



Dislocations of arbitrary topology in Coulomb eigenfunctions

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Abstract. For any finite link L in \mathbb{R}^3 we prove the existence of a complex-valued eigenfunction of the Coulomb Hamiltonian such that its nodal set contains a union of connected components diffeomorphic to L . This problem goes back to Berry, who constructed such eigenfunctions in the case where L is the trefoil knot or the Hopf link and asked the question about the general result.

1. Introduction

Dislocations were introduced in quantum mechanics by Berry and Nye in 1974 by analogy with the classical wavefront dislocations studied in optics and have found numerous applications in very diverse areas of science including chemistry, water waves and the theory of liquid crystals [11]. If one writes the quantum mechanical wavefunction in terms of its amplitude and phase as $\psi = \rho e^{i\chi}$, we recall that a *dislocation* is a connected component of the zero set $\{\rho = 0\}$ such that the phase changes by a nonzero multiple of 2π on a closed circuit around it.

Motivated by problems in the theory of dislocations, in [2] Berry constructs eigenfunctions of the Coulomb Hamiltonian in \mathbb{R}^3 whose nodal set contains a trefoil knot or a Hopf link as a union of connected components. He then raises the question as to whether there exist eigenfunctions of a quantum system whose nodal set has components with higher order linking. The existence of knotted structures, both from theoretical [5], [7] and applied [3], [9], [8], [13] viewpoints, has recently attracted considerable attention, especially in optics and in fluid mechanics.

This question was answered in the affirmative by the authors in [4], where the quantum system considered was the harmonic oscillator. In this paper we return to the Coulomb potential, as the original setting considered by Berry, and prove that any link is the union of connected components of the nodal set for some eigenfunction. As we will see later on, the singularity of the potential and

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the fact that one can no longer employ bound states of arbitrarily high energy introduce serious technical complications in the problem that must be dealt with using new ideas.

Eigenfunctions of the Coulomb Hamiltonian are the functions ψ in the Sobolev space $H^1(\mathbb{R}^3)$ that satisfy the equation

$$(1.1) \quad \left(\Delta + \frac{2}{|x|} + \lambda\right)\psi = 0,$$

in \mathbb{R}^3 . It is well-known that the eigenvalues are given by

$$\lambda_n := -\frac{1}{n^2},$$

for $n \in \mathbb{N}$ and that a basis of the eigenspace corresponding to λ_n is given by

$$\begin{aligned} \psi_{nlm} &:= f_{nl}(r) Y_{lm}(\theta, \phi), \\ f_{nl} &:= A_{nl} e^{-r/n} r^l L_{n-l-1}^{2l+1}\left(\frac{2r}{n}\right), \end{aligned}$$

where $0 \leq l \leq n - 1$, and $-l \leq m \leq l$. Here A_{nl} is a normalisation factor that we take to be

$$A_{nl} = \frac{2^{l-1}(n-l-1)!}{(n+l)!},$$

the L_k^α are the associated Laguerre polynomials, and Y_{lm} are the spherical harmonics on S^2 . The degeneracy of the eigenspace of the λ_n energy level is therefore seen to be n^2 .

The main result of the paper is the following theorem, which shows that, as conjectured by Berry, there are Coulomb eigenfunctions having dislocations of arbitrary topology.

Theorem 1.1. *Let L be any finite link in \mathbb{R}^3 . Then there is some positive constant $E = E(L)$ such that, for any Coulomb eigenvalue with $\lambda_n > -E$, there exist a complex-valued eigenfunction ψ of energy λ_n and a diffeomorphism Φ of \mathbb{R}^3 such that $\Phi(L)$ is a union of connected components of the zero set $\psi^{-1}(0)$.*

It should be emphasized that the deformed link $L' := \Phi(L)$ is a bona fide dislocation set, meaning that there is a nonzero change of phase (actually, of 2π) along a circuit around any component of L' . This is an immediate consequence of the fact that L' satisfies the nondegeneracy condition

$$\text{rank}(\nabla \text{Re } \psi(x), \nabla \text{Im } \psi(x)) = 2$$

for all $x \in L'$, which means that L' arises as the transverse intersection of the zero sets of the real and imaginary parts of the eigenfunction ψ . It is also worth mentioning that, as Berry conjectured, the link L' is *structurally stable*, that is, there exists an $\epsilon > 0$ such that for any C^1 complex-valued function φ with $\|\varphi - \psi\|_{C^1} < \epsilon$, there is a diffeomorphism Φ_1 of \mathbb{R}^3 close to the identity such that $\Phi_1(L')$ is a union of connected components of the zero set $\varphi^{-1}(0)$.

Heuristically, the idea of the proof of the theorem is the following. As the energy levels λ_n tend to 0 for large n , it is clear that the formal limit of equation (1.1) as $n \rightarrow \infty$ is

$$(1.2) \quad \left(\Delta + \frac{2}{|x|}\right)\varphi = 0.$$

For this equation, one can prove that if the link L is contained in a ball B_{R_0} of a certain fixed radius (and this can always be ensured upon deforming the link with a suitable diffeomorphism), then there is a solution of (1.2) whose zero set contains a union of connected components that is a small deformation of the link L . Furthermore, this union is a structurally stable set. The point now is that one can prove that there is a sequence of eigenfunctions of energy λ_n that approximate the above function φ in B_{R_0} as $n \rightarrow \infty$, so by structural stability we infer that for all large enough n there are eigenfunctions with energy λ_n and whose zero set contains a union of connected components diffeomorphic to L .

This approach is completely different from Berry's, which hinges in controlling explicit perturbations of a fixed eigenfunction of the Coulomb Hamiltonian whose zero set contains a degenerate circle (i.e., a circle on which the gradient of the eigenfunction is zero identically). Berry's construction suggests that one probably needs eigenfunctions with high n to realize complicated knots; e.g., for the trefoil knot he needs at least $n = 7$. Concerning the relationship between Theorem 1.1 and our previous result for the harmonic oscillator [4], and leaving technicalities aside, an intriguing fact is that whereas for the harmonic oscillator one shows the existence of arbitrary links only if they are contained in arbitrarily small balls (specifically, of radius of order $n^{-1/2}$), in the case of the Coulomb potential any link can be realized with a diameter of order 1. In a way, this can be understood as evidence of the fact that the key ingredient of the proof is only the degeneracy of the eigenfunctions, rather than having arbitrarily large energies.

The structure of the paper is as follows. In Section 2 we consider the behavior of the eigenfunctions ψ_{nlm} for large values of n by computing the high-order asymptotics of the Laguerre polynomials, and prove that they become C^1 -close to a continuous solution of (1.2). Section 3 details the existence of continuous solutions to (1.2) in \mathbb{R}^3 that approximate functions satisfying (1.2) in a compact subset of the ball B_{R_0} , for a certain fixed radius R_0 . Finally, in Section 4, a function satisfying (1.2) in \mathbb{R}^3 with a nodal set containing a small deformation of the link $L \subset B_{R_0}$ is constructed and the two previous approximation results are used to prove Theorem 1.1.

2. Large n asymptotics for Coulomb eigenfunctions

In this section we will present a large n asymptotic expansion for the radial part of the eigenfunctions of the Coulomb Hamiltonian, which will be useful in the proof of the main theorem. The result can be stated as follows, where J_ν denotes the Bessel function of the first kind.

Proposition 2.1. *For any fixed l and R ,*

$$\lim_{n \rightarrow \infty} \left\| f_{nl} - \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} \right\|_{C^0((0,R))} = 0.$$

Proof. The proposition follows from Theorem 8.22.4 in [12], which gives the asymptotic expansion for fixed $\alpha > 0$:

$$(2.1) \quad e^{-x/2} x^{\alpha/2} L_k^\alpha(x) = \frac{\Gamma(k + \alpha + 1)}{(k + \frac{\alpha+1}{2})^{\alpha/2} k!} J_\alpha(\sqrt{(4k + 2\alpha + 2)x}) + x^{5/4} O(k^{\frac{2\alpha-3}{4}}),$$

where the bound holds uniformly in $0 \leq x \leq R$ for any $R > 0$. Using the substitutions $x = 2r/n$, $k = n - l - 1$, and $\alpha = 2l + 1$ we are able to obtain the asymptotic expansion of f_{nl} :

$$\begin{aligned} f_{nl}(r) &= \frac{A_{nl} n^{l+1/2}}{2^{l+1/2} \sqrt{r}} e^{-r/n} \left(\frac{2r}{n}\right)^{\frac{2l+1}{2}} L_{n-l-1}^{2l+1}\left(\frac{2r}{n}\right) \\ &= \frac{A_{nl} n^{l+1/2}}{2^{l-1} \sqrt{8r}} \left(n^{-l-1/2} \frac{(n+l)!}{(n-l-1)!} J_{2l+1}(\sqrt{8r}) + r^{5/4} n^{-5/4} O(n^{\frac{4l-1}{4}})\right) \\ &= \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} + r^{3/4} O(n^{-2}), \end{aligned}$$

from which the proposition follows. To show that the error is of order n^{-2} we have used the definition of the factors A_{nl} and Stirling’s asymptotic formula for the factorial, which yields

$$\frac{(n-l-1)!}{(n+l)!} = n^{-2l-1} + O(n^{-2l-2}). \quad \square$$

Of course, since the eigenfunctions satisfy the radial equation

$$\left(\partial_r^2 + \frac{2}{r} \partial_r - \frac{l(l+1)}{r^2} + \frac{2}{r} + \lambda_n\right) f_{nl}(r) = 0,$$

this uniform estimate can be easily promoted to a C^k bound. For our purposes, it is enough to state the result as follows.

Corollary 2.2. *For any fixed l and any $0 < R_1 < R_2$,*

$$\lim_{n \rightarrow \infty} \left\| f_{nl} - \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} \right\|_{C^1((R_1,R_2))} = 0.$$

Proof. Since the difference

$$g_{nl} := f_{nl} - \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}}$$

satisfies the ODE

$$\left(\partial_r^2 + \frac{2}{r} \partial_r - \frac{l(l+1)}{r^2} + \frac{2}{r} + \lambda_n\right) g_{nl}(r) = \lambda_n \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}},$$

where the coefficients and the source term are uniformly bounded for fixed l and all n on any finite interval (R_1, R_2) with $R_1 > 0$, it is standard that

$$\begin{aligned} \left\| f_{nl} - \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} \right\|_{C^1((R_1, R_2))} &\leq C \left\| f_{nl} - \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} \right\|_{C^0((R_1, R_2))} \\ &\quad + C |\lambda_n| \left\| \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} \right\|_{C^0((R_1, R_2))}, \end{aligned}$$

where the constant depends on R_1 and R_2 but not on n . Since $\lambda_n \rightarrow 0$ as $n \rightarrow \infty$, the result then follows from the C^0 bound given in Proposition 2.1. \square

3. An approximation theorem for the Coulomb problem with zero energy

In this section we shall prove an approximation theorem for the zero-energy Coulomb Hamiltonian $\Delta + 2/|x|$. In comparison with other Runge theorems, a significant technical difference is that the operator is not negative, which will force us to establish an approximation theorem for sets contained in balls of radii not larger than

$$R_0 := \frac{\sqrt{\pi}}{4}.$$

For this, a first step is to show the existence of a suitable Green’s function for the Coulomb Hamiltonian.

Lemma 3.1. *Let us consider the open ball $B_R \subset \mathbb{R}^3$ centered at the origin and of radius $R < R_0$. There exists a symmetric Dirichlet Green’s function on $B_R \times B_R$, bounded as*

$$|G_R(x, y)| \leq \frac{C}{|x - y|}$$

and continuous outside of the diagonal, which satisfies

$$\left(\Delta_x + \frac{2}{|x|} \right) G_R(x, y) = -\delta(x - y), \quad G_R(\cdot, y)|_{\partial B_R} = 0$$

on $B_R \times B_R$.

Proof. The result will follow from the fact that the Dirichlet spectrum of $-\Delta - 2/|x|$ is positive on B_R . To prove this we can use the min-max principle since its eigenfunctions are in H_0^1 . We first note that for all $\varphi \in C_0^\infty(B_R)$ we have

$$\begin{aligned} \int_{B_R} \frac{\varphi^2}{|x|} &\leq \left(\int_{B_R} \varphi^2 \right)^{1/2} \left(\int_{B_R} \frac{\varphi^2}{|x|^2} \right)^{1/2} \\ &\leq \left(\int_{B_R} \varphi^2 \right)^{1/2} \left(4 \int_{B_R} |\nabla \varphi|^2 \right)^{1/2} \leq \frac{2}{\sqrt{\lambda_1(B_R)}} \int_{B_R} |\nabla \varphi|^2, \end{aligned}$$

where

$$\lambda_1(B_R) := \frac{\pi}{R^2}$$

is the first Dirichlet eigenvalue of the Laplacian in the ball of radius R and we have used Hardy’s inequality followed by Poincaré’s inequality. Therefore the first Dirichlet eigenvalue of $-\Delta - \frac{2}{|x|}$ on B_R satisfies:

$$\begin{aligned} \lambda &= \min_{\varphi \in H_0^1(B_R) \setminus \{0\}} \frac{\int_{B_R} \left(|\nabla \varphi|^2 - \frac{2}{|x|} \varphi^2 \right)}{\int_{B_R} \varphi^2} \\ &\geq \left(1 - \frac{4R}{\sqrt{\pi}} \right) \min_{\varphi \in H_0^1(B_R) \setminus \{0\}} \frac{\int_{B_R} |\nabla \varphi|^2}{\int_{B_R} \varphi^2} > 0. \end{aligned}$$

Reference [14], Theorems 2 and 8, then ensures the existence of a Green’s function G_R as above. □

We are now ready to prove the approximation theorem for sets contained in the ball B_{R_0} . Notice that the series $\tilde{\varphi}$ defined in the statement below satisfies the equation

$$\left(\Delta + \frac{2}{|x|} \right) \tilde{\varphi} = 0$$

in the whole space \mathbb{R}^3 , is continuous at the origin and falls off at infinity as $|x|^{-3/4}$.

Theorem 3.2. *Let us consider a radius $R < R_0$ and fix an integer k and a positive real ϵ . Suppose that K is a compact subset of the ball B_R which does not contain the origin and whose complement $B_R \setminus K$ is connected. If a function φ satisfies the equation*

$$\left(\Delta + \frac{2}{|x|} \right) \varphi = 0$$

in a neighborhood U of K , then for any large enough N there is a finite series of the form

$$\tilde{\varphi} := \sum_{l=0}^N \sum_{m=-l}^l c_{lm} \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} Y_{lm}(\theta, \phi),$$

where c_{lm} are constants, which approximates φ in the sense that

$$\|\varphi - \tilde{\varphi}\|_{C^k(K)} < \epsilon.$$

Proof. Let us take a smooth function $\chi: \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $\chi = 1$ in a neighborhood of K and $\chi = 0$ outside the neighborhood U of K , whose closure can be assumed to lie in $B_R \setminus \{0\}$, and define a smooth extension of φ to \mathbb{R}^3 by setting $\varphi_1 := \chi \varphi$. Since $\varphi_1|_{\partial B_R} = 0$, we can use the Green’s function in Lemma 3.1 to represent φ_1 as the integral

$$(3.1) \quad \varphi_1(x) = \int_{B_R} G_R(x, y) \rho(y) dy,$$

where $\rho(x) = -\Delta\varphi_1(x) - \frac{2}{|x|}\varphi_1(x)$ is supported in $U \setminus K$. In order to see this, notice that the difference

$$v := \varphi_1(x) - \int_{B_R} G_R(x, y) \rho(y) dy$$

is zero on ∂B_R and satisfies the equation

$$\left(\Delta + \frac{2}{|x|}\right)v = 0.$$

Since we saw in the proof of Lemma 3.1 that the operator $\Delta + 2/|x|$ in B_R with Dirichlet boundary conditions and $R < \sqrt{\pi}/4$ is negative definite, we infer that $v = 0$, thereby proving (3.1).

Since the support of ρ is contained in $U \setminus K$, a continuity argument ensures we can approximate the integral uniformly in K by a finite Riemann sum, i.e., for any $\delta > 0$ there is a large enough integer J , points $x_j \in U \setminus K$ and constants a_j such that

$$\varphi_2(x) = \sum_{j=1}^J a_j G_R(x, x_j),$$

satisfies

$$(3.2) \quad \|\varphi_2 - \varphi\|_{C^0(K)} = \|\varphi_2 - \varphi_1\|_{C^0(K)} < \delta.$$

Next we choose $0 < R_1 < R$ such that the closure of U is contained in B_{R_1} and consider the infinite dimensional space

$$\mathcal{V} = \text{span}\{G_R(z, x) : z \in B_R \setminus \overline{B_{R_1}}\} \subset C^0(K).$$

By the Riesz–Markov theorem, the dual space to $C^0(K)$, which we will denote by $C^0(K)^*$, is the space of finite signed Borel measures on B_R whose support is contained in the set K . We let $\mu \in C^0(K)^*$ be any measure such that \mathcal{V} is in its null space, i.e., $\int_{B_R} f(x) d\mu(x) = 0$ for all $f \in \mathcal{V}$, and define

$$F(x) := \int_{B_R} G_R(x, y) d\mu(y).$$

Notice that F is well defined since the Green’s function is bounded as $|G_R(x, y)| \leq C/|x - y|$. In particular, F is continuous in $B_R \setminus K$.

By the definition of μ , $F(x) = 0$ for all $x \in B_R \setminus \overline{B_{R_1}}$ and, by the properties of the Green’s function,

$$\Delta F + \frac{2}{|x|}F = -\mu.$$

In particular, $\Delta F(x) + \frac{2}{|x|}F(x) = 0$ distributionally for all $x \in B_R \setminus K$. Hence, F is analytic in the region $B_R \setminus (K \cup \{0\})$, which is connected and contains an open set, $B_R \setminus \overline{B_{R_1}}$, on which F is identically zero. Therefore $F(x) = 0$ for all $x \in B_R \setminus K$ and it follows that φ_2 is also in the null space of μ :

$$\int_{B_R} \varphi_2(x) d\mu(x) = \sum_{j=1}^J a_j \int_{B_R} G_R(x_j, x) d\mu(x) = \sum_{j=1}^J a_j F(x_j) = 0.$$

Notice that in the last identity one gets zero because each x_j lies in $U \setminus K \subset B_R \setminus K$. We have also used that the Green's function is symmetric.

Therefore φ_2 cannot be separated from the subspace \mathcal{V} . Hence by the Hahn–Banach theorem φ_2 can be uniformly approximated in K by functions in \mathcal{V} , i.e., for any $\delta > 0$ there exists $\varphi_3 \in \mathcal{V}$ such that

$$(3.3) \quad \|\varphi_3 - \varphi_2\|_{C^0(K)} < \delta.$$

Now, by construction, φ_3 satisfies

$$\Delta\varphi_3 + \frac{2}{|x|}\varphi_3 = 0,$$

inside B_{R_1} . By the properties of $G_R(x, y)$ in Lemma 3.1, φ_3 is continuous at the origin, so it can be expanded as

$$\varphi_3(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^l c_{lm} \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} Y_{lm}(\theta, \phi),$$

where c_{lm} are constants.

Since the series converges in $L^2(B_{R_1})$, for any $\delta > 0$ there is an integer N such that the finite sum

$$\tilde{\varphi}(x) = \sum_{l=0}^N \sum_{m=-l}^l c_{lm} \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} Y_{lm}(\theta, \phi),$$

satisfies

$$\|\tilde{\varphi} - \varphi_3\|_{L^2(B_{R_1})} < \delta.$$

As

$$\Delta(\tilde{\varphi} - \varphi_3) + \frac{2}{|x|}(\tilde{\varphi} - \varphi_3) = 0,$$

in B_{R_1} , the fact that the closure of U is contained in $B_{R_1} \setminus \{0\}$ allows us to use standard elliptic estimates to promote the above L^2 bound to the uniform estimate

$$\|\tilde{\varphi} - \varphi_3\|_{C^0(U)} < C\delta.$$

From this inequality and the bounds in (3.2) and (3.3) we obtain

$$\|\tilde{\varphi} - \varphi\|_{C^0(K)} < (C + 2)\delta.$$

Finally, since

$$\Delta(\tilde{\varphi} - \varphi) + \frac{2}{|x|}(\tilde{\varphi} - \varphi) = 0,$$

in a neighborhood of K , standard elliptic estimates then allow us to promote the C^0 bound to a C^k bound, i.e.,

$$\|\tilde{\varphi} - \varphi\|_{C^k(K)} < C'\delta.$$

The theorem then follows by choosing δ small enough so that $C'\delta < \epsilon$. □

4. Coulomb eigenfunctions with dislocations of arbitrary topology

This section will be devoted to the proof of Theorem 1.1. For this, we start by taking a diffeomorphism of \mathbb{R}^3 , Φ_0 , such that the transformed link $L_0 := \Phi_0(L)$ is contained in a punctured ball $B_R \setminus \{0\}$ for some $R < R_0$. Whitney’s approximation theorem guarantees that, by perturbing it if necessary, we can assume that L_0 is a real analytic submanifold of \mathbb{R}^3 .

Let $L_{0,a}$, $a = 1, \dots, M$, denote the connected components of L_0 (which are just closed curves in \mathbb{R}^3) and construct surfaces $\Sigma_a^1 \subset B_R \setminus \{0\}$ and $\Sigma_a^2 \subset B_R \setminus \{0\}$ that intersect transversally at $L_{0,a}$. This can be achieved by considering an analytic submersion $\Theta_a: W_a \rightarrow \mathbb{R}^2$, where W_a is a small tubular neighborhood of $L_{0,a}$ and $\Theta_a^{-1}(0) = L_{0,a}$. We recall that the map Θ_a is a *submersion* if the rank of the differential $(D\Theta_a)_x$ is 2 for all $x \in W_a$. The existence of such a submersion is guaranteed since any closed curve in \mathbb{R}^3 has trivial normal bundle [10]. The surfaces can then be taken as $\Sigma_a^1 := \Theta_a^{-1}((-1, 1) \times \{0\})$ and $\Sigma_a^2 := \Theta_a^{-1}(\{0\} \times (-1, 1))$, so they are small cylinders embedded in W_a .

With $b = 1, 2$, we now consider the Cauchy problems

$$\Delta \varphi_a^b + \frac{2}{|x|} \varphi_a^b = 0, \quad \varphi_a^b|_{\Sigma_a^b} = 0, \quad \partial_\nu \varphi_a^b|_{\Sigma_a^b} = 1,$$

where ∂_ν denotes a normal derivative at the relevant surface. Because the equations are analytic inside $B_R \setminus \{0\}$, we can use the Cauchy–Kowalewski theorem to obtain solutions in the closure of small neighborhoods of each surface, $U_a^b \supset \Sigma_a^b$. By shrinking the neighborhoods if necessary we can assume that $U_a^b \subset B_R \setminus \{0\}$ and the tubular neighborhoods $U_a^1 \cap U_a^2$ of $L_{0,a}$ are disjoint.

We now take the union of these tubular neighborhoods,

$$U := \bigcup_{a=1}^M U_a^1 \cap U_a^2,$$

and define a complex valued function φ on the set U as

$$\varphi|_{U_a^1 \cap U_a^2} := \varphi_a^1 + i\varphi_a^2.$$

By construction, φ has the following properties:

- (i) It satisfies the equation $\Delta \varphi + \frac{2}{|x|} \varphi = 0$ in the tubular neighborhood U of the link L_0 ,
- (ii) By taking U small enough its nodal set is precisely L_0 , i.e., $\varphi^{-1}(0) = L_0$, and
- (iii) The intersection of the zero sets of the real and imaginary parts of φ on L_0 is transverse, i.e., $\text{rank}(\nabla \text{Re } \varphi(x), \nabla \text{Im } \varphi(x)) = 2$ for all $x \in L_0$. This is an immediate consequence of the Cauchy data used to construct φ_a^b (namely, $\varphi_a^b|_{\Sigma_a^b} = 0$ and $\partial_\nu \varphi_a^b|_{\Sigma_a^b} = 1$) and of the fact that Σ_a^1 and Σ_a^2 intersect transversally.

We denote by $K \subset U$ a compact set containing L_0 whose complement $B_R \setminus K$ is connected. We can now use Theorem 3.2 to obtain constants $c_{lm} \in \mathbb{C}$ and $N \in \mathbb{N}$ such that the function

$$\tilde{\varphi} = \sum_{l=0}^N \sum_{m=-l}^l c_{lm} \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} Y_{lm}(\theta, \phi),$$

satisfies

$$(4.1) \quad \|\tilde{\varphi} - \varphi\|_{C^1(K)} < \epsilon,$$

for an $\epsilon > 0$ to be chosen later. With n a large number to be fixed later, we now define

$$\psi_n := \sum_{l=0}^N \sum_{m=-l}^l c_{lm} f_{nl}(r) Y_{lm}(\theta, \phi),$$

and note that

$$\begin{aligned} \|\psi_n - \tilde{\varphi}\|_{C^1(K)} &\leq \sum_{l=0}^N \sum_{m=-l}^l |c_{lm}| \left\| \left(f_{nl} - \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} \right) Y_{lm} \right\|_{C^1(K)} \\ &\leq C \sum_{l=0}^N \left\| f_{nl} - \frac{J_{2l+1}(\sqrt{8r})}{\sqrt{8r}} \right\|_{C^1(K)}. \end{aligned}$$

By Corollary 2.2 there exists $N' \in \mathbb{N}$ such that for all $n \geq N'$ we have

$$\|\psi_n - \tilde{\varphi}\|_{C^1(K)} < \epsilon,$$

and combining this with equation (4.1) we obtain

$$(4.2) \quad \|\psi_n - \varphi\|_{C^1(K)} < 2\epsilon,$$

for all $n \geq N'$

Item (iii) above and Thom’s isotopy theorem, Theorem 20.2 in [1], ensure that the link L_0 is structurally stable. We can therefore choose $\epsilon > 0$ small enough so that for each $n \geq N'$ there exists a diffeomorphism, Φ_1 , of \mathbb{R}^3 that is C^1 -close to the identity and different from the identity just in a small neighborhood of L_0 , such that $\Phi_1(L_0)$ is a union of connected components of the zero set $\psi_n^{-1}(0)$. Moreover, we can assume $\Phi_1(L_0) \subset B_R \setminus \{0\}$ and it is structurally stable since the $C^1(K)$ -closeness of ψ_n to φ implies that $\text{rank}(\nabla \text{Re } \psi_n(x), \nabla \text{Im } \psi_n(x)) = 2$ for all $x \in \Phi_1(L_0)$. The theorem follows by setting $E := -\lambda_{N'}$ and $\Phi := \Phi_1 \circ \Phi_0$.

Remark 4.1. It follows from the proof of Theorem 1.1 that if the link L is contained in a punctured ball $B_R \setminus \{0\}$ for some $R < R_0$, the diffeomorphism Φ can be chosen as close to the identity as one wishes in the C^1 norm.

References

- [1] ABRAHAM, R. AND ROBBIN, J.: *Transversal mappings and flows*. Benjamin, New York, 1967.
- [2] BERRY, M.: Knotted zeros in the quantum states of hydrogen. *Found. Phys.* **31** (2001), no. 4, 659–667.
- [3] DENNIS, M. R., KING, R. P., JACK, B., O’HOLLERAN, K. AND PADGETT, M. J.: Isolated optical vortex knots. *Nature Phys.* **6** (2010), 118–121.
- [4] ENCISO, A., HARTLEY, D. AND PERALTA-SALAS, D.: A problem of Berry and knotted zeros in the eigenfunctions of the harmonic oscillator. *J. Eur. Math. Soc. (JEMS)* **20** (2018), no. 2, 301–314.
- [5] ENCISO, A. AND PERALTA-SALAS, D.: Knots and links in steady solutions of the Euler equation. *Ann. of Math. (2)* **175** (2012), no. 1, 345–367.
- [6] ENCISO, A. AND PERALTA-SALAS, D.: Submanifolds that are level sets of solutions to a second-order elliptic PDE. *Adv. Math.* **249** (2013), 204–249.
- [7] ENCISO, A. AND PERALTA-SALAS, D.: Existence of knotted vortex tubes in steady Euler flows. *Acta Math.* **214** (2015), no. 1, 61–134.
- [8] IRVINE, W. T. M. AND KLECKNER, D.: Liquid crystals: tangled loops and knots. *Nature Materials* **13** (2014), no. 3, 229–231.
- [9] KLECKNER, D. AND IRVINE, W. T. M.: Creation and dynamics of knotted vortices. *Nature Phys.* **9** (2013), no. 4, 253–258.
- [10] MASSEY, W. S.: On the normal bundle of a sphere imbedded in Euclidean space. *Proc. Amer. Math. Soc.* **10** (1959), 959–964.
- [11] NYE, J. F.: *Natural focusing and fine structure of light*. Institute of Physics Publishing, Bristol, 1999.
- [12] SZEGŐ, G.: *Orthogonal polynomials*. Fourth edition. Colloquium Publications 23, American Mathematical Society, Providence, RI, 1975.
- [13] TKALEC, U., RAVNIK, M., ČOPAR, S., ŽUMER, S. AND MUŠEVIČ, I.: Reconfigurable knots and links in chiral nematic colloids. *Science* **333** (2011), no. 6038, 62–65.
- [14] ZHAO, Z.: Green function for Schrödinger operator and conditioned Feynman–Kac gauge. *J. Math. Anal. Appl.* **116** (1986), no. 2, 309–334.

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