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C_0-semigroups of 2-isometries and Dirichlet spaces

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Abstract. In the context of a theorem of Richter, we establish a similarity between C_0-semigroups of analytic 2-isometries $\{T(t)\}_{t\geq 0}$ acting on a Hilbert space $\mathcal{H}$ and the multiplication operator semigroup $\{M_{\phi_t}\}_{t\geq 0}$ induced by $\phi_t(s) = \exp(-st)$ for s in the right-half plane $\mathbb{C}_+$ acting boundedly on weighted Dirichlet spaces on $\mathbb{C}_+$. As a consequence, we derive a connection with the right shift semigroup $\{S_t\}_{t\geq 0}$ given by

$$S_t f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq t, \\ f(x-t) & \text{if } x > t, \end{cases}$$

acting on a weighted Lebesgue space on the half line $\mathbb{R}_+$ and address some applications regarding the study of the invariant subspaces of C_0-semigroups of analytic 2-isometries.

1. Introduction

The concept of a 2-isometry was introduced by Agler in the early eighties (cf. [1]); this is related to notions due to J. W. Helton (see [8] and [9]) and characterized in terms of their extension properties (see [2]). Recall that a bounded linear operator T on a separable, infinite dimensional complex Hilbert space $\mathcal{H}$ is called a 2-*isometry* if it satisfies

$$T^{*2}\, T^2 - 2\, T^* T + I = 0\,,$$

where I denotes the identity operator. In addition, such operators are called *analytic* if no nonzero vector is in the range of every power of T. It turns out that M_z, i.e., the multiplication operator by z, acting on the classical Dirichlet space, is a cyclic analytic 2-isometry. But, moreover, in [14] (see also [13]) Richter proved that any cyclic analytic 2-isometry is unitarily equivalent to M_z acting on a generalized Dirichlet space $D(\mu)$.

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More precisely, let μ be a finite non-negative Borel measure on the unit circle $\mathbb{T}$, and let $D(\mu)$ be the *generalized Dirichlet space* associated to μ, that is, the Hilbert space consisting of analytic functions on the unit disc $\mathbb{D}$ such that the integral

$$\int_{\mathbb{D}} |f'(z)|^2 \Big(\int_{|\xi|=1} \frac{1-|z|^2}{|\xi - z|^2}\, d\mu(\xi) \Big) \frac{dm(z)}{\pi}$$

is finite (here $dm(z)$ denotes the Lebesgue area measure in $\mathbb{D}$). Note that if $\mu = 0$, the space $D(\mu)$ is defined to be the classical Hardy space H^2 and for non-zero, finite, non-negative Borel measures μ on $\mathbb{T}$, the space $D(\mu)$ is contained in the Hardy space (see [7], Chapter 7). Then Richter's theorem reads as follows.

Theorem (Richter). *Let T be a bounded linear operator on an infinite dimensional complex Hilbert space $\mathcal{H}$. Then the following condition are equivalent:*

(i) *T is an analytic 2-isometry with* $\dim \operatorname{Ker} T^* = 1$,

(ii) *T is unitarily equivalent to $(M_z, D(\mu))$ for some finite non-negative Borel measure on $\mathbb{T}$, where*

$$\|f\|^2_{D(\mu)} = \|f\|^2_{H^2} + \int_{\mathbb{D}} |f'(z)|^2 \Big(\int_{|\xi|=1} \frac{1-|z|^2}{|\xi - z|^2}\, d\mu(\xi) \Big) \frac{dm(z)}{\pi}.$$

One of the main applications of Richter's theorem concerns the study of the invariant subspaces for the multiplication operator M_z in the spaces $D(\mu)$ and its relationship with the classical Beurling theorem for the Hardy space H^2 (see [3]). For instance, regarding the Dirichlet space $D = D\big(\frac{|d\xi|}{2\pi}\big)$, Richter and Sundberg [15] proved that any closed, invariant subspace $\mathcal{M}$ under M_z satisfies that $\dim(\mathcal{M} \ominus z\mathcal{M}) = 1$. Moreover, if $\varphi \in \mathcal{M} \ominus z\mathcal{M}$ with $\|\varphi\|_D = 1$, then $|\varphi(z)| \leq 1$ for $|z| \leq 1$ and $\mathcal{M} = \varphi D(m_\varphi)$, where dm_φ is the measure on $\mathbb{T}$ given by $dm_\varphi(\xi) = |\varphi(\xi)|^2 \frac{|d\xi|}{2\pi}$. For general $D(\mu)$ spaces, an analogous result holds. We refer the reader to Chapters 7 and 8 in the recent monograph *"A primer on the Dirichlet space"* [7] for more on the subject.

Motivated by the Beurling–Lax theorem and the work carried out by Richter, the aim of this work is taking further the study of the 2-isometries and considering C_0-semigroups of 2-isometric operators. In particular, we will establish a similarity between C_0-semigroups of analytic 2-isometries $\{T(t)\}_{t\geq 0}$ acting on a Hilbert space $\mathcal{H}$ and the multiplication operator semigroup $\{M_{\phi_t}\}_{t\geq 0}$ induced by $\phi_t(s) = \exp(-st)$ for s in the right-half plane $\mathbb{C}_+$ acting boundedly on weighted Dirichlet spaces $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$ on $\mathbb{C}_+$ (see Definition 2.3). As a consequence, by means of the Laplace transform, we derive a connection with the right shift semigroup $\{S_t\}_{t\geq 0}$

$$S_t f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq t, \\ f(x-t) & \text{if } x > t, \end{cases}$$

acting on a weighted Lebesgue space on the half line $\mathbb{R}_+$. Finally, some applications regarding the study of the invariant subspaces of C_0-semigroups of analytic 2-isometries are also discussed in Section 3.

2. C_0-semigroups of analytic 2-isometries

First, we introduce some basic concepts and terminology regarding C_0-semigroups of bounded linear operators. For more on this topic, we refer the reader to the Engel–Nagel monograph [6].

A C_0-semigroup $\{T(t)\}_{t\geq 0}$ of operators on a Hilbert space $\mathcal{H}$ is a family of bounded linear operators on $\mathcal{H}$ satisfying the functional equation

$$\begin{cases} T(t+s) = T(t)T(s) & \text{for all } t, s \geq 0, \\ T(0) = I, \end{cases}$$

and such that $T(t) \to I$ in the strong operator topology as $t \to 0^+$. Given a C_0-semigroup $\{T(t)\}_{t\geq 0}$, there exists a closed and densely defined linear operator A that determines the semigroup uniquely, called the generator of $\{T(t)\}_{t\geq 0}$, defined by means of

$$Ax := \lim_{t\to 0^+} \frac{T(t)x - x}{t},$$

where the domain $D(A)$ of A consists of all $x \in \mathcal{H}$ for which this limit exists (see Chapter II in [6], for instance). Although the generator is, in general, an unbounded operator, it plays an important role in the study of a C_0-semigroup, reflecting many of its properties.

However, if 1 is in the resolvent of A, that is, in the set

$$\rho(A) = \{\lambda \in \mathbb{C} : \ (A - \lambda I) : D(A) \subset \mathcal{H} \to \mathcal{H} \text{ is bijective}\},$$

then $(A-I)^{-1}$ is a bounded operator on $\mathcal{H}$ by the closed graph theorem, and the Cayley transform of A defined by

$$V := (A+I)(A-I)^{-1}$$

is a bounded operator on $\mathcal{H}$, since $V - I = 2(A-I)^{-1}$. Therefore V determines the semigroup uniquely, since A does. This operator is called the *cogenerator* of the C_0-semigroup $\{T(t)\}_{t\geq 0}$. Observe that 1 is not an eigenvalue of V.

Recall that if A is a closed operator, then the *spectral bound* $s(A)$ of A is defined by

$$s(A) := \sup\{\operatorname{Re}\lambda : \ \lambda \in \sigma(A)\},$$

where $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is the spectrum of A, and in case that A is the generator of a C_0-semigroup, then $s(A)$ is always dominated by the *growth bound of the semigroup*, that is,

$$-\infty \leq s(A) \leq w_0 = \inf\left\{ w \in \mathbb{R} : \begin{array}{c} \text{there exists } M_w \geq 1 \text{ such that} \\ \|T(t)\| \leq M_w\, e^{wt} \text{ for all } t \geq 0 \end{array} \right\}.$$

Indeed, if $r(T(t))$ denotes the spectral radius of $T(t)$, it follows that, for each $t > 0$, $w_0 = \frac{1}{t}\log r(T(t))$ (see [6], Section 2, Chapter IV, for instance). The following lemma will be useful in the context of our main result later.

Lemma 2.1. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a separable, infinite dimensional complex Hilbert space $\mathcal{H}$ consisting of 2-isometries and A its generator. Then $1 \in \rho(A)$ and therefore, the cogenerator V of $\{T(t)\}_{t\geq 0}$ is well-defined.*

Proof. By induction it follows that, for any $n \geq 1$ and $t \geq 0$, $T(t)$ satisfies

$$T(t)^{*n}\, T(t)^n - nT(t)^*\, T(t) + (n-1)I = 0,$$

and so

$$\|T(t)^n x\|^2 = n\|T(t)x\|^2 - (n-1)\|x\|^2$$

for $x \in \mathcal{H}$. From here, it follows that $\|T(t)^n\| \leq C\sqrt{n}$, where C is a constant independent of n, and therefore the spectral radius $r(T(t)) \leq 1$ for any t. Therefore, $s(A) \leq 0$; and therefore $1 \in \rho(A)$. □

The next result consists of a particular instance of Theorem 1 in [11], where C_0-semigroups of hypercontractions are considered. We state it for C_0-semigroups of 2-isometries and include its proof for the sake of completeness.

Proposition 2.2. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a separable, infinite dimensional complex Hilbert space $\mathcal{H}$. Then the following conditions are equivalent:*

(i) *$T(t)$ is a 2-isometry for every $t \geq 0$.*

(ii) *The mapping $t \in \mathbb{R}_+ \mapsto \|T(t)x\|^2$ is affine for each $x \in \mathcal{H}$.*

(iii) $\mathrm{Re}\langle A^2 y, y\rangle + \|Ay\|^2 = 0 \quad (y \in \mathcal{D}(A^2))$.

(iv) *The cogenerator V of $\{T(t)\}_{t\geq 0}$ exists and is a 2-isometry.*

Proof. (i) $\Longleftrightarrow$ (ii): If each $T(t)$ is a 2-isometry, then for $t \geq 0$ and $\tau > 0$ we have

$$\langle T(t+2\tau)x, T(t+2\tau)x\rangle - 2\langle T(t+\tau)x, T(t+\tau)x\rangle + \langle T(t)x, T(t)x\rangle = 0,$$

so that

$$\|T(t+\tau)x\|^2 = \frac{1}{2}(\|T(t)x\|^2 + \|T(t+2\tau)x\|^2). \tag{2.1}$$

Since $t \in \mathbb{R}_+ \to \|T(t)x\|^2$ is continuous, the mapping is affine.

Conversely, taking $t = 0$ we see that (2.1) implies that $T(\tau)$ is a 2-isometry.

(ii) $\Longleftrightarrow$ (iii): For $t > 0$, we calculate the second derivative of the function $g : t \mapsto \|T(t)y\|^2$ for $y \in \mathcal{D}(A^2)$. We have

$$\begin{aligned} g''(t) &= \frac{d^2}{dt^2}\langle T(t)y, T(t)y\rangle \\ &= \langle A^2T(t)y, T(t)y\rangle + 2\langle AT(t)y, AT(t)y\rangle + \langle T(t)y, A^2T(t)y\rangle. \end{aligned}$$

For g affine, g'' is zero, and Condition (iii) follows on letting $t \to 0$. Conversely, Condition (iii) implies Condition (ii) for $y \in \mathcal{D}(A^2)$, and hence for all y by density.

(iii) $\Longleftrightarrow$ (iv): We calculate

$$\langle (I - 2V^*V + V^{*2}V^2)x, x\rangle$$

for $x = (A-I)^2 y$ (note that $(A-I)^{-2} : H \to H$ is defined everywhere and has dense range). We obtain

$$\langle (A-I)^2 y, (A-I)^2 y\rangle - 2\langle (A^2-I)y, (A^2-I)y\rangle + \langle (A+I)^2 y, (A+I)^2 y\rangle \\ = 4\langle A^2 y, y\rangle + 8\langle Ay, Ay\rangle + 4\langle y, A^2 y\rangle.$$

Thus V is a 2-isometry if and only if Condition (iii) holds. □

Before stating the main result of the section, let us introduce the following definition.

Definition 2.3. Let ν be a finite positive Borel measure supported on the imaginary axis. The Dirichlet space $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$ is defined as the space of analytic functions F on right half-plane $\mathbb{C}_+$ such that

$$\|F\|^2 = |F(1)|^2 + \frac{1}{\pi}\int_{\mathbb{C}_+} |F'(s)|^2 \Big(x + \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x}{x^2+(y-\tau)^2}\, d\nu(\tau)\Big)\, dx\, dy < \infty,$$

where $s = x + iy$.

The spaces $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$ arise, in a natural way, when we analyze C_0-semigroups of analytic 2-isometries in Hilbert spaces, as it is stated in our main result:

Theorem 2.4. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a separable, infinite dimensional complex Hilbert space $\mathcal{H}$ consisting of analytic 2-isometries for every $t>0$ such that*

$$\dim \bigcap_{t>0} \ker \big(T^*(t) - e^{-t} I\big) = 1. \tag{2.2}$$

Then there exists a finite positive Borel measure ν supported on the imaginary axis such that $\{T(t)\}_{t\geq 0}$ is similar to the semigroup of multiplication operators induced by $\exp(-ts)$ acting on the space $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$. Moreover, if the multiplication operators induced by $\exp(-ts)$ act continuously for every $t>0$ on a Dirichlet space $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\tilde{\nu})$ where $\tilde{\nu}$ is a finite positive Borel measure supported on the imaginary axis, then the corresponding semigroup consists of analytic 2-isometries and satisfies (2.2).

Before proceeding further, let us remark that our main result yields similarity for the semigroup $\{T(t)\}_{t\geq 0}$ because of the definition of the norm in $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$. In addition, as we shall see later, condition (2.2) is a way of expressing the property that $\dim\ker V^* = 1$, where V is the cogenerator of the semigroup $\{T(t)\}_{t\geq 0}$.

In order to prove Theorem 2.4, we need the following auxiliary results.

Proposition 2.5. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a separable, infinite dimensional complex Hilbert space $\mathcal{H}$ consisting of analytic 2-isometries. Then the cogenerator V is an analytic 2-isometry.*

Proof. First, we observe that V is well-defined by Lemma 2.1 and, it is a 2-isometry by Proposition 2.2. So, we are required to show that V is analytic.

The Wold decomposition theorem for 2-isometries (see [12], for instance), yields that V can be decomposed as $V = S \oplus U$ with respect to $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, where U is the unitary part on $\mathcal{H}_2 = \bigcap_n V^n \mathcal{H}$ and S is an analytic 2-isometry. We will show that $U = 0$.

Let us assume, on the contrary, that $U \neq 0$.

First, we observe that since 1 is not an eigenvalue of V, the generator A of the semigroup $\{T(t)\}_{t\geq 0}$ may be expressed as the (possibly) unbounded operator

$$(V + I)(V - I)^{-1}.$$

Moreover, since $T(t)$ commutes with $(A-I)^{-1}$ and hence with V, it holds that $\mathcal{H}_2$ is invariant under $T(t)$ for every $t \geq 0$. In addition, the generator B of the restricted semigroup $\{T(t)_{|\mathcal{H}_2}\}_{t\geq 0}$ is the restriction of A to the $D(A) \cap \mathcal{H}_2$ (see Ch. 2, Sec. 2 in [6], for instance); and the cogenerator is U.

Now, taking into account the fact that U is unitary, one deduces that B is skew-adjoint (i.e., $B^\star = -B$). Then the restriction of $T(t)$ to $\mathcal{H}_2$ is unitary for every $t \geq 0$ and, therefore, every vector in $\mathcal{H}^2$ is in the range of (powers of) $T(t)$. Since $T(t)$ is analytic, it follows that $\mathcal{H}^2 = \{0\}$, a contradiction. Hence, $U = 0$ and the proof is completed. □

Lemma 2.6. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on a separable, infinite dimensional complex Hilbert space $\mathcal{H}$ and A its generator. The following conditions are equivalent:*

(1) *$Ax_0 = -x_0$ for some $x_0 \in D(A)$.*

(2) *$T(t)x_0 = e^{-t}x_0$ for all $t \geq 0$ and $x_0 \in D(A)$.*

In addition, if $1 \in \rho(A)$ and V is the cogenerator, any of the previous conditions is equivalent to

(3) *$Vx_0 = 0$ for some $x_0 \in D(A)$.*

Note that the equivalence between (1) and (2) in Lemma 2.6 just follows from the relationship between the eigenspaces of A and the semigroup $\{T(t)\}_{t\geq 0}$, that is,

$$\mathrm{Ker}(\mu I - A) = \bigcap_{t\geq 0} \mathrm{Ker}(e^{\mu t} - T(t)),$$

with $\mu \in \mathbb{C}$ (see Corollary 3.8, Section IV in [6], for instance). The last statement follows from the definition of V.

We are now in position to prove Theorem 2.4.

Proof of Theorem 2.4. Assume that $\{T(t)\}_{t>0}$ consists of analytic 2-isometries. Let V denote its cogenerator; this is well-defined by Lemma 2.1, and it is an analytic 2-isometry by Proposition 2.5.

In addition, the hypotheses

$$\dim \bigcap_{t>0} \ker\left(T^*(t) - e^{-t}\, I\right) = 1,$$

along with Lemma 2.6 applied to the adjoint semigroup $\{T^*(t)\}_{t\geq 0}$, yields that $\dim \mathrm{Ker}\, V^* = 1$.

By means of Richter's theorem, it follows that V is similar to M_z acting on the space $D(\mu)$ for some finite non-negative Borel measure μ on $\mathbb{T}$ considered with the equivalent norm

$$\begin{aligned} \|f\|^2_{D(\mu)} &\approx |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \Big(\int_{|\xi|=1} \frac{1-|z|^2}{|\xi - z|^2}\, d\mu(\xi) \Big) \frac{dm(z)}{\pi} \\ &= |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 P_\mu(z) \frac{dm(z)}{\pi}. \end{aligned} \tag{2.3}$$

Observe that the similarity is the price paid when we consider the equivalent norm. Hence, for any $t \geq 0$, it follows that $T(t)$ is unitarily equivalent to the multiplication operator induced by $\exp(-t(1+z)/(1-z))$ on $D(\mu)$. Now, we migrate to the right half-plane $\mathbb{C}_+ = \{\operatorname{Re} s > 0\}$ applying the change of variables $s = (1+z)/(1-z)$, or $z = (s-1)/(s+1)$.

First, we observe that

$$P_\mu\Big(\frac{s-1}{s+1}\Big) = \int_{|\xi|=1} \frac{1 - |\frac{s-1}{s+1}|^2}{|\xi - \frac{s-1}{s+1}|^2}\, d\mu(\xi) \quad (s \in \mathbb{C}_+)$$

is a positive harmonic function in $\mathbb{C}_+$; so there exists a non-negative constant ρ and a finite positive Borel measure ν supported on the imaginary axis such that

$$P_\mu\Big(\frac{s-1}{s+1}\Big) = \rho\, x + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y-\tau)^2}\, d\nu(\tau), \quad (s = x+iy) \tag{2.4}$$

(see Exercise 6, p. 134 in [10], for instance).

We can express ν in terms of μ, since with $\xi = (u-1)/(u+1)$ for $u = i\tau \in i\mathbb{R}$, we have

$$\begin{aligned} &P_\mu\Big(\frac{s-1}{s+1}\Big) \\ &= \mu(1)\, \frac{|s+1|^2 - |s-1|^2}{|(s+1)-(s-1)|^2} \quad + \int_{\xi \in \mathbb{T}\setminus\{1\}} \frac{(|s+1|^2 - |s-1|^2)|u+1|^2}{|(u-1)(s+1) - (u+1)(s-1)|^2}\, d\mu(\xi) \\ &= \mu(1)\, x + \int_{\xi\in\mathbb{T}\setminus\{1\}} \frac{x|u+1|^2}{|u-s|^2}\, d\mu(\xi) = \mu(1)\, x + \int_{\xi\in\mathbb{T}\setminus\{1\}} \frac{x(1+\tau^2)}{x^2 + (y-\tau)^2}\, d\mu(\xi), \end{aligned}$$

where $s = x+iy \in \mathbb{C}_+$. So in (2.4) we have

$$\rho = \mu(1) \quad \text{and} \quad \frac{d\nu(\tau)}{\pi(1+\tau^2)} = d\mu(\xi). \tag{2.5}$$

Then, upon applying the change of variables $s = (1+z)/(1-z)$ in (2.3), we deduce that $T(t)$ is similar to the multiplication operator induced by $\exp(-ts)$ acting on the space $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$ consisting of analytic functions F on $\mathbb{C}_+$ such that

$$\frac{1}{\pi} \int_{\mathbb{C}_+} |F'(s)|^2 \Big(x + \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{x^2 + (y-\tau)^2}\, d\nu(\tau) \Big)\, dx\, dy < \infty, \tag{2.6}$$

where $s = x+iy$ and $F(s) = f(z)$. This proves the first half of Theorem 2.4.

In order to conclude the proof, let us assume that the multiplication operators induced by $\exp(-ts)$ act continuously for every $t > 0$ on a Dirichlet space $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\tilde{\nu})$ where $\tilde{\nu}$ is a finite positive Borel measure supported on the imaginary axis. Reversing the steps above and taking into account the fact that (2.5) defines a measure $\tilde{\mu}$ on $\mathbb{T}$, where $\tilde{\mu}(1) = \tilde{\nu}(0) = \rho$, we deduce that the given semigroup is similar to the semigroup of multiplication operators induced by $\phi_t(z) = \exp(-t(1+z)/(1-z))$ on $D(\tilde{\mu})$. Since the cogenerator of such a semigroup is M_z, which is a 2-isometry, it follows by Proposition 2.2 that $\{M_{\phi_t}\}_{t\geq 0}$ consists of 2-isometries.

It remains to show that M_{ϕ_t} is analytic for every $t > 0$. If not, then there are a $t_0 > 0$ and a $F \in \widetilde{\mathcal{D}}_{\mathbb{C}_+}(\tilde{\nu})$ such that the function $s \mapsto e^{nt_0 s}F(s)$ lies in $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\tilde{\nu})$ for $n = 1, 2, 3, \dots$

In particular,

$$\int_{\mathbb{C}_+} |(F(s)e^{nt_0 s})'|^2\, x\, dx\, dy < \infty.$$

Transferring to the disc by letting $s = (1+z)/(1-z)$ and $F(s) = f(z)$, we have

$$\int_{\mathbb{D}} \left| [f(z)\exp(nt_0(1+z)/(1-z))]' \right|^2 \frac{1-|z|^2}{|1-z|^2}\, dA(z) < \infty,$$

so that the function $z \mapsto f(z)\exp(nt_0(1+z)/(1-z))$ lies in the weighted Dirichlet space $D(\delta_1)$ corresponding to a Dirac measure at 1, and hence in $H^2(\mathbb{D})$, by Theorem 7.1.2 in [7]. We conclude that f is identically zero, since no nontrivial H^2 function can be divisible by an arbitrarily large power of a nonconstant inner function. Hence the analyticity is also established. □

A connection with the right-shift semigroup in weighted $L^2(\mathbb{R}_+)$

Now, by means of the Laplace transform, we will establish a connection of C_0-semigroups of analytic 2-isometries $\{T(t)\}_{t\geq 0}$ acting on a Hilbert space $\mathcal{H}$ and the the right shift semigroup $\{S_t\}_{t\geq 0}$

$$S_t f(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq t, \\ f(x-t) & \text{if } x > t, \end{cases}$$

acting on a weighted Lebesgue space on the half line $\mathbb{R}_+$.

First, let us begin by recalling a result asserting that for each $\alpha > -1$, a function G analytic in $\mathbb{C}_+$ belongs to the *weighted Bergman space* $\mathcal{A}^2_\alpha(\mathbb{C}_+)$, that is, the space consisting of analytic functions on $\mathbb{C}_+$ for which

$$\|G\|^2_{\mathcal{A}^2_\alpha(\Pi^+)} = \int_{\mathbb{C}_+} |G(x+iy)|^2\, x^\alpha\, dx\, dy < \infty\,,$$

if and only if it has the form

$$G(s) := \mathcal{L}g(s) = \int_0^\infty e^{-st}\, g(t)\, dt\,, \quad s \in \mathbb{C}_+\,,$$

where g is a measurable function on $\mathbb{R}^+$ with

$$\int_0^\infty |g(t)|^2\, t^{-1-\alpha}\, dt < \infty\,.$$

Moreover,

$$\|G\|^2_{\mathcal{A}^2_\alpha(\mathbb{C}^+)} = \frac{\pi\,\Gamma(1+\alpha)}{2^\alpha} \int_0^\infty |g(t)|^2\, t^{-1-\alpha}\, dt\,,$$

(see [4] or Theorem 1 in [5], for instance). In other words, the Laplace transform is an isometric isomorphism between $\mathcal{A}^2_\alpha(\mathbb{C}_+)$ and $L^2(\mathbb{R}_+, \big(\frac{\pi\,\Gamma(1+\alpha)}{2^\alpha}\big)^{1/2} t^{-1-\alpha}\, dt)$. Hence, by means of a density argument, and taking $\alpha = 1$, it follows that for any $F \in \widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$, there exists $f \in L^2(\mathbb{R}_+)$ (unique in the usual sense of equivalence classes), such that

(i) $\mathcal{L}(tf(t)) = F'(s)$;

(ii)

$$\frac{1}{\pi}\int_{\mathbb{C}_+} |F'(s)|^2\, x\, dx\, dy = \frac{1}{2}\int_0^\infty |f(t)|^2 dt, \quad (s = x+iy),$$

which corresponds to the first sum in (2.6); and

(iii)

$$\frac{1}{\pi^2}\int_{\mathbb{C}_+} |F'(s)|^2 \frac{x}{x^2+(y-\tau)^2}\, dx\, dy = \frac{1}{2\pi}\int_0^\infty \Big|\int_0^t u\, f(u) e^{-i\tau u}\, du\Big|^2 \frac{dt}{t^2},$$

with $s = x+iy$.

These three items, along with the fact that, for any $t \geq 0$, $T(t)$ is similar to the multiplication operator induced by $\exp(-ts)$ acting on the space $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$, yield, by means of the Laplace transform, that $\{T(t)\}_{t\geq 0}$ is transformed to the right-shift semigroup $\{S_t\}_{t\geq 0}$ acting on the Hilbert space $\mathfrak{H}$ which consists of functions f defined on $\mathbb{R}_+$ such that

$$\int_0^\infty |f(t)|^2\, dt + \int_0^\infty \int_{-\infty}^\infty \Big|\int_0^t f(u) e^{-i\tau u} u\, du\Big|^2 d\nu(\tau)\, \frac{dt}{t^2} < \infty.$$

3. A final remark on invariant subspaces of C_0-semigroups of analytic 2-isometries

As an application of our main result, we deal with the study of the lattice of the closed invariant subspaces of a C_0-semigroup $\{T(t)\}_{t\geq 0}$ of analytic 2-isometries.

Here we shall use the following result from [14], Theorem 7.1, and [15], Theorem 3.2.

Theorem 3.1. *Let $\mathcal{M}$ be a non-zero invariant subspace of $(M_z, D(\mu))$. Then $\mathcal{M} = \phi D_{\mu_\phi}$ where $\phi \in \mathcal{M} \ominus z\mathcal{M}$ is a multiplier of $D(\mu)$ and $d\mu_\phi = |\phi|^2 d\mu$.*

In the continuous case we have the following result.

Theorem 3.2. *Let* $\{T(t)\}_{t\geq 0}$ *denote the semigroup of multiplication operators induced by* $\exp(-ts)$ *on the space* $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$*, as in Theorem 2.4, and let* $\mathcal{M}$ *be a non-zero closed subspace invariant under all the operators* $T(t)$*. Then there is a function* $\psi \in \mathcal{M}$ *such that* $\mathcal{M} = \psi\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu_\psi)$*.*

Proof. If $\mathcal{M}$ is invariant under the semigroup, then it is also invariant under the cogenerator V, and after transforming to the disc as in the proof of Theorem 2.4, we may apply Theorem 3.1.

Note that under the equivalence between $D(\mu)$ and $\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu)$, as detailed in (2.4) and (2.5), the subspace ϕD_{μ_ϕ} maps to a space $\psi\widetilde{\mathcal{D}}_{\mathbb{C}_+}(\nu_\psi)$, where

$$\psi(s) = \phi((s-1)/(s+1)) \quad \text{and} \quad d\nu_\psi = |\psi|^2 d\nu\,. \qquad \square$$

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