



The Dirichlet problem for p -harmonic functions with respect to arbitrary compactifications

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Abstract. We study the Dirichlet problem for p -harmonic functions on metric spaces with respect to arbitrary compactifications. A particular focus is on the Perron method, and as a new approach to the invariance problem we introduce Sobolev–Perron solutions. We obtain various resolvability and invariance results, and also show that most functions that have earlier been proved to be resolutive are in fact Sobolev-resolutive. We also introduce (Sobolev)–Wiener solutions and harmonizability in this nonlinear context, and study their connections to (Sobolev)–Perron solutions, partly using Q -compactifications.

1. Introduction

The Dirichlet problem asks for a solution of a partial differential equation in a bounded domain $\Omega \subset \mathbb{R}^n$ with prescribed boundary values f on $\partial\Omega$. Even for harmonic functions, i.e., for solutions of $\Delta u = 0$, and with continuous f , it is not always possible to solve this boundary value problem so that the solution $u \in \mathcal{C}(\overline{\Omega})$. To overcome this problem, one is forced to formulate the boundary value problem in a generalized sense.

Perhaps the most fruitful approach to solving the Dirichlet problem in very general situations is the Perron method, which always produces an upper and a lower Perron solution. When they coincide, this gives a nice solution $P_\Omega f$ of the Dirichlet problem and f is called *resolutive*. One of the major problems in the theory is to determine which functions are resolutive. In the linear potential theory, Brelot [19] showed that f is resolutive if and only if $f \in L^1(d\omega)$, where ω is the harmonic measure. A similar characterization in the nonlinear theory is impossible since there is no equivalent of the harmonic measure suitable for this task.

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The aim of this article is to study the Dirichlet problem and related questions about boundary values in a very general setting for the nonlinear potential theory associated with p -harmonic functions on metric spaces. Nevertheless, most of our results are new even in \mathbb{R}^n , even though we formulate them in metric spaces, and all our (counter)examples are in Euclidean domains. A p -harmonic function on \mathbb{R}^n is a continuous solution of the p -Laplace equation

$$(1.1) \quad \Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0.$$

On metric spaces, p -harmonic functions are defined through an energy minimizing problem (see Definition 5.1), which on \mathbb{R}^n is equivalent to the definition above.

The Dirichlet problem and the closely related boundary regularity for p -harmonic functions defined by (1.1) have been studied by e.g. Granlund–Lindqvist–Martio [25], Heinonen–Kilpeläinen [29], Kilpeläinen [35], Kilpeläinen–Malý [36], [37], [38], Lindqvist–Martio [46], Maeda–Ono [49], [50] and Maz’ya [52] in \mathbb{R}^n , by Heinonen–Kilpeläinen–Martio [30] in weighted \mathbb{R}^n , by Holopainen [33] and Lucia–Puls [48] on Riemannian manifolds and in [7]–[14], [16], [27], [28] and [56] on metric spaces.

It is well known that two p -harmonic functions on Ω which coincide outside a compact subset of Ω , coincide in all of Ω . This can be rephrased as saying that a p -harmonic function is uniquely determined by its boundary values, whenever there are some natural such values to attach to the function. However, in many situations, such as for the slit disc in the plane, the metric boundary is too small to be able to attach natural boundary values even to quite well behaved harmonic functions, since their behaviour can be very different on each side of the slit. Therefore, it is natural to study the Dirichlet problem for other compactifications as well. In the linear potential theory, many such compactifications have been studied, most notably the one by Martin [51], to produce boundaries allowing more harmonic functions to have natural boundary values attached to them.

On the contrary, most of the nonlinear papers so far only deal with the Dirichlet problem with respect to the metric boundary. In [16], the Dirichlet problem was studied with respect to the Mazurkiewicz boundary in the case when it is a compactification. In this case the Mazurkiewicz boundary coincides with the prime end boundary introduced in [1]. Estep–Shanmugalingam [24] and A. Björn [8] obtained partial results also when the prime end boundary is noncompact. The Dirichlet problem on the whole space, based on so-called p -Royden boundaries, was pursued in Lucia–Puls [48]. In this paper, we generalize the treatment from [16] in a different direction by studying the Dirichlet problem with respect to arbitrary compactifications (which do not have to be metrizable). For domains in \mathbb{R}^n some results in this direction were obtained by Maeda–Ono [49], [50].

A particular problem is to determine when resolutive functions are *invariant* under perturbations on small sets, e.g. if f is resolutive and $k = f$ outside a set of zero capacity, is then $P_\Omega k = P_\Omega f$? In the linear case this is well known (and holds when the set $\{x : k(x) \neq f(x)\}$ has zero harmonic measure), but in the nonlinear case this is only known under additional assumptions on f . Such results were first obtained by Avilés–Manfredi [3] and Kurki [45] who considered invariance of upper

Perron solutions for characteristic functions (also called p -harmonic measures). Invariance for continuous f was proved in [13] (covering also the metric space case). Further invariance results have been obtained in [7], [8], [13], [14], [16], [28].

As a new approach to the invariance problem, we introduce *Sobolev–Perron solutions* $S_\Omega f$. They are defined using upper (superharmonic) functions which are also required to belong to a suitable Sobolev space (or more precisely to the Dirichlet space $D^p(\Omega)$ of functions with finite p -energy). It turns out that all Sobolev-resolutive functions are resolutive (Corollary 6.3) but not vice versa (Examples 6.5 and 6.6). At the same time, we are also able to show that most of the functions that have earlier been proved to be resolutive are in fact Sobolev-resolutive. Summarizing (parts of) Theorem 6.4 and Proposition 7.1 we get the following result, which holds true for arbitrary compactifications. We denote by $\partial^1\Omega$ the boundary of Ω induced by a compactification $\bar{\Omega}^1$ of Ω , and we will also use the somewhat abusive notation Ω^1 to denote the set Ω with the intended boundary $\partial^1\Omega$.

Theorem 1.1. *Let $\bar{\Omega}^1$ be a compactification of Ω .*

- (a) *If $f \in C(\bar{\Omega}^1)$ is such that $f|_\Omega \in D^p(\Omega)$, then $f|_{\partial^1\Omega}$ is Sobolev-resolutive.*
- (b) *If $f: \partial^1\Omega \rightarrow \bar{\mathbb{R}}$ is Sobolev-resolutive and $k = f$ outside a set of zero \bar{C}_p -capacity, then*

$$S_{\Omega^1} k = S_{\Omega^1} f.$$

Here, and for the other theorems below, we assume the standing assumptions given in the beginning of Section 5. In particular, these results hold for the usual p -harmonic functions in bounded Euclidean domains.

The capacity \bar{C}_p is an interior version of the Sobolev capacity, which sees the boundary $\partial^1\Omega$ only from inside Ω and is thus well adapted to the Dirichlet problem. In particular, it makes sense also for subsets of the boundary $\partial^1\Omega$, which are not in general subsets of the metric space we start from. Similar capacities were earlier used in [16], [24], [28] and Kilpeläinen–Malý [36].

Our next aim is to construct compactifications by the general method of embeddings into product spaces, and to see which such constructions lead to *resolutive boundaries* (i.e., boundaries for which all continuous functions are resolutive). The idea of Q -compactifications is to prescribe a set $Q \subset \mathcal{C}(\Omega)$ with the aim of defining the smallest boundary $\partial^Q\Omega$ for which every function in Q has a continuous extension to the boundary. The fundamental existence and uniqueness (up to homeomorphism) theorem for such compactifications is due to Constantinescu–Cornea [21].

A fundamental concept used to study which sets Q lead to resolutive boundaries is *harmonizability*. It is based on the fact that a p -harmonic function is uniquely determined by its behaviour close to the boundary. In the linear potential theory on Riemann surfaces this concept was also introduced and studied by Constantinescu–Cornea [21] and is closely related to the Wiener solutions defined by Wiener [60]. We generalize this approach to the nonlinear case, which has earlier been considered by Maeda–Ono [49], [50] on \mathbb{R}^n . This time, one defines upper and lower Wiener

solutions for functions f defined inside Ω (and not on the boundary as for Perron solutions). When they coincide, f is *harmonizable* and we denote the common Wiener solution by Wf . Two functions which are equal outside some compact subset of Ω give, by definition, the same upper and lower Wiener solutions, and thus it is only the values of f near the boundary that are relevant.

Also here we introduce the alternative (new) concepts of Sobolev-harmonizability and Sobolev–Wiener solutions Zf which turn out to satisfy similar perturbation properties as Sobolev–Perron solutions. We observe that Sobolev-harmonizability implies harmonizability, while Example 8.6 shows that the converse is not true. We also relate (Sobolev)–Wiener and (Sobolev)–Perron solutions, as well as resolutivity and harmonizability. (Parts of) Theorem 8.9 and Proposition 9.1 can be summarized in the following way.

Theorem 1.2. *Let $\overline{\Omega}^1$ be a compactification of Ω .*

- (a) *If $f \in \mathcal{C}(\overline{\Omega}^1)$ then $f|_{\Omega}$ is (Sobolev)-harmonizable if and only if $f|_{\partial^1\Omega}$ is (Sobolev)-resolutive. In this case we have*

$$Wf = P_{\Omega^1} f \quad (\text{resp. } Zf = S_{\Omega^1} f).$$

- (b) *If $f: \Omega \rightarrow \overline{\mathbb{R}}$ is Sobolev-harmonizable and $k = f$ outside a set of capacity zero, then $Zk = Zf$.*

Theorem 8.14 and Corollary 9.3 give the following characterization of resolutive.

Theorem 1.3. *Assume that $Q \subset \mathcal{C}_{\text{bdd}}(\Omega)$.*

- (a) *If Q is a vector lattice containing the constant functions, then the boundary $\partial^Q\Omega$ is (Sobolev)-resolutive if and only if every function in Q is (Sobolev)-harmonizable.*
- (b) *If $Q \subset D^p(\Omega)$, then $\partial^Q\Omega$ is Sobolev-resolutive.*

The outline of the paper is as follows: in Section 2 we discuss compactifications, and obtain some results which will be needed in the sequel. In Sections 3–5 we introduce the relevant background in nonlinear potential theory on metric spaces, as well as the capacity \overline{C}_p . Next, in Sections 6 and 7, we turn to Perron and Sobolev–Perron solutions, while harmonizability, Wiener solutions and their connections to Perron solutions are studied in Sections 8 and 9.

We end the paper with Section 10 on Sobolev-resolutivity and Sobolev-harmonizability for quasi(semi)continuous functions. For these results it is essential that we use Sobolev–Perron and Sobolev–Wiener solutions.

2. Compactifications

For a topological space T , we let $\mathcal{C}(T)$ be the space of real-valued continuous functions, $\mathcal{C}_{\text{bdd}}(T)$ be the space of bounded continuous functions, and $\mathcal{C}_c(T)$ be the space of real-valued continuous functions with compact support in T , all equipped

with the supremum norm, and the induced topology. We also let $\mathcal{C}(T, \overline{\mathbb{R}})$ be the space of extended real-valued continuous functions, where $\overline{\mathbb{R}} := [-\infty, \infty]$ (with the usual topology).

Definition 2.1. Let Ω be a locally compact noncompact Hausdorff space with topology $\tilde{\tau}$. A couple $(\partial\Omega, \tau)$ is said to *compactify* Ω if $\partial\Omega$ is a set with $\partial\Omega \cap \Omega = \emptyset$ and τ is a Hausdorff topology on $\overline{\Omega} := \Omega \cup \partial\Omega$ such that

- (a) $\overline{\Omega}$ is compact with respect to τ ;
- (b) Ω is dense in $\overline{\Omega}$ with respect to τ ;
- (c) the topology induced on Ω by τ is $\tilde{\tau}$.

The space $\overline{\Omega}$ with the topology τ is a *compactification* of Ω .

In this section $\partial\Omega$ will denote the boundary of an arbitrary compactification, while in later sections $\partial\Omega$ will be reserved for the given metric boundary. We will denote the compactification by $\overline{\Omega} := \Omega \cup \partial\Omega$ and similarly $\overline{\Omega}^1 := \Omega \cup \partial^1\Omega$, $\overline{\Omega}^Q := \Omega \cup \partial^Q\Omega$, etc.

Usually we will not specify τ , but it should be clear from the context what it is. We first show that Ω is automatically open in $\overline{\Omega}$, so we do not need to require this as an extra axiom.

Lemma 2.2. *Let $(\Omega, \tilde{\tau})$ be a locally compact noncompact Hausdorff space, and let $\overline{\Omega}$ with the topology τ be a compactification of Ω . Then Ω is τ -open.*

Recall that $E \Subset \Omega$ if \overline{E} is a compact subset of Ω . Moreover, in this paper neighbourhoods are always open.

Proof. Let $x \in \Omega$. By the local compactness of Ω , there is a $\tilde{\tau}$ -open $G \Subset \Omega$ containing x . We shall show that G is also τ -open. By compactness (and (c)) the $\tilde{\tau}$ -closure $\overline{G} \subset \Omega$ of G equals the τ -closure of G . By (c), there is a τ -open set $\widehat{G} \subset \overline{\Omega}$ such that $\widehat{G} \cap \Omega = G$. If there were a point $y \in \widehat{G} \cap \partial\Omega$, then by (b), any τ -neighbourhood $\widetilde{G} \subset \widehat{G}$ of y would contain a point $z \in \Omega$. But then $z \in \widehat{G} \cap \Omega = G$, and thus y would belong to $\overline{G} \subset \Omega$, a contradiction. Hence $G = \widehat{G}$ is τ -open. Since $x \in \Omega$ was arbitrary it follows that Ω is τ -open. □

We will denote the set of boundaries compactifying Ω by $\mathcal{L}(\Omega)$. A fundamental concept for us will be the natural order that $\mathcal{L}(\Omega)$ carries. It is defined as follows: For two boundaries $\partial^1\Omega$ and $\partial^2\Omega$ in $\mathcal{L}(\Omega)$ we define

$$\partial^1\Omega \prec \partial^2\Omega$$

to mean that there is a continuous mapping

$$\Phi : \overline{\Omega}^2 \longrightarrow \overline{\Omega}^1 \quad \text{with } \Phi|_{\Omega} = \text{id}.$$

(This order is rather between compactifications, but for notational convenience we let $\mathcal{L}(\Omega)$ denote the boundaries instead of the actual compactifications.) Note that $\partial^1\Omega \prec \partial^2\Omega \prec \partial^1\Omega$ if and only if $\overline{\Omega}^1 \simeq \overline{\Omega}^2$ (with $\Omega \xrightarrow{\text{id}} \Omega$ and where \simeq

denotes homeomorphism). This is so because the continuous mapping $\overline{\Omega}^1 \rightarrow \overline{\Omega}^2$ must equal Φ^{-1} , by the denseness of Ω in both compactifications. We will usually consider homeomorphic compactifications as identical.

The most important tool for studying the space $\mathcal{L}(\Omega)$ is the method of constructing compactifications by making embeddings into product spaces. The idea goes back to Tikhonov [59], but the theorem on existence and uniqueness of Q -compactifications given below is from Constantinescu–Cornea [21] (see also Theorem 13 in BreLOT [20]). For the reader’s convenience and to set the notation and terminology, we provide a complete proof of the existence result, which more or less follows [21].

Definition 2.3. For $Q \subset \mathcal{C}(\Omega, \overline{\mathbb{R}})$ we say that a compactification $\overline{\Omega}$ is a Q -compactification of Ω if

- (a) for every $f \in Q$ there is $\hat{f} \in \mathcal{C}(\overline{\Omega}, \overline{\mathbb{R}})$ such that $\hat{f}|_{\Omega} = f$;
- (b) the functions $\{\hat{f} : f \in Q\}$ separate the points of $\partial\Omega$.

Note that any element in $\mathcal{C}_c(\Omega)$ extends as zero on the boundary of every compactification, so we may always add $\mathcal{C}_c(\Omega)$ to Q without changing anything. Similarly, constant functions can be added without change.

We should first realize that each compactification is a Q -compactification for some suitable $Q \subset \mathcal{C}(\Omega, \overline{\mathbb{R}})$.

Lemma 2.4. *Let $\overline{\Omega}$ be a compactification of Ω and let Q be dense in $\{f|_{\Omega} : f \in \mathcal{C}(\overline{\Omega})\}$ (with respect to the supremum norm). Then $\overline{\Omega}$ is a Q -compactification of Ω .*

Proof. For distinct $x, y \in \partial\Omega$, the function $u = \chi_{\{x\}}$ is continuous on $\{x, y\}$. As $\overline{\Omega}$ is a compact Hausdorff space it is normal (see e.g. Theorem 32.3 in Munkres [54]). Thus Tietze’s extension theorem (see e.g. Theorem 35.1 in [54]) shows that there is a $\tilde{u} \in \mathcal{C}(\overline{\Omega})$ such that $\tilde{u}|_{\{x,y\}} = u$, and hence \tilde{u} separates x and y . By the density of Q there must also be a function $v \in Q$ which separates these points.

Condition (a) is directly fulfilled. □

Lemma 2.5. *Let $\overline{\Omega}$ be a compactification of Ω , and assume that $Q \subset \mathcal{C}(\overline{\Omega}, \overline{\mathbb{R}})$ separates the points of $\overline{\Omega}$. Also let $I_f = \overline{\mathbb{R}}$ for each $f \in Q$ and define*

$$\varphi : \overline{\Omega} \longrightarrow \prod_{f \in Q} I_f \quad \text{by } \varphi(x) := \{f(x)\}_{f \in Q} \text{ for } x \in \overline{\Omega},$$

where $\prod_{f \in Q} I_f$ is equipped with the product topology. If we let $K = \varphi(\overline{\Omega})$ then φ , seen as a map from $\overline{\Omega}$ to K , is a homeomorphism and the set $\varphi(\Omega)$ is an open dense subset of K .

Proof. Let $\pi_f : \prod_{f \in Q} I_f \rightarrow I_f$ denote the projection onto the f -th coordinate. Since $\pi_f \circ \varphi = f$ for each $f \in Q$ it follows that φ is continuous (this property is what characterizes the product topology), and since Q separates the points in $\overline{\Omega}$ we conclude that φ is also injective (so it is a continuous bijection between $\overline{\Omega}$ and K).

Since both $\overline{\Omega}$ and K are compact it follows that φ is a homeomorphism. In particular, φ is an open map, so $\varphi(\Omega)$ is an open subset of K . Since homeomorphisms also trivially map dense subsets to dense subsets the result follows. \square

Proposition 2.6. *If $Q_1 \subset Q_2 \subset \mathcal{C}(\Omega, \overline{\mathbb{R}})$ and $\overline{\Omega}^1$ and $\overline{\Omega}^2$ are Q_1 - and Q_2 -compactifications of Ω , respectively, then $\partial^1\Omega \prec \partial^2\Omega$.*

In particular, if $Q_1 = Q_2$ then $\overline{\Omega}^1 \simeq \overline{\Omega}^2$, i.e., Q -compactifications are unique up to homeomorphism.

Proof. If we let $Q'_i = Q_i \cup \mathcal{C}_c(\Omega)$, and regard all these functions as extended to the whole compact space $\overline{\Omega}^i$, then according to Lemma 2.5 we see that the map

$$\varphi_i : \overline{\Omega}^i \longrightarrow \prod_{f \in Q'_i} I_f, \quad \text{defined by } \varphi_i(x) := \{f(x)\}_{f \in Q'_i} \text{ for } x \in \overline{\Omega}^i,$$

is a homeomorphism from $\overline{\Omega}^i$ to $\varphi_i(\overline{\Omega}^i) \subset \prod_{f \in Q'_i} I_f$. As $Q'_1 \subset Q'_2$, the result is an immediate consequence of the fact that the mapping from $\prod_{f \in Q'_2} I_f$ to $\prod_{f \in Q'_1} I_f$, defined by

$$\{y_f\}_{f \in Q'_2} \longmapsto \{y_f\}_{f \in Q'_1},$$

is continuous. \square

We need to prove the existence of Q -compactifications.

Theorem 2.7. *For $Q \subset \mathcal{C}(\Omega, \overline{\mathbb{R}})$ there is a Q -compactification $\overline{\Omega}^Q = \Omega \cup \partial^Q\Omega$ of Ω .*

Together with Proposition 2.6 this shows that the Q -compactification $\overline{\Omega}^Q$ exists and is unique (up to homeomorphism).

Proof. As suggested by Lemma 2.5, the construction builds on embedding Ω into the product space

$$\prod_{f \in Q'} I_f,$$

where $Q' = Q \cup \mathcal{C}_c(\Omega)$ and $I_f = \overline{\mathbb{R}}$ for each $f \in Q'$. We give this space the product topology, and we let π_f denote the projection onto the f -th coordinate. Let

$$\psi : \Omega \longrightarrow \prod_{f \in Q'} I_f, \quad \text{where } \psi(x) = \{f(x)\}_{f \in Q'} \text{ for } x \in \Omega.$$

Since $\pi_f \circ \psi = f$ for each $f \in Q'$ it follows that ψ is continuous, and as Q' separates the points in Ω it is also injective. Set

$$R = \psi(\Omega).$$

If we prove that the map ψ is open then it follows that, seen as a map from Ω to R , it is a homeomorphism. To do so, let $y \in \psi(G)$, where $G \Subset \Omega$ is open. Then

$y = \{f(x)\}_{f \in Q'}$ for some $x \in G$. Choose $g \in C_c(\Omega)$ with $\text{supp } g \subset G$ and $g(x) \neq 0$. If we put

$$V = \{z \in \bar{R} : \pi_g(z) \neq 0\},$$

then V is relatively open in \bar{R} , and so is $V \setminus \psi(\bar{G})$, by the compactness of $\psi(\bar{G})$. Thus, $V \setminus \psi(\bar{G})$ must either intersect R or be empty, because of the density of R in \bar{R} . Since $(V \setminus \psi(\bar{G})) \cap R = \emptyset$, by the choice of g , we see that $V \subset \psi(\bar{G}) \subset R$, and V is open in \bar{R} . As $g = 0$ on ∂G we conclude that $V \subset \psi(G)$. This proves that ψ is an open mapping, which (together with the continuity and bijectivity of $\psi: \Omega \rightarrow R$) yields that it is a homeomorphism.

We now identify Ω with R and let $\partial^Q \Omega := \bar{R} \setminus R$, where the closure is with respect to $\prod_{f \in Q'} I_f$. We recall that this identifies $f \in Q$ with π_f in the following sense: any $y \in R$ is by construction of the form $y = \{g(x)\}_{g \in Q'}$ for a unique $x \in \Omega$ (i.e., y is the element we identify x with in the product space), and the projection $\pi_f(y)$ is the f -th coordinate of y , i.e., $\pi_f(y) = f(x)$. Or what amounts to the same thing, $\pi_f \circ \psi = f$. Thus, we may set $\hat{f} := \pi_f$ in the notation of Definition 2.3. Since the projections π_f are continuous on \bar{R} , and also separate the points of $\partial^Q \Omega$, it follows that \bar{R} is a Q -compactification of Ω . □

The theorem above is a very convenient tool for introducing the lattice structure on $\mathcal{L}(\Omega)$. For a family $\{\partial^i \Omega\}_{i \in I} \subset \mathcal{L}(\Omega)$ we first put $Q_i = \{f|_\Omega : f \in \mathcal{C}(\bar{\Omega}^i)\}$, and then note that $\bar{\Omega}^i \simeq \bar{\Omega}^{Q_i}$, by Lemma 2.4. To introduce the least upper bound of the family $\{\partial^i \Omega\}_{i \in I}$ with respect to \prec we put

$$Q = \bigcup_{i \in I} Q_i.$$

It is easy to see that $\partial^Q \Omega$ is the least upper bound in the order \prec , because any boundary larger than each $\partial^i \Omega$ must by definition have the property that each element in Q has a continuous extension to this boundary.

Similarly $\partial^{\tilde{Q}} \Omega$, where $\tilde{Q} = \bigcap_{i \in I} Q_i$, is the greatest lower bound of the family $\{\partial^i \Omega\}_{i \in I}$. Note that the least upper and the greatest lower bounds are only defined up to homeomorphism, and the above construction gives canonical representatives.

Example 2.8. If we take $Q = \emptyset$, then the Q -compactification of Ω is simply the *one-point compactification*. This is hence the least element in $\mathcal{L}(\Omega)$. This also explains why (b) in Definition 2.3 is only required for points in $\partial \Omega$.

At the other extreme we may take $Q = \mathcal{C}(\Omega)$. Then the Q -compactification of Ω is the *Stone-Ćech compactification*. This is the largest element in $\mathcal{L}(\Omega)$.

Later on we will look at the role of compactifications in relation to the Dirichlet problem in potential theory. In this situation the first boundary is too small to be of any real interest, since the only functions on the boundary are constant. Typically when working in potential theory one wants a boundary which is *resolutive* (roughly speaking for which continuous functions on the boundary have well defined solutions to the Dirichlet problem). The Stone-Ćech compactification is too large in general to satisfy this.

The following lemma makes it possible to further reduce the set of functions defining a compactification.

Lemma 2.9. *Assume that Q_1 is dense in $Q_2 \supset Q_1$. Then $\overline{\Omega}^{Q_1} \simeq \overline{\Omega}^{Q_2}$.*

Proof. In view of Proposition 2.6 it suffices to show that $\overline{\Omega}^{Q_2}$ is a Q_1 -compactification of Ω . If $f \in Q_1 \subset Q_2$ then, by definition, there exists $\hat{f} \in \mathcal{C}(\overline{\Omega}^{Q_2})$ such that $\hat{f}|_\Omega = f$, which verifies (a) of Definition 2.3.

To show that the functions $\{\hat{f} : f \in Q_1\}$ separate the points of $\partial^{Q_2}\Omega$, let $y, z \in \partial^{Q_2}\Omega$ be arbitrary and find $f_2 \in Q_2$ such that $\hat{f}_2(y) \neq \hat{f}_2(z)$. By denseness, there exists $f_1 \in Q_1$ such that $\|f_1 - f_2\| < \frac{1}{4}|\hat{f}_2(y) - \hat{f}_2(z)|$ (in the supremum norm), which implies that $\hat{f}_1(y) \neq \hat{f}_1(z)$. □

The following theorem characterizes the metrizable compactifications. This is probably well known to the experts in the field, but as we have not been able to find a reference we include it here.

Recall that a compact metric space is totally bounded and hence separable. We note that any second countable space is automatically separable. The converse is not true in general, but it is true for metric spaces (see e.g. Theorem 2 in Kuratowski [44], p. 177). Thus, for a metrizable compactification to exist it is necessary that Ω is second countable.

Theorem 2.10. *Assume that Ω is a locally compact noncompact second countable Hausdorff space. For a compactification $\overline{\Omega}$ of Ω the following are equivalent:*

- (a) $\overline{\Omega}$ is metrizable;
- (b) $\overline{\Omega}$ is second countable;
- (c) $\mathcal{C}(\overline{\Omega})$ is second countable;
- (d) $\mathcal{C}(\overline{\Omega})$ is separable;
- (e) there is a countable set $Q \subset \mathcal{C}(\Omega)$ such that $\overline{\Omega} \simeq \overline{\Omega}^Q$.

The above result can be compared to the Urysohn metrization theorem which states that any second countable regular Hausdorff space (in particular any locally compact second countable Hausdorff space) is metrizable, see e.g. Theorem 34.1 in Munkres [54] and Theorem 2 in Kuratowski [43], p. 42.

For $\mathcal{C}(\overline{\Omega})$ separability and second countability are equivalent (as it is a metric space), but the same is not true for $\overline{\Omega}$ itself, since it need not be metrizable. In fact, if Ω is assumed to be second countable, then Ω is necessarily separable. Hence also $\overline{\Omega}$ is separable, as Ω is dense in $\overline{\Omega}$. Since not all compactifications are metrizable it follows from Theorem 2.10 that $\overline{\Omega}$ is not always second countable.

Proof. (c) \Leftrightarrow (d) As $\mathcal{C}(\overline{\Omega})$ is a metric space this follows from the remarks above.

(a) \Rightarrow (b) Since $\overline{\Omega}$ is homeomorphic to a compact metric space, it is second countable by the remarks above.

(b) \Rightarrow (d) If $\overline{\Omega}$ is second countable, then we choose a countable base $\{G_n\}_{n=1}^\infty$ for the topology on $\overline{\Omega}$. For each pair (n, m) , such that $\overline{G}_n \cap \overline{G}_m = \emptyset$, choose a

function $f_{nm} \in \mathcal{C}(\overline{\Omega})$ which is 1 on G_n and 0 on G_m (this is possible by Tietze’s extension theorem). Together with the constant function 1, these functions form a countable set $A \subset \mathcal{C}(\overline{\Omega})$ separating the points of $\overline{\Omega}$. Let Q denote the algebra over \mathbb{Q} generated by A . Then Q is countable, and \overline{Q} is a closed algebra (over \mathbb{R}) of continuous functions which contains the constant functions and separates the points of $\overline{\Omega}$. Hence $\overline{Q} = \mathcal{C}(\overline{\Omega})$ by the Stone–Weierstrass theorem (see e.g. Corollary 1 in Stone [58], p. 179). Thus $\mathcal{C}(\overline{\Omega})$ is separable.

(d) \Rightarrow (e) By assumption there is a countable dense subset $Q' \subset \mathcal{C}(\overline{\Omega})$. If we let $Q = \{f|_{\Omega} : f \in Q'\}$, then Q is dense in $\{f|_{\Omega} : f \in \mathcal{C}(\overline{\Omega})\}$, and hence $\overline{\Omega}$ is a Q -compactification, by Lemma 2.4. The conclusion now follows from Proposition 2.6.

(e) \Rightarrow (a) Let Q' be a countable dense subset of $\mathcal{C}_c(\Omega)$, which exists as Ω is second countable. Then $Q_1 = Q \cup Q'$, extended as continuous functions on the whole space $\overline{\Omega}^Q$, is a countable set which separates the points of $\overline{\Omega}^Q$. It thus follows from Lemma 2.5 that $\overline{\Omega} \simeq \overline{\Omega}^Q \simeq \overline{\Omega}^{Q_1}$ is homeomorphic to a subset of the product space $\prod_{f \in Q_1} I_f$. As Q_1 is countable, this product space is metrizable (to see this, let $d((x_1, x_2, \dots), (y_1, y_2, \dots)) = \sum_{j=1}^{\infty} 2^{-j} |\arctan x_j - \arctan y_j|$), and hence also $\overline{\Omega}$ is metrizable. □

In particular we note that if both $\mathcal{C}(\overline{\Omega}^1)$ and $\mathcal{C}(\overline{\Omega}^2)$ are second countable, then the same is true of both their union and intersection (seen as restrictions to Ω). So both the least upper bound and the greatest lower bound of $\partial^1\Omega$ and $\partial^2\Omega$ are metrizable if $\partial^1\Omega$ and $\partial^2\Omega$ are metrizable.

Proposition 2.11. *For any set $Q \subset \mathcal{C}(\Omega)$ we have that $\overline{\Omega}^Q$ is metrizable if and only if Q contains a countable dense subset.*

Proof. If $Q_1 \subset Q$ is countable and dense, then $\overline{\Omega}^Q \simeq \overline{\Omega}^{Q_1}$, by Lemma 2.9, and Theorem 2.10 implies that $\overline{\Omega}^{Q_1}$ is metrizable.

Conversely, if $\overline{\Omega}^Q$ is metrizable then Q (seen as functions extended to elements in $\mathcal{C}(\overline{\Omega}^Q)$) is, according to Theorem 2.10, a subset of a second countable metric space (with the induced topology). Thus it is itself a second countable metric space, and hence separable. □

Above we characterized the metrizable compactifications. Assume that the space Ω has a topology given by a metric d . A natural class of continuous functions to compactify Ω with in this situation is given by

$$Q = \{d(x, \cdot) : x \in \Omega\}.$$

If Ω is separable, then it follows from Proposition 2.11 that this leads to a metrizable compactification $\overline{\Omega}^Q$, for some metric d_Q which is locally equivalent to the metric d inside Ω (i.e., it gives the same topology). However it is not always possible to choose $d_Q = d$ on $\Omega \times \Omega$. In particular this is never possible if Ω is unbounded. We do however have the following result.

Proposition 2.12. *Assume that (K, d) is a compact metric space, and that Ω is an open dense subset of K . Then $K \simeq \overline{\Omega}^Q$, where $Q = \{d(x, \cdot) : x \in \Omega\}$.*

By Lemma 2.9, it is enough to take the distance functions only with respect to a countable dense subset of Ω .

Proof. Let $\partial\Omega = K \setminus \Omega$ be the boundary induced by the metric d . Then, by definition and denseness, K is a Q -compactification of Ω . Thus, Proposition 2.6 shows that $K \simeq \overline{\Omega}^Q$. □

Proposition 2.13. *Assume that (K, d) is a compact metric space, and that Ω is an open dense subset of K . If $\widehat{Q} \subset \mathcal{C}(K)$ and we let $Q = \{f|_{\Omega} : f \in \widehat{Q}\}$, then $\partial^Q\Omega \prec \partial\Omega$.*

Proof. This is an immediate consequence of Lemma 2.4 and Proposition 2.6. □

Another way to introduce a boundary on Ω is to complete it (with respect to a given metric). If Ω is unbounded then this will of course never lead to a compact space. On the other hand, if the completion is compact, then it must be homeomorphic to the Q -compactification with $Q = \{d(x, \cdot) : x \in \Omega\}$, by Proposition 2.12.

When proving Theorem 8.14 we will need the following result.

Theorem 2.14. *Let Q be a sublattice of $\mathcal{C}_{\text{bdd}}(\Omega)$ which contains the constant functions. Then*

$$\widehat{Q} := \{f|_{\partial^Q\Omega} : f \in \mathcal{C}(\overline{\Omega}^Q) \text{ and } f|_{\Omega} \in Q\}$$

is dense in $\mathcal{C}(\partial^Q\Omega)$.

The lattice structure is with respect to max and min. To prove this we need the following fundamental theorem due to Stone, see Corollary 1 in [58], p. 170.

Theorem 2.15. *If T is a compact Hausdorff space and L is a sublattice of $\mathcal{C}(T)$ which contains the constant functions and separates the points of T , then L is dense in $\mathcal{C}(T)$.*

Proof of Theorem 2.14. It is easy to see that \widehat{Q} is a sublattice of $\mathcal{C}(\partial^Q\Omega)$ which separates the points of the compact space $\partial^Q\Omega$ by construction. Hence the statement follows from Theorem 2.15. □

3. Metric measure spaces

We assume throughout the rest of the paper that $1 < p < \infty$ and that $X = (X, d, \mu)$ is a metric space equipped with a metric d and a positive complete Borel measure μ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$ (we adopt the convention that balls are nonempty and open). It follows that X is separable. We also assume that $\Omega \subset X$ is a nonempty open set. Further standing assumptions will be given at the beginning

of Section 5. Proofs of the results in this section can be found in the monographs Björn–Björn [10] and Heinonen–Koskela–Shanmugalingam–Tyson [32].

A *curve* is a continuous mapping from an interval, and a *rectifiable* curve is a curve with finite length. We will only consider curves which are nonconstant, compact and rectifiable, and thus each curve can be parameterized by its arc length ds . A property is said to hold for *p -almost every curve* if it fails only for a curve family Γ with zero p -modulus, i.e., there exists $0 \leq \rho \in L^p(X)$ such that $\int_\gamma \rho ds = \infty$ for every curve $\gamma \in \Gamma$.

Following Heinonen–Koskela [31], we introduce upper gradients as follows (they called them very weak gradients).

Definition 3.1. A Borel function $g: X \rightarrow [0, \infty]$ is an *upper gradient* of a function $f: X \rightarrow \overline{\mathbb{R}}$ if for all curves $\gamma: [0, l_\gamma] \rightarrow X$,

$$(3.1) \quad |f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g ds,$$

where the left-hand side is considered to be ∞ whenever at least one of the terms therein is infinite. If $g: X \rightarrow [0, \infty]$ is measurable and (3.1) holds for p -almost every curve, then g is a *p -weak upper gradient* of f .

The p -weak upper gradients were introduced in Koskela–MacManus [42]. It was also shown therein that if $g \in L^p_{\text{loc}}(X)$ is a p -weak upper gradient of f , then one can find a sequence $\{g_j\}_{j=1}^\infty$ of upper gradients of f such that $g_j - g \rightarrow 0$ in $L^p(X)$. If f has an upper gradient in $L^p_{\text{loc}}(X)$, then it has an a.e. unique *minimal p -weak upper gradient* $g_f \in L^p_{\text{loc}}(X)$ in the sense that for every p -weak upper gradient $g \in L^p_{\text{loc}}(X)$ of f we have $g_f \leq g$ a.e., see Shanmugalingam [56]. Following Shanmugalingam [55], we define a version of Sobolev spaces on the metric space X .

Definition 3.2. For a measurable function $f: X \rightarrow \overline{\mathbb{R}}$, let

$$\|f\|_{N^{1,p}(X)} = \left(\int_X |f|^p d\mu + \inf_g \int_X g^p d\mu \right)^{1/p},$$

where the infimum is taken over all upper gradients g of f . The *Newtonian space* on X is

$$N^{1,p}(X) = \{f : \|f\|_{N^{1,p}(X)} < \infty\}.$$

The quotient space $N^{1,p}(X)/\sim$, where $f \sim h$ if and only if $\|f - h\|_{N^{1,p}(X)} = 0$, is a Banach space and a lattice, see Shanmugalingam [55]. We also define

$$D^p(X) = \{f : f \text{ is measurable and has an upper gradient in } L^p(X)\}.$$

In this paper we assume that functions in $N^{1,p}(X)$ and $D^p(X)$ are defined everywhere (with values in $\overline{\mathbb{R}}$), not just up to an equivalence class in the corresponding function space.

For a measurable set $E \subset X$, the Newtonian space $N^{1,p}(E)$ is defined by considering $(E, d|_E, \mu|_E)$ as a metric space in its own right. We say that $f \in N^{1,p}_{\text{loc}}(E)$ if for every $x \in E$ there exists a ball $B_x \ni x$ such that $f \in N^{1,p}(B_x \cap E)$.

The spaces $D^p(E)$ and $D^p_{\text{loc}}(E)$ are defined similarly. If $f, h \in D^p_{\text{loc}}(X)$, then $g_f = g_h$ a.e. in $\{x \in X : f(x) = h(x)\}$, in particular $g_{\min\{f,c\}} = g_f \chi_{\{f < c\}}$ for $c \in \mathbb{R}$.

Definition 3.3. Let $\Omega \subset X$ be an open set. The (Sobolev) *capacity* (with respect to Ω) of a set $E \subset \Omega$ is the number

$$C_p(E; \Omega) = \inf_u \|u\|_{N^{1,p}(\Omega)}^p,$$

where the infimum is taken over all $u \in N^{1,p}(\Omega)$ such that $u = 1$ on E .

Observe that the above capacity is *not* the so-called variational capacity. For a given set E we will consider the capacity taken with respect to different sets Ω . When the capacity is taken with respect to the underlying metric space X , we usually drop X from the notation and merely write $C_p(E)$. The capacity is countably subadditive.

We say that a property holds *quasieverywhere* (q.e.) if the set of points for which the property does not hold has capacity zero. The capacity is the correct gauge for distinguishing between two Newtonian functions. If $u \in N^{1,p}(X)$, then $u \sim v$ if and only if $u = v$ q.e. Moreover, if $u, v \in D^p_{\text{loc}}(X)$ and $u = v$ a.e., then $u = v$ q.e.

The space of Newtonian functions with zero boundary values is defined by

$$N^{1,p}_0(\Omega) = \{f|_{\Omega} : f \in N^{1,p}(X) \text{ and } f = 0 \text{ in } X \setminus \Omega\}.$$

The space $D^p_0(\Omega)$ is defined analogously.

We say that μ is *doubling* if there exists a *doubling constant* $C > 0$ such that for all balls $B = B(x_0, r) := \{x \in X : d(x, x_0) < r\}$ in X ,

$$0 < \mu(2B) \leq C\mu(B) < \infty,$$

where $\lambda B = B(x_0, \lambda r)$. Recall that X is *proper* if all closed bounded subsets of X are compact. If μ is doubling then X is proper if and only if it is complete.

Definition 3.4. We say that X supports a (q, p) -*Poincaré inequality* if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B \subset X$, all integrable functions f on X and all $(p$ -weak) upper gradients g of f ,

$$\left(\int_B |f - f_B|^q d\mu \right)^{1/q} \leq C \text{diam}(B) \left(\int_{\lambda B} g^p d\mu \right)^{1/p},$$

where $f_B := \int_B f d\mu / \mu(B)$.

If X is complete and supports a $(1, p)$ -Poincaré inequality and μ is doubling, then Lipschitz functions are dense in $N^{1,p}(X)$, see Shanmugalingam [55], and the functions in $N^{1,p}(\Omega)$ are *quasicontinuous*, i.e., for every $\varepsilon > 0$ there is an open set U such that $C_p(U) < \varepsilon$ and $f|_{\Omega \setminus U}$ is real-valued continuous, see Björn–Björn–Shanmugalingam [15]. This means that in the Euclidean setting, $N^{1,p}(\mathbb{R}^n)$ is the refined Sobolev space as defined in Heinonen–Kilpeläinen–Martio [30], p. 96; see Appendix A.2 in [10] for a proof of this fact valid in weighted \mathbb{R}^n .

4. The capacity $\overline{C}_p(\cdot; \Omega^1)$

In Björn–Björn–Shanmugalingam [16] the capacity $\overline{C}_p(\cdot; \Omega)$ was introduced. A similar capacity was considered in Kilpeläinen–Malý [36]. It can also be compared to the reduction up to the boundary of superharmonic functions as defined in Section 1.III.4 in Doob [23]. In [16], such a capacity was also defined with respect to the Mazurkiewicz distance

$$(4.1) \quad d_M(x, y) := \inf_E \text{diam } E,$$

where the infimum is over all connected sets $E \subset \Omega$ containing $x, y \in \Omega$. The completion of Ω with respect to d_M is compact if and only if Ω is finitely connected at the boundary, by Theorem 1.1 in Björn–Björn–Shanmugalingam [17] or Theorem 1.3.8 in Karmazin [34]. (See [16] or [17] for the definition of finite connectedness at the boundary.) In this paper we consider the same capacity for arbitrary compactifications of Ω , which will turn out useful in the study of Perron solutions later in the paper. As in Section 2, we let $\overline{\Omega}^1 = \Omega \cup \partial^1 \Omega$ be an arbitrary compactification of Ω with topology τ^1 . Sometimes we will also use the somewhat abusive but convenient notation Ω^1 to denote the open set Ω with indication of the particular compactification $\overline{\Omega}^1$.

Throughout the paper, the Newtonian space $N^{1,p}(\Omega)$ is always taken with respect to the underlying metric d .

Definition 4.1. For $E \subset \overline{\Omega}^1$ let

$$\overline{C}_p(E; \Omega^1) = \inf_{u \in \mathcal{A}_E(\Omega^1)} \|u\|_{N^{1,p}(\Omega)}^p,$$

where $u \in \mathcal{A}_E(\Omega^1)$ if $u \in N^{1,p}(\Omega)$ satisfies both $u \geq 1$ on $E \cap \Omega$ and

$$\liminf_{\Omega \ni y \xrightarrow{\tau^1} x} u(y) \geq 1 \quad \text{for all } x \in E \cap \partial^1 \Omega.$$

By truncation it is easy to see that one may as well take the infimum over all $u \in \widetilde{\mathcal{A}}_E(\Omega^1) := \{u \in \mathcal{A}_E(\Omega^1) : 0 \leq u \leq 1\}$. For $E \subset \Omega$ we have $\overline{C}_p(E; \Omega^1) = C_p(E; \Omega)$.

The capacity $\overline{C}_p(\cdot; \Omega^1)$ is easily shown to be monotone and countably subadditive, cf. Proposition 3.2 in [16]. It is also an outer capacity, which will be important for us.

Proposition 4.2. *Assume that all functions in $N^{1,p}(\Omega)$ are quasicontinuous. Then $\overline{C}_p(\cdot; \Omega^1)$ is an outer capacity, i.e., for all $E \subset \overline{\Omega}^1$,*

$$\overline{C}_p(E; \Omega^1) = \inf_{\substack{G \supseteq E \\ G \text{ is } \tau^1\text{-open in } \overline{\Omega}^1}} \overline{C}_p(G; \Omega^1).$$

Proof. The proof of Proposition 3.3 in Björn–Björn–Shanmugalingam [16], i.e., of the corresponding result for the metric boundary, applies almost verbatim. The only slight difference is that instead of selecting r_x , and thus implicitly the neighbourhood $B(x, r_x) \cap \overline{\Omega}$ of x , one should choose a τ^1 -neighbourhood of x . \square

For the sake of clarity we make the following explicit definition.

Definition 4.3. A function $f \in \overline{\Omega}^1 \rightarrow \overline{\mathbb{R}}$ is $\overline{C}_p(\cdot; \Omega^1)$ -quasicontinuous if for every $\varepsilon > 0$ there is a τ^1 -open set $U \subset \overline{\Omega}^1$ such that $\overline{C}_p(U; \Omega^1) < \varepsilon$ and $f|_{\overline{\Omega}^1 \setminus U}$ is real-valued continuous.

For the Dirichlet problem in this paper it is important to know when functions in $N_0^{1,p}(\Omega)$ are quasicontinuous.

Proposition 4.4. Let $f: \Omega \rightarrow \overline{\mathbb{R}}$ and let

$$f_j = \begin{cases} f, & \text{in } \Omega, \\ 0, & \text{on } \partial^j \Omega, \end{cases} \quad j = 1, 2.$$

Then f_1 is $\overline{C}_p(\cdot; \Omega^1)$ -quasicontinuous if and only if f_2 is $\overline{C}_p(\cdot; \Omega^2)$ -quasicontinuous.

In particular, if X is proper, Ω bounded and all functions in $N^{1,p}(X)$ are $C_p(\cdot)$ -quasicontinuous, then every $f \in N_0^{1,p}(\Omega)$ is $\overline{C}_p(\cdot; \Omega^1)$ -quasicontinuous for every compactification $\overline{\Omega}^1$.

Proof. Let $\varepsilon > 0$ and assume that f_1 is $\overline{C}_p(\cdot; \Omega^1)$ -quasicontinuous. Then there is a τ^1 -open set $G \subset \overline{\Omega}^1$ such that $\overline{C}_p(G, \Omega^1) < \varepsilon$ and $f_1|_{\overline{\Omega}^1 \setminus G}$ is real-valued continuous. Let $U = G \cap \Omega$. It is easy to see that $\mathcal{A}_U(\Omega^1) = \mathcal{A}_U(\Omega^2)$ since $U \subset \Omega$, and hence $\overline{C}_p(U, \Omega^2) = \overline{C}_p(U, \Omega^1) \leq \overline{C}_p(G, \Omega^1) < \varepsilon$.

It is immediate that $f_2|_{\overline{\Omega}^2 \setminus U}$ is continuous at all $x \in \Omega \setminus U$, so we need to prove continuity at points in $\partial^2 \Omega$. Since f_1 is continuous on the compact set $\overline{\Omega}^1 \setminus U$, the set $K = \{x \in \overline{\Omega}^1 \setminus U : |f_1(x)| \geq \alpha\}$ is compact for any $\alpha > 0$. As $f_1 = 0$ on $\partial^1 \Omega$, we see that $K \subset \Omega$. It follows that $|f_2| = |f_1| < \alpha$ in $(\Omega \setminus U) \setminus K$ and $f_2 = 0$ on $\partial^2 \Omega$. Since $\overline{\Omega}^2 \setminus K$ is a τ^2 -neighbourhood of every point y in $\partial^2 \Omega$ we conclude that $f_2|_{\overline{\Omega}^2 \setminus U}$ is continuous at all $y \in \partial^2 \Omega$. The converse direction follows by swapping the roles of Ω^1 and Ω^2 .

For the last part, extend f by zero in $X \setminus \Omega$. Then $f \in N^{1,p}(X)$ and is thus $C_p(\cdot)$ -quasicontinuous, by assumption. By Lemma 5.2 in Björn–Björn–Shanmugalingam [16], $\overline{C}_p(\cdot, \Omega)$ is majorized by $C_p(\cdot)$, and thus the restriction $f|_{\overline{\Omega}}$ to the metric closure $\overline{\Omega}$ of Ω (induced by X) is $\overline{C}_p(\cdot; \Omega)$ -quasicontinuous. As X is proper, the set $\overline{\Omega}$ is compact. Thus the $\overline{C}_p(\cdot; \Omega^1)$ -quasicontinuity of f follows from the first part. \square

We conclude this section with the following modification of Lemma 5.3 in Björn–Björn–Shanmugalingam [13], which will be useful later.

Lemma 4.5. *Let $\{U_k\}_{k=1}^\infty$ be a decreasing sequence of τ^1 -open sets in $\overline{\Omega}^1$ such that $\overline{C}_p(U_k; \Omega^1) < 2^{-kp}$. Then there exists a decreasing sequence of nonnegative functions $\{\psi_j\}_{j=1}^\infty$ on Ω such that $\|\psi_j\|_{N^{1,p}(\Omega)} < 2^{-j}$ and $\psi_j \geq k - j$ in $U_k \cap \Omega$.*

Proof. Let $\psi_j = \sum_{k=j+1}^\infty f_k$, where $f_k \in \mathcal{A}_{U_k}(\Omega^1)$ with $\|f_k\|_{N^{1,p}(\Omega)} < 2^{-k}$. □

Remark 4.6. The results in this section hold also when $p = 1$.

5. p -harmonic and superharmonic functions

We assume from now on that X is a complete metric space supporting a $(1, p)$ -Poincaré inequality, that μ is doubling, and that $1 < p < \infty$. We also assume that Ω is a bounded domain such that $C_p(X \setminus \Omega) > 0$, and that $\overline{\Omega}^j = \Omega^j \cup \partial^j \Omega$, $j = 1, 2$, are compactifications of Ω , where $\Omega^j = \Omega$ with the intended boundary $\partial^j \Omega$. Furthermore, we reserve $\partial \Omega$ and $\overline{\Omega}$ for the metric boundary and closure induced by X on Ω .

Recall that a *domain* is a nonempty connected open set.

Before introducing p -harmonic and superharmonic functions, we will draw some conclusions from our standing assumptions above. First of all, functions in $N_{\text{loc}}^{1,p}(\Omega)$ are quasicontinuous and the C_p capacity is an outer capacity, by Theorem 1.1 and Corollary 1.3 in Björn–Björn–Shanmugalingam [15] (or Theorems 5.29 and 5.31 in [10]). Next, observe that X supports a (p, p) -Poincaré inequality, by Theorem 5.1 in Hajlasz–Koskela [26] (or Corollary 4.24 in [10]). Thus, by Proposition 4.14 in [10], $D_{\text{loc}}^p(\Omega) = N_{\text{loc}}^{1,p}(\Omega)$ and, by Proposition 2.7 in Björn–Björn [11], $D_0^p(\Omega) = N_0^{1,p}(\Omega)$.

Definition 5.1. A function $u \in N_{\text{loc}}^{1,p}(\Omega)$ is a (*super*)*minimizer* in Ω if

$$\int_{\varphi \neq 0} g_u^p d\mu \leq \int_{\varphi \neq 0} g_{u+\varphi}^p d\mu \quad \text{for all (nonnegative) } \varphi \in N_0^{1,p}(\Omega).$$

A *p*-harmonic function is a continuous minimizer.

For various characterizations of (super)minimizers see A. Björn [6]. It was shown in Kinnunen–Shanmugalingam [40] that under the above assumptions, a minimizer can be modified on a set of zero capacity to obtain a p -harmonic function. For a superminimizer u , it was shown by Kinnunen–Martio [39] that its *lsc-regularization*

$$u^*(x) := \text{ess lim inf}_{y \rightarrow x} u(y) = \lim_{r \rightarrow 0} \text{ess inf}_{B(x,r)} u$$

is also a superminimizer and $u^* = u$ q.e.

In this paper we will use the following obstacle problem.

Definition 5.2. For $f \in D^p(\Omega)$ and $\psi: \Omega \rightarrow \overline{\mathbb{R}}$, we set

$$\mathcal{K}_{\psi,f}(\Omega) = \{v \in D^p(\Omega) : v - f \in N_0^{1,p}(\Omega) \text{ and } v \geq \psi \text{ q.e. in } \Omega\}.$$

A function $u \in \mathcal{K}_{\psi,f}(\Omega)$ is a *solution of the $\mathcal{K}_{\psi,f}(\Omega)$ -obstacle problem* if

$$\int_{\Omega} g_u^p d\mu \leq \int_{\Omega} g_v^p d\mu \quad \text{for all } v \in \mathcal{K}_{\psi,f}(\Omega).$$

A solution of the $\mathcal{K}_{\psi,f}(\Omega)$ -obstacle problem is easily seen to be a superminimizer in Ω . Conversely, a superminimizer u in Ω is a solution of the $\mathcal{K}_{u,u}(G)$ -obstacle problem for any open $G \Subset \Omega$.

If $\mathcal{K}_{\psi,f}(\Omega) \neq \emptyset$, then there is a solution u of the $\mathcal{K}_{\psi,f}(\Omega)$ -obstacle problem, which is unique up to sets of capacity zero. Moreover, u^* is the unique lsc-regularized solution. If the obstacle ψ is continuous, then u^* is also continuous. See Björn–Björn [11], Hansevi [27] and Kinnunen–Martio [39]. The obstacle ψ , as a continuous function, is even allowed to take the value $-\infty$.

Given $f \in D^p(\Omega)$, we let $H_{\Omega}f$ denote the continuous solution of the $\mathcal{K}_{-\infty,f}(\Omega)$ -obstacle problem; this function is p -harmonic in Ω and takes on the same boundary values (in the Sobolev sense) as f on $\partial\Omega$, and hence it is also called the solution of the Dirichlet problem with Sobolev boundary values.

Definition 5.3. A function $u: \Omega \rightarrow (-\infty, \infty]$ is *superharmonic* in Ω if

- (i) u is lower semicontinuous;
- (ii) u is not identically ∞ ;
- (iii) for every nonempty open set $G \Subset \Omega$ and all functions $v \in \text{Lip}(X)$, we have $H_G v \leq u$ in G whenever $v \leq u$ on ∂G .

A function $u: \Omega \rightarrow [-\infty, \infty)$ is *subharmonic* in Ω if $-u$ is superharmonic.

This definition of superharmonicity is equivalent to the ones in Heinonen–Kilpeläinen–Martio [30] and Kinnunen–Martio [39], see Theorem 6.1 in A. Björn [5]. A superharmonic function which is either locally bounded or belongs to $N_{\text{loc}}^{1,p}(\Omega) = D_{\text{loc}}^p(\Omega)$ is a superminimizer, and all superharmonic functions are lsc-regularized. Conversely, any lsc-regularized superminimizer is superharmonic.

We will also need the following proposition, which in this generality is due to Hansevi, see Theorem 3.4 and Proposition 4.5 in his paper [28].

Proposition 5.4. *Let $\{f_j\}_{j=1}^{\infty}$ be a q.e. decreasing sequence of functions in $D^p(\Omega)$ such that $g_{f_j-f} \rightarrow 0$ in $L^p(\Omega)$ as $j \rightarrow \infty$. Then $H_{\Omega}f_j$ decreases to $H_{\Omega}f$ locally uniformly in Ω .*

Moreover, if u and u_j are solutions of the $\mathcal{K}_{f,f}(\Omega)$ - and $\mathcal{K}_{f_j,f_j}(\Omega)$ -obstacle problems, $j = 1, 2, \dots$, then $\{u_j\}_{j=1}^{\infty}$ decreases q.e. in Ω to u .

For further discussion and references on these topics see [10].

6. Perron solutions with respect to Ω^1

We are now going to introduce the Perron solutions with respect to Ω^1 . The solution $H_{\Omega}f$ of the Dirichlet problem with Sobolev boundary values f was introduced

in Section 5. It is defined without use of the boundary, and hence is independent of which compactification of Ω we use. Similarly, the obstacle problem $\mathcal{K}_{\psi,f}(\Omega)$ and its solutions are independent of the compactification. For this reason we will often drop Ω from the notation and write $Hf = H_{\Omega}f$ and $\mathcal{K}_{\psi,f} = \mathcal{K}_{\psi,f}(\Omega)$.

The Perron solutions are however highly dependent on the compactification, since that is where their boundary values are defined.

Definition 6.1. Given $f: \partial^1\Omega \rightarrow \overline{\mathbb{R}}$, let $\mathcal{U}_f(\Omega^1)$ be the set of all superharmonic functions u on Ω , bounded from below, such that

$$(6.1) \quad \liminf_{\Omega \ni y \xrightarrow{\tau^1} x} u(y) \geq f(x)$$

for all $x \in \partial^1\Omega$. The *upper Perron solution* of f is then defined to be

$$\overline{P}_{\Omega^1} f(x) = \inf_{u \in \mathcal{U}_f(\Omega^1)} u(x), \quad x \in \Omega,$$

while the *lower Perron solution* of f is defined by

$$\underline{P}_{\Omega^1} f = -\overline{P}_{\Omega^1}(-f).$$

(It can equivalently be defined using subharmonic functions bounded from above.) If $\overline{P}_{\Omega^1} f = \underline{P}_{\Omega^1} f$ and it is real-valued, then we let $P_{\Omega^1} f := \overline{P}_{\Omega^1} f$ and f is said to be *resolutive* with respect to Ω^1 .

Furthermore, let $\mathcal{DU}_f(\Omega^1) = \mathcal{U}_f(\Omega^1) \cap D^p(\Omega)$, and define the *Sobolev–Perron solutions* of f by

$$(6.2) \quad \overline{S}_{\Omega^1} f(x) = \inf_{u \in \mathcal{DU}_f(\Omega^1)} u(x), \quad x \in \Omega, \quad \text{and} \quad \underline{S}_{\Omega^1} f = -\overline{S}_{\Omega^1}(-f).$$

If $\overline{S}_{\Omega^1} f = \underline{S}_{\Omega^1} f$ and it is real-valued, then we let $S_{\Omega^1} f := \overline{S}_{\Omega^1} f$ and f is said to be *Sobolev-resolutive* with respect to Ω^1 .

Note that, as Ω is bounded, the upper and lower Sobolev–Perron solutions for *bounded* f can equivalently be defined taking the infimum in (6.2) over $u \in \mathcal{U}_f(\Omega^1) \cap N^{1,p}(\Omega)$.

There are two main reasons for why we have chosen to introduce the Sobolev–Perron solutions. First of all, in most of our resolitivity results we are able to deduce Sobolev-resolitivity (which is a stronger property by Corollary 6.3 and Examples 6.5 and 6.6). Secondly, in Proposition 7.1 we show that Sobolev–Perron solutions are always invariant under perturbations on sets of capacity zero, while for the standard Perron solutions this is only known for some functions. Moreover, for the results in Section 10 it is essential that we use Sobolev–Perron solutions (at least for our methods).

For the metric boundary, $\overline{P}_{\Omega} f$ is p -harmonic unless it is identically $\pm\infty$, see Theorem 4.1 in Björn–Björn–Shanmugalingam [13] (or Theorem 10.10 in [10]). The proof therein applies also to $\overline{P}_{\Omega^1} f$ without any change, whereas for $\overline{S}_{\Omega^1} f$ one also needs to observe that the Poisson modification of a superharmonic function in $D^p(\Omega)$ belongs to $D^p(\Omega)$, which is rather immediate.

A direct consequence of the above definition is that if $f_1 \leq f_2$, then

$$(6.3) \quad \overline{P}_{\Omega^1} f_1 \leq \overline{P}_{\Omega^1} f_2 \quad \text{and} \quad \overline{S}_{\Omega^1} f_1 \leq \overline{S}_{\Omega^1} f_2.$$

The following comparison principle makes it possible to compare the upper and lower Perron solutions.

Proposition 6.2. *Assume that u is superharmonic and that v is subharmonic in Ω . If*

$$\infty \neq \limsup_{\Omega \ni y \xrightarrow{\tau^1} x} v(y) \leq \liminf_{\Omega \ni y \xrightarrow{\tau^1} x} u(y) \neq -\infty \quad \text{for all } x \in \partial^1 \Omega,$$

then $v \leq u$ in Ω .

The corresponding result with respect to the given metric d was obtained in Theorem 7.2 in Kinnunen–Martio [39], while for the Mazurkiewicz distance d_M (when it gives a compactification, i.e., when Ω is finitely connected at the boundary) it was given as Proposition 7.2 in Björn–Björn–Shanmugalingam [16]. See also Proposition 7.3 in Estep–Shanmugalingam [24].

Proof. The proof of Proposition 7.2 in [16] applies almost verbatim. The only slight difference is that instead of taking a ball $B_x^M \ni x$ one should choose a τ^1 -neighbourhood of x . (The proof in [39] does not generalize so easily to arbitrary compactifications.) \square

Corollary 6.3. *If $f: \partial^1 \Omega \rightarrow \mathbb{R}$, then*

$$\underline{S}_{\Omega^1} f \leq \underline{P}_{\Omega^1} f \leq \overline{P}_{\Omega^1} f \leq \overline{S}_{\Omega^1} f.$$

In particular, if f is Sobolev-resolutive, then f is resolutive.

The following is one of the main results in this theory.

Theorem 6.4. *Let $f: \overline{\Omega}^1 \rightarrow \overline{\mathbb{R}}$ be a $\overline{C}_p(\cdot; \Omega^1)$ -quasicontinuous function such that $f|_{\Omega} \in D^p(\Omega)$. Then f is Sobolev-resolutive with respect to Ω^1 and*

$$S_{\Omega^1} f = P_{\Omega^1} f = Hf.$$

There are several earlier versions of this result, the first in Theorem 5.1 in Björn–Björn–Shanmugalingam [13]. This was subsequently generalized in Theorem 10.12 in [10], Theorem 9.1 in Björn–Björn–Shanmugalingam [16] and Theorem 8.1 in Hansevi [28] for the metric boundary. Hansevi’s result also applies to some unbounded domains. In Theorem 7.4 in [16] a similar result was obtained for the Mazurkiewicz boundary of domains which are finitely connected at the boundary. All of these results give resolutive, but the proofs actually also show Sobolev-resolutive, a fact that has not been noticed earlier, as the Sobolev–Perron solutions have not been introduced before this paper.

The following examples show that Sobolev-resolutive can be quite restrictive compared with resolutive, nevertheless we obtain Sobolev-resolutive in Theorem 6.4 (whose proof we postpone until after the examples).

Example 6.5. Let Ω be a ball in \mathbb{R}^n , $n \geq 2$, and let $p > n$. Let $E \subset \partial\Omega$ be a dense subset such that also $\partial\Omega \setminus E$ is dense in $\partial\Omega$, and let $f = \chi_E$. As Ω is an extension domain, the Sobolev embedding theorem shows that any bounded $u \in \mathcal{DU}_f$ has a continuous extension to $\overline{\Omega}$. Hence $u \geq 1$ everywhere in Ω , so $\overline{S}_\Omega f \geq 1$. Similarly, $\underline{S}_\Omega f \leq 0$. Thus $f = \chi_E$ is not Sobolev-resolutive. In particular we may choose E to be countable, in which case f is resolutive and $P_\Omega f \equiv 0$ by Theorem 1.3 in A. Björn [7].

Example 6.6. Let $\Omega = (0, 1)^n$ be the open unit cube in \mathbb{R}^n , $n \geq 2$, and let $1 < p \leq n$. We shall construct a subset $E \subset \partial\Omega$ with zero $(n - 1)$ -dimensional measure, such that $\overline{S}_\Omega \chi_E \equiv 1$.

In the case when $p = 2$, it follows directly that E has zero harmonic measure, so that $P_\Omega \chi_E \equiv 0$, and thus also $\underline{S}_\Omega \chi_E \equiv 0$. Hence, χ_E is resolutive but not Sobolev-resolutive.

The last deduction, using the harmonic measure, only works in the linear case, but the construction of E up to that point is as valid in the nonlinear situation as in the linear. To emphasize this we give the construction for general $1 < p \leq n$.

Let $\tilde{C} \subset [0, 1]$ be a selfsimilar Cantor set with endpoints 0 and 1 and selfsimilarity constant $0 < \alpha < 1/2$, i.e., α is the largest value such that $\alpha\tilde{C} = \tilde{C} \cap [0, \alpha]$. Next, let $C = \tilde{C}^{n-1}$,

$$\tilde{E} = [0, 1]^{n-1} \cap (C + \mathbb{Q}^{n-1}) := \{x + q \in [0, 1]^{n-1} : x \in C \text{ and } q \in \mathbb{Q}^{n-1}\}$$

and let E be the union of \tilde{E} and the affine copies of \tilde{E} to the other faces of $[0, 1]^n$. As \mathbb{Q} is countable, $\dim_{\mathbb{H}} E = \dim_{\mathbb{H}} C = -(n - 1) \log 2 / \log \alpha$. From now on, we require α to be so large that $\dim_{\mathbb{H}} E > n - p$.

Next, let $\tilde{u} \in \mathcal{DU}_{\chi_E}(\Omega)$. Then $u = \min\{\tilde{u}, 1\} \in \mathcal{U}_{\chi_E}(\Omega) \cap N^{1,p}(\Omega)$. As Ω is an extension domain we may assume that $u \in N^{1,p}(\mathbb{R}^n)$. We shall show below that $u = 1$ q.e. on $\partial\Omega$. Since u is an lsc-regularized solution of the $\mathcal{K}_{u,u}(\Omega)$ -obstacle problem (by Proposition 7.15 in [10]), the comparison principle (Lemma 8.30 in [10]) then implies that $u \geq H1 = 1$ in Ω , and thus $\overline{S}_\Omega \chi_E \equiv 1$.

To show that $u = 1$ q.e. on $\partial\Omega$, we will use fine continuity, and we therefore need to introduce some more terminology. A set $A \subset \mathbb{R}^n$ is *thin* at x if

$$\int_0^1 \left(\frac{\text{cap}_p(A \cap B(x, t), B(x, 2t))}{\text{cap}_p(B(x, t), B(x, 2t))} \right)^{1/(p-1)} \frac{dt}{t} < \infty,$$

where cap_p is the *variational capacity* defined by

$$\text{cap}_p(A, B) = \inf \int_B g_u^p dx,$$

with the infimum taken over all $u \in N_0^{1,p}(B)$ such that $u = 1$ on A . A set $U \subset \mathbb{R}^n$ is *finely open* if $X \setminus U$ is thin at every $x \in U$. A function $u : U \rightarrow \overline{\mathbb{R}}$ is *finely continuous* at $x \in U$ if for every $\varepsilon > 0$ there is a finely open set $V \ni x$ such that $|u(y) - u(x)| < \varepsilon$ for $y \in V$ (this should be suitably modified if $u(x) = \pm\infty$). By Theorem 4.6 in J. Björn [18] or Corollary 4.4 in Korte [41] (or Theorem 11.40 in [10]), any Newtonian function, and in particular u , is finely continuous q.e. Thus

there is a set A_u with zero capacity such that u is finely continuous at all $x \in \mathbb{R}^n \setminus A_u$. We claim that E is nonthin at every $x \in \partial\Omega$. From this it follows that every fine neighbourhood of x contains points in E . Hence $u(x) = \lim_{E \ni y \rightarrow x} u(y) = 1$ if $x \in \partial\Omega \setminus A_u$.

To show that E is nonthin at every $x \in \partial\Omega$, assume without loss of generality that $x \in [0, 1/2]^{n-1}$. Let $B_s = B(0, s)$. By the selfsimilarity of C and Theorem 2.27 in Heinonen–Kilpeläinen–Martio [30], we see that

$$(6.4) \quad \text{cap}_p(C \cap B_{\alpha t}, B_{5\alpha t}) = \alpha^{n-p} \text{cap}_p(C \cap B_t, B_{5t}) > 0 \quad \text{if } 0 < t < 1,$$

since $\dim_{\mathbb{H}} E > n - p$. Also

$$(6.5) \quad \text{cap}_p(B(x, \alpha t), B(x, 2\alpha t)) = \alpha^{n-p} \text{cap}_p(B_t, B_{2t}) \quad \text{if } t > 0.$$

If $0 < t < 1/2$, then we can find $q \in \mathbb{Q}^{n-1} \cap [0, 1/2]^{n-1}$ such that $|q - x| < t/2$. By the monotonicity and translation invariance of cap_p ,

$$\begin{aligned} \text{cap}_p(E \cap B(x, t), B(x, 2t)) &\geq \text{cap}_p(E \cap B(q, t/2), B(q, 5t/2)) \\ &\geq \text{cap}_p(C \cap B_{t/2}, B_{5t/2}). \end{aligned}$$

It follows from this and the scaling invariances (6.4) and (6.5) that for $m = 1, 2, \dots$,

$$\begin{aligned} \int_{\alpha^{m+1}}^{\alpha^m} \left(\frac{\text{cap}_p(E \cap B(x, t), B(x, 2t))}{\text{cap}_p(B(x, t), B(x, 2t))} \right)^{1/(p-1)} \frac{dt}{t} \\ &\geq \int_{\alpha^{m+1}}^{\alpha^m} \left(\frac{\text{cap}_p(C \cap B_{t/2}, B_{5t/2})}{\text{cap}_p(B_t, B_{2t})} \right)^{1/(p-1)} \frac{dt}{t} \\ &= \int_{\alpha^2}^{\alpha} \left(\frac{\text{cap}_p(C \cap B_{t/2}, B_{5t/2})}{\text{cap}_p(B_t, B_{2t})} \right)^{1/(p-1)} \frac{dt}{t} \\ &> 0. \end{aligned}$$

Hence

$$\int_0^1 \left(\frac{\text{cap}_p(E \cap B(x, t), B(x, 2t))}{\text{cap}_p(B(x, t), B(x, 2t))} \right)^{1/(p-1)} \frac{dt}{t} = \infty,$$

i.e., E is nonthin at x , which concludes the argument.

Proof of Theorem 6.4. The proof is very close to the one given for Theorem 7.4 in [16]. Therein, properties related to $\overline{C}_p(\cdot; \Omega)$ and $\overline{\Omega}$ need to be replaced by similar ones for $\overline{C}_p(\cdot; \Omega^1)$ and $\overline{\Omega}^1$, which are provided by Proposition 4.4, Lemma 4.5 and Corollary 6.3 here. As we also want to use functions merely in D^p we need to use Proposition 5.4. Moreover, the open sets and neighbourhoods need to be taken with respect to the τ^1 -topology on $\overline{\Omega}^1$.

To deduce Sobolev-resolutivity, it suffices to observe that the solutions φ_j of the obstacle problems in the proof belong to $\mathcal{DU}_f(\Omega^1)$ (which shows that $\varphi_j \geq \overline{S}_{\Omega^1} f$), and to replace $\overline{P}_{\Omega^M} f$ and $\underline{P}_{\Omega^M} f$ by $\overline{S}_{\Omega^1} f$ and $\underline{S}_{\Omega^1} f$ in the remaining part of the proof. Finally, $P_{\Omega^1} f = S_{\Omega^1} f$, by Corollary 6.3. \square

We end this section by comparing Perron solutions with respect to different compactifications.

Theorem 6.7. *Let $\partial^1\Omega \prec \partial^2\Omega$, $\Phi: \overline{\Omega}^2 \rightarrow \overline{\Omega}^1$ denote the projection, and let $f: \partial^1\Omega \rightarrow \overline{\mathbb{R}}$. Then*

$$(6.6) \quad \overline{P}_{\Omega^1} f = \overline{P}_{\Omega^2}(f \circ \Phi) \quad \text{and} \quad \underline{P}_{\Omega^1} f = \underline{P}_{\Omega^2}(f \circ \Phi).$$

In particular, f is resolutive if and only if $f \circ \Phi: \partial^2\Omega \rightarrow \mathbb{R}$ is resolutive, and in that case $P_{\Omega^1} f = P_{\Omega^2}(f \circ \Phi)$.

Similarly,

$$\overline{S}_{\Omega^1} f = \overline{S}_{\Omega^2}(f \circ \Phi) \quad \text{and} \quad \underline{S}_{\Omega^1} f = \underline{S}_{\Omega^2}(f \circ \Phi),$$

and f is Sobolev-resolutive if and only if $f \circ \Phi$ is Sobolev-resolutive, in which case $S_{\Omega^1} f = S_{\Omega^2}(f \circ \Phi)$.

Proof. Let $u \in \mathcal{U}_{f \circ \Phi}(\Omega^2)$. We shall prove that $u \in \mathcal{U}_f(\Omega^1)$, i.e., that for any $\varepsilon > 0$ and any $y \in \partial^1\Omega$ there is a τ^1 -neighbourhood U of y in $\overline{\Omega}^1$ such that

$$(6.7) \quad u(x) + \varepsilon > f(y) \quad \text{for all } x \in U \cap \Omega.$$

(If $f(y) = \infty$ we instead require that $u(x) > 1/\varepsilon$, while if $f(y) = -\infty$ there is nothing to prove.) To do so, note that by definition there is for every $z \in \Phi^{-1}(y)$ a τ^2 -neighbourhood G_z of z in $\overline{\Omega}^2$ such that $u(x) + \varepsilon > f(\Phi(z)) = f(y)$ for all $x \in G_z \cap \Omega$. Let

$$K = \Phi(\overline{\Omega}^2 \setminus G), \quad \text{where } G = \bigcup_{z \in \Phi^{-1}(y)} G_z.$$

Then K is a compact subset of $\overline{\Omega}^1$, and $U = \overline{\Omega}^1 \setminus K$ is a τ^1 -neighbourhood of y . Now clearly $\Phi^{-1}(U) \subset G$ and hence (6.7) holds, as required.

We have thus shown that $\mathcal{U}_{f \circ \Phi}(\Omega^2) \subset \mathcal{U}_f(\Omega^1)$. The converse inclusion can be shown similarly, and thus the first identity in (6.6) holds. The other three identities are shown similarly. \square

7. Invariance for Sobolev-resolutive functions

Proposition 7.1. *Let $f, h: \partial^1\Omega \rightarrow \overline{\mathbb{R}}$ and assume that h is zero $\overline{C}_p(\cdot; \Omega^1)$ -q.e., i.e., that*

$$\overline{C}_p(\{x \in \partial^1\Omega : h(x) \neq 0\}; \Omega^1) = 0.$$

Then $\overline{S}_{\Omega^1} f = \overline{S}_{\Omega^1}(f + h)$.

In particular, if f is Sobolev-resolutive then so is $f + h$ and $S_{\Omega^1}(f + h) = S_{\Omega^1} f$.

Proof. First assume that $h \geq 0$. Let $E = \{x \in \partial^1\Omega : h(x) \neq 0\}$. By assumption, $\overline{C}_p(E; \Omega^1) = 0$ and thus Proposition 4.2 shows that for each j there is a τ^1 -open set $G_j \subset \overline{\Omega}^1$ such that $E \subset G_j$ and $\overline{C}_p(G_j; \Omega^1) < 2^{-jp}$. Letting $U_k = \bigcup_{j=k+1}^\infty G_j$ we see that $\overline{C}_p(U_k; \Omega^1) < 2^{-kp}$. Let $\{\psi_j\}_{j=1}^\infty$ be the decreasing sequence of nonnegative functions given by Lemma 4.5 with respect to $\{U_k\}_{k=1}^\infty$.

Let $u \in \mathcal{DU}_f(\Omega^1)$. Since $u \in D^p(\Omega)$, u is a superminimizer in Ω and it is the lsc-regularized solution of the $\mathcal{K}_{u,u}$ -obstacle problem. Let $v_j = u + \psi_j \in D^p(\Omega)$ and let φ_j be the lsc-regularized solution of the \mathcal{K}_{v_j,v_j} -obstacle problem. Then $\varphi_j \geq u$ in Ω , by the comparison principle in Corollary 4.3 in Björn-Björn [11]. (As both functions are lsc-regularized the inequality holds everywhere.) It follows that if $x \in \partial^1\Omega \setminus E$, then

$$\liminf_{\Omega \ni y \xrightarrow{\tau^1} x} \varphi_j(y) \geq \liminf_{\Omega \ni y \xrightarrow{\tau^1} x} u(y) \geq f(x) = (f + h)(x).$$

On the other hand, if $x \in E$, then for positive integers m , by Lemma 4.5,

$$\varphi_j \geq \psi_j^* + \inf_{\Omega} u \geq m + \inf_{\Omega} u \quad \text{on } U_{m+j} \cap \Omega,$$

which implies that

$$\liminf_{\Omega \ni y \xrightarrow{\tau^1} x} \varphi_j(y) = \infty \geq (f + h)(x).$$

As $\varphi_j \in D^p(\Omega)$, we see that $\varphi_j \in \mathcal{DU}_{f+h}(\Omega^1)$ and hence that $\varphi_j \geq \bar{S}_{\Omega^1}(f + h)$.

By Proposition 5.4, the sequence $\{\varphi_j\}_{j=1}^{\infty}$ decreases q.e. to u . It follows that $\bar{S}_{\Omega^1}(f + h) \leq u$ q.e. in Ω , and hence everywhere in Ω , since both functions are lsc-regularized. As $u \in \mathcal{DU}_f(\Omega^1)$ was arbitrary, we conclude that $\bar{S}_{\Omega^1}(f + h) \leq \bar{S}_{\Omega^1}f$. The converse inequality is trivial, and thus $\bar{S}_{\Omega^1}(f + h) = \bar{S}_{\Omega^1}f$ if $h \geq 0$.

For a general h , we then get that $\bar{S}_{\Omega^1}(f + h) = \bar{S}_{\Omega^1}(f + h_+) = \bar{S}_{\Omega^1}f$. The last part follows by applying this also to $-f$ and $-h$. \square

Corollary 7.2. *In the definition of the Sobolev-Perron solutions, it is enough if condition (6.1) is satisfied only $\bar{C}_p(\cdot; \Omega^1)$ -q.e.*

The proof of the following result is rather straightforward using (6.3), cf. Theorem 10.25 in [10].

Proposition 7.3. *Let $f_j: \partial^1\Omega \rightarrow \bar{\mathbb{R}}$, $j = 1, 2, \dots$, be (Sobolev)-resolutive functions and assume that $f_j \rightarrow f$ uniformly on $\partial^1\Omega$. Then f is (Sobolev)-resolutive and $P_{\Omega^1}f_j \rightarrow P_{\Omega^1}f$ (resp. $S_{\Omega^1}f_j \rightarrow S_{\Omega^1}f$) uniformly in Ω .*

We also obtain the following corollary.

Corollary 7.4. *Let $f_j: \bar{\Omega}^1 \rightarrow \bar{\mathbb{R}}$ be $\bar{C}_p(\cdot; \Omega^1)$ -quasicontinuous functions such that $f_j|_{\Omega} \in D^p(\Omega)$, $j = 1, 2, \dots$. Assume also that $f_j \rightarrow f$ uniformly on $\partial^1\Omega$ as $j \rightarrow \infty$. Let $h: \partial^1\Omega \rightarrow \bar{\mathbb{R}}$ be a function which is zero $\bar{C}_p(\cdot; \Omega^1)$ -q.e. on $\partial^1\Omega$. Then f and $f + h$ are Sobolev-resolutive and $S_{\Omega^1}f = S_{\Omega^1}(f + h)$.*

Proof. This follows directly from Theorem 6.4 and Propositions 7.1 and 7.3. \square

8. Wiener solutions and harmonizability

In the previous sections we developed the theory for the Dirichlet problem on general compactifications. In this section we will look at the question of constructing resolutive compactifications.

Definition 8.1. A compactification of Ω is (Sobolev)-resolutive if every continuous function on the boundary is (Sobolev)-resolutive.

A compactification is internally (Sobolev)-resolutive if every bounded p -harmonic function on Ω is the (Sobolev)–Perron solution of some resolutive boundary function.

We will focus on (Sobolev)-resolutivity in this article. Internal resolutivity seems to be essentially untouched in the nonlinear potential theory (even on unweighted \mathbb{R}^n), but also seems to be quite difficult in this situation. In linear potential theory both these concepts are very well understood. For instance the Martin compactification (see Example 8.15) is always internally resolutive. (For linear potential theory on \mathbb{R}^n see e.g. Armitage–Gardiner [2] or Doob [23]. See also Bishop [4] and Mountford–Port [53] where the relation between internal resolutivity and harmonic measure is discussed.)

We have the following fundamental result.

Proposition 8.2. *Assume that $\partial^1\Omega \prec \partial^2\Omega$. Then the following are true:*

- (a) *If $\partial^2\Omega$ is (Sobolev)-resolutive, then so is $\partial^1\Omega$.*
- (b) *If $\partial^1\Omega$ is internally (Sobolev)-resolutive, then so is $\partial^2\Omega$.*

Proof. Let $\Phi: \overline{\Omega}^2 \rightarrow \overline{\Omega}^1$ denote the projection.

(a) Let $f \in C(\partial^1\Omega)$. Then $f \circ \Phi \in C(\partial^2\Omega)$ and it is resolutive by assumption. By Theorem 6.7, $P_{\Omega^1} f = P_{\Omega^2}(f \circ \Phi)$, and hence $\partial^1\Omega$ is resolutive.

(b) Let u be a bounded p -harmonic function on Ω . By assumption there is a resolutive $f: \partial^1\Omega \rightarrow \mathbb{R}$ such that $u = P_{\Omega^1} f$. By Theorem 6.7 again, $u = P_{\Omega^2}(f \circ \Phi)$, and thus $\partial^2\Omega$ is internally resolutive.

The Sobolev cases are shown similarly. □

It follows from Proposition 8.5 and Theorem 8.9 below that the Stone–Čech compactification (see Example 2.8) is internally resolutive.

To see which Q -compactifications are resolutive we introduce the fundamental concept of harmonizability due to Constantinescu–Cornea [21] who studied it in the case of linear potential theory on Riemann surfaces (see also Section 2 in their article [22]). In the nonlinear theory, harmonizability and Wiener solutions have (as far as we know) only been studied by Maeda–Ono [49], [50] who studied them on weighted \mathbb{R}^n for more general equations with right-hand sides than those under consideration here. Sobolev–Wiener solutions and Sobolev-harmonizability have not been considered earlier.

Perhaps a more appropriate name for the solutions below would be Wiener–Poincaré solutions, since they combine Poincaré’s notion of balayage (i.e., the infimum of superharmonic majorants) with Wiener’s method of solving the Dirichlet problem by using exhaustions [60]. Though no name is given to these solutions in [21] or [22], Wiener’s legacy is reflected therein in the notation \mathcal{W}_f as well as in the definition of Wiener functions as those which are harmonizable for some domain. For simplicity, we use the name Wiener solutions here, following the path indicated in [21].

A different notion of Wiener solutions in the context of metric spaces was for p -harmonic functions considered in Björn–Björn [9]. Similar to Wiener’s original approach in [60], it obtains solutions to the Dirichlet problem with continuous boundary data on a nonregular domain as a limit of solutions, which are found by other methods on regular exhaustions of the domain. (In [60], Wiener actually constructs the approximating solutions as Riemann sums of the boundary values with respect to what is nowadays called the harmonic measure.)

Definition 8.3. For an arbitrary function $f: \Omega \rightarrow \overline{\mathbb{R}}$ we let \mathcal{U}_f^W be the set of all superharmonic functions u in Ω which are bounded from below and such that there is a compact set $K \Subset \Omega$ with $u \geq f$ in $\Omega \setminus K$.

Then the *upper* and *lower Wiener solutions* are defined by

$$\overline{W}f(x) = \inf_{u \in \mathcal{U}_f^W} u(x), \quad x \in \Omega, \quad \text{and} \quad \underline{W}f = -\overline{W}(-f).$$

If $\overline{W}f = \underline{W}f$ and it is real-valued, then we denote the common value by Wf and say that f is *harmonizable*.

Similarly we let $\mathcal{D}\mathcal{U}_f^W = \mathcal{U}_f^W \cap D^p(\Omega)$ and define the *Sobolev–Wiener solutions* by

$$(8.1) \quad \overline{Z}f(x) = \inf_{u \in \mathcal{D}\mathcal{U}_f^W} u(x), \quad x \in \Omega, \quad \text{and} \quad \underline{Z}f = -\overline{Z}(-f).$$

If $\overline{Z}f = \underline{Z}f$ and it is real-valued, then we denote the common value by Zf and say that f is *Sobolev-harmonizable*.

As we will only consider these solutions on the set Ω , we have omitted Ω from the notation. It can be shown in the same way as for the (Sobolev)–Perron solutions that $\overline{W}f, \underline{W}f, \overline{Z}f$ and $\underline{Z}f$ are p -harmonic in Ω , unless they are identically $\pm\infty$.

Note that, as Ω is bounded, the Sobolev–Wiener solutions for *bounded* f can equivalently be defined by taking the infimum in (8.1) over $u \in \mathcal{U}_f^W \cap N^{1,p}(\Omega)$.

The following inequalities are fundamental.

Proposition 8.4. *Let $f: \Omega \rightarrow \overline{\mathbb{R}}$ be arbitrary. Then $\underline{Z}f \leq \underline{W}f \leq \overline{W}f \leq \overline{Z}f$.*

It will be convenient for us to define the classes $\mathcal{L}_f := \{u : -u \in \mathcal{U}_{-f}\}$ and $\mathcal{L}_f^W := \{u : -u \in \mathcal{U}_{-f}^W\}$ of subharmonic minorants.

Proof. Let $u \in \mathcal{U}_f^W$ and $v \in \mathcal{L}_f^W$. Then $v \leq f \leq u$ in $\Omega \setminus K$ for some $K \Subset \Omega$. Let G be open and such that $K \subset G \Subset \Omega$. Then by Proposition 6.2 applied to \overline{G} , we see that $v \leq u$ in G , and hence in all of Ω . Taking infimum over all $u \in \mathcal{U}_f^W$ and supremum over all $v \in \mathcal{L}_f^W$ yields $\underline{W}f \leq \overline{W}f$. The remaining inequalities are trivial. \square

Proposition 8.5. *If $f: \Omega \rightarrow \mathbb{R}$ is a bounded superharmonic function, then f is harmonizable and $Wf \leq f$.*

In particular if $f: \Omega \rightarrow \mathbb{R}$ is a bounded p -harmonic function then $Wf = f$, and if $f \in D^p(\Omega)$ is a bounded p -harmonic function then $Zf = Wf = f$.

See Theorem 9.2 below for a substantial generalization of the last part.

Proof. Since f is bounded, we directly have $\overline{W}f \leq f$ by definition. But this also implies that

$$\underline{W}f \geq \overline{W}f,$$

since $\overline{W}f$ is a bounded p -harmonic function which belongs to \mathcal{L}_f^W . Together with Proposition 8.4 this shows that f is harmonizable and $Wf \leq f$.

If f is a bounded p -harmonic function, then so is $-f$, and hence $f \leq -W(-f) = Wf \leq f$, so $Wf = f$. The last part follows similarly. \square

Proposition 8.4 shows that being Sobolev-harmonizable is a stronger concept than being harmonizable. That it is strictly stronger follows from the following example.

Example 8.6. Let $\Omega = (0, 1)^n$ and $E \subset \partial\Omega$ be as in Example 6.6. For $p = 2$, it is shown therein that $P_\Omega \chi_E \equiv 0$ and hence there exists a bounded superharmonic function $v \in \mathcal{U}_{\chi_E}(\Omega)$ such that $v(0) < 1$. By Proposition 8.5, v is harmonizable and $Wv \leq v$.

At the same time, if $u \in \mathcal{D}\mathcal{U}_v^W$ then

$$\liminf_{\Omega \ni y \rightarrow x} u(y) \geq \liminf_{\Omega \ni y \rightarrow x} v(y) \geq \chi_E(x)$$

for all $x \in \partial\Omega$ and since E is nonthin at every $x \in \partial\Omega$, it follows as in Example 6.6 that $u \geq 1$ in Ω . Consequently, $\overline{Z}v \equiv 1 \neq Wv$, since $v(0) < 1$.

A similar example can be based on Example 6.5, this time for a ball $\Omega \subset \mathbb{R}^n$ with $p > n \geq 2$.

The following example shows that the boundedness assumption cannot be dropped from Proposition 8.5.

Example 8.7. Let $\Omega = B(0, 1) \setminus \{0\} \subset \mathbb{R}^n$ be the punctured ball with $1 < p \leq n$ and let $f(x) = |x|^{(p-n)/(p-1)} - 1$ (or $f(x) = -\log|x|$ if $p = n$) be the Green function in $B = B(0, 1)$ with pole at 0. Then $\overline{W}f \leq f$ and using Proposition 6.2 as in the proof of Proposition 8.4, we see that $\overline{W}f = f$. (The above argument shows that $\overline{W}f = f$ for any p -harmonic f bounded from below.) On the other hand, if $u \in \mathcal{L}_f^W$ then also $u_+ \in \mathcal{L}_f^W$, and u_+ (being bounded) has an extension as a subharmonic function to B , by Theorem 7.35 in Heinonen–Kilpeläinen–Martio [30] (or Theorem 12.3 in [10]). Comparing with a constant function in $B(0, 1 - \delta)$, $\delta > 0$, using Proposition 8.4 and letting $\delta \rightarrow 0$, shows that $u \leq 0$. Hence $\underline{W}f = 0$.

Next, let $u \in \mathcal{D}\mathcal{U}_f^W$ and let $g \in L^p(\Omega)$ be an upper gradient of u in Ω . Extend g and u by letting $g(0) = u(0) = \infty$. Since $C_p(\{0\}) = 0$, p -almost no curve hits the origin (by Proposition 1.48 in [10]), and thus g is a p -weak upper gradient of u also in B . Hence $u \in D^p(B) \subset D_{\text{loc}}^p(B) = N_{\text{loc}}^{1,p}(B)$. The Sobolev inequality implies that $u \in L_{\text{loc}}^{p^*}(B)$, where $p^* = np/(n - p)$. Since $u \geq f$ in a neighbourhood of the origin, an easy calculation shows that this is impossible. Hence no such u exists, and $\overline{Z}f = \infty$, thus providing another example when $\overline{Z}f \neq \overline{W}f$.

We also have $P_\Omega f = S_\Omega f = 0$, by e.g. Proposition 7.1.

Proposition 8.8. *If $f: \overline{\Omega}^1 \rightarrow \overline{\mathbb{R}}$ is upper semicontinuous, then $\underline{W}f \leq \underline{P}_{\Omega^1}f$ and $\underline{Z}f \leq \underline{S}_{\Omega^1}f$. If moreover $f < \infty$ on $\partial^1\Omega$, then also $\overline{W}f \leq \overline{P}_{\Omega^1}f$ and $\overline{Z}f \leq \overline{S}_{\Omega^1}f$.*

Note that the inequalities may be strict, e.g. for $f = \chi_{\partial^1\Omega}$ we have $Wf = Zf \equiv 0$ and $P_{\Omega^1}f = S_{\Omega^1}f \equiv 1$. Example 8.7 shows that the condition $f < \infty$ cannot be dropped from the last part. This result will be partially extended to $\overline{C}_p(\cdot; \Omega^1)$ -quasiseicontinuous functions in Propositions 10.3 and 10.4.

Proof. To prove the first inequality, let $v \in \mathcal{L}_f^W$ be arbitrary. Then $v \leq f$ in $\Omega \setminus K$ for some $K \Subset \Omega$ and hence, by the upper semicontinuity of f ,

$$\limsup_{\Omega \ni y \xrightarrow{\tau^1} x} v(y) \leq \limsup_{\Omega \ni y \xrightarrow{\tau^1} x} f(y) \leq f(x) \quad \text{for all } x \in \partial^1\Omega.$$

It follows that $v \in \mathcal{L}_f$, and hence $v \leq \underline{P}_{\Omega^1}f$. Taking supremum over all $v \in \mathcal{L}_f^W$ shows that $\underline{W}f \leq \underline{P}_{\Omega^1}f$. The second inequality is proved similarly.

For the third inequality, let $u \in \mathcal{U}_f$ and $\varepsilon > 0$ be arbitrary. As $f < \infty$ on $\partial^1\Omega$ and

$$\liminf_{\Omega \ni y \xrightarrow{\tau^1} x} u(y) \geq f(x) \geq \limsup_{\Omega \ni y \xrightarrow{\tau^1} x} f(y)$$

for every $x \in \partial^1\Omega$, there are τ^1 -neighbourhoods $V_x \ni x$ such that $u + \varepsilon > f$ in V_x . Thus, $u + \varepsilon > f$ in $\Omega \setminus K$, where $K := \Omega \setminus \bigcup_{x \in \partial^1\Omega} V_x$ is compact, i.e., $u + \varepsilon \in \mathcal{U}_f^W$ and consequently $u + \varepsilon \geq \overline{W}f$. Taking infimum over all $u \in \mathcal{U}_f$ and letting $\varepsilon \rightarrow 0$ proves the third inequality $\overline{W}f \leq \overline{P}_{\Omega^1}f$. The fourth inequality is proved similarly. \square

The following theorem is now a direct consequence of Proposition 8.8. The Sobolev part will be generalized to quasicontinuous functions in Theorem 10.5.

Theorem 8.9. *Let $f \in \mathcal{C}(\overline{\Omega}^1)$. Then $\underline{W}f = \underline{P}_{\Omega^1}f$, $\overline{W}f = \overline{P}_{\Omega^1}f$, $\underline{Z}f = \underline{S}_{\Omega^1}f$ and $\overline{Z}f = \overline{S}_{\Omega^1}f$.*

Moreover, $f|_{\Omega}$ is (Sobolev)-harmonizable if and only if $f|_{\partial^1\Omega}$ is (Sobolev)-resolutive, and when this happens we have $Wf = P_{\Omega^1}f$ (resp. $Zf = S_{\Omega^1}f$).

For Wiener solutions and harmonizability, the corresponding result on weighted \mathbb{R}^n was obtained in Proposition 2.2 in Maeda-Ono [50].

Corollary 8.10. $\overline{\Omega}^1$ is (Sobolev)-resolutive if and only if every $f \in \mathcal{C}(\overline{\Omega}^1)$ is (Sobolev)-harmonizable.

Example 8.11. Let $\Omega = B(0, 1)$ be the unit ball in \mathbb{R}^n , $n \geq 1$, and let $f(x) = \sin(1/(1 - |x|))$. Then $f \in \mathcal{C}(\Omega)$. By the minimum principle for superharmonic functions, every $u \in \mathcal{U}_f^W$ must satisfy $u \geq 1$ in Ω (since this holds on spheres accumulating towards $\partial\Omega$). It follows that $\overline{W}f \equiv 1$.

Similarly, $\underline{W}f \equiv -1$, so f is a bounded continuous function which is not harmonizable in the unit ball. Let $Q = \{f\}$. Then f has a continuous extension $\tilde{f} \in \mathcal{C}(\overline{\Omega}^Q)$, showing (because of Proposition 2.11) that $\overline{\Omega}^Q$ is a metrizable nonresolutive compactification. Observe that $f \notin D^p(\Omega)$, cf. Theorem 9.2 below.

Note that in the proof of Proposition 8.8 we only used that f is upper semicontinuous at every $x \in \partial^1\Omega$, not in Ω . A similar observation concerning continuity thus applies to Theorem 8.9 as well. This is even true more generally, as seen by the following result.

Proposition 8.12. *Let $f_1, f_2: \Omega \rightarrow \overline{\mathbb{R}}$ be such that*

$$\lim_{\Omega \ni y \xrightarrow{\tau^1} x} (f_1(y) - f_2(y)) = 0 \quad \text{for all } x \in \partial^1\Omega.$$

Then $\overline{W}f_1 = \overline{W}f_2$ and $\overline{Z}f_1 = \overline{Z}f_2$.

Note that we do not require f_1 and f_2 to have limits at $\partial^1\Omega$, it is enough that the difference has limits. In fact, if f_1 and f_2 do have equal limits everywhere on $\partial^1\Omega$, but some of the limits are infinite, then the first conclusion does not necessarily hold. Let e.g. $f_1 = f$ and $f_2 = 2f$ in Example 8.7. Then f_1 and f_2 have the same limits at all boundary points, but $\overline{W}f_1 = f_1 \neq f_2 = \overline{W}f_2$.

Open problem 8.13. Are there functions f_1 and f_2 with the same (necessarily not only finite) limits at all boundary points, but with $\overline{Z}f_1 \neq \overline{Z}f_2$?

Proof of Proposition 8.12. Let $\varepsilon > 0$. Then for each $x \in \partial^1\Omega$, there is a τ^1 -neighbourhood U_x of x such that $|f_1 - f_2| < \varepsilon$ in U_x . Let $\tilde{K} = \Omega \setminus \bigcup_{x \in \partial^1\Omega} U_x$ which is a compact subset of Ω .

For each $u \in \mathcal{U}_{f_1}^W$ there is $K \Subset \Omega$ such that $u \geq f_1$ in $\Omega \setminus K$. Hence $u \geq f_2 - \varepsilon$ in $\Omega \setminus (K \cup \tilde{K})$, and thus $u \in \mathcal{U}_{f_2 - \varepsilon}^W$. Taking infimum over all $u \in \mathcal{U}_{f_1}^W$, shows that $\overline{W}(f_2 - \varepsilon) \leq \overline{W}f_1$. Letting $\varepsilon \rightarrow 0$ shows that $\overline{W}f_2 \leq \overline{W}f_1$. Similarly $\overline{Z}f_2 \leq \overline{Z}f_1$. The converse inequalities follow by swapping the roles of f_1 and f_2 . \square

Let

$$\begin{aligned} \mathcal{W}(\Omega) &= \{f : \Omega \rightarrow \overline{\mathbb{R}} : f \text{ is harmonizable}\}, \\ \mathcal{SW}(\Omega) &= \{f : \Omega \rightarrow \overline{\mathbb{R}} : f \text{ is Sobolev-harmonizable}\}. \end{aligned}$$

It is straightforward to show that $\mathcal{W}(\Omega)$ and $\mathcal{SW}(\Omega)$ are closed in the supremum norm, cf. Proposition 7.3.

Theorem 8.14. *Let Q be a sublattice of $\mathcal{C}_{\text{bdd}}(\Omega)$ which contains the constant functions. Then $\partial^Q\Omega$ is resolutive if and only if $Q \subset \mathcal{W}(\Omega)$, and Sobolev-resolutive if and only if $Q \subset \mathcal{SW}(\Omega)$.*

Proof. First assume that $Q \subset \mathcal{W}(\Omega)$. Let $\widehat{Q} = \{f|_{\partial^Q\Omega} : f \in \mathcal{C}(\overline{\Omega}^Q) \text{ and } f|_{\Omega} \in Q\}$. Then all functions in \widehat{Q} are resolutive, by Theorem 8.9. As \widehat{Q} is dense in $\mathcal{C}(\partial^Q\Omega)$, by Theorem 2.14, it follows from Proposition 7.3 that all functions in $\mathcal{C}(\partial^Q\Omega)$ are resolutive, i.e., $\partial^Q\Omega$ is resolutive.

Conversely, assume that $\partial^Q\Omega$ is resolutive, i.e., the functions in $\mathcal{C}(\partial^Q\Omega)$ are resolutive. It then follows from Theorem 8.9 that $\{f|_{\Omega} : f \in \mathcal{C}(\overline{\Omega}^Q)\} \subset \mathcal{W}(\Omega)$. By the definition of $\partial^Q\Omega$, we see that $Q \subset \{f|_{\Omega} : f \in \mathcal{C}(\overline{\Omega}^Q)\} \subset \mathcal{W}(\Omega)$.

The second equivalence is proved similarly. \square

There is one technical difficulty here in which the nonlinear potential theory differs from the linear. In the linear potential theory it is immediate that $Q_{\mathcal{W}} = \mathcal{W}(\Omega) \cap \mathcal{C}_{\text{bdd}}(\Omega)$ is a vector lattice. Hence the $Q_{\mathcal{W}}$ -compactification, which in this case is usually called the Wiener compactification, is well known to be resolutive (e.g. by Theorem 8.14), and it is the largest resolutive compactification. In the nonlinear setting we do not know whether this compactification becomes resolutive or not, as we do not know if $Q_{\mathcal{W}}$ is a vector lattice. The problem is that the construction of a Q -compactification only guarantees that the functions in Q separate the points of the boundary, not that the set is dense among all continuous functions on the boundary. However, in case $Q_{\mathcal{W}}$ is also a vector lattice then Theorem 2.14 guarantees that this is indeed the case.

At the same time, although not directly applicable here, the main result (Theorem 1.1) in Llorente–Manfredi–Wu [47] indicates that in general $Q_{\mathcal{W}}$ is not likely to be a vector lattice. Their result shows that for the upper half plane (which is unbounded and thus not included here) whenever $1 < p < \infty$ and $p \neq 2$, there are finitely many sets E_1, E_2, \dots, E_n such that $\mathbb{R} = \bigcup_{j=1}^n E_j$ while $P\chi_{E_j} \equiv 0$ for all j and $P\chi_{\mathbb{R}} \equiv 1$. Thus if $E \subset \mathbb{R}$ is a set so that χ_E is nonresolutive (such sets can be constructed in the same way as nonmeasurable sets), and we let $f_j = \chi_{E \cap E_j}$, then $Pf_j = 0$ (so f_j is resolutive), but $\sum_{j=1}^n f_j = \chi_E$ is nonresolutive. Hence $Q_{\mathcal{W}}$ is *not* a vector space.

Example 8.15. We end this section by recalling some results from the linear potential theory for the Laplacian in \mathbb{R}^n , $n \geq 2$. Assume that Ω is a bounded domain in (unweighted) \mathbb{R}^n . Apart from the Euclidean boundary, which is always resolutive, there is in particular one boundary which is of large interest for the Dirichlet problem, the Martin boundary. For sets which have nice (e.g. Lipschitz) boundaries these two compactifications are homeomorphic. However if Ω has for instance a large part of the boundary which is not one-sided (such as a slit disc in the plane), then the Euclidean boundary is not internally resolutive. Hence there is no chance of having a Poisson-type representation of all bounded harmonic functions such as for balls. To remedy this problem, Martin introduced his compactification in [51], which is a metrizable compactification with many useful properties.

To understand how the Martin boundary is constructed, we fix a ball $B \Subset \Omega$ and introduce the functions

$$M(x, y) = \frac{G_{\Omega}(x, y)}{\int_B G_{\Omega}(z, y) d\mu(z)},$$

where G_{Ω} is the Green function for Ω . The function M is the *Martin kernel*. The Q_M -compactification with $Q_M = \{M(x, \cdot) : x \in \Omega\}$ is called the *Martin compactification* and the *Martin boundary* is $\partial^{Q_M}\Omega$.

It is well known that the Martin boundary is always both resolutive and internally resolutive (see Section 1.XII.10 in Doob [23]). Furthermore, there is a Poisson-type representation for every bounded harmonic function of the form

$$h(x) = \int_{\partial^{Q_M}\Omega} M(x, y)f(y) d\nu(y),$$

where ν is the harmonic measure on $\partial^{Q_M}\Omega$.

For simply connected planar domains this boundary is reasonably simple to understand. Indeed, it is the compactification induced by relating the domain to the unit disc via the Riemann map (and is hence homeomorphic to the prime end boundary of Carathéodory). This implies that even for simply connected planar domains the following cases are possible: (i) $\partial\Omega \simeq \partial^{Q_M}\Omega$; (ii) $\partial\Omega \prec \partial^{Q_M}\Omega \not\asymp \partial\Omega$; (iii) $\partial^{Q_M}\Omega \prec \partial\Omega \not\asymp \partial^{Q_M}\Omega$; and (iv) $\partial\Omega \not\asymp \partial^{Q_M}\Omega \not\asymp \partial\Omega$, see Example 10.2 in Björn–Björn–Shanmugalingam [16]. Thus in general there is no immediate topological relation between the metric boundary $\partial\Omega$ and the Martin boundary $\partial^{Q_M}\Omega$. (There is however a measure-theoretic relation in the sense that there is a “measurable” projection from $\partial^{Q_M}\Omega$ to $\partial\Omega$ defined on a set of full harmonic measure and such that the harmonic measure on $\partial\Omega$ is the image of the harmonic measure on $\partial^{Q_M}\Omega$ for this map. So from the point of view of harmonic measure, the Martin boundary is always larger than the Euclidean boundary. See for instance Mountford–Port [53].)

In higher dimensions the situation is more subtle even for reasonably simple domains. For instance, let $\Omega = B_2 \setminus \overline{B_1}$, where $B_1 \subset B_2 \subset \mathbb{R}^3$ are balls such that $\partial B_1 \cap \partial B_2 = \{0\}$. Then every point of the Euclidean boundary apart from 0 corresponds to one point on the Martin boundary, but 0 corresponds to infinitely many points. (By a suitable Kelvin transformation, the problem can be transformed to the unbounded region between two parallel planes and with 0 corresponding to ∞ . Then the Martin compactification “identifies” 0 with a circle attached at infinity.)

For the above results we refer to Chapter 8 in Armitage–Gardiner [2]. The construction of the compactification is however more direct both in [2] and Martin’s original article [51] introducing the Martin metric and making a completion, which is then proved to be compact.

9. Sobolev-harmonizability

Proposition 9.1. *Let $f, h : \Omega \rightarrow \overline{\mathbb{R}}$ be arbitrary functions. If $h = 0$ q.e. in Ω , then $\overline{Z}f = \overline{Z}(f + h)$.*

In particular, if f is Sobolev-harmonizable, then so is $f + h$ and $Zf = Z(f + h)$.

Proof. First assume that $h \geq 0$. We have $\overline{C}_p(\{x \in \Omega : h(x) \neq 0\}; \Omega) = 0$, since $h = 0$ q.e. By Proposition 4.2, $\overline{C}_p(\cdot; \Omega)$ is an outer capacity and thus we can find a decreasing sequence of open sets $U_j \subset \Omega$ such that $h = 0$ in $\Omega \setminus U_j$ and $\overline{C}_p(U_j; \Omega) < 2^{-jp}$, $j = 1, 2, \dots$. Consider the decreasing sequence of nonnegative functions $\{\psi_j\}_{j=1}^\infty$ given by Lemma 4.5 with respect to $\{U_j\}_{j=1}^\infty$.

Let $u \in \mathcal{DU}_f^W$ be arbitrary, set $f_j = u + \psi_j$ and let φ_j be the lsc-regularized solution of the \mathcal{K}_{f_j, f_j} -obstacle problem. Then

$$\varphi_j = \varphi_j^* \geq f_j^* \geq u^* = u \geq f$$

in $\Omega \setminus K$ for some $K \Subset \Omega$. Moreover, in U_{j+m} we have

$$\varphi_j = \varphi_j^* \geq f_j^* \geq \psi_j^* \geq m.$$

Thus, if $h(x) \neq 0$, then $\varphi_j(x) \geq m$ for all $m = 1, 2, \dots$, i.e., $\varphi_j(x) = \infty \geq f(x) + h(x)$. It follows that $\varphi_j \geq f + h$ in $\Omega \setminus K$ and hence $\varphi_j \geq \bar{Z}(f + h)$ in Ω . Since $u \in D^p(\Omega)$ is a solution of the $\mathcal{K}_{u,u}(\Omega)$ -obstacle problem and $g_{f_j-u} \rightarrow 0$ in $L^p(\Omega)$, Proposition 5.4 implies that the sequence $\{\varphi_j\}_{j=1}^\infty$ decreases q.e. to u , showing that $u \geq \bar{Z}(f + h)$ q.e. in Ω . As both functions are lsc-regularized, the inequality holds everywhere in Ω . Taking infimum over all $u \in \mathcal{DU}_f^W$ gives $\bar{Z}f \geq \bar{Z}(f + h)$, while the opposite inequality is trivial as $h \geq 0$.

Using this we see that for a general h , we have $\bar{Z}(f + h) = \bar{Z}(f + h_+) = \bar{Z}f$. The last part follows by applying this also to $-f$ and $-h$. □

Theorem 9.2. *Every $f \in D^p(\Omega)$ is Sobolev-harmonizable and*

$$(9.1) \quad Wf = Zf = Hf.$$

On weighted \mathbb{R}^n , Maeda and Ono showed that every $f \in D^p(\Omega) \cap \mathcal{C}_{\text{bdd}}(\Omega)$ is harmonizable and $Wf = Hf$, see Theorem 2.2 in [50]. Their proof is quite different from ours.

Proof. It is enough to prove that $\bar{Z}f \leq Hf$. Then also

$$Zf = -\bar{Z}(-f) \geq -H(-f) = Hf,$$

and we get (9.1) using Proposition 8.4.

To prove that $\bar{Z}f \leq Hf$, let $\Omega_1 \subset \Omega_2 \subset \dots \Subset \Omega$ be an exhaustion of Ω by open sets. Let u_n be the lsc-regularized solution of the $\mathcal{K}_{\psi_n,f}(\Omega)$ -obstacle problem with

$$\psi_n = \begin{cases} f, & \text{in } \Omega \setminus \Omega_n, \\ -\infty, & \text{in } \Omega_n, \end{cases} \quad n = 1, 2, \dots$$

Then u_n is superharmonic in Ω , p -harmonic in Ω_n and belongs to $D^p(\Omega)$. Moreover, $u_n \geq \psi_n$ in $\Omega \setminus E_n$, where $C_p(E_n) = 0$. It follows that $u_n \geq \tilde{f}$ in $\Omega \setminus \Omega_n$, where

$$\tilde{f} = \begin{cases} f, & \text{in } \Omega \setminus \bigcup_{n=1}^\infty E_n, \\ -\infty, & \text{in } \bigcup_{n=1}^\infty E_n, \end{cases}$$

and $\tilde{f} = f$ q.e. Thus, $u_n \in \mathcal{DU}_{\tilde{f}}^W(\Omega)$ and, in view of Proposition 9.1, this implies that $u_n \geq \bar{Z}\tilde{f} = \bar{Z}f$.

By the comparison principle in Corollary 4.3 in Björn–Björn [11], we see that $\{u_n\}_{n=1}^\infty$ is a decreasing sequence and $u_n \geq Hf \not\equiv -\infty$. (As these functions are lsc-regularized this holds everywhere.) Harnack’s convergence theorem, see Proposition 5.1 in Shanmugalingam [57] (or Corollary 9.38 in [10]), implies that $u_n \searrow h \in N_{\text{loc}}^{1,p}(\Omega)$, where h is p -harmonic in Ω . By construction, $\|g_{u_n}\|_{L^p(\Omega)}$ is decreasing. As $u_n \in N_{\text{loc}}^{1,p}(\Omega)$, we see that for each j , $\|u_n\|_{L^p(\Omega_j)} \leq \|u_1\|_{L^p(\Omega_j)} + \|Hf\|_{L^p(\Omega_j)}$ and thus $\{u_n\}_{n=1}^\infty$ is a bounded sequence in $N^{1,p}(\Omega_j)$. Hence, by Corollary 6.3 in [10],

$$\int_\Omega g_h^p \, d\mu \leq \liminf_{n \rightarrow \infty} \int_\Omega g_{u_n}^p \, d\mu,$$

showing that $h \in D^p(\Omega)$. Since $Hf \leq h \leq u_1$, $Hf - f \in N_0^{1,p}(\Omega)$ and $u_1 - f \in N_0^{1,p}(\Omega)$, it follows from Lemma 2.8 in Hansevi [27] that $h - f \in N_0^{1,p}(\Omega)$. Hence $h = Hf$, by the uniqueness of Hf . Finally, as $u_n \geq \bar{Z}f$ for all n , we conclude that $Hf \geq \bar{Z}f$, and the result follows. \square

Corollary 9.3. *If $Q \subset \widehat{Q} := C_{\text{bdd}}(\Omega) \cap D^p(\Omega)$, then $\partial^Q\Omega$ is Sobolev-resolutive.*

For resolutive, the corresponding result on weighted \mathbb{R}^n was obtained in Theorem 3.2 in Maeda–Ono [49] (and is the main result therein). Their proof is not based on the equality $Wf = Hf$, which they did not have at their disposal at that time. In Maeda–Ono [50], bottom p. 519, they do note that this corollary can be obtained in a way similar to our proof.

Note that $\widehat{Q} = C_{\text{bdd}}(\Omega) \cap N^{1,p}(\Omega)$ as Ω is bounded. We do not know if $\partial^{\widehat{Q}}\Omega$ is metrizable in general. To see that it can be metrizable, let $\Omega = B(0, 1)$ be the unit ball in (unweighted) \mathbb{R}^n and let $p > n$. As Ω is an extension domain it follows that

$$\{f|_{\Omega} : f \in \text{Lip}(\overline{\Omega})\} \subset \widehat{Q} \subset \{f|_{\Omega} : f \in C(\overline{\Omega})\}$$

and hence $\overline{\Omega^{\widehat{Q}}} \simeq \overline{\Omega}$ is metrizable.

Proof. By Theorems 8.14 and 9.2, $\partial^{\widehat{Q}}\Omega$ is Sobolev-resolutive. Hence, by Proposition 8.2, $\partial^Q\Omega$ is Sobolev-resolutive. \square

Remark 9.4. Let d_1 be a metric on Ω and assume that its completion leads to a compact space $\overline{\Omega}^1$. If $d_1(x, \cdot) \in D^p(\Omega)$ for each $x \in \Omega$, then by Corollary 9.3 together with Proposition 2.12 we see that $\partial^1\Omega$ is Sobolev-resolutive.

One particular application is the case of the Mazurkiewicz distance d_M , see (4.1). The completion with respect to d_M is compact if and only if Ω is finitely connected at the boundary, by Theorem 1.1 in Björn–Björn–Shanmugalingam [17] or Theorem 1.3.8 in Karmazin [34]. Moreover, for every curve γ connecting x and y , we have that $d_M(x, y) \leq \int_{\gamma} 1 ds$, which shows that for each $x \in \Omega$, the constant function 1 is an upper gradient of $d_M(x, \cdot)$. As Ω is assumed to be bounded with respect to the given metric, it is also bounded with respect to d_M . Hence, if Ω is finitely connected at the boundary, then $d_M(x, \cdot) \in N^{1,p}(\Omega)$, and the above applies in this case. Perron solutions with respect to the Mazurkiewicz compactification were studied in Björn–Björn–Shanmugalingam [16].

Similarly, Sobolev-resolutivity is guaranteed by Corollary 9.3 for every metrizable compactification such that the distance functions $d'(x, \cdot) \in N^{1,p}(\Omega)$, for all x belonging to a countable dense subset of Ω , where d' is a metric associated with the compactification.

Remark 9.5. In Definition 6.1 we required that (6.1) should hold for all $x \in \partial^1\Omega$, and similarly we required that $u \geq f$ everywhere in $\Omega \setminus K$ in Definition 8.3. In the linear case it is well known that one can equivalently require these inequalities to hold q.e., and one may ask if this is also true in the nonlinear case. For Sobolev–Perron and Sobolev–Wiener solutions this is indeed equivalent, by Corollary 7.2 and Proposition 9.1, but for ordinary Perron solutions this is an open question in

the nonlinear potential theory. Nonlinear Perron solutions $\overline{Q}f$ and $\underline{Q}f$ with (6.1) holding q.e. were studied in Section 10 in Björn–Björn–Shanmugalingam [13] and a major open question is if $\underline{Q}f \leq \overline{Q}f$ always holds.

On the other hand, if we denote the upper and lower q.e.-Wiener solutions of f by $\overline{Y}f$ and $\underline{Y}f$, respectively, then it is always true that $\underline{Y}f \leq \overline{Y}f$. To see this, let u be a superharmonic function bounded from below, v be a subharmonic function bounded from above, and $K \Subset \Omega$ be such that $v \leq f \leq u$ q.e. in $\Omega \setminus K$. If now G is an open set such that $K \subset G \Subset \Omega$, then $v \leq u$ everywhere in G , by Proposition 10.28 in [10] (as $u, v \in N_{loc}^{1,p}(\Omega) \subset N^{1,p}(\overline{G})$). Since G was arbitrary, $v \leq u$ everywhere in Ω , and hence $\underline{Y}f \leq \overline{Y}f$.

Moreover, if f is lower semicontinuous, then $\overline{W}f = \overline{Y}f$, as if u is superharmonic and $u \geq f$ q.e. in $\Omega \setminus K$ for some compact $K \Subset \Omega$, then

$$f(x) \leq \operatorname{ess\,lim\,inf}_{y \rightarrow x} f(y) \leq \operatorname{ess\,lim\,inf}_{y \rightarrow x} u(y) = u(x) \quad \text{for } x \in \Omega \setminus K,$$

and thus $u \in \mathcal{U}_f^W$ yielding $\overline{W}f \leq \overline{Y}f$, while the converse inequality is trivial.

Lucia–Puls [48] studied resolutivity for the Q_R -compactification of the whole space X (when it is unbounded) with respect to the p -Royden algebra $Q_R = D^p(X) \cap \mathcal{C}_{\text{bdd}}(X)$, cf. Corollary 9.3. Instead of the lattice property and Theorem 2.15, they used the Stone–Weierstrass theorem.

10. $\overline{C}_p(\cdot; \Omega^1)$ -quasisemicontinuous functions

In this section we partially extend some of the results from Section 8 to quasisemicontinuous functions. In particular, we relate Sobolev-harmonizability to Sobolev-resolutivity for quasicontinuous functions. For the results in this section it is important that we deal with Sobolev–Perron and Sobolev–Wiener solutions.

Definition 10.1. A function $f: \overline{\Omega}^1 \rightarrow \overline{\mathbb{R}}$ is lower (upper) $\overline{C}_p(\cdot; \Omega^1)$ -quasisemicontinuous if for every $\varepsilon > 0$ there exists a τ^1 -open set U such that $\overline{C}_p(U; \Omega^1) < \varepsilon$ and $f|_{\overline{\Omega}^1 \setminus U}$ is lower (upper) semicontinuous.

Since $\overline{C}_p(\cdot; \Omega^1)$ is an outer capacity, by Proposition 4.2, it is immediate that $u + h$ is $\overline{C}_p(\cdot; \Omega^1)$ -quasisemicontinuous whenever u is $\overline{C}_p(\cdot; \Omega^1)$ -quasisemicontinuous and $h = 0$ $\overline{C}_p(\cdot; \Omega^1)$ -q.e.

Lemma 10.2. For all functions f it is true that $\overline{W}f = \lim_{m \rightarrow -\infty} \overline{W} \max\{f, m\}$, and similar statements hold for $\overline{Z}f$, $\overline{P}_{\Omega^1}f$ and $\overline{S}_{\Omega^1}f$.

Proof. This follows directly from the definition, since the classes \mathcal{U}_f^W , $\mathcal{D}\mathcal{U}_f^W$, \mathcal{U}_f and $\mathcal{D}\mathcal{U}_f$ only contain functions which are bounded from below. □

Proposition 10.3. If $f: \overline{\Omega}^1 \rightarrow \overline{\mathbb{R}}$ is lower $\overline{C}_p(\cdot; \Omega^1)$ -quasisemicontinuous then $\overline{Z}f \geq \overline{S}_{\Omega^1}f$.

Proof. First assume that $f \geq 0$. Since f is lower $\overline{C}_p(\cdot; \Omega^1)$ -quasisemicontinuous on $\overline{\Omega}^1$, we can find a decreasing sequence of τ^1 -open subsets U_k of $\overline{\Omega}^1$ such that $\overline{C}_p(U_k; \Omega^1) < 2^{-kp}$ and $f|_{\overline{\Omega}^1 \setminus U_k}$ is lower semicontinuous. Consider the decreasing sequence of nonnegative functions $\{\psi_j\}_{j=1}^\infty$ given by Lemma 4.5 with respect to this sequence of sets.

Let $u \in \mathcal{DU}_f^W$, set $f_j = u + \psi_j$ and let φ_j be the lsc-regularized solution of the \mathcal{K}_{f_j, f_j} -obstacle problem. For $m = 1, 2, \dots$, we have by the lsc-regularity of φ_j that

$$(10.1) \quad \varphi_j = \varphi_j^* \geq f_j^* \geq \psi_j^* \geq m \quad \text{in } U_{m+j} \cap \Omega.$$

Let $\varepsilon > 0$ and $x \in \partial^1 \Omega$. If $x \notin U_{m+j}$, then by the lower semicontinuity of $f|_{\overline{\Omega}^1 \setminus U_{m+j}}$ there is a τ^1 -neighbourhood V_x of x in $\overline{\Omega}^1$ such that

$$(10.2) \quad \varphi_j = \varphi_j^* \geq f_j^* \geq u^* = u \geq f \geq \min\{f(x) - \varepsilon, m\} \quad \text{in } (V_x \cap \Omega) \setminus U_{m+j},$$

where the equality $u^* = u$ is due to the lsc-regularity of u . (For the last inequality, we either have $f \geq f(x) - \varepsilon$ if $f(x) < \infty$, or otherwise $f(x) \geq m$.) Combining (10.1) and (10.2) we see that for $x \in \partial^1 \Omega \setminus U_{m+j}$,

$$(10.3) \quad \varphi_j \geq \min\{f(x) - \varepsilon, m\} \quad \text{in } V_x \cap \Omega.$$

On the other hand, if $x \in U_{m+j} \cap \partial^1 \Omega$, then setting $V_x = U_{m+j}$, we see by (10.1) that (10.3) holds as well. Hence

$$\liminf_{\Omega \ni y \xrightarrow{\tau^1} x} \varphi_j(y) \geq \min\{f(x) - \varepsilon, m\}.$$

Letting $\varepsilon \rightarrow 0$ and $m \rightarrow \infty$ yields

$$\liminf_{\Omega \ni y \xrightarrow{\tau^1} x} \varphi_j(y) \geq f(x) \quad \text{for all } x \in \partial^1 \Omega.$$

As φ_j is superharmonic, it follows that $\varphi_j \in \mathcal{DU}_f(\Omega^1)$, and hence that $\varphi_j \geq \overline{S}_{\Omega^1} f$.

Since u clearly is a solution of the $\mathcal{K}_{u, u}$ -obstacle problem, we see by Proposition 5.4 that $\{\varphi_j\}_{j=1}^\infty$ decreases q.e. to u . Hence $u \geq \overline{S}_{\Omega^1} f$ q.e. in Ω . As both functions are lsc-regularized, the inequality holds everywhere in Ω . Taking infimum over all $u \in \mathcal{DU}_f^W$ shows that $\overline{Z}f \geq \overline{S}_{\Omega^1} f$.

Finally, let f be arbitrary. The above argument applied to $\max\{f, m\}$, together with Lemma 10.2, implies that

$$\overline{S}_{\Omega^1} f = \lim_{m \rightarrow -\infty} \overline{S}_{\Omega^1} \max\{f, m\} \leq \lim_{m \rightarrow -\infty} \overline{Z} \max\{f, m\} = \overline{Z}f \quad \text{in } \Omega. \quad \square$$

Proposition 10.4. *If $f: \overline{\Omega}^1 \rightarrow [-\infty, \infty)$ is upper $\overline{C}_p(\cdot; \Omega^1)$ -quasisemicontinuous and bounded from above then $\overline{Z}f \leq \overline{S}_{\Omega^1} f$.*

Proof. First, assume that f is bounded, say $0 \leq f \leq 1$. Since f is upper $\overline{C}_p(\cdot; \Omega^1)$ -quasisemicontinuous on $\overline{\Omega}^1$, we can find a decreasing sequence of τ^1 -open subsets U_k of $\overline{\Omega}^1$ such that $\overline{C}_p(U_k; \Omega^1) < 2^{-kp}$ and $f|_{\overline{\Omega}^1 \setminus U_k}$ is upper semicontinuous. Consider the decreasing sequence of nonnegative functions $\{\psi_j\}_{j=1}^\infty$ given by Lemma 4.5 with respect to this sequence of sets.

Let $u \in \mathcal{DU}_f$, set $f_j = u + \psi_j$ and let φ_j be the lsc-regularized solution of the \mathcal{K}_{f_j, f_j} -obstacle problem. Then $\varphi_j \geq u$ q.e. in Ω , and as both functions are lsc-regularized the inequality holds everywhere in Ω .

Let $\varepsilon > 0$ and $j = 1, 2, \dots$ be arbitrary. As φ_j is lsc-regularized, we see that

$$\varphi_j = \varphi_j^* \geq f_j^* \geq \psi_j^* \geq 1 \geq f - \varepsilon \quad \text{in } U_{j+1} \cap \Omega.$$

Since

$$\liminf_{\Omega \ni y \xrightarrow{\tau^1} x} u \geq f(x) \quad \text{for all } x \in \partial^1 \Omega,$$

and $f|_{\overline{\Omega^1} \setminus U_{j+1}}$ is upper semicontinuous, there exists for every $x \in \partial^1 \Omega \setminus U_{j+1}$ a τ^1 -open set $V_x \ni x$ such that $u \geq f - \varepsilon$ in $(V_x \cap \Omega) \setminus U_{j+1}$. Let

$$V = U_{j+1} \cup \bigcup_{x \in \partial^1 \Omega \setminus U_{j+1}} V_x.$$

As $\varphi_j \geq u$, we obtain that $\varphi_j \geq f - \varepsilon$ in $V \cap \Omega$. Since $\Omega \setminus V$ is compact, it follows that $\varphi_j + \varepsilon \in \mathcal{DU}_f^W$, and hence $\varphi_j + \varepsilon \geq \overline{Z}f$. Letting $\varepsilon \rightarrow 0$ shows that $\varphi_j \geq \overline{Z}f$.

Since u clearly is a solution of the $\mathcal{K}_{u, u}$ -obstacle problem, we see by Proposition 5.4 that $\{\varphi_j\}_{j=1}^\infty$ decreases q.e. to u . Hence $u \geq \overline{Z}f$ q.e. in Ω . As both functions are lsc-regularized, the inequality holds everywhere in Ω . Taking infimum over all $u \in \mathcal{DU}_f$ proves the statement for bounded functions. If f is merely bounded from above then applying the above argument to $\max\{f, m\}$, together with Lemma 10.2, completes the proof. \square

The following result is a direct consequence of Propositions 10.3 and 10.4, cf. Theorem 8.9.

Theorem 10.5. *Let $f: \overline{\Omega^1} \rightarrow \mathbb{R}$ be a bounded $\overline{C}_p(\cdot; \Omega^1)$ -quasicontinuous function. Then $\underline{Z}f = \underline{S}_{\Omega^1} f$ and $\overline{Z}f = \overline{S}_{\Omega^1} f$.*

Moreover, $f|_\Omega$ is Sobolev-harmonizable if and only if $f|_{\partial^1 \Omega}$ is Sobolev-resolutive, and when this happens $Zf = S_{\Omega^1} f$.

Example 8.7 shows that the boundedness assumptions in Proposition 10.4 and Theorem 10.5 cannot be omitted.

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