, and in the Appendix we will give a self-contained prove in that context (see the proof of (A.15)).

Every u_δ is a bounded function; now we prove that the sequence $\{u_\delta\}$ is bounded in $L^\infty(\Omega)$. In doing this, we will often use the functions T_k, $G_k : \mathbb{R} \mapsto \mathbb{R}$ defined, for every $k > 0$, as

$$T_k(s) = \max(-k, \min(k,s)), \quad G_k(s) = s - T_k(s). \tag{2.13}$$

Proposition 2.5. *There exists a positive constant M such that*

$$\|u_\delta\|_\infty \le M. \tag{2.14}$$

Proof. Condition (1.2) allows us to choose a positive number λ such that

$$\max\left\{0, p - \frac{N+2}{N-2}\right\} < \lambda < \min\left\{p-1, 2-(p-1)\frac{N-2}{N+2}\right\}. \tag{2.15}$$

Taking $G_k(u_\delta)$ as test function in (2.10) and using Sobolev inequality, yields

$$\alpha\mathcal{S}\Big[\int_\Omega G_k(u_\delta)^{2^*}\Big]^{2/2^*} \le \|u_\delta\|_\infty^\lambda \int_\Omega {u_\delta}^{(p-1-\lambda)} G_k(u_\delta).$$

Applying Hölder's inequality on the right-hand side with exponents

$$\frac{2^*}{p-1-\lambda}, \quad 2^*, \quad \Big(1 - \frac{p-\lambda}{2^*}\Big)^{-1}$$

one obtains

$$\alpha\mathcal{S}\Big[\int_\Omega G_k(u_\delta)^{2^*}\Big]^{2/2^*} \le \|u_\delta\|_\infty^\lambda\,\|u_\delta\|_{2^*}^{p-\lambda-1}\Big[\int_\Omega G_k(u_\delta)^{2^*}\Big]^{1/2^*}|A_k|^{1-(p-\lambda)/2^*},$$

where $A_k = \{x \in \Omega : u_\delta(x) > k\}$.

From (2.12) and using the Hölder inequality one deduces that

$$\int_\Omega G_k(u_\delta) \le \Big[\int_\Omega G_k(u_\delta)^{2^*}\Big]^{1/2^*} |A_k|^{1-1/2^*} \le B_\delta\, |A_k|^{1-(p-\lambda)/2^*+1/2+1/N}, \tag{2.16}$$

where

$$B_\delta = C_L\, \|u_\delta\|_\infty^\lambda. \tag{2.17}$$

It is easy to see that the function

$$y(k) := \|G_k(u_\delta)\|_1 = \int_\Omega G_k(u_\delta) \tag{2.18}$$

satisfies $y'(k) = -|A_k|$. Here we follow [16] and we write inequality (2.16) as

$$y \le\ B_\delta(-y')^{1+\mu}, \quad \mu = -\frac{(p-\lambda)}{2^*} + \frac{1}{2} + \frac{1}{N}. \tag{2.19}$$

From (2.15) we get that $p-\lambda < (N+2)/(N-2)$, so that $\mu > 0$ and $1/(\mu+1) \in (0,1)$. Integrating for $t \in [0,k]$ we get

$$k \le\ C_1\, B_\delta^{\frac{1}{1+\mu}}\, [y(0)^{\frac{\mu}{1+\mu}} - y(k)^{\frac{\mu}{1+\mu}}]$$

and, taking into account (2.18) and (2.12) one obtains

$$C_1\, B_\delta^{\frac{1}{1+\mu}} \Big[\int_\Omega G_k(u_\delta)\Big]^{\frac{\mu}{1+\mu}} \le -k + B_\delta^{\frac{1}{1+\mu}}\, \tilde C_L.$$

When we choose $k = k_\delta = B_\delta^{1/\mu+1} \tilde C_L$, we obtain $y(k_\delta) = 0$, that is equivalent to say that

$$0 \le u_\delta \le k_\delta = B_\delta^{1/\mu+1}\, \tilde C_L.$$

Passing to the supremum and using (2.17) yield

$$0 \le \|u_\delta\|_\infty \le \|u_\delta\|_\infty^{\lambda/(\mu+1)}\, \overline{C}_L. \tag{2.20}$$

Finally, (2.15) implies that $\lambda < \mu + 1$, because, from (2.19) it follows that this is equivalent to

$$\lambda < \frac{2N}{N+2} - p\frac{N-2}{N+2} + 1 = 2 - (p-1)\frac{N-2}{N+2} \tag{2.21}$$

which is exactly assumed in (2.15). Then (2.20) yields (2.14). □

Propositions 2.4 and 2.5 imply that there exists a function u in $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $u \ge 0$, such that, up to a subsequence,

$$u_\delta \rightharpoonup u \ \text{ weakly in } W_0^{1,2}(\Omega), \tag{2.22}$$

$$u_\delta \to u \ \text{ strongly in } L^q(\Omega) \text{ for } q \in [1,\infty) \text{ and almost everywhere.} \tag{2.23}$$

Because of the presence of a singular term in the derivative of J, we need to know that $u > 0$ in Ω in order to show that u is a critical point of J. This will be proved in the next proposition, the proof of which relies on an argument similar to [7].

Proposition 2.6. *The weak limit u is positive in Ω.*

Proof. Let us define the real function $H_\delta : [0, +\infty) \mapsto [0, +\infty)$ by

$$H_\delta(t) = \int_0^t \frac{s}{(\delta + s)^{2-\theta}}\, ds, \tag{2.24}$$

and consider $v = e^{-bH_\delta(u_\delta)}\phi$, with $\phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\phi \geq 0$, and b chosen as $b = \theta\beta/2\alpha$. Notice that Proposition 2.5 yields

$$|\nabla v| = \left|e^{-bH_\delta(u_\delta)}\left[-bH'_\delta(u_\delta)\nabla u_\delta \phi + \nabla\phi\right]\right| \leq |\nabla\phi| + \frac{bM}{\delta^{2-\theta}}\|\phi\|_\infty |\nabla u_\delta|,$$

so that $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and we can take it as test function in (2.10), obtaining

$$\begin{aligned} &\int_\Omega a_\delta(x, u_\delta)\nabla u_\delta \nabla\phi\, e^{-bH_\delta(u_\delta)} \\ &= \int_\Omega u_\delta^{p-1} e^{-bH_\delta(u_\delta)}\phi + \int_\Omega |\nabla u_\delta|^2 e^{-bH_\delta(u_\delta)}\phi\Big[ba_\delta(x, u_\delta)H'_\delta(u_\delta) - \frac{1}{2}\partial_s a_\delta(x, u_\delta)\Big]. \end{aligned} \tag{2.25}$$

Since $H'_\delta(u_\delta) > 0$, from (1.3), (2.5) and (2.24), it follows

$$\begin{aligned} ba_\delta(x, u_\delta)H'_\delta(u_\delta) - \frac{1}{2}\partial_s a_\delta(x, u_\delta) &\geq \alpha b H'_\delta(u_\delta) - \frac{\theta}{2}\frac{\beta u_\delta}{(\delta + u_\delta)^{2-\theta}} \\ &= \frac{u_\delta}{(\delta + u_\delta)^{2-\theta}}\left[\alpha b - \frac{\theta}{2}\beta\right] = 0, \end{aligned} \tag{2.26}$$

where the last equality follows from the choice of b. Using this information in (2.25) one gets

$$\int_\Omega a_\delta(x, u_\delta)\nabla u_\delta \nabla\phi\, e^{-bH_\delta(u_\delta)} \geq \int_\Omega u_\delta^{p-1} e^{-bH_\delta(u_\delta)}\phi. \tag{2.27}$$

Condition (2.4) and (2.23) imply that $a_\delta(x, u_\delta)$ converges to $a(x) + b(x)u^\theta$ almost everywhere in Ω. Moreover, taking into account Proposition 2.5 and that $u_\delta \geq 0$, one can apply Lebesgue dominated convergence theorem to get that $a_\delta(x, u_\delta)\nabla\phi\, e^{-bH_\delta(u_\delta)}$ converges strongly to $\left[a(x) + b(x)u^\theta\right]\nabla\phi\, e^{-bH_0(u)}$ in $L^2(\Omega)$, where $H_0(t)$ is defined in (2.24) choosing $\delta = 0$. Then, taking limit in (2.27) we obtain

$$\int_\Omega [a(x) + b(x)\, u^\theta] e^{-bH_0(u)}\, \nabla u\, \nabla\phi \geq \int_\Omega u^{p-1} e^{-bH_0(u)}\phi. \tag{2.28}$$

Define

$$w := P(u), \quad \text{where} \quad P(s) = \int_0^s e^{-bH_0(t)} dt,$$

so that $w \in H_0^1(\Omega)$ is a super-solution of the linear problem

$$\begin{cases} -\mathrm{div}(B(x)\nabla w) = g(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.29}$$

with

$$B(x) = a(x) + b(x)(u(x))^\theta, \quad g(x) = (u(x))^{p-1} e^{-bH_0(u(x))}.$$

Condition (1.3), Proposition 2.5 and the positiveness of u imply that

$$\alpha \leq B(x) \leq \beta[1 + M^{\theta}], \quad 0 \leq g(x) \in L^{\infty}(\Omega).$$

Therefore, the strong maximum principle for linear operators with bounded coefficients (see [13]) implies that, either $w > 0$ in Ω or $w \equiv 0$. Assume, by contradiction, that $w \equiv 0$. Then, as P is a strictly increasing function, it follows that $u \equiv 0$ and (2.23) becomes

$$u_\delta \to 0 \text{ strongly in } L^q(\Omega) \text{ for } q \in [1, \infty) \text{ and a.e..} \tag{2.30}$$

Let us take $v = u_\delta$ as test function in (2.10) and use (2.5) and the positivity of u_δ to obtain

$$J_\delta(u_\delta) + \frac{1}{p}\int_\Omega (u_\delta)^p = \frac{1}{2}\int_\Omega a_\delta(x, u_\delta)|\nabla u_\delta|^2 \leq \frac{1}{2}\int_\Omega u_\delta^p.$$

Taking into account (2.30), we deduce that $J_\delta(u_\delta) \to 0$ which contradicts (2.11). Then, $w > 0$ and, as the map $s \mapsto P(s)$ is strictly increasing, $u > 0$ in Ω. □

Remark 2.7. Notice that, since $w > 0$, the strong maximum principle for linear operators with bounded coefficients also states that, for every compact set $K \subset \Omega$, there exists $m_K > 0$ such that $w(x) \geq m_K > 0$ on$K$; and then $u(x) \geq P^{-1}(m_K) > 0$ on K.

As a consequence of the previous results we have obtained that u belongs to $W_0^{1,2}(\Omega) \cap L^{\infty}(\Omega)$ and it is positive in Ω. It is left to show that it satisfies all the requirements in Definition 2.1.

Lemma 2.8. *The sequence $\nabla u_\delta(x)$ converges almost everywhere to $\nabla u(x)$.*

Proof. First of all, we show that the following estimate holds:

$$0 \leq \frac{\theta}{2}\int_\Omega \frac{b(x)\, u_\delta}{(\delta + u_\delta)^{2-\theta}}\, |\nabla u_\delta|^2 \leq \int_\Omega (u_\delta)^{p-1} \leq M^{p-1}|\Omega|. \tag{2.31}$$

Indeed, following [7], let $h > 0$ and choose $v = T_h(u_\delta)/h$ in (2.10). Using (1.3) and (2.13), we obtain, dropping a positive term,

$$\frac{1}{2}\int_\Omega \partial_s a_\delta(x, u_\delta)\, |\nabla u_\delta|^2\, \frac{T_h(u_\delta)}{h} \leq \int_\Omega u_\delta^{p-1}.$$

Letting h tend to zero and exploiting (2.5), (2.13), and (2.23) yield (2.31). Now, u_δ is a distributional solution of the equation

$$-\mathrm{div}\Big(\Big[a(x) + b(x)(\delta + u_\delta)^{\theta} + \frac{\delta\theta}{(1-\theta)(\delta + u_\delta)^{1-\theta}}\Big]\nabla u_\delta\Big) = f_\delta + g_\delta,$$

where $f_\delta = u_\delta^{p-1}$ converges strongly in $W^{-1,p'}(\Omega)$, and (2.31) says that $g_\delta = b(x)\, u_\delta(\delta + u_\delta)^{\theta - 1}|\nabla u_\delta|^2$ is a sequence bounded in $L^1(\Omega)$. Condition (1.3), and Proposition 2.5 yield

$$a(x) + b(x)(\delta + u_\delta)^{\theta} + \frac{\delta\theta}{(1-\theta)(\delta + u_\delta)^{1-\theta}} \leq \beta\Big[1 + (\delta + M)^{\theta} + \frac{\theta\delta^{\theta}}{1-\theta}\Big],$$

then, we can apply Theorem 2.1 in [8] to achieve the conclusion. □

Proposition 2.9. *The weak limit u is a critical point of J.*

Proof. Applying Fatou's lemma in (2.31), it follows

$$\frac{\theta}{2}\int_\Omega \frac{b(x)}{u^{1-\theta}}|\nabla u|^2 \le M^{p-1}|\Omega|, \tag{2.32}$$

as requested in Definition 2.1. Now, let us take $v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, with $v \ge 0$ as test function in (2.10), and obtain

$$\frac{1}{2}\int_\Omega \partial_s a_\delta(x,u_\delta)|\nabla u_\delta|^2 v = -\int_\Omega a(x,u_\delta)\nabla u_\delta \nabla v + \int_\Omega u_\delta^{p-1} v.$$

Note that the sequence in the integral in the left-hand side is positive and, thanks to (2.23) and Lemma 2.8, it converges almost everywhere; in addition, in view of (2.2), (2.6), (2.22) and (2.14), both the integral terms in the right-hand side converge. As a consequence, applying Fatou's lemma, it results

$$\int_\Omega \left[a(x)+b(x)u^\theta\right]\nabla u \nabla v + \frac{\theta}{2}\int_\Omega \frac{b(x)}{u^{1-\theta}}|\nabla u|^2 v \le \int_\Omega u^{p-1} v, \tag{2.33}$$

for every $v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, $v \ge 0$.

In order to prove the reverse inequality in (2.33), let us consider the function $v = e^{-bH_\delta(u_\delta)}\, e^{bH_0(u)}\,\phi$, where H_δ is defined in (2.24), $b = \theta\beta/2\alpha$ and $\phi \ge 0$, $\phi \in C_c^\infty(\Omega)$. Proposition 2.5 yields the existence of a constant C_M such that

$$|\nabla v|^2 \le C_M\left\{|\nabla \phi|^2 + b^2\phi^2\frac{|\nabla u|^2}{u^{2(1-\theta)}} + \|\phi\|_\infty^2 b^2|\nabla u_\delta|^2\left[\frac{u_\delta}{(\delta+u_\delta)^{2-\theta}}\right]^2\right\}.$$

Thanks to Proposition 2.6 and to Remark 2.7, we have that $u \ge C_\phi > 0$ in the support of ϕ, so that $v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, and it can be chosen as test function in (2.10). It results

$$\begin{aligned} &b\int_\Omega a_\delta(x,u_\delta)\nabla u_\delta \nabla u e^{-bH_\delta(u_\delta)}\, e^{bH_0(u)}\,\phi H_0'(u) \\ &\quad + \int_\Omega a_\delta(x,u_\delta)\nabla u_\delta \nabla\phi e^{-bH_\delta(u_\delta)}\, e^{bH_0(u)} \\ &= \int_\Omega |\nabla u_\delta|^2\, e^{-bH_\delta(u_\delta)}\, e^{bH_0(u)}\,\phi\Big[b\, a_\delta(x,u_\delta)H_\delta'(u_\delta) - \frac{1}{2}\partial_s a_\delta(x,u_\delta)\Big] \\ &\quad + \int_\Omega u_\delta^{p-1} e^{-bH_\delta(u_\delta)}\, e^{bH_0(u)}\,\phi. \end{aligned} \tag{2.34}$$

Inequality (2.26) and Lemma 2.8 imply that we can use Fatou's lemma to pass to the limit in the right-hand side, while the left-hand side converges thanks to conditions (2.22), (2.23) and to Proposition 2.5. It follows

$$\begin{aligned} &\int_\Omega \left[a(x)+b(x)\,u^\theta\right]\nabla u\left[b\nabla u\,\phi H_0'(u) + \nabla\phi\right] \\ &\qquad \ge \int_\Omega |\nabla u|^2\,\phi\Big[\left[a(x)+b(x)u^\theta\right]bH_0'(u) - \frac{\theta}{2}\frac{b(x)}{u^{1-\theta}}\Big] + \int_\Omega u^{p-1}\,\phi. \end{aligned} \tag{2.35}$$

Proposition 2.5 and (2.32) imply that the function

$$\big[a(x)+b(x)u^{\theta}\big]\,|\nabla u|^2 H_0'(u)=\big[a(x)+b(x)u^{\theta}\big]\,|\nabla u|^2\frac{1}{u^{1-\theta}}$$

belongs to $L^1(\Omega)$. Thus we can simplify (2.35) to obtain

$$\int_\Omega \big[a(x)+b(x)u^{\theta}\big]\nabla u\nabla\phi+\frac{\theta}{2}\int_\Omega b(x)\,\frac{|\nabla u|^2}{u^{1-\theta}}\,\phi\geq\int_\Omega u^{p-1}\,\phi.$$

From this and from (2.33) we get that the equality

$$\int_\Omega \big[a(x)+b(x)u^{\theta}\big]\nabla u\nabla\phi+\frac{\theta}{2}\int_\Omega b(x)\,\frac{|\nabla u|^2}{u^{1-\theta}}\,\phi=\int_\Omega u^{p-1}\,\phi \tag{2.36}$$

holds for every $\phi\in C_c^\infty(\Omega)$, with $\phi\geq 0$. Then, we can obtain that (2.36) holds for every $\phi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ with $\phi\geq 0$ by density. Finally, for every $\varphi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, writing $\varphi=\varphi^+-\varphi^-$ and using (2.36) with $\phi=\varphi^+$ and with $\phi=\varphi^-$, one obtains that u satisfies all the requirements in Definition 2.1, i.e., u is a critical point of J □

Remark 2.10. Notice that, as a byproduct of the previous results, we obtain the strong convergence in $W_0^{1,2}(\Omega)$ of u_δ to u.

Proof of Theorem 1.1 for $\gamma\leq 1$. When $\gamma<1$ we apply Theorem A.3 to the functional J_δ obtaining u_δ satisfying (2.9) and (2.10). Propositions 2.4, 2.5 and 2.6 imply that there exists $u\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, $u>0$ in Ω, such that (2.22) and (2.23) hold. Finally, from Proposition 2.9 it follows that u is a critical point in terms of the Definition 2.1.

The case $\gamma=1$ can be handled with the same argument, by using the approximating functionals

$$I_\delta(v)=\frac{1}{2}\int_\Omega\big[a(x)+b(x)\sqrt{\delta+(v^+)^2}\,\big]\,|\nabla v|^2-\frac{1}{p}\int_\Omega(v^+)^p.$$

Indeed, all the hypotheses of Theorem A.3 are satisfied choosing

$$\beta(s)=\beta\big[\,1+\sqrt{\delta+(s^+)^2}\,\big],\quad \eta(s)=s^+\delta^{-1/2}.$$

Propositions 2.4, 2.5 and Lemma 2.8 follow in the same way as before and Propositions 2.6 and 2.9 can be proved by means of the function $K_\delta(t)\colon[0,+\infty)\mapsto[0,+\infty)$ defined as $K_\delta(t):=\sqrt{\delta+t^2}$. □

A. Appendix

In this section we will handle the case in which $a(x,s)$ is a C^1 function with respect to s, not supposed to be uniformly bounded for every $s\in\mathbb{R}$.

Functionals with this kind of coefficients has been studied in [4], [5], [6], [9], [22], by means of different techniques, here we will report for the reader's convenience an alternative, new, proof. In doing this we will be able to underline the analogies and differences with the case of singular coefficients. As done in the introduction, let us consider the functional $J\colon W_0^{1,2}(\Omega) \to \mathbb{R}\cup\{+\infty\}$ defined in (1.7). In order to prove Theorem 1.1, with $\gamma > 1$, it would be enough to consider $a(x,s)$ defined by

$$a(x,s) = a(x) + b(x)(s^+)^\gamma \tag{A.1}$$

so that $\partial_s a(x,s) \equiv 0$ for every $s < 0$, and it is clear that a positive critical point of J will be a positive critical point of I. However, in this section, we will consider more general coefficients, in order to obtain a critical point u_δ of the approximating functional J_δ. Since we look for positive solutions, we consider the coefficient $a(x,s)$ defined only for $s \geq 0$, i.e., $a\colon \Omega \times \mathbb{R}^+ \mapsto \mathbb{R}$ such that it is measurable with respect to $x \in \Omega$, and continuously derivable with respect to s, and satisfying the following conditions for almost every x in Ω and for every s in $\mathbb{R}^+$, for $\beta, \eta, \in C^0(\mathbb{R}^+)$ monotone increasing, and α, $\delta \in \mathbb{R}^+$:

$$0 < \alpha \leq a(x,s) \leq \beta(s), \tag{A.2}$$

$$\lim_{s\to+\infty} \beta(s) - s^{p-2} < 0, \quad 2 < p < \frac{2N}{N-2}, \tag{A.3}$$

$$a_s(x,s)\, s \geq 0, \quad |\partial_s a(x,s)| \leq \eta(s), \tag{A.4}$$

$$(p-2)a(x,s) - a_s(x,s)s \geq \delta > 0, \tag{A.5}$$

$$\lim\nolimits_{s\to 0^+} \partial_s a(x,s) = 0. \tag{A.6}$$

We define, for $s < 0$,

$$a(x,s) \equiv a(x,0) \ \Rightarrow a'_s(x,s) \equiv 0, \quad \forall\ s \leq 0. \tag{A.7}$$

Remark A.1. Notice that our model case, namely, the function defined in (A.1), satisfies all the above assumptions when (1.3) holds and the following condition is satisfied

$$\gamma \in (1, +\infty), \quad \gamma + 2 < p < \frac{2N}{N-2}$$

which implies $N < 6$. In order to treat the case $N \geq 6$, one should allow p to overcome the critical Sobolev exponent. This case for our functional I is still open.

Conditions (A.2), (A.4) imply that J is derivable along directions $v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$, so that a natural definition of a critical point is the following.

Definition A.2. A function u is a critical point of J if $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and it satisfies

$$\int_\Omega a(x,u)\nabla u \nabla\varphi + \frac{1}{2}\int_\Omega a_s(x,u)|\nabla u|^2\varphi = \int_\Omega (u^+)^{p-1}\varphi,$$

for all $\varphi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Theorem A.3. *Under the assumptions (A.2), (A.3), (A.4), (A.5), there exists a positive, bounded, critical point of J.*

Proof. Assumption (A.2) implies that $J(v) \geq \frac{\alpha}{2}\|v\|^2 - \frac{1}{p}\|v\|_p^p$, so that there exist $\rho, \overline{R} \in \mathbb{R}^+$ such $J(v) \geq \rho > 0$, for every v with $\|v\| = \overline{R} > 0$. Moreover, denoting by φ_1 the first eigenfunction of the Laplace operator with homogeneous Dirichlet boundary conditions in Ω, and using assumption (A.3), we can find $T > 0$ such that $\|T\varphi_1\|_\infty \geq 2\overline{R}$. and $J(T\varphi_1) < 0$. Then, having defined

$$c = \inf_{\Gamma} \max_{[0,1]} J(\gamma(t)), \tag{A.8}$$

where $\Gamma = \{\gamma \in C^0([0,1], Y) : \gamma(0) = 0, \gamma(1) = T\varphi_1\}$, it results

$$c \geq \rho. \tag{A.9}$$

Therefore, all the geometrical assumptions of Theorem 2.1 in [3] are satisfied. Moreover, setting $X = W_0^{1,2}(\Omega)$ and $Y = W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, endowed with the norm $\|\cdot\|_Y = \|\cdot\| + \|\cdot\|_\infty$, conditions (A.2) and (A.4) imply that for every u, $v \in Y$ there exists the directional derivative $\langle J'(u), v\rangle$; moreover for every fixed $u \in Y$ $\langle J'(u), v\rangle$ is linear and continuous for $v \in Y$, and for every fixed $v \in Y$ the map $u \to \langle J'(u), v\rangle$ is continuous for every $u \in Y$. This regularity properties of the functional J are sufficient to apply Theorem 2.1 in [3]. Then (see also the beginning of the proof of Theorem 3.3 in [3]) we obtain the existence of sequences $u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and $M_n \in \mathbb{R}^+$ satisfying

$$u_n \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \quad \|u_n\|_\infty \leq 2M_n, \quad M_n \geq T\|\varphi_1\|_\infty, \tag{A.10}$$

$$J(u_n) \to c, \tag{A.11}$$

and

$$\begin{cases} |\langle J'(u_n), v\rangle| \leq \varepsilon_n \big[\|v\|_{L^\infty(\Omega)}/M_n + \|v\|_{W_0^{1,2}(\Omega)}\big], \\ \forall\, v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega), \quad \{\varepsilon_n\} \subset \mathbb{R}^+ : \varepsilon_n \to 0. \end{cases} \tag{A.12}$$

From now on we will show that u_n converges weakly to a positive critical point of J. We will do this in several steps.

STEP 1. In this step we show that the sequence $\{u_n\}$ is bounded in $W_0^{1,2}(\Omega)$. In order to do this, it is enough to write $J(u_n) - \frac{1}{p}\langle J'(u_n), u_n\rangle$ and use (A.12), (A.10), (A.11) and (A.5), to get the existence of a positive constant L such that

$$\|u_n\| \leq L. \tag{A.13}$$

Notice that (A.13) also implies that

$$\int_\Omega a(x, u_n)|\nabla u_n|^2 + \frac{1}{2}\int_\Omega \partial_s a(x, u_n) u_n |\nabla u_n|^2 \leq \|u\|_p^p + \varepsilon_n(2 + L) \leq Q. \tag{A.14}$$

As a consequence of (A.13), we get that there exist a function $u \in W_0^{1,2}(\Omega)$ and a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that u_n converges to u weakly in $W_0^{1,2}(\Omega)$, strongly in $L^q(\Omega)$ for every $q \in [1, 2N/(N-2))$ and almost everywhere. Now, even if we do not know that u solves an equation, we shall prove that the function u is bounded.

Step 2. In this step we will denote with C possibly different positive constants and with ε_n' possibly different positive sequences converging to zero as n tends to infinity. We will prove that there exists a positive constant $M = M(L, p, \Omega, \alpha, N)$ such that

$$\|u\|_\infty \leq M. \tag{A.15}$$

Indeed, taking $v = G_k(u_n)$ as test function in (A.12), and using (2.13) and (A.4) we deduce that

$$C \int_\Omega |\nabla G_k(u_n)|^2 \leq \varepsilon_n + \Big[\int_{A_k^n} |u_n|^{2^*}\Big]^{p/2^*} |A_k^n|^{1-p/2^*}, \tag{A.16}$$

where $A_k^n := \{x \in \Omega : |u_n| > k\}$. Definition (2.13) and Hölder's inequality yield

$$\int_{A_k^n} |u_n|^{2^*} = \int_{A_k^n} |u_n - k + k|^{2^*} \leq C \Big[\int_{A_k^n} |G_k(u_n)|^{2^*} + k^{2^*}|A_k^n|\Big]$$
$$\leq C \Big\{\Big[\int_{A_k^n} |\nabla G_k(u_n)|^2\Big]^{2^*/2} + k^{2^*}|A_k^n|\Big\}.$$

So that

$$\Big[\int_{A_k^n} |u_n|^{2^*}\Big]^{p/2^*} |A_k^n|^{1-p/2^*} \leq C\big[\,\|\nabla G_k(u_n)\|_2^{2^*} + k^{2^*}|A_k^n|\big]^{p/2^*} |A_k^n|^{1-p/2^*}.$$

Using this inequality in (A.16) one deduces the following inequality:

$$\int_\Omega |\nabla G_k(u_n)|^2 \leq \varepsilon_n' + C\Big[\int_{A_k^n} |\nabla G_k(u_n)|^2\Big]^{p/2} |A_k^n|^{1-p/2^*} + C\, k^p\, |A_k^n|.$$

Using the fact that $2 < p$, we get

$$\int_\Omega |\nabla G_k(u_n)|^2 \leq C\Big[\int_{A_k^n} |\nabla G_k(u_n)|^2\Big]^{p/2-1} \Big[\int_{A_k^n} |\nabla G_k(u_n)|^2\Big] |A_k^n|^{1-p/2^*}$$
$$+ \varepsilon_n' + C\, k^p |A_k^n|.$$

From (A.13) it follows

$$\int_\Omega |\nabla G_k(u_n)|^2 \leq C\Big[\int_\Omega |\nabla G_k(u_n)|^2\Big] |A_k^n|^{1-p/2^*} + \varepsilon_n' + C\, k^p |A_k^n|. \tag{A.17}$$

Moreover, thanks to (A.13) and using the Sobolev embedding, we deduce that there exists $C_L > 0$ such that

$$|A_k^n|^{1-p/2^*} \leq \Big(\frac{1}{k}\Big)^{1-p/2^*} \Big[\int_{A_k^n} |u_n|\Big]^{1-p/2^*} \leq C_L \Big(\frac{1}{k}\Big)^{1-p/2^*},$$

implying that there exists $k_0 \in \mathbb{R}$ such that

$$C\,|A_k^n|^{1-p/2^*} \leq 1/2, \quad \forall\, k \geq k_0 \;\text{ uniformly with respect to } n. \tag{A.18}$$

Using this information in (A.17), we obtain that

$$\int_\Omega |\nabla G_k(u_n)|^2 \leq \varepsilon_n' + C\, k^p |A_k^n| \quad \text{for every } k \geq k_0.$$

Let $K = \{k \geq k_0 : |\{u = k\}| = 0\}$ and observe that letting n tend to infinity, for every $k \in K$, $|A_k^n|$ converges to $|A_k|$ with $A_k = \{|u| \geq k\}$. Hence, since the norm is weakly lower semicontinuous, it results

$$\int_\Omega |\nabla G_k(u)|^2 \leq C\,k^p |A_k|, \quad \forall\, k \in K,\, k \geq k_0. \tag{A.19}$$

Since $u \in L^{2^*}(\Omega)$, it holds $|A_k| \leq C_L/k^{2^*}$; so that $k^{p-2} \leq C_L |A_k|^{(2-p)/2^*}$ and (A.19) becomes, thanks to the Sobolev inequality,

$$\Big[\int_\Omega |G_k(u)|^{2^*}\Big]^{1/2^*} \leq C\,k\,|A_k|^{\frac{1}{2}-\frac{p}{22^*}+\frac{1}{2^*}}$$

and Hölder's inequality yields

$$\int_\Omega |G_k(u)| \leq C\,k |A_k|^{\frac{1}{2}-\frac{p}{22^*}+1} = C\,k|A_k|^{1+\frac{1}{2}(1-\frac{p}{2^*})}.$$

As done in (2.18) we consider $y(k)$ the $L^1(\Omega)$ norm of $G_k(u)$ and we observe that $y(k)$ satisfies

$$y \leq C\,k(-y')^{1+\nu}, \quad \nu = \frac{1}{2}\Big(1 - \frac{p}{2^*}\Big).$$

Notice that (1.2) implies that $\nu > 0$. Integrating, we obtain

$$k^{\frac{\nu}{1+\nu}} \leq C\,[y(0)^{\frac{\nu}{1+\nu}} - y(k)^{\frac{\nu}{1+\nu}}],$$

that is

$$k^{\frac{\nu}{1+\nu}} \leq C\,\|u\|_1^{\frac{\nu}{1+\nu}} - C\,\|G_k(u)\|_1^{\frac{\nu}{1+\nu}}.$$

Taking into account (A.13), it results

$$\|G_k(u)\|_1^{\frac{\nu}{1+\nu}} \leq C_L - k^{\frac{\nu}{1+\nu}}.$$

Choosing k_L such that $k_L^{\nu/(1+\nu)} = C_L$, we obtain that the norm of $G_{k_L}(u)$ in $L^1(\Omega)$ is zero. Thus the above inequality says that (A.15) is satisfied with $M = k_L$.

STEP 3. Taking $v = -u_n^-$ as test function in (A.12), using (A.7) and (A.13) immediately gives that $a(x,u_n)|\nabla u_n^-|^2$ converges strongly to zero in $L^1(\Omega)$. Then (A.2) implies that the sequence $\{u_n^-\}$ converges strongly to zero in $W_0^{1,2}(\Omega)$, so that

$$u \geq 0. \tag{A.20}$$

STEP 4. Now we prove that the sequence ∇u_n converges to ∇u almost everywhere in Ω.

First of all, notice that thanks to (A.2), (A.10) and (2.13) we can argue as in Lemma 2.8 to obtain

$$\int_\Omega |\partial_s a(x,u_n)|\,|\nabla u_n|^2 \leq A. \tag{A.21}$$

Note that, in order to obtain the almost everywhere convergence of ∇u_n, we cannot apply Theorem 2.1 in [8] as u_n does not satisfy an equation; however, we can prove it directly by fixing $h \in (0, \|u\|_\infty)$ and using $v = T_h(u_n - u)$ as test function in (A.12). Since u_n strongly converges to u in $L^p(\Omega)$, and thanks to (A.13), one deduces that there exists a positive constant C_1 such that

$$\text{(A.22)}\quad \int_\Omega a(x,u_n)\nabla u_n \nabla T_h(u_n - u) + \frac{1}{2}\int_\Omega \partial_s a(x,u_n)|\nabla u_n|^2 T_h(u_n - u) \le \omega_n + C_1\varepsilon_n,$$

with $\omega_n \to 0$ as $n \to +\infty$. Taking into account (2.13) and (A.21), one observes that

$$\frac{1}{2}\int_\Omega \partial_s a(x,u_n)|\nabla u_n|^2\, T_h(u_n - u) \le \frac{A}{2}\, h.$$

From this, (A.2) and (A.22), it results

$$\alpha \int_\Omega |\nabla T_h(u_n - u)|^2 \le \int_\Omega a(x,u_n)\nabla u\, \nabla T_h(u_n - u) + \frac{A}{2}\, h + \omega_n + C_1\, \varepsilon_n,$$

which implies, in view of (A.13), and using the Hölder inequality,

$$\begin{aligned}
\|\nabla(u_n - u)\|_1 &= \int_\Omega |\nabla\, T_h(u_n - u)| + \int_{\{h<|u_n - u|\}} |\nabla(u_n - u)| \\
&\le \Big[\int_\Omega |\nabla\, T_h(u_n - u)|^2\Big]^{1/2} |\Omega|^{1/2} + C_L\, \big|\{h < |u_n - u|\}\big|^{1/2} \\
&\le \Big[\int_\Omega a(x,u_n)\nabla u \nabla T_h(u_n - u) + \frac{A}{2} h + \omega_n + C_1\varepsilon_n\Big]^{1/2} \frac{|\Omega|^{1/2}}{\alpha} \\
&\quad + C_L\, |\{h < |u_n - u|\}|^{1/2}.
\end{aligned}$$

Taking into account that u_n weakly converges to u in $W_0^{1,2}(\Omega)$, the limit as n tends to plus infinity gives

$$\limsup_{n\to+\infty} \int_\Omega |\nabla(u_n - u)| \le \frac{|\Omega|}{\alpha}^{1/2} \Big[\frac{A}{2} h\Big]^{1/2}.$$

By passing to the limit as $h \to 0$ one gets that ∇u_n converges to ∇u strongly in $L^1(\Omega)$, so that, up to a subsequence, it converges almost everywhere.

STEP 5. Now we prove that u is a critical point of J. Let us consider the functions $H\colon \mathbb{R} \mapsto \mathbb{R}$ and $F \in C^\infty(\mathbb{R})$, with $|F'(t)| \le 2$, defined by

$$\text{(A.23)}\qquad H(t) = \begin{cases} \displaystyle\int_0^t \eta(s)\, ds & \text{for } t \ge 0, \\ 0 & \text{for } t < 0, \end{cases} \qquad F(t) = \begin{cases} 1 & \text{for } |t| \le 1, \\ 0 & \text{for } |t| > 2, \end{cases}$$

where η is given in assumption (A.4). We take $\phi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, $\phi \ge 0$ and define the function

$$\text{(A.24)}\qquad v = \phi\, e^{-bH(u_n)} e^{bH(u)} F\Big(\frac{u_n}{k}\Big), \quad b = \frac{1}{2\alpha}, \quad k > M,$$

with M introduced in (A.15). Then, thanks to (A.13), (A.23) and A.15, one deduces that $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$, so that it can be chosen as test function in (A.12) to obtain

$$\begin{aligned}\int_\Omega & a(x,u_n)\nabla u_n \nabla\phi\, e^{-bH(u_n)} e^{bH(u)} F\Big(\frac{u_n}{k}\Big)\\ &+ \int_\Omega a(x,u_n)\nabla u_n\nabla u\Big[\phi\, e^{-bH(u_n)} e^{bH(u)} F\Big(\frac{u_n}{k}\Big) bH'(u)\Big]\\ &\geq -\frac{1}{k}\int_\Omega a(x,u_n)|\nabla u_n|^2 e^{-bH(u_n)} e^{bH(u)} F'\Big(\frac{u_n}{k}\Big)\\ &+ \int_\Omega |\nabla u_n|^2 \phi\, e^{-bH(u_n)} e^{bH(u)} F\Big(\frac{u_n}{k}\Big)\Big[bH'(u_n)a(x,u_n) - \frac{1}{2}\partial_s a(x,u_n)\Big]\\ &+ \int_\Omega (u_n^+)^{p-1}\phi\, e^{-bH(u_n)} e^{bH(u)} F\Big(\frac{u_n}{k}\Big) - C\varepsilon_n.\end{aligned}$$

Note that (A.14), (A.7) and (A.13) imply

$$\Big|\frac{1}{k}\int_\Omega a(x,u_n)|\nabla u_n|^2\, e^{-bH(u_n)} e^{bH(u)} F'\Big(\frac{u_n}{k}\Big)\Big| \leq C(M,Q)\,\frac{1}{k}.$$

Moreover, the definition of the function H and (A.24) imply that

$$bH'(u_n)a(x,u_n) - \frac{1}{2}\partial_s a(x,u_n) \geq 0.$$

Taking the limit in the left-hand side, the inferior limit on the right, and applying Fatou's lemma (using Step 4), one gets

$$\begin{aligned}&\int_\Omega a(x,u)\nabla u\nabla\phi + \int_\Omega a(x,u)|\nabla u|^2[\phi bH'(u)]\\ &\quad\geq -C(M,Q)\frac{1}{k} + \int_\Omega u^{p-1}\phi + \int_\Omega |\nabla u|^2\phi\Big[bH'(u)a(x,u) - \frac{1}{2}\partial_s a(x,u)\Big],\end{aligned}\tag{A.25}$$

where we have taken into account that $F(u/k) = 1$ as $k \geq M$. Since $u \in L^\infty(\Omega)$, it follows that $a(x,u)|\nabla u|^2 H'(u) \in L^1(\Omega)$, so that we can cancel the equal terms in the left and right sides to obtain that

$$\int_\Omega a(x,u)\nabla u\nabla\phi + \frac{1}{2}\int_\Omega \partial_s a(x,u)|\nabla u|^2\phi \geq -C(M,Q)\frac{1}{k} + \int_\Omega u^{p-1}\phi.$$

Taking the limit for $k \to +\infty$, we get

$$\int_\Omega a(x,u)\nabla u\nabla\phi + \frac{1}{2}\int_\Omega \partial_s a(x,u)|\nabla u|^2\phi \geq \int_\Omega u^{p-1}\phi.\tag{A.26}$$

The proof of the reverse inequality is simpler. Indeed, take

$$v = \phi\, F\Big(\frac{u_n}{k}\Big), \quad \phi \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega),\ \phi \geq 0, \quad k \geq M,$$

with F defined in (A.23) and use (A.12) to get

$$\begin{aligned}\frac{1}{2}\int_\Omega \partial_s a(x,u_n)|\nabla u_n|^2\phi\, F\Big(\frac{u_n}{k}\Big) \le &-\int_\Omega a(x,u_n)\nabla u_n\nabla\phi\, F\Big(\frac{u_n}{k}\Big)\\ &-\frac{1}{k}\int_\Omega a(x,u_n)|\nabla u_n|^2 F'\Big(\frac{u_n}{k}\Big)\phi\\ &+\int_\Omega (u_n^+)^{p-1}\phi\, F\Big(\frac{u_n}{k}\Big)+C\varepsilon_n.\end{aligned}$$

The sequence in the integral in the left-hand side is positive, while, in view of (A.14), there exists a positive constant C_Q such that

$$\frac{1}{k}\Big|\int_\Omega a(x,u_n)|\nabla u_n|^2 F'\Big(\frac{u_n}{k}\Big)\phi\Big| \le \frac{C_Q}{k}.$$

Moreover, using Step 4, we can apply Fatou's lemma, obtaining

$$\frac{1}{2}\int_\Omega \partial_s a(x,u)|\nabla u|^2\phi \le -\int_\Omega a(x,u)\nabla u\nabla\phi + \frac{C_Q}{k} + \int_\Omega u^{p-1}\phi.$$

Letting $k\to+\infty$ one obtains $\langle J'(u),\phi\rangle \le 0$. This and (A.26) imply that

$$\langle J'(u),\phi\rangle = 0, \quad \forall\,\phi\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega),\ \phi\ge 0. \tag{A.27}$$

Since any $w\in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)$ can be decomposed in its positive and negative part, both positive functions, we have shown that u is a critical point of J.

STEP 6. Finally, we can show that $J(u)=c$. Choosing $v=u_n$ in (A.12) and using (A.4) and (A.2), we can apply the results proved in Steps 4 and 5 to obtain

$$\begin{aligned}&\int_\Omega a(x,u)|\nabla u|^2 + \int_\Omega \partial_s a(x,u)|\nabla u|^2 u\\ &\quad\le \liminf_{n\to+\infty}\Big[\int_\Omega a(x,u_n)|\nabla u_n|^2 + \int_\Omega \partial_s a(x,u_n)|\nabla u_n|^2 u_n\Big]\\ &\quad= \lim_{n\to+\infty}\int_\Omega (u_n^+)^p = \int_\Omega u^p = \int_\Omega a(x,u)|\nabla u|^2 + \int_\Omega \partial_s a(x,u)|\nabla u|^2\, u.\end{aligned}$$

Then, the positivity of the sequences $a(x,u_n)|\nabla u_n|^2$, $\partial_s a(x,u_n)u_n|\nabla u_n|^2$ imply that $a(x,u_n)|\nabla u_n|^2$ converges to $a(x,u)|\nabla u|^2$, strongly in $L^1(\Omega)$, so that (A.11) and (A.9) imply

$$0<\rho<c=\lim_{n\to+\infty} J(u_n) = J(u)$$

yielding that u is not trivial. □

Remark A.4. As a byproduct of Step 6, and using (A.2) we obtain that u_n converges to u strongly in $W_0^{1,2}(\Omega)$. Moreover, notice that the convergence of $J(u_n)$ to $J(u)$ cannot be seen as a trivial consequence of the strong convergence of u_n to u (as in the semi-linear case), as $a(x,t)$ is not supposed to be uniformly bounded by a positive constant.

Remark A.5. Notice that, differently from Section 2, here we cannot show that all the sequence u_n satisfies a uniform $L^\infty(\Omega)$ bound, but we can obtain this information on the weak limit u. This is a consequence of the fact that u_δ are critical points, while u_n only satisfies (A.12).

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