p-Fourier algebras on compact groups

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Abstract. Let G be a compact group. For $1 \leq p \leq \infty$ we introduce a class of Banach function algebras $\mathcal{A}^p(G)$ on G which are the Fourier algebras in the case $p = 1$, and for $p = 2$ are certain algebras discovered by Forrest, Samei and Spronk. In the case $p \neq 2$ we find that $\mathcal{A}^p(G) \cong \mathcal{A}^p(H)$ if and only if G and H are isomorphic compact groups. These algebras admit natural operator space structures, and also weighted versions, which we call p-Beurling–Fourier algebras. We study various amenability and operator amenability properties, Arens regularity and representability as operator algebras. For a connected Lie G and $p > 1$, our techniques of estimation of when certain p-Beurling–Fourier algebras are operator algebras rely more on the fine structure of G, than in the case $p = 1$. We also study restrictions to subgroups. In the case that $G = \mathrm{SU}(2)$, restrict to a torus and obtain some exotic algebras of Laurent series. We study amenability properties of these new algebras, as well.

1. Introduction and plan

For a compact group G, the Fourier algebra $\mathrm{A}(G)$ goes back to Kreĭn (see §34 in [27], for example). For a continuous function u on G

$$u \in \mathrm{A}(G) \quad \text{if and only if} \quad \|u\|_{\mathrm{A}} = \sum_{\pi \in \widehat{G}} d_\pi \|\hat{u}(\pi)\|_{\mathrm{S}^1_{d_\pi}} < \infty.$$

Here, each $\hat{u}(\pi)$ is the matricial Fourier coefficient, and each $\|\cdot\|_{\mathrm{S}^1_d}$ denotes the trace norm. This is a special version of the Fourier algebra as subsequently defined by Stinespring ([54]) and Eymard ([15]). In his investigation of non-amenability of $\mathrm{A}(\mathrm{SO}(3))$, Johnson ([32]) created a new algebra $\mathrm{A}_\gamma(G)$ by averaging over the diagonal subgroup. More precisely, he considered the map from the projective

tensor product of Fourier algebras into continuous functions on a homogeneous space modulo the diagonal subgroup $\Delta = \{(s,s) : s \in G\}$:

$$\Gamma : A(G) \otimes_\gamma A(G) \to \mathcal{C}(G \times G/\Delta) \cong \mathcal{C}(G)$$

$$\Gamma(u \otimes v)((s,t)\Delta) = \int_G u(sr)v(tr)\,dr = u * \check{v}(st^{-1}),$$

where $\check{v}(r) = v(r^{-1})$. Letting $A_\gamma(G) = \Gamma(A(G) \otimes_\gamma A(G))$, we have

$$u \in A_\gamma(G) \quad \text{if and only if} \quad \|u\|_{A_\gamma} = \sum_{\pi \in \widehat{G}} d_\pi^2 \,\|\hat{u}(\pi)\|_{S^1_{d_\pi}} < \infty$$

and $\|\cdot\|_\gamma$ is exactly that norm which makes Γ a quotient map. In [22] Forrest and two of the present authors followed the same philosophy to consider the operator space version of Johnson's algebra, $A_\Delta(G) = \Gamma(A(G \times G))$, where we recall that $A(G \times G) = A(G) \hat\otimes A(G)$ (operator projective tensor product). A characterization of the algebra in terms of its natural quotient norm is given by

$$u \in A_\Delta(G) \quad \text{if and only if} \quad \|u\|_{A_\Delta} = \sum_{\pi \in \widehat{G}} d_\pi^{3/2} \,\|\hat{u}(\pi)\|_{S^2_{d_\pi}} < \infty,$$

where each $\|\cdot\|_{S^2_d}$ denotes the Hilbert–Schmidt norm. This result was mysterious to the authors at the time, and there were questions about the underlying meaning of the formula. We remark that the space $A_\Delta(G)$ seems to have appeared before. It arises as a host space for sufficiently differentiable functions on compact Lie groups; see [55] or [16], 12.2.2, for example.

We provide an answer to this question of understanding $A_\gamma(G)$ and $A_\Delta(G)$, in a uniform context with $A(G)$, by developing a family of algebras $\mathcal{A}^p(G, \omega)$, $1 \le p \le \infty$, where $\omega \colon \widehat{G} \to (0,\infty)$ is a weight in the sense of [37] or [38]. We recover the Fourier algebra $A(G)$ when $p = 1$ and $\omega \equiv 1$, the algebra $A_\gamma(G)$ when $p = 1$ and $\omega(\pi) = d_\pi$, and the algebra $A_\Delta(G)$ when $p = 2$ and $\omega \equiv 1$. We may simply refer to each of these as the *p-Beurling–Fourier algebra* of G and the *p-Fourier algebra* of G for the case of $\omega \equiv 1$, which we simply write $\mathcal{A}^p(G)$. We immediately warn the reader that the algebras $\mathcal{A}^p(G)$ are distinct from, and not to be confused with, the Figà–Talamanca–Herz algebras $A_p(G)$, $1 < p < \infty$. For example, we observe for virtually abelian groups that $\mathcal{A}^p(G) = A(G)$ isomorphically (Proposition 2.9), while this does not hold for $A_p(G)$, even for general compact abelian groups. In particular, our class of algebras is new only for sufficiently non-abelian compact groups.

Having constructed this class of algebras $\mathcal{A}^p(G, \omega)$, we embark on a systematic program of understanding how its members behave as Banach algebras. In general, the algebras tend to reflect the structure of G in a similar way as $A(G)$ does. For example, they still determine the underlying compact group G for most p and for some natural weights ω (Section 2.3). The Gelfand spectrum of $\mathcal{A}^p(G, \omega)$ is still identical to G for sufficiently slowly growing weights, and for dimension weights these algebras are regular (Section 2.2). As is noted in Section 3.2, the algebras $\mathcal{A}^p(G)$ enjoy exactly the same amenability properties of the Fourier algebras.

However, the algebras $\mathcal{A}^p(G,\omega)$ show significant differences from Fourier algebras, in particular when we look at reults in the context of operator spaces. Each $\mathcal{A}^p(G,\omega)$ comes equipped with a natural operator space structure, generalizing the usual "predual" operator space structure of A(G), via the complex interpolation. For $p = 2$ other operator space structures are available, which arise naturally in some contexts. As has been amply demonstrated in the case of Fourier algebras the use of operator space structure makes analysis on non-abelian groups more tractable: compare the results of [32] and [20] with those of [49] and [53], [52]. We examine operator weak amenability and operator amenability in Section 3.1, where we can observe that the corresponding properties usually become worse than the the case of Fourier algebra. More intriguing differences lie in the functoriality of restriction to subgroups. It turns out that the functoriality holds in a very restricted case, namely, the case where the subgroup is more or less a direct summand of the supergroup (Section 4.1). This limited functoriality already gives us additional results on operator weak amenability and operator amenability of $\mathcal{A}^p(G,\omega)$. On the other hand the restriction procedure to subgroups is not functorial in general. We demonstrate its failure focusing on the case of 2×2 special unitary group, SU(2) in Section 4.2, since its representation theory allows more direct computations. Interestingly, we gain an exotic class of Banach algebras of Laurent series as a result of restriction to the maximal torus. We characterize various amenabilities of these new algebras, which show unexpected coincidence of amenabilities in Banach space category and operator space category.

Finally in Section 5.1 we conduct a study of Arens regularity of algebras $\mathcal{A}^p(G,\omega)$ and in Section 5.2 we study, in the case of connected Lie groups, some conditions which allow $\mathcal{A}^p(G,\omega)$ to be represented as operator algebras. The latter property in particular shows some similarity and difference at the same time from the case of $p = 1$ in [24]. It is similar in the sense that the algebras $\mathcal{A}^p(G,\omega)$ may still be represented as operator algebras for the weights growing quickly enough. However, it is quite different from the case of [24], where our estimates for the growth rate of the corresponding weight relied only on the dimension $d(G)$ of the connected Lie group G, but the present work shows the connection to a finer structure information of G such as semisimple rank $s(G)$ of G and the dimension $z(G)$ of the center of G.

Since this paper features a wide variety of different results, many of them quantitative, we include two tables in the last section which summarize most of the results of this paper in the context of known results. We also point out some questions which arise naturally from the present investigation.

1.1. Fourier series on compact groups

For a compact group G, we let $\widehat{G}$ denote its dual object, the set of unitary equivalence classes of all irreducible unitary representations. By standard abuse of notation, we shall identify elements of $\widehat{G}$ with representatives of the equivalence classes. Each π in $\widehat{G}$ acts on a space $\mathcal{H}_\pi$ of dimension d_π. Let M_d denote the space of $d \times d$ complex matrices. Given π in $\widehat{G}$ we let $\mathrm{Trig}_\pi = \{\mathrm{Tr}(A\pi(\cdot)) : A \in \mathrm{M}_{d_\pi}\}$ and $\mathrm{Trig}(G) = \bigoplus_{\pi \in \widehat{G}} \mathrm{Trig}_\pi$, where the sum may be regarded as an internal direct sum

in the space of continuous functions on G. Schur's orthogonality formula provides for u in $\mathrm{Trig}(G)$ that

$$u = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{Tr}(\hat{u}(\pi)^{\mathrm{t}} \pi(\cdot)), \quad \text{where each} \quad \hat{u}(\pi) = \int_G u(s) \bar{\pi}(s) \, ds \in \mathrm{M}_{d_\pi}.$$

Here, the transpose and complex conjugation are realized with respect to the same hermitian matrix unit on each M_{d_π}. The linear space $\mathrm{Trig}(G)$ admits algebraic dual space $\mathrm{Trig}(G)^\dagger = \prod_{\pi \in \widehat{G}} \mathrm{M}_{d_\pi}$ via the duality

$$(1.1) \qquad \langle u, (T_\pi)_{\pi \in \widehat{G}} \rangle = \sum_{\pi \in \widehat{G}} d_\pi \mathrm{Tr}(\hat{u}(\pi)^{\mathrm{t}} T_\pi).$$

This duality is constructed specifically to recognize the functionals of evaluation at points: for s in G, let $\lambda(s) = (\pi(s))_{\pi \in \widehat{G}}$, and we have $u(s) = \langle u, \lambda(s) \rangle$ for each u in $\mathrm{Trig}(G)$. Furthermore, a consequence of Schur's lemma is that $\mathrm{span}\,\lambda(G)$ is weak† dense in $\mathrm{Trig}(G)^\dagger$, i.e., dense with respect to the initial topology $\sigma(\mathrm{Trig}(G)^\dagger, \mathrm{Trig}(G))$ with respect to the dual pairing (1.1). We let $m \colon \mathrm{Trig}(G) \otimes \mathrm{Trig}(G) \cong \mathrm{Trig}(G \times G) \to \mathrm{Trig}(G)$ denote the pointwise product, and define the *coproduct* $M \colon \mathrm{Trig}(G)^\dagger \to \mathrm{Trig}(G \times G)^\dagger$ to be the adjoint of m. Hence M is defined by the relation

$$M\lambda(s) = \lambda(s) \otimes \lambda(s) \text{ for } s \text{ in } G$$

which is the familiar co-commutative coproduct from the theory of (compact) quantum groups. We shall require a form of this which is more suitable for certain norm computations, however. If σ is equivalent to a subrepresentation of $\pi \otimes \pi'$ (we write $\sigma \subset \pi \otimes \pi'$ in this case), we let $U^{(i)}_{\sigma, \pi \otimes \pi'} \colon \mathcal{H}_\sigma \to \mathcal{H}_\pi \otimes^2 \mathcal{H}_{\pi'}$, for $i = 1, \ldots, m(\sigma, \pi \otimes \pi')$, denote a maximal family of isometric intertwiners with pairwise disjoint ranges. Then for suitably sized matrices A_π and $A_{\pi'}$ we see that

$$m(\mathrm{Tr}(A_\pi \pi(\cdot)) \otimes \mathrm{Tr}(A_{\pi'} \pi'(\cdot))) = \sum_{\sigma \subset \pi \otimes \pi'} \sum_{i=1}^{m(\sigma, \pi \otimes \pi')} \mathrm{Tr}(A_\pi \otimes A_{\pi'} U^{(i)}_{\sigma, \pi \otimes \pi'} \sigma(\cdot) U^{(i)*}_{\sigma, \pi \otimes \pi'}).$$

We note that there is a natural linear isomorphism $\mathrm{Trig}(G \times G) = \mathrm{Trig}(G) \otimes \mathrm{Trig}(G)$, and that $\mathrm{Trig}(G \times G)^\dagger = \prod_{\pi, \pi' \in \widehat{G}} \mathrm{M}_{d_\pi} \otimes \mathrm{M}_{d_{\pi'}}$. Hence by the weak† density of $\mathrm{span}\,\lambda(G)$ in $\mathrm{Trig}(G)^\dagger$, and the weak† density of $\bigoplus_{\pi, \pi' \in \widehat{G}} \mathrm{M}_{d_\pi} \otimes \mathrm{M}_{d_{\pi'}}$ in $\mathrm{Trig}(G \times G)^\dagger$, we have that

$$(1.2) \qquad M W_{\pi, \pi'} = \bigoplus_{\sigma \subset \pi \otimes \pi'} U^*_{\sigma, \pi \otimes \pi'} W_\sigma U_{\sigma, \pi \otimes \pi'}$$

for W in $\mathrm{Trig}(G \times G)^\dagger$, where the sum now counts multiplicity, i.e., we absorb indices $i = 1, \ldots, m(\sigma, \pi \otimes \pi')$ so as to simplify the notation in our future computations. It is well known, see, for example, [54], that $M(\mathrm{A}(G)^*) \subset \mathrm{A}(G)^* \bar{\otimes} \mathrm{A}(G)^*$, where $\bar{\otimes}$ denotes the normal spatial, or von Neumann, tensor product.

We note that for any norm $\|\cdot\|_{\mathcal{B}}$ on $\mathrm{Trig}(G)$, the completion $\mathcal{B}(G)$ of $\mathrm{Trig}(G)$ with respect to that norm may be understood to have its continuous dual space $\mathcal{B}(G)^*$ as a subspace of $\mathrm{Trig}(G)^\dagger$. For example, the Fourier algebra, mentioned earlier, satisfies

$$\mathrm{A}(G) \cong \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi \, \mathrm{S}^1_{d_\pi} \quad \text{and} \quad \mathrm{A}(G)^* \cong \ell^\infty\text{-}\bigoplus_{\pi \in \widehat{G}} \mathrm{S}^\infty_{d_\pi},$$

i.e., Banach space direct sum of weighted trace-class matrix spaces and direct product of operators on d_π-dimensional Hilbert spaces, respectively, so $\mathrm{A}(G)^*$ is a von Neumann algebra.

1.2. Operator spaces

Our standard references for the theory of operator spaces are [13] and [46]. An operator space is a complex vector space $\mathcal{V}$ admitting a family of norms $\|\cdot\|_{\mathrm{M}_n(\mathcal{V})}$, one on each of the spaces of $n \times n$ matrices with entries in $\mathcal{V}$, satisfying certain compatibility conditions of Ruan (which we shall not explicitly use), and for which each $\mathrm{M}_n(\mathcal{V})$ is complete. For us, the space S^∞_d will always have the canonical operator space structure it admits by virtue of being a C*-algebra. The space S^1_d will come by its operator space structure as the dual space of S^∞_d via the dual pairing

$$\langle A, B \rangle = \mathrm{Tr}(A^{\mathrm{t}} B).$$

We shall make use of direct products and direct sums (the latter of which is described nicely in [46]), denoted $\oplus^\infty$ or $\ell^\infty\text{-}\bigoplus$, and $\oplus^1$ or $\ell^1\text{-}\bigoplus$, respectively; as well as operator projective, operator injective (or spatial), and Haagerup tensor products, denoted $\hat{\otimes}$, $\check{\otimes}$ and $\otimes^h$, respectively. We shall also often use the normal spatial tensor product $\mathcal{V}^* \bar{\otimes} \mathcal{W}^*$ of operator dual spaces. We note that $\mathcal{V}^* \bar{\otimes} \mathcal{W}^*$ is isometrically isomorphic to the weak*-closure of the algebraic tensor product $\mathcal{V}^* \otimes \mathcal{W}^*$ in $(\mathcal{V} \hat{\otimes} \mathcal{W})^*$. Furthermore, there are distributive laws over infinite families of operator space of operator direct sums and the operator projective tensor product, and over operator dual spaces of operator direct products and the normal spatial tensor product.

The spaces S^p_d, with Schatten p-norms, will have their operator space structures realized through Pisier's complex interpolation theory ([45]): $\mathrm{S}^p_d = [\mathrm{S}^\infty_d, \mathrm{S}^1_d]_{1/p}$. Hence the Hilbert–Schmidt space S^2_d will be understood with the operator Hilbert space structure. However, we will also have occasions to consider this space as a column Hilbert space, $\mathrm{S}^2_{d,C}$, or a row Hilbert space, $\mathrm{S}^2_{d,R}$. We shall do the same with all of the d-dimensional spaces ℓ^p_d. In this case ℓ^∞_d, being a commutative C*-algebra, is a minimal operator space, while ℓ^1_d, being its dual, is maximal. In the case $p = 2$, we let $\mathrm{C}_d = \ell^2_{d,C}$ and $\mathrm{R}_d = \ell^2_{d,R}$ denote the d-dimensional column and row spaces. We shall use the bilinear identifications $\mathrm{C}^*_d \cong \mathrm{R}_d$ and $\mathrm{R}^*_d \cong \mathrm{C}_d$, given by $(\xi, \eta) \mapsto \eta\xi, \xi\eta$, respectively. Thus we obtain completely isometric identifications

$$\mathrm{S}^\infty_d = \mathrm{C}_d \otimes^h \mathrm{R}_d \quad \text{and} \quad \mathrm{S}^1_d = \mathrm{R}_d \otimes^h \mathrm{C}_d,$$

which are given for a hermitian matrix unit $(e_{ij})_{i,j=1}^d$ for matrices, and an orthonormal basis $(e_i)_{i=1^d}$ for ℓ_d^2 by

(1.3) $$e_{ij} \mapsto e_i \otimes e_j.$$

Then, keeping with standard notation of [46], we obtain interpolated Hilbertian operator spaces

$$C_d^p = [C_d, R_d]_{1/p} \quad \text{and} \quad R_d^p = [R_d, C_d]_{1/p}.$$

In this notation we have $C_d^\infty = C_d$ and $C_d^1 = R_d$, for example. This notation proves itself in the remarkable tensorial factorizations

$$S_d^p = C_d^p \otimes^h R_d^p,$$

which hold thanks to the stability of Haagerup tensor product under complex interpolation ([45]). Furthermore, with respect to the dual pairing $(A, B) \mapsto \text{Tr}(A^t B)$, we have completely isometric identifications $(S_d^p)^* \cong S_d^{p'}$, where $1/p + 1/p' = 1$.

It is well known that the space of completely bounded maps $\mathcal{CB}(C_d, R_d)$ is isometrically isomorphic to S_d^2. Hence the identity $\text{id} \colon C_d \to R_d$ has completely bounded norm $d^{1/2}$. We abbreviate this by writing that $\text{id} \colon d^{1/2}C_d \to R_d$ is a complete contraction.

We realize p-direct sums through interpolation: $\mathcal{V} \oplus^p \mathcal{W} = [\mathcal{V} \oplus^\infty \mathcal{W}, \mathcal{V} \oplus^1 \mathcal{W}]_{1/p}$, completely isometrically.

2. The algebras $\mathcal{A}^p(G)$

Let G be a compact group.

2.1. Definition and elementary properties of $\mathcal{A}^p(G)$

Let $1 \leq p \leq \infty$ and p' denote the conjugate index so $1/p + 1/p' = 1$. We consider on $\text{Trig}(G)$ the norm

$$\|u\|_{\mathcal{A}^p} = \sum_{\pi \in \widehat{G}} d_\pi^{1+1/p'} \|\hat{u}(\pi)\|_{S_{d_\pi}^p}.$$

Let $\mathcal{A}^p(G)$ denote the completion of $\text{Trig}(G)$ with respect to this norm, hence we have isometric identification

(2.1) $$\mathcal{A}^p(G) \cong \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^{1+1/p'} S_{d_\pi}^p.$$

We may further put the canonical operator space structures on the individual spaces $S_{d_\pi}^p$ and use operator space direct sum, thus making (2.1) a completely isometric identification. We note that for any positive integer d, (dS_d^∞, S_d^1) forms a natural compatible couple of operator spaces with interpolated spaces $[dS_d^\infty, S_d^1]_{1/p}$

$= d^{1/p'} \mathrm{S}^p_d$. Thus since the formal identity $d\mathrm{S}^\infty_d \cong d^{1/2}\mathrm{C}_d \otimes^h d^{1/2}\mathrm{R}_d \hookrightarrow \mathrm{R}_d \otimes^h \mathrm{C}_d \cong \mathrm{S}^1_d$ is a complete contraction, it follows by standard operator space interpolation theory we get for $1 \le p \le q \le \infty$ completely contractive identity maps $d\mathrm{S}^\infty_d \hookrightarrow d^{1/q'}\mathrm{S}^q_d \hookrightarrow d^{1/p'}\mathrm{S}^p_d \hookrightarrow \mathrm{S}^1_d$. These give rise to completely contractive inclusions

$$(2.2) \qquad \mathcal{A}^\infty(G) \subseteq \mathcal{A}^q(G) \subseteq \mathcal{A}^p(G) \subseteq \mathcal{A}^1(G) = \mathrm{A}(G).$$

Hence each space is a space of continuous functions on G.

Let us see that each $\mathcal{A}^p(G)$ is a completely contractive Banach algebra. Hence we are justified in calling it the *p-Fourier algebra* of G. We first observe that with respect to the dual pairing (1.1) the dual space is the direct product space

$$(2.3) \qquad \mathcal{A}^p(G)^* \cong \ell^\infty\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^{-1/p'}\, \mathrm{S}^{p'}_{d_\pi}.$$

Also, since the operator projective tensor product distributes over operator space direct sums, we have that

$$(2.4) \qquad \mathcal{A}^p(G) \,\hat{\otimes}\, \mathcal{A}^p(G) \cong \ell^1\text{-}\bigoplus_{\pi,\pi' \in \widehat{G}\times\widehat{G}} (d_\pi d_{\pi'})^{1+1/p'}\, \mathrm{S}^p_{d_\pi} \hat{\otimes} \mathrm{S}^p_{d_{\pi'}}$$

and hence we obtain

$$(2.5) \qquad (\mathcal{A}^p(G) \,\hat{\otimes}\, \mathcal{A}^p(G))^* \cong \ell^\infty\text{-}\bigoplus_{\pi,\pi' \in \widehat{G}\times\widehat{G}} (d_\pi d_{\pi'})^{-1/p'}\, \mathrm{S}^{p'}_{d_\pi} \check{\otimes} \mathrm{S}^{p'}_{d_{\pi'}}.$$

Theorem 2.1. *The space $\mathcal{A}^p(G)$ is a completely contractive Banach algebra under pointwise multiplication.*

Proof. It suffices to show that the coproduct M defined in (1.2) takes $\mathrm{A}(G)^*$ into $(\mathcal{A}^p(G)\hat{\otimes}\mathcal{A}^p(G))^*$, completely contractively. Now if $[W_{ij}] \in \mathrm{M}_n(\mathcal{A}^p(G)^*)$ then we wish to estimate

(2.6)

$$\|[MW_{ij}]\| = \sup_{\pi,\pi' \in \widehat{G}\times\widehat{G}} (d_\pi d_{\pi'})^{-1/p'} \left\| \left[\bigoplus_{\sigma \subset \pi\otimes\pi'} U^*_{\sigma,\pi\otimes\pi'} W_{ij,\sigma} U_{\sigma,\pi\otimes\pi'} \right] \right\|_{\mathrm{M}_n(\mathrm{S}^{p'}_{d_\pi}\check{\otimes}\mathrm{S}^{p'}_{d_{\pi'}})}.$$

Let us make three general observations.

(I) If d and d' be two positive integers, the formal identity $\mathrm{S}^{p'}_{dd'} \hookrightarrow \mathrm{S}^{p'}_d \check{\otimes} \mathrm{S}^{p'}_{d'}$ is a complete contraction. Indeed, the desired map is the adjoint of the map $\mathrm{S}^p_d \hat{\otimes} \mathrm{S}^p_{d'} \hookrightarrow \mathrm{S}^p_{dd'}$, which is completely contractive since we may recognize it as

$$(\dagger) \qquad \mathrm{S}^p_d \hat{\otimes} \mathrm{S}^p_{d'} \cong (\mathrm{C}^p_d \otimes^h \mathrm{R}^p_d) \hat{\otimes} (\mathrm{C}^p_{d'} \otimes^h \mathrm{R}^p_{d'}) \hookrightarrow (\mathrm{C}^p_d \hat{\otimes} \mathrm{C}^p_{d'}) \otimes^h (\mathrm{R}^p_d \hat{\otimes} \mathrm{R}^p_{d'})$$
$$\hookrightarrow (\mathrm{C}^p_d \otimes^h \mathrm{C}^p_{d'}) \otimes^h (\mathrm{R}^p_d \otimes^h \mathrm{R}^p_{d'}) \cong \mathrm{C}^p_{dd'} \otimes^h \mathrm{R}^p_{dd'} \cong \mathrm{S}^p_{dd'}$$

where the complete contraction $c_1 \otimes r_1 \otimes c_2 \otimes r_2 \mapsto c_1 \otimes c_2 \otimes r_1 \otimes r_2$ at ($\dagger$) is provided by Theorem 6.1 in [14].

(II) If $d_1, \ldots, d_m$ are positive integers and $d = d_1 + \cdots + d_m$, then the block-diagonal embedding of the operator space $\ell^{p'}\text{-}\bigoplus_{j=1}^m S_{d_j}^{p'}$ into $S_d^{p'}$ is a complete isometry. Indeed when $p' = \infty$, this is a completely contractively complemented subspace, whence, by duality, the same holds when $p' = 1$. The case for general p then follows from the generalized Riesz–Thorin theorem, using the facts that $\ell^{p'}\text{-}\bigoplus_{j=1}^m S_{d_j}^{p'} = [\ell^\infty\text{-}\bigoplus_{j=1}^m S_{d_j}^\infty, \ell^1\text{-}\bigoplus_{j=1}^m S_{d_j}^1]_{1/p'}$.

(III) If E and F are operator spaces then the formal identity $\mathrm{M}_n(E) \oplus^{p'} \mathrm{M}_n(F) \hookrightarrow \mathrm{M}_n(E \oplus^{p'} F)$ is a contraction. Indeed, this map is an isometry if $p' = \infty$, and, thanks to the universal property of direct sums, is a contraction when $p' = 1$. The case for general p' follows from the generalized Riesz–Thorin theorem, using the facts that $\mathrm{M}_n(E) \oplus^{p'} \mathrm{M}_n(F) = [\mathrm{M}_n(E) \oplus^\infty \mathrm{M}_n(F), \mathrm{M}_n(E) \oplus^1 \mathrm{M}_n(F)]_{1/p'}$ and $\mathrm{M}_n(E \oplus^{p'} F) = [\mathrm{M}_n(E \oplus^\infty F), \mathrm{M}_n(E \oplus^1 F)]_{1/p'}$.

Using observations (I), (II) and (III) above, in order, and then a rudimentary estimate we obtain

$$\left\| \left[\bigoplus_{\sigma \subset \pi \otimes \pi'} U_{\sigma, \pi \otimes \pi'}^* W_{ij,\sigma} U_{\sigma, \pi \otimes \pi'} \right] \right\|_{\mathrm{M}_n(S_{d_\pi}^{p'} \check\otimes S_{d_{\pi'}}^{p'})}$$

$$\leq \left\| \left[\bigoplus_{\sigma \subset \pi \otimes \pi'} U_{\sigma, \pi \otimes \pi'}^* W_{ij,\sigma} U_{\sigma, \pi \otimes \pi'} \right] \right\|_{\mathrm{M}_n(S_{d_\pi d_{\pi'}}^{p'})}$$

$$= \left\| \bigoplus_{\sigma \subset \pi \otimes \pi'} [W_{ij,\sigma}] \right\|_{\mathrm{M}_n(\ell^{p'}\text{-}\bigoplus_{\sigma \subset \pi \otimes \pi'} S_{d_\sigma}^{p'})} \leq \left(\sum_{\sigma \subset \pi \otimes \pi'} \|[W_{ij,\sigma}]\|_{\mathrm{M}_n(S_{d_\sigma}^{p'})}^{p'} \right)^{1/p'}$$

$$\leq \left(\sum_{\sigma \subset \pi \otimes \pi'} d_\sigma \cdot \sup_{\tau \subset \pi \otimes \pi'} \frac{1}{d_\tau} \|[W_{ij,\tau}]\|_{\mathrm{M}_n(S_{d_\tau}^{p'})}^{p'} \right)^{1/p'}$$

$$= (d_\pi d_{\pi'})^{1/p'} \sup_{\tau \subset \pi \otimes \pi'} d_\tau^{-1/p'} \|[W_{ij,\tau}]\|_{\mathrm{M}_n(S_{d_\tau}^{p'})}.$$

It follows that the quantity of (2.6) is no greater than

$$(2.7) \qquad \sup_{\tau \in \widehat{G}} d_\tau^{-1/p'} \|[W_{ij,\tau}]\|_{\mathrm{M}_n(S_{d_\tau}^{p'})} = \|[W_{ij}]\|_{\mathrm{M}_n(\mathcal{A}^p(G)^*)}.$$

Hence M satisfies the desired complete contractivity property. $\qquad\square$

We shall also make use of the *p-Beurling–Fourier algebras* which we define below. As defined in [37], [38], a *weight* is a function $\omega: \widehat{G} \to \mathbb{R}^{>0}$ which satisfies

$$\omega(\sigma) \leq \omega(\pi)\omega(\pi') \quad \text{whenever } \sigma \subset \pi \otimes \pi'.$$

We will always assume that ω is *bounded away from zero*:

$$\inf_{\pi \in \widehat{G}} \omega(\pi) > 0.$$

Notice that this is automatic if ω is *symmetric*, i.e., $\omega(\bar\pi) = \omega(\pi)$; indeed $\pi \otimes \bar\pi \supseteq 1$ so $\omega(\pi) \geq \omega(1)^{1/2}$ in this case. We let

$$(2.8) \qquad \mathcal{A}^p(G, \omega) = \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} \omega(\pi) \, d_\pi^{1+1/p'} \, S_{d_\pi}^p.$$

As before we give this the weighted direct sum operator space structure, with usual interpolated structure on each $\mathrm{S}^p_{d_\pi}$. Boundedness away from zero of ω ensures that $\mathcal{A}^p(G,\omega) \hookrightarrow \mathcal{A}^p(G)$, completely boundedly – completely contractively provided $\inf_{\pi \in \widehat{G}} \omega(\pi) \geq 1$. We observe that

$$\mathcal{A}^1(G,\omega) = \mathrm{A}(G,\omega),$$

which are the *Beurling–Fourier algebras* defined in [37], [38], and we shall typically denote them in the original notation.

Corollary 2.2. *For any weight ω, the space $\mathcal{A}^p(G,\omega)$ is a completely contractive Banach algebra under pointwise multiplication.*

Proof. We make the obvious changes to (2.5) and hence to (2.6). Then the computation follows exactly as in the proof of the last theorem. In place of (2.7) we obtain

$$\sup_{\pi,\pi' \in \widehat{G} \times \widehat{G}} \frac{1}{\omega(\pi)\omega(\pi')} \sup_{\tau \subset \pi \otimes \pi'} d_\tau^{-1/p'} \left\| [W_{ij,\tau}] \right\|_{\mathrm{S}^{p'}_{d_\tau}}$$

$$\leq \sup_{\pi,\pi' \in \widehat{G} \times \widehat{G}} \sup_{\tau \subset \pi \otimes \pi'} \frac{1}{\omega(\tau) d_\tau^{1/p'}} \left\| [W_{ij,\tau}] \right\|_{\mathrm{S}^{p'}_{d_\tau}}$$

$$= \sup_{\tau \in \widehat{G}} \frac{1}{\omega(\tau) d_\tau^{1/p'}} \left\| [W_{ij,\tau}] \right\|_{\mathrm{S}^{p'}_{d_\tau}} = \left\| [W_{ij}] \right\|_{\mathrm{M}_n(\mathcal{A}^p(G,\omega)^*)}. \qquad \square$$

Let us close this section by noting that the family of algebras $\mathcal{A}^p(G)$ form an interpolation scale, i.e., $\mathcal{A}^p(G) = [\mathcal{A}^\infty(G), \mathrm{A}(G)]_{1/p}$. Let us explain this fact, briefly, and generalize it. Let ω and $\tilde{\omega}$ each be weights on $\widehat{G}$. We remark that complex interpolation is isometrically stable for completely contractively complemented subspaces. See, for example, [46], 2.7.6. Hence, using (2.8) we see that

$$[\mathcal{A}^\infty(G,\omega), \mathrm{A}(G,\tilde{\omega})]_{1/p} = \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi \left[d_\pi \omega(\pi) \mathrm{S}^\infty_{d_\pi}, \tilde{\omega}(\pi) \mathrm{S}^1_{d_\pi} \right]_{1/p}$$

$$(2.9) \qquad = \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^{1+1/p'} \omega(\pi)^{1/p'} \tilde{\omega}(\pi)^{1/p} \mathrm{S}^p_{d_\pi} = \mathcal{A}^p(G, \omega^{1/p'}\tilde{\omega}^{1/p}).$$

It is evident that positive powers of single weights, and products of multiple weights, remain weights.

2.2. Spectrum of $\mathcal{A}^p(G)$

We find it desirable to compute the spectra of elements of a certain class of Beurling–Fourier algebras. Given a commutative Banach algebra $\mathcal{B}$, the *spectrum* is the space of multiplicative characters on $\mathcal{B}$. We say that $\mathcal{B}$ is *regular* on its spectrum if given two disjoint compact sets E and F of the spectrum, there is an element of $\mathcal{B}$ taken to 1 by each element of E and to 0 by each element of F.

Let $\alpha > 0$ and $d^\alpha \colon \widehat{G} \to \mathbb{C}$ be the α-power of the dimension weight: $d^\alpha(\pi) = d_\pi^\alpha$.

We shall also make use of the basic polynomial weights for a compact Lie group G. In this case $\widehat{G}$ is finitely generated, i.e., there is a finite $S \subset \widehat{G}$ for which $\bigoplus_{\pi \in S} \pi$ is faithful. We may and shall suppose that S is symmetric: $\pi \in S$ implies $\bar{\pi} \in S$. Let $S^{\otimes n} = \{\sigma \in \widehat{G} : \sigma \subset \pi_1 \otimes \cdots \otimes \pi_n : \pi_1, \ldots, \pi_n \in S\}$ and $S^{\otimes 0} = \{1\}$. Then S generates $\hat{G}$ in the sense that $\widehat{G} = \bigcup_{n=1}^{\infty} S^{\otimes n}$. See §4.2 in [38] for details. It is easy to check that $\tau_S(\sigma) = \min\{n : \sigma \in S^{\otimes n}\}$ is subadditive. We let $\omega_S^\alpha(\pi) = (1 + \tau_S(\pi))^\alpha$, for $\alpha \geq 0$, and call these the *polynomial weights*. Given another symmetric generating set S', it is each to see that for some constants c, C that $c\omega_S \leq \omega_{S'} \leq C\omega_S$, so all polynomial weights are equivalent.

Suppose now that G is a general compact group. For any finite symmetric set $S \subset \widehat{G}$ we may consider $\tau_S : \langle S \rangle \to \mathbb{R}^{\geq 0}$ to be defined as above, and hence define a weight $\omega_S^\alpha : \langle S \rangle \to \mathbb{R}^{>0}$, as above. Notice that ω_S^α is really a weight on $\widehat{G/N_S} \circ q$ where $N_S = \bigcup_{\pi \in S} \ker \pi$ and $q : G \to G/N_S$ is the quotient map. A weight ω on a general compact group G is called *weakly polynomial* if for any S as above, there is a constant C_S and $\alpha_S \geq 0$ for which

$$\omega|_{\langle S \rangle} \leq C_S \omega_S^{\alpha_S}.$$

(For a connected group, this was termed a "polynomial weight" in Definition 5.1 of [38]. Compare the results (5.5) and Theorem 5.4 in [38].) Note that if G is totally disconnected, then any weight is weakly polynomial. Indeed, any finite subset S of $\widehat{G}$ has that N_S is open, so $\langle S \rangle$ is finite.

Proposition 2.3. *If ω is a symmetric weakly polynomial weight on $\widehat{G}$, then $\mathrm{A}(G, \omega)$ has spectrum G.*

Proof. In Proposition 5.5 of [38], this is shown for connected groups. Let us adapt the proof for general G. Fix π in $\widehat{G}$, and let $S = \{\pi, \bar{\pi}\}$. Then for any $\sigma \subset \pi^{\otimes n}$, the definition of τ_S provides that $\tau_S(\sigma) \leq n$, hence $\omega_S(\sigma) \leq 1 + n$. Hence using our assumption that ω is weakly polynomial we have

$$\left(\sup_{\sigma \subset \pi^{\otimes n}} \omega(\sigma)\right)^{1/n} \leq \left(\sup_{\sigma \subset \pi^{\otimes n}} C_S \omega_S(\sigma)^{\alpha_S}\right)^{1/n} \leq C_S^{1/n}(1+n)^{\alpha_S/n} \overset{n \to \infty}{\longrightarrow} 1.$$

The result now follows from Proposition 4.19 in [38]. $\qquad\square$

If ω is a weight on $\widehat{G}$ and H is a closed subgroup, we follow §4.1 in [38], or Proposition 3.5 in [37], and define the *restricted weight* $\omega_G|_H$ on $\widehat{H}$ by

$$(2.10) \qquad \omega_G|_H(\sigma) = \inf\{\omega(\pi) : \pi \in \widehat{G}, \sigma \subset \pi|_H\},$$

where $\pi|_H$ refers to the restricted representation.

Remark 2.4. For a connected Lie group G, and a weakly polynomial weight, $\mathrm{A}(G, \omega_S^\alpha)$ is shown to be regular in [38], Theorem 5.11. We do not know how to extend this result to non-connected groups. For us to do this, it would be sufficient to see that for a Lie group G, with symmetric generating set S for $\widehat{G}$, that $(\omega_S)_G|_{G_e}$ is weakly polynomial on $\widehat{G_e}$. We observe, however, that the easy estimate to show is in the wrong direction.

Let $S_e = \{\sigma \in \widehat{G_e} : \sigma \subset \pi|_{G_e}$ for some π in $S\}$. If $\sigma \subset \pi|_{G_e}$, where $\tau_S(\pi) = n$, then there are $\pi_1, \ldots, \pi_n$ in S for which $\sigma \subset \pi_1 \otimes \cdots \otimes \pi_n|_{G_e} = \pi_1|_{G_e} \otimes \cdots \otimes \pi_n|_{G_e}$. It follows that $\tau_{S_e}(\sigma) \leq n = \tau_S(\pi)$. Hence

$$(2.11) \qquad\qquad \omega_{S_e}(\sigma) \leq (\omega_S)_G|_{G_e}(\sigma).$$

We shift our attention to dimension weights. We can obtain regularity.

Proposition 2.5. (i) *The Gelfand spectrum of* $A(G, d^\alpha)$ *is* G.

(ii) *The algebra* $A(G, d^\alpha)$ *is regular on* G.

Proof. (i) For $\alpha = 1$, this result is stated as Corollary 5.6 in [38], and erroneously attributed to [44]. Regrettably, the proof of Corollary 5.6 in [38] only deals with the case of connected groups. It is sufficient to see that that the dimension weight d on G is a weakly polynomial weight, and then appeal to Proposition 2.3.

First, suppose that G is a connected Lie group. Then Example 5.2 in [38] shows that d is a weakly polynomial weight. (This uses ideas we shall use in proving Theorem 5.11.)

Now suppose that G is a Lie group, so the connected component of the identity G_e is open, hence the index $[G : G_e]$ is finite. Then for π in $\widehat{G}$ and σ in $\widehat{G_e}$ the Frobenius reciprocity formula of Theorem 8.2 in [39] (see also [34], 2.61) yields equality of multiplicities

$$m(\sigma : \pi|_{G_e}) = m(\pi : \sigma\uparrow^G),$$

where $\sigma\uparrow^G$ denotes the induced representation. Thus if $\sigma \subset \pi|_{G_e}$, then $\pi \subset \sigma\uparrow^G$, so

$$d_\pi \leq d_{\sigma\uparrow^G} = [G : G_e]d_\sigma.$$

Fix any symmetric generating set S of $\widehat{G}$. Then for any π in $\widehat{G}$ and σ in $\widehat{G_e}$ with $\sigma \subset \pi|_{G_e}$ we have

$$d_\pi \leq [G : G_e]d_\sigma \leq C\,\omega_{S_e}^{\alpha'}(\sigma) \leq C(\omega_S)_G^{\alpha'}|_{G_e}(\sigma) \leq C\,\omega_S^{\alpha'}(\pi)$$

for some constant C and $\alpha' \geq 0$, where the second inequality follows from the fact that the dimension weight on G_e is weakly polynomial, as noted in the prior paragraph, and the fact that for any two generating sets, the respective polynomial weights are equivalent; and the third inequality is provided by (2.11).

We finally consider the case of a general compact G. For any finite symmetric set S of $\widehat{G}$, with $N_S = \bigcap_{\pi \in S} \ker \pi$, we see that G/N_S, being isomorphic to a compact subgroup of $\bigoplus_{\pi \in S} \pi(G)$ is a Lie group, with $\langle S \rangle = \widehat{G/N_S} \circ q$, where $q \colon G \to G/N_S$ is the quotient map. Hence the result of the last paragraph shows that $d|_{\langle S \rangle} \leq C_S\,\omega_S^{\alpha_S}$ on $\langle S \rangle$, for some C_S and α_S, i.e., d is weakly polynomial.

(ii) Let us now show the regularity. First, let us fix $\alpha = 1$. We see from Theorem 4.1 in [21] that the map $\check{\Gamma} \colon A(G \times G) \to A(G, d)$ given by $\check{\Gamma}u(s) = \int_G u(st, t^{-1})\, dt$ is a surjection. If E and F are non-empty disjoint closed subsets of G then $\check{E}$ and $\check{F}$ are non-empty disjoint closed subsets of $G \times G$, where $\check{S} = \{(s, t) : st \in S\}$. Then (3.2) in [15] provides u in $A(G \times G)$ such that $u|_{\check{E}} = 1$ and $u|_{\check{F}} = 0$. Then $\check{\Gamma}(u)|_E = 1$ and $\check{\Gamma}(u)|_F = 0$.

It is now straightforward to verify the (completely) isometric identification

$$\mathrm{A}(G, d^\alpha) \,\hat{\otimes}\, \mathrm{A}(G, d^\alpha) \cong \mathrm{A}(G \times G, (d \otimes d)^\alpha),$$

where $d \otimes d(\pi, \sigma) = d_\pi d_\sigma$. Hence the proof of Proposition 2.6 in [48] can be followed to show that

$$\check{\Gamma}(\mathrm{A}(G \times G, (d \otimes d)^\alpha)) = \mathrm{A}(G, d^{2\alpha+1}).$$

The recursion $\alpha_0 = 0$, $\alpha_{n+1} = 2\alpha_n + 1$ admits solution $\alpha_n = 2^n - 1$. See also [37], p. 189. Hence, applying induction to the paragraph above yields that each algebra $\mathrm{A}(G, d^{2^n-1})$ is regular. The contractive embedding $\mathrm{A}(G, d^{2^{\lceil \log_2(\alpha+1)\rceil}-1}) \hookrightarrow \mathrm{A}(G, d^\alpha)$ yields the regularity of the latter algebra. $\qquad\square$

We move from $p = 1$ to all $1 \le p \le \infty$.

Proposition 2.6. *The Gelfand spectrum of $\mathcal{A}^p(G, d^\alpha)$ is G, and $\mathcal{A}^p(G, d^\alpha)$ is regular on G.*

Proof. For any positive integer d the formal identity map $\mathrm{S}_d^1 \to \mathrm{S}_d^p$ is a contraction, hence so too is $d\mathrm{S}_d^1 \to d\mathrm{S}_d^p \to d^{1/p'}\mathrm{S}_d^p$. Thus the inclusion map $\mathrm{A}(G, d^{\alpha+1}) \hookrightarrow \mathcal{A}^p(G, d^\alpha)$ is a contraction. The desired results are immediate from Proposition 2.5 above. $\qquad\square$

We observe that for s in G, the unitary $\pi(s)$ in $\mathrm{S}_{d_\pi}^{p'}$ necessarily has norm $d_\pi^{1/p'}$. Hence by by (2.3) and the result above, each element of the spectrum has norm 1. Thus the choice of exponent $1+1/p'$ in (2.1) is minimal for allowing the space $\mathcal{A}^p(G)$ to be a Banach function algebra on G. Indeed, characters need necessarily contractive.

Let $\theta \colon \mathbb{R}^{\ge 0} \to \mathbb{R}^{>0}$ be a non-decreasing weight (i.e., $\theta(s + t) \le \theta(s)\theta(t)$) and let $\omega_\theta(\pi) = \theta(\log d_\pi)$. Such weights are symmetric.

Example 2.7. The weight $\theta^\alpha(t) = e^{\alpha t}$, where $\alpha > 0$, gives the dimension weight $\omega_{\theta^\alpha}(\pi) = d_\pi^\alpha$. The weight $w^\alpha(t) = (1+t)^\alpha$, on $\mathbb{R}^{\ge 0}$, leads to $\omega_{w^\alpha}(\pi) = (1+\log d_\pi)^\alpha$.

We say that a weight ω on $\widehat{G}$ is a *weakly dimension* weight if there is C and $\alpha \ge 0$ for which $\omega(\pi) \le C d_\pi^\alpha$. Notice that for θ as above, $\theta(s) \le \theta(1)^{1+\lfloor s \rfloor} \le \theta(1)e^{\log \theta(1)s}$, from which it follows that ω_θ is a weakly dimension weight.

Corollary 2.8. *Let ω be a weakly dimension weight on $\widehat{G}$. Then the algebra $\mathcal{A}^p(G, \omega)$ has Gelfand spectrum G and is regular on G. In particular, this holds for ω_θ where θ is any non-decreasing weight on $\mathbb{R}^{\ge 0}$.*

Proof. The bounded inclusions $\mathcal{A}^p(G, d^\alpha) \subseteq \mathcal{A}^p(G, \omega) \subseteq \mathcal{A}^p(G)$ give the first conclusion. The second conclusion is immediate from the comments above. $\qquad\square$

The following is a straightforward adaptation of Corollary 2.4 in [21], which we leave to the reader.

Proposition 2.9. (i) *Given a weight ω on $\widehat{G}$ and $1 < p \leq \infty$, we have that $\mathcal{A}^p(G, \omega) = \mathrm{A}(G, \omega)$ isomorphically if and only if G admits an open abelian subgroup.*

(ii) *For any weakly dimension weight ω, $\mathrm{A}(G, \omega) = \mathrm{A}(G)$ isomorphically if and only if G admits an open abelian subgroup.*

2.3. Isometric isomorphisms

The main theorem of [60] tells us that any isometric isomorphism $\Phi\colon \mathrm{A}(G) \to \mathrm{A}(H)$ is of the form $\Phi u = u(s_0\varphi(\cdot))$ where $\varphi\colon H \to G$ is a homeomorphism which is either an isomorphism or anti-isomorphism of the groups. In particular, $\mathrm{A}(G) \cong \mathrm{A}(H)$ isometrically, only if $G \cong H$ as topological groups.

With compact groups, the addition of certain weights does not change this result. Furthermore, we can obtain this result for most indices p. We retain our convention that G and H denote compact groups. The weights ω_θ are defined at the end of the last section.

Theorem 2.10. *Let $1 \leq p \leq \infty$ with $p \neq 2$. Fix a non-decreasing weight $\theta\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{>0}$. If $\mathcal{A}^p(G, \omega_\theta) \cong \mathcal{A}^p(H, \omega_\theta)$ isometrically isomorphically, then $G \cong H$ as compact groups.*

Proof. Given any Banach space E, we let $B(E)$ denote the closed unit ball, and $S(E)$ the unit sphere. We identify the spaces $\mathrm{Trig}_\pi = d_\pi^{1+1/p'}\omega_\theta(\pi)\mathrm{S}_{d_\pi}^p$, for π in $\widehat{G}$, as subspaces of $\mathcal{A}^p(G, \omega_\theta)$. Then we have the following routine identification of sets of extreme points:

$$\mathrm{ext}B(\mathcal{A}^p(G, \omega_\theta)) = \mathrm{ext}B\Big(\ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^{1+1/p'}\omega_\theta(\pi)\mathrm{S}_{d_\pi}^p\Big) = \bigcup_{\pi \in \widehat{G}} \mathrm{ext}B\big(d_\pi^{1+1/p'}\omega_\theta(\pi)\mathrm{S}_{d_\pi}^p\big),$$

where each $\mathrm{ext}B\big(d_\pi^{1+1/p'}\omega_\theta(\pi)\mathrm{S}_{d_\pi}^p\big)$ is open in $\mathrm{ext}B(\mathcal{A}^p(G, \omega_\theta))$.

We remark that we have for any d in $\mathbb{N}$, $1 < p < \infty$,

$$\mathrm{ext}B(\mathrm{S}_d^1) = \{a \in S(\mathrm{S}_d^1) : \mathrm{rank}\, a = 1\}, \;\; \mathrm{ext}B(\mathrm{S}_d^p) = S(\mathrm{S}_d^p), \text{ and } \mathrm{ext}B(\mathrm{S}_d^\infty) = \mathrm{U}(d),$$

where $\mathrm{U}(d)$ is the unitary group. Indeed, $\mathrm{S}_d^1 = \ell_d^2 \otimes^\gamma \ell_d^2$ and the description of the projective tensor product norm shows that $\mathrm{ext}B(\mathrm{S}_d^1)$ must consist of rank one elements. The fact that any two norm one rank one elements a and b admit unitaries u, v for which $uav = b$ show that all such rank one elements are achieved. For $1 < p < \infty$, it is shown by [41] that S_d^p is uniformly convex, hence strictly convex. The description of $\mathrm{ext}B(\mathrm{S}_d^\infty)$ may be found in [33] or [17], Chapter 10. Hence we observe that each set $\mathrm{ext}B(\mathrm{S}_d^p)$, $1 \leq p \leq \infty$ is connected.

Since Φ is an isometry, for each $u \in \mathrm{ext}B\big(d_\pi^{1+1/p'}\omega_\theta(\pi)\mathrm{S}_{d_\pi}^p\big)$ we have $\Phi(u) \in \mathrm{ext}B\big(d_{\pi'}^{1+1/p'}\omega_\theta(\pi')\mathrm{S}_{d_{\pi'}}^p\big)$ for some $\pi' \in \widehat{H}$. We observe that, given $\pi \in \widehat{G}$, the sets $X_{\pi,\pi'} = \Phi^{-1}(\mathrm{ext}B(\mathrm{S}_{d_{\pi'}}^p)) \cap \mathrm{ext}B(\mathrm{S}_{d_\pi}^p)$, $\pi' \in \widehat{H}$, comprise a cover of $\mathrm{ext}B(\mathrm{S}_{d_\pi}^p)$ by pairwise disjoint open sets. Hence, by connectedness, we have that there is a unique π' for which $X_{\pi,\pi'} \neq \varnothing$. Thus we obtain a bijection $\hat{\Phi}\colon \widehat{G} \to \widehat{H}$, for

which $\Phi(\mathrm{Trig}_\pi) = \mathrm{Trig}_{\hat{\Phi}(\pi)}$. Clearly $d_{\hat{\Phi}(\pi)} = d_\pi$ for each π, so this map induces an isometry $\mathrm{S}^p_{d_\pi} \to \mathrm{S}^p_{d_{\hat{\Phi}(\pi)}}$.

Thanks to 10.2.2 or 10.3.5 in [17] (based on results of [33], [40]) in the case $p = 1, \infty$, and [2] in the case $1 < p < \infty$ but $p \neq 2$, each isometry $\mathrm{S}^p_{d_\pi} \to \mathrm{S}^p_{d_{\hat{\Phi}(\pi)}}$ is of the form $a \mapsto uav$ or $a \mapsto ua^{\mathrm{T}}v$, where u and v are unitaries and a^{T} denotes the transpose with respect to some orthonormal basis. Hence we see that $\|\Phi u\|_{\mathrm{A}} = \|u\|_{\mathrm{A}}$ for u in $\mathcal{A}^p(G, \omega_\theta)$ extends to an isometry $\mathrm{A}(G) \to \mathrm{A}(H)$ with dense range. The structure of Φ follows from [60], accordingly. $\qquad\square$

2.4. Different operator space structures on $\mathcal{A}^2(G)$

The construction above gives

$$\mathcal{A}^2(G) = \ell^1\text{-}\bigoplus_{\pi\in\widehat{G}} d_\pi^{3/2}\, \mathrm{S}^2_{d_\pi, OH},$$

where the subscript OH denotes the operator Hilbert space structure on each space $\mathrm{S}^2_{d_\pi}$. However, we wish to observe that other choices of operator space structure allow $\mathcal{A}^2(G)$ to be a completely contractive Banach algebra.

A *Hilbertian* operator space structure is an operator space structure $\mathcal{H} \mapsto \mathcal{H}_E$ which may be assigned to any Hilbert space. Such a structure is called *homogeneous* if $\mathcal{B}(\mathcal{H}) = \mathcal{CB}(\mathcal{H}_E)$ isometrically for all $\mathcal{H}$. Furthermore, $\mathcal{H} \mapsto \mathcal{H}_E$ is *subquadratic* if for any projection on $\mathcal{H}$ we have $\|[x_{ij}]\|^2_{\mathrm{M}_m(\mathcal{H}_E)} \leq \|[Px_{ij}]\|^2_{\mathrm{M}_m(\mathcal{H}_E)} + \|[(I-P)x_{ij}]\|^2_{\mathrm{M}_m(\mathcal{H}_E)}$ for any n and $[x_{ij}]$ in $\mathrm{M}_n(\mathcal{H})$. This is equivalent to saying that for (finite dimensional) $\mathcal{H}$ and $\mathcal{K}$, the formal identity

$$\mathrm{M}_n(\mathcal{H}_E) \oplus_2 \mathrm{M}_n(\mathcal{K}_E) \hookrightarrow \mathrm{M}_n((\mathcal{H} \oplus^2 \mathcal{K})_E)$$

is a contraction for every n. Finally, we will say that $\mathcal{H} \mapsto \mathcal{H}_E$ is *subcross* if the identity map on the algebraic tensor product of any two Hilbert spaces $\mathcal{H} \otimes \mathcal{K}$ extends to a complete contraction $\mathcal{H}_E \hat{\otimes} \mathcal{K}_E \to (\mathcal{H} \otimes_2 \mathcal{K})_E$. By duality, this is equivalent to having the dual structure $\mathcal{H} \mapsto \mathcal{H}_{E^*}$ satisfy that the identity map on any $\mathcal{H} \otimes \mathcal{K}$ extends to a complete contraction $(\mathcal{H} \otimes_2 \mathcal{K})_{E^*} \to \mathcal{H}_{E^*} \check{\otimes} \mathcal{K}_{E^*}$.

We note that the standard homogeneous Hilbertian operator space structures OH, C (column), R (row), $R+C$, and max are subcross, and that the dual structures $OH = OH^*$, $R = C^*$, $C = R^*$, $R \cap C = (R+C)^*$ and $\min = \max^*$ are subquadratic. The structures max and $R+C$, themselves, however, are not subquadratic. See the discussion in p. 81 of [45]. Let us also consider the interpolated structures, $\mathcal{H}_{C^p} = [\mathcal{H}_C, \mathcal{H}_R]_{1/p}$ for $1 \leq p \leq \infty$, and likewise for row structure. We have that $C^2 = OH = R^2$. By interpolation, each of these structures is homogeneous. Moreover, stability of the Haagerup tensor products under interpolation gives us that

$$\mathcal{H}_{C^p} \otimes^h \mathcal{K}_{C^p} = [\mathcal{H}_C \otimes^h \mathcal{K}_C, \mathcal{H}_R \otimes^h \mathcal{K}_R]_{1/p} = [(\mathcal{H} \otimes^2 \mathcal{K})_C, (\mathcal{H} \otimes^2 \mathcal{K})_R]_{1/p} = (\mathcal{H} \otimes^2 \mathcal{K})_{C^p},$$

from which it is immediate that each C^p is subcross. We observe the duality $(C^p)^* = C^{p'} = R^p$. Finally, subquadraticity of C^p follows from that for C, R, and interpolation.

Theorem 2.11. *Let $\mathcal{H} \mapsto \mathcal{H}_E$ be a subcross homogeneous operator space structure whose dual structure is subquadratic. Then the operator space*

$$\mathcal{A}_E^2(G) = \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^{3/2}\, S_{d_\pi, E}^2$$

is a completely contractive Banach algebra under pointwise multiplication.

Proof. The proof is essentially the same as that of Theorem 2.1. The subcross, homogeneity and subquadratic conditions of E, provide, respectively, the observations (I), (II) and (III) required in that proof.   □

Of course $\mathcal{A}_{\max}^2(G)$ is simply $\mathcal{A}^2(G)$ with its maximal operator space structure, hence it is no surprise based on Theorem 2.1, that it is a completely contractive Banach algebra. As in Corollary 2.2, we can see that the 2-Beurling–Fourier algebras $\mathcal{A}_E^2(G, \omega)$, for weights ω, are completely contractive Banach algebras.

The completely contractive Banach algebra $\mathcal{A}_R^2(G)$ has actually been observed in [22]. Let us review this briefly. Let $\Gamma \colon A(G \times G) \to A(G)$ be given by

(2.12) $$\Gamma u(s) = \int_G u(st, t)\, dt$$

so Γ averages elements of $A(G \times G)$ over left cosets of the diagonal subgroup $\Delta = \{(s, s) : s \in G\}$. For an elementary tensor, $u \otimes v \in A(G) \otimes A(G) \subset A(G \times G)$, $\Gamma(u \otimes v) = u * \check{v}$ where $\check{v}(s) = v(s^{-1})$.

We record the following fact from Theorem 2.1 in [48], which is partially generalized by Theorem 3.5, below. See Remark 3.6.

Proposition 2.12. *Let $A_\Delta(G) = \Gamma(A(G \times G))$. If we assign $A_\Delta(G)$ the operator space structure which makes Γ a complete quotient map, then $A_\Delta(G) = \mathcal{A}_R^2(G)$, completely isometrically.*

We remark that in [48], the operator space structure for $A(G) \cong \ell^1\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi S_{d_\pi}^1$ is the opposite, i.e., transposed, structure on each $S_{d_\pi}^1$. However, the proof of Theorem 2.1 in [48] is on $A(G)^*$, which has the same operator space we use.

3. Amenability properties

We shall purposely limit our definitions to a commutative unital Banach algebra $\mathcal{B}$, even when they may be more broadly made. After [31], we say that $\mathcal{B}$ is *amenable* if there is a net (w_i) in the projective tensor product $\mathcal{B} \otimes^\gamma \mathcal{B}$ which is bounded, satisfies that $(m(w_i))$ is an approximate identity on $\mathcal{B}$ (m is the multiplication map on $\mathcal{B}$), and for each u in $\mathcal{B}$, $(u \otimes 1 - 1 \otimes u)w_i \xrightarrow{\;i\;} 0$ in the norm of $\mathcal{B} \otimes^\gamma \mathcal{B}$. This is equivalent to having any bounded derivation $D \colon \mathcal{B} \to \mathcal{X}^*$, where $\mathcal{X}$ is a Banach $\mathcal{B}$-bimodule with dual module $\mathcal{X}^*$, be inner, i.e., $D(u) = u \cdot f - f \cdot u$ for some f in $\mathcal{X}^*$.

We say that $\mathcal{B}$ is *weakly amenable*, if for every symmetric $\mathcal{B}$-module $\mathcal{X}$ (i.e., for which $a \cdot x = x \cdot a$, for a in $\mathcal{B}$ and x in $\mathcal{X}$) we have that the only bounded derivation from $\mathcal{B}$ into $\mathcal{X}$ is zero. This definition was given in [3], where it was also noted that it is equivalent to seeing that the only bounded derivation from $\mathcal{B}$ into the symmetric dual bimodule $\mathcal{B}^*$ is inner. We note that weak amenability implies that $\mathcal{B}$ admits no bounded point derivations. We recall that if $\mathcal{B}$ is a function algebra on a compact Hausdorff space X, then a point derivation at x in X is a linear functional $d: \mathcal{B} \to \mathbb{C}$ which satisfies $d(uv) = d(u)v(x) + u(x)d(v)$.

Both definitions above admit obvious analogues in the setting of completely contractive Banach algebras, where we substitute operator projective tensor product for projective, and study only completely bounded derivations. See, for example, [49]. It is obvious that (weak) amenability implies operator (weak) amenability.

Let us begin with a synthesis of well-known facts. We adopt the perspective of [21] and let $\mathcal{B}(G)$ be a unital (regular) Banach algebra of functions on G, which

(a) contains $\mathrm{Trig}(G)$ as a dense subspace,

(b) admits an operator space structure with respect to which it is completely contractive,

(c) translations are complete isometries on $\mathcal{B}(G)$ and are continuous on G; and

(d) $\mathcal{B}(G)\hat{\otimes}\mathcal{B}(G)$ is a (regular) function algebra on $G \times G$.

We let $\mathcal{B}_\Delta(G)$ denotes the image of $\mathcal{B}(G)\hat{\otimes}\mathcal{B}(G)$ under the map Γ of (2.12), and let its operator space structure is given to make Γ a complete quotient map.

Proposition 3.1. *If $\mathcal{B}(G)$ satisfies* (a)–(d) *above, the algebra is operator weakly amenable if and only if $\mathcal{B}_\Delta(G)$ admits a bounded point derivation at e.*

Proof. We first note that the unital algebra $\mathcal{B}(G)$ is operator weakly amenable if and only if the ideal $\mathrm{I}_{\mathcal{B}\hat{\otimes}\mathcal{B}}(\Delta) = \{u \in \mathcal{B}(G)\hat{\otimes}\mathcal{B}(G) : u|_\Delta = 0\}$ satisfies that it is *essential*: $\overline{\mathrm{I}_{\mathcal{B}\hat{\otimes}\mathcal{B}}(\Delta)^2} = \mathrm{I}_{\mathcal{B}\hat{\otimes}\mathcal{B}}(\Delta)$. Indeed see Theorem 3.2 in [25] (which is shown to hold in the operator space setting in [53]). A more illuminating proof may be found in Theorem 2.2 of [50]; being mostly functorial, this proof is straightforward to modify into the operator space setting with help of the infinite matrix techniques of [13], 10.2.1.

By Corollary 1.5 in [21] essentiality of $\mathrm{I}_{\mathcal{B}\hat{\otimes}\mathcal{B}}(\Delta)$ is equivalent to essentiality of $\mathrm{I}_{\mathcal{B}_\Delta}(e) = \{u \in \mathcal{B}_\Delta(G) : u(e) = 0\}$. Any bounded linear functional on $\mathcal{B}_\Delta(G)$ which vanishes on $\mathrm{I}_{\mathcal{B}_\Delta}(e)^2$, but not on $\mathrm{I}_{\mathcal{B}_\Delta}(e)$, is a bounded point derivation. Thus by the Hahn–Banach theorem, $\mathrm{I}_{\mathcal{B}_\Delta}(e)$ is essential if and only if $\mathcal{B}_\Delta(G)$ admits a bounded point derivation at e. $\qquad\square$

Let us now consider the nature of point derivations at e on $\mathcal{B}(G)$. We wish to use the perspective of [7]. We observe that $\mathrm{Trig}(G)^\dagger \cong \prod_{\pi \in \widehat{G}} \mathrm{M}_{d_\pi}$ is an involutive algebra with the usual conjugate transpose on each M_{d_π} and the coproduct $M : \mathrm{Trig}(G)^\dagger \to \mathrm{Trig}(G \times G)^\dagger$ is a $*$-homomorphism. We further assume that

(e) $\mathcal{B}(G)^* \subset \mathrm{Trig}(G)^\dagger$ is closed under this involution.

A point derivation D at e on $\mathcal{B}(G)$ clearly restricts to such on $\mathrm{Trig}(G)$. Moreover, we have the derivation law for the coproduct is $MD = 1 \otimes D + D \otimes 1$.

Hence D^* is also a derivation. Hence we may write as a linear combination of two skew-hermitian derivations: $D = \frac{1}{2}(D - D^*) + \frac{1}{2i}(iD + iD^*)$. Then Theorem 1 in [7] provides us with the following.

Proposition 3.2. *Let G be a connected Lie group and let $\mathcal{B}(G)$ satisfy the conditions* (a)–(e) *above. Then if there exists a bounded point derivation on $\mathcal{B}(G)$, then there is necessarily a bounded skew-symmetric point derivation. Furthermore, each bounded skew-symmetric point derivation D on $\mathcal{B}(G)$ is a classical Lie derivative, i.e., of the form $D(u) = \frac{d}{d\theta}\big|_{\theta=0} u(t_\theta)$ where $\theta \mapsto t_\theta \colon \mathbb{R} \to G$ is a one-parameter subgroup.*

Let recall how $\mathcal{B}_\Delta(G)$ witnesses the operator amenability of $\mathcal{B}(G)$.

Proposition 3.3. *Let $\mathcal{B}(G)$ satisfy* (a)–(d) *above. Then $\mathcal{B}(G)$ is operator amenable if and only if $I_{\mathcal{B}_\Delta}(e) = \{u \in \mathcal{B}_\Delta(G) : u(e) = 0\}$ admits a bounded approximate identity.*

Proof. A splitting result of [26] (see [10]) shows that $\mathcal{B}(G)$ is operator amenable exactly when $I_{\mathcal{B}\hat{\otimes}\mathcal{B}}(\Delta) = \{w \in \mathcal{B}(G)\hat{\otimes}\mathcal{B}(G) : w(s,s) = 0 \text{ for all } s \text{ in } G\}$ admits a bounded approximate identity. While this result is stated in the Banach space category, its proof moves easily to operator spaces. Then [22], Section 1.2 and Corollary 1.5, shows that the latter statement is equivalent to $I_{\mathcal{B}_\Delta}(e)$ admitting a bounded approximate identity. (There is also a beautifully "hands-on" proof of this in [32], Theorem 3.2, which can be easily modified to operator spaces and to our general setting.) $\square$

Remark 3.4. Let ω be any weight. Then each $\mathcal{A}^p(G,\omega)$ and $\mathcal{A}^2_{R^q}(G,\omega)$ satisfies (a)–(c) and (e) above. Indeed, (a) and (e) are obvious; while (b) is Theorem 2.1; and (c) is straightforward as multiplication by a unitary $\pi(s)$ is an isometry on each $S^p_{d_\pi}$, and continuous for s in G. If we further assume that ω is a weakly dimension weight then (d) holds by virtue of Corollary 2.8 and [56].

3.1. Operator weak amenability and operator amenability

Motivated by the considerations above, we may immediately launch into the first major result of this section. As above we let $\mathcal{A}^p_\Delta(G,\omega) = \Gamma(\mathcal{A}^p(G,\omega)\hat{\otimes}\mathcal{A}^p(G,\omega))$.

Theorem 3.5. *The space $\mathcal{A}^p_\Delta(G,\omega)$, qua quotient of $\mathcal{A}^p(G,\omega)\hat{\otimes}\mathcal{A}^p(G,\omega)$ via Γ, is given by*

$$\mathcal{A}^p_\Delta(G,\omega) = \mathcal{A}^{r(p)}(G, d^{\beta(p)}\Omega)$$

where $\frac{1}{r(p)} + \frac{|p-2|}{2p} = 1$, $\beta(p) = \begin{cases} 4 - 4/p & \text{if } 1 \leq p < 2, \\ 2 & \text{if } p \geq 2, \end{cases}$ and $\Omega(\pi) = \omega(\pi)\omega(\bar{\pi})$.

We call Ω the *symmetrization* of ω.

Remark 3.6. The identification is not, in general, a complete isometry. Indeed the second identification in (3.2), below, is not generally a complete isometry.

However if $p = 1$, then we obtain the $\mathrm{S}_{d,C}^2$ in (3.2), from which we can deduce Proposition 2.12. We leave the details to the reader; or see [48], Theorem 2.1.

Proof. We first note that

$$\Gamma^*(\lambda(s)) = \int_G \lambda(st) \otimes \lambda(t)\,dt = (\lambda(s) \otimes I) \int_G \lambda(t) \otimes \lambda(t)\,dt,$$

where the integral is a weak* integral in $(\mathcal{A}^p(G)\hat{\otimes}\mathcal{A}^p(G))^*$. Since $\mathrm{Trig}(G)$ is dense in $\mathcal{A}^p(G, \omega)$, and $\mathrm{span}\,\lambda(G)$ is weak†-dense in $\mathrm{Trig}(G)^\dagger$, we have for T in $\mathrm{Trig}(G)^\dagger$ that

$$(3.1) \qquad\qquad \Gamma^*(T) = (T \otimes I) \int_G \lambda(t) \otimes \lambda(t)\,dt.$$

Using the decomposition $\lambda = \bigoplus_{\pi \in \widehat{G}} \pi$, the Schur orthogonality relations tell us that $\int_G \lambda(s) \otimes \lambda(s)\,ds \cong \bigoplus_{\pi,\pi' \in \widehat{G} \times \widehat{G}} \int_G \pi(s) \otimes \pi'(s)\,ds$, where each integral is zero unless $\pi' = \bar{\pi}$. Thus for $T = (T_\pi)_{\pi \in \widehat{G}}$ we obtain

$$\Gamma^*(T) = \bigoplus_{\pi \in \widehat{G}} (T_\pi \otimes I) \int_G \pi(t) \otimes \bar{\pi}(t)\,dt.$$

Further, select for each π an orthonormal basis $\{e_1^\pi, \ldots, e_{d_\pi}^\pi\}$ for $\mathcal{H}_\pi$, and let $e_{ij}^\pi = e_i^\pi \otimes e_j^\pi$ in $\mathrm{S}_{d_\pi}^{p'} \cong \mathrm{C}_{d_\pi}^{p'} \otimes^h \mathrm{R}_{d_\pi}^{p'}$ as in (1.3). For each π, we consider both π and $\bar{\pi}$ as acting on one and the same Hilbert space, and we depict their matrices with the same basis. Then a more refined use of the Schur orthogonality relations shows that

$$\int_G \pi(t) \otimes \bar{\pi}(t)\,dt = \frac{1}{d_\pi} \sum_{i,j=1}^{d_\pi} e_{ij}^\pi \otimes e_{ij}^{\bar{\pi}}.$$

We write each $T_\pi = \sum_{k,l=1}^{d_\pi} T_{\pi,kl} e_{kl}^\pi$ to obtain

$$(T_\pi \otimes I) \int_G \pi(t) \otimes \bar{\pi}(t)\,dt = \frac{1}{d_\pi} \sum_{i,j,k=1}^{d_\pi} T_{\pi,ki} e_{kj}^\pi \otimes e_{ij}^{\bar{\pi}} \cong \frac{1}{d_\pi} \sum_{i,j,k=1}^{d_\pi} T_{\pi,ki} e_k^\pi \otimes e_j^\pi \otimes e_i^\pi \otimes e_j^\pi.$$

Consider the sequence of maps applied to each of the elements above:

$$\sum_{i,j,k=1}^{d_\pi} T_{\pi,ki} e_k^\pi \otimes e_j^\pi \otimes e_i^\pi \otimes e_j^\pi \qquad\qquad \in (\mathrm{C}_{d_\pi}^{p'} \otimes^h \mathrm{R}_{d_\pi}^{p'})\check{\otimes}(\mathrm{C}_{d_\pi}^{p'} \otimes^h \mathrm{R}_{d_\pi}^{p'})$$

$$\mapsto \sum_{i,j,k=1}^{d_\pi} T_{\pi,ki} e_k^\pi \otimes e_j^\pi \otimes e_i^\pi \otimes e_j^\pi \qquad\qquad \in \mathrm{C}_{d_\pi}^{p'} \check{\otimes} \mathrm{R}_{d_\pi}^{p'} \check{\otimes} \mathrm{C}_{d_\pi}^{p'} \check{\otimes} \mathrm{R}_{d_\pi}^{p'}$$

$$\mapsto \sum_{k,i=1}^{d_\pi} T_{\pi,ki} e_k^\pi \otimes e_i^\pi \otimes \sum_{j=1}^{d_\pi} e_j^\pi \otimes e_j^\pi \qquad\qquad \in \mathrm{C}_{d_\pi}^{p'} \check{\otimes} \mathrm{C}_{d_\pi}^{p'} \check{\otimes} \mathrm{R}_{d_\pi}^{p'} \check{\otimes} \mathrm{R}_{d_\pi}^{p'}$$

$$\mapsto \Big(\sum_{k,i=1}^{d_\pi} T_{\pi,ki} e_k^\pi \otimes e_i^\pi \Big) \otimes \Big(\sum_{j=1}^{d_\pi} e_j^\pi \otimes e_j^\pi \Big) \quad \in (\mathrm{C}_{d_\pi}^{p'} \check{\otimes} \mathrm{C}_{d_\pi}^{p'}) \otimes^h (\mathrm{R}_{d_\pi}^{p'} \check{\otimes} \mathrm{R}_{d_\pi}^{p'})$$

$$\mapsto \sum_{i,j,k=1}^{d_\pi} T_{\pi,ki} e_k^\pi \otimes e_j^\pi \otimes e_i^\pi \otimes e_j^\pi \qquad\qquad \in (\mathrm{C}_{d_\pi}^{p'} \otimes^h \mathrm{R}_{d_\pi}^{p'})\check{\otimes}(\mathrm{C}_{d_\pi}^{p'} \otimes^h \mathrm{R}_{d_\pi}^{p'}).$$

Each map is a contraction on the elements to which it is applied. The map at the second from last line is contractive by virtue of the fact that it is being applied to an elementary tensor and all tensor norms are cross-norms. The map in the last line is a contraction thanks to the shuffle theorem of [14]. Since the composition of the entire sequence of maps is the identity, and each map is a contraction, each map must be an isometry. We now observe that we have isometric identifications

$$(3.2) \qquad \mathrm{C}_d^{p'} \check{\otimes} \mathrm{C}_d^{p'} \cong \mathcal{CB}(\mathrm{C}_d^p, \mathrm{C}_d^{p'}) = \mathrm{S}_d^{\frac{2pp'}{|p-p'|}}$$

where the first identification is standard, and the second is observed in [61], Lemma 5.9. We further note that $2pp'/|p-p'| = 2p/|p-2|$. Since $(\mathrm{R}_d^{p'})^* \cong \mathrm{C}_d^p$, we likewise obtain $\mathrm{R}_d^{p'} \check{\otimes} \mathrm{R}_d^{p'} \cong \mathrm{S}_d^{2p/|p-2|}$, as well. Collecting together the results of this paragraph we obtain

$$\frac{1}{d_\pi} \left\| \sum_{i,j,k=1}^{d_\pi} T_{\pi,ki} e_{kj}^\pi \otimes e_{ij}^\pi \right\|_{\mathrm{S}_{d_\pi}^{p'} \check{\otimes} \mathrm{S}_{d_\pi}^{p'}} = \frac{1}{d_\pi} \|T_\pi \otimes I\|_{\mathrm{S}_{d_\pi}^{\frac{2p}{|p-2|}} \otimes^h \mathrm{S}_{d_\pi}^{\frac{2p}{|p-2|}}} = d_\pi^{\frac{|p-2|}{2p}-1} \|T_\pi\|_{\mathrm{S}_{d_\pi}^{\frac{2p}{|p-2|}}}.$$

We observe, in analogy to (2.4), that

$$(\mathcal{A}^p(G,\omega) \hat{\otimes} \mathcal{A}^p(G,\omega))^* \cong \ell^\infty\text{-}\bigoplus_{\pi,\pi'\in\widehat{G}} \frac{(d_\pi d_{\pi'})^{-1/p'}}{\omega(\pi)\,\omega(\pi')} \mathrm{S}_{d_\pi}^{p'} \check{\otimes} \mathrm{S}_{d_{\pi'}}^{p'}.$$

By results of the last paragraph, we find for T in $\mathrm{Trig}(G)^\dagger$ that

$$\Gamma^*(T) \in (\mathcal{A}^p(G,\omega) \hat{\otimes} \mathcal{A}^p(G,\omega))^* \quad \Leftrightarrow \quad \sup_{\pi\in\widehat{G}} \frac{d_\pi^{-\frac{2}{p'}+\frac{|p-2|}{2p}-1}}{\omega(\pi)\omega(\bar{\pi})} \|T_\pi\|_{\mathrm{S}_{d_\pi}^{\frac{2p}{|p-2|}}} < \infty.$$

Hence we obtain

$$(3.3) \quad \mathcal{A}_\Delta^p(G,\omega)^* \cong \ell^\infty\text{-}\bigoplus_{\pi\in\widehat{G}} \frac{d_\pi^{-\frac{2}{p'}+\frac{|p-2|}{2p}-1}}{\omega(\pi)\omega(\bar{\pi})} \mathrm{S}_{d_\pi}^{\frac{2p}{|p-2|}} = \ell^\infty\text{-}\bigoplus_{\pi\in\widehat{G}} \frac{d_\pi^{-\beta(p)-1/r(p)'}}{\Omega(\pi)} \mathrm{S}_{d_\pi}^{r'(p)},$$

where $r(p)' = 2p/|p-2|$ and $\beta(p) = 1 + 2/p' - |2-p|/p$. The desired result follows by duality. □

In particular, we obtain an isometric identification

$$\mathcal{A}_\Delta^2(G) = \mathrm{A}(G, d^2).$$

This stands in contrast to Theorem 2.6 in [21], where the isometric identification

$$\mathcal{A}_{R,\Delta}^2(G) = \mathcal{A}^2(G, d)$$

is obtained. Let us compare these two further. In the notation of Section 2.4, we let for $1 \le q \le \infty$, $\mathcal{A}_{R^q,\Delta}^2(G,\omega) = \Gamma(\mathcal{A}_{R^q}^2(G,\omega) \hat{\otimes} \mathcal{A}_{R^q}^2(G,\omega))$.

Theorem 3.7. *The space* $\mathcal{A}^2_{R^q,\Delta}(G,\omega)$, *qua quotient of* $\mathcal{A}^2_{R^q}(G,\omega)\hat{\otimes}\mathcal{A}^2_{R^q}(G,\omega)$ *via* Γ, *is given by*

$$\mathcal{A}^2_{R^q,\Delta}(G,\omega) = \mathcal{A}^{r(q)}(G, d^{\gamma(q)}\Omega)$$

where $\frac{1}{r(q)} + \frac{|q-2|}{2q} = 1$, $\gamma(q) = \begin{cases} 3 - 2/q & \text{if } 1 \le q < 2, \\ 1 + 2/q & \text{if } q \ge 2, \end{cases}$ *and* $\Omega(\pi) = \omega(\pi)\omega(\bar{\pi})$.

Remark 3.8. As before, we do not generally obtain a complete isometry. However, when $q = \infty$ we will, in fact, get $\mathcal{A}^2_{R,\Delta}(G,\omega) = \mathcal{A}^2_R(G, d\Omega)$ completely isometrically, as we comment in the proof, below. In particular, we generalize and refine Theorem 2.1 in [22]: we see that $\mathcal{A}^2_{R,\Delta}(G, d^{2^n-1}) = \mathcal{A}^2_R(G, d^{2^{n+1}-1})$, completely isometrically.

Proof. We use the template of the proof of Theorem 3.5. First recall that we get isometric identifications

$$S^2_{d,R^{q'}} = R^{q'}_d \otimes^h R^{q'}_d \quad \text{and} \quad R^{q'}_d \check{\otimes} R^{q'}_d \cong \mathcal{CB}(R^q_d, R^{q'}_d) = S^{2q/|q-2|}_d$$

which are both completely isometric if $q = 1, \infty$ and we let $S^2_d = S^2_{d,R^{q'}}$. We may then compute, as in the last proof, that for T in $\mathrm{Trig}(G)^\dagger$ and each π in $\widehat{G}$ we have

$$\|\Gamma^*(T)_{\pi,\bar{\pi}}\|_{S^2_{d_\pi,R^{q'}}\check{\otimes}S^2_{d_\pi,R^{q'}}} = \frac{1}{d_\pi}\Big\|\sum_{i,j,k=1}^{d_\pi} T_{\pi,ki}e^\pi_{kj}\otimes e^\pi_{ij}\Big\|_{S^2_{d_\pi,R^{q'}}\check{\otimes}S^2_{d_\pi,R^{q'}}}$$

$$= \frac{1}{d_\pi}\Big\|\sum_{k,i=1}^{d_\pi} T_{\pi,ki}e^\pi_k\otimes e^\pi_i \otimes \sum_{j=1}^{d_\pi} e^\pi_j \otimes e^\pi_j\Big\|_{(R^{q'}_{d_\pi}\check{\otimes}R^{q'}_{d_\pi})\otimes^h(R^{q'}_{d_\pi}\check{\otimes}R^{q'}_{d_\pi})}$$

$$= \frac{1}{d_\pi}\|T_\pi \otimes I\|_{S^{\frac{2q}{|q-2|}}_{d_\pi}\otimes^h S^{\frac{2q}{|q-2|}}_{d_\pi}} = d_\pi^{\frac{|q-2|}{2q}-1}\|T_\pi\|_{S^{\frac{2q}{|q-2|}}_{d_\pi}}.$$

Hence we see for such T that

$$\Gamma^*(T) \in (\mathcal{A}^2_{R^q}(G,\omega)\hat{\otimes}\mathcal{A}^2_{R^q}(G,\omega))^* \quad \Leftrightarrow \quad \sup_{\pi\in\widehat{G}} \frac{d_\pi^{-1+\frac{|q-2|}{2q}-1}}{\omega(\pi)\omega(\bar{\pi})}\|T_\pi\|_{S^{\frac{2q}{|q-2|}}_{d_\pi}} < \infty.$$

Hence we obtain

$$\mathcal{A}^2_{R^q,\Delta}(G,\omega)^* \cong \ell^\infty\text{-}\bigoplus_{\pi\in\widehat{G}} \frac{d_\pi^{-2+\frac{|q-2|}{2q}}}{\omega(\pi)\omega(\bar{\pi})}S^{\frac{2p}{|p-2|}}_{d_\pi} = \ell^\infty\text{-}\bigoplus_{\pi\in\widehat{G}} \frac{d_\pi^{-\gamma(p)-1/r'}}{\Omega(\pi)}S^{r'}_{d_\pi},$$

where $r(q)' = 2q/|q-2|$ and $\gamma(q) = 2-|2-q|/q$. If $q = 1, \infty$, then $r(q) = r(q)' = 2$ and the description above is completely isometric provided we use structure $R^{q'}$ on each $S^2_{d_\pi}$. $\qquad\square$

We can now apply our results to the 2×2 special unitary group SU(2). Observe that in the case $p = 1$, we improve Theorem 4.2 (v) in [37]. (We note that in [37], the operator space structure used on each $S^1_{d_\pi}$ is the opposite from the structure used here, as their choice of Fourier coefficients, and hence duality is transposed from ours. However, this does not affect their proof of the quoted result.)

Theorem 3.9. *For $p \geq 1$, the completely contractive Banach algebra $\mathcal{A}^p(\mathrm{SU}(2), d^\alpha)$ admits a bounded point derivation if and only if $\alpha \geq 1$. The algebra $\mathcal{A}^p(\mathrm{SU}(2), d^\alpha)$ is operator weakly amenable if and only if*

$$1 \leq p < \frac{4}{3 + 2\alpha}.$$

In particular, if $p = 1$, $\mathrm{A}(\mathrm{SU}(2), d^\alpha)$ is operator weakly amenable if and only if $\alpha < 1/2$.

Proof. Up to conjugation, the only one-parameter subgroups in $G = \mathrm{SU}(2)$ are $\theta \mapsto t_\theta = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$. By Remark 3.4 and Proposition 3.1, $D(u) = \frac{d}{d\theta}\big|_{\theta=0} u(t_\theta)$ is, up to conjugacy and scalar, the only candidate bounded skew-symmetric point derivation at e on either of $\mathcal{A}^p(G, d^\alpha)$ or on $\mathcal{A}^p_\Delta(G)$ and hence its boundedness determines the only occasions in which such a derivation may exist.

We recall that $\widehat{\mathrm{SU}}(2) = \{\pi_n : n = 0, 1, 2, \dots\}$ and $d_{\pi_n} = n + 1$, and $\pi_n(t_\theta) = \mathrm{diag}(e^{in\theta}, e^{i(n-2)\theta}, \dots, e^{-i(n-2)\theta}, e^{-in\theta})$. Thus in the dual pairing (1.1) we have for u in $\mathrm{Trig}(\mathrm{SU}(2))$ that

$$\langle u, D \rangle = \sum_{n=0}^{\infty} (n+1) \, \mathrm{Tr}\Big(\hat{u}(\pi_n) \frac{d}{d\theta}\Big|_{\theta=0} \mathrm{diag}(e^{in\theta}, e^{i(n-2)\theta}, \dots, e^{-i(n-2)\theta}, e^{-in\theta})\Big)$$

$$= \sum_{n=0}^{\infty} (n+1) \, \mathrm{Tr}\big(\hat{u}(\pi_n)[i\,\mathrm{diag}(n, n-2, \dots, -(n-2), -n)]\big)$$

so $D_{\pi_n} = i\,\mathrm{diag}(n, n-2, \dots, 2-n, -n)$. Hence for any $s \geq 1$ we find

$$\|D_{\pi_n}\|_{\mathrm{S}^s_{n+1}} = \Big(\sum_{j=0}^{n} |n - 2j|^s\Big)^{1/s}.$$

Elementary integral estimates yield constants c, C (depending only on r') for which

$$(3.4) \qquad c\,(n+1)^{1+1/s} \leq \Big(\sum_{j=0}^{n} |n - 2j|^s\Big)^{1/s} \leq C\,(n+1)^{1+1/s}.$$

Thus setting $s = p'$, we see from the weighted analogue of (2.3) that $D \in \mathcal{A}^p(G, d^\alpha)^*$ if and only if

$$\sup_{n=0,1,2,\dots} (n+1)^{-(\alpha + 1/p') + (1 + 1/p')} < \infty$$

i.e., $-\alpha + 1 \leq 0$, so $\alpha \geq 1$. This characterizes when we get a bounded point derivation at e. Since $\mathcal{A}^p(G, d^\alpha)$ is evidently isometrically translation invariant, we get a bounded point derivation at any point if and only if we get one at e.

We now wish to determine operator weak amenability. Hence we need to establish those p which allow $D \in (\mathcal{A}^p_\Delta(G))^*$. We use (3.3), with $\Omega(\pi_n) = d^{2\alpha}_{\pi_n} = (n+1)^{2\alpha}$, and $s = r(p)' = 2p/|p - 2|$. We see that $D \in (\mathcal{A}^p_\Delta(G))^*$ if and only if

$$\sup_{n=0,1,2,\dots} (n+1)^{-(\beta(p) + 2\alpha + 1/r(p)') + (1 + 1/r(p)')} < \infty$$

i.e., exactly when $1 - \beta(p) - 2\alpha \leq 0$. For $p \geq 2$ this always holds; for $1 \leq p < 2$ we obtain $-3 + 4/p \leq 2\alpha$, which gives the desired result. $\qquad \square$

With less precise data we gain a more general result, but with weaker control.

Theorem 3.10. *Let $\alpha \geq 0$. Let G be a non-commutative connected Lie group. Then $\mathcal{A}^p(G, d^\alpha)$ admits a bounded point derivation if $\alpha \geq 1$, and is not operator weakly amenable if $p \geq \frac{4}{3+2\alpha}$. Moreover, $\mathcal{A}^2_{R^q}(G)$ is never operator weakly amenable.*

Proof. According to [47], there is a one-parameter subgroup which, for each π in $\widehat{G}$ satisfies $\theta \mapsto \pi(t_\theta) = \mathrm{diag}(e^{in_1\theta}, \ldots, e^{in_{d_\pi}\theta})$ (up to conjugacy) where each $|n_j| < d_\pi$. Thus we obtain the following estimates for the point derivation given by $D(u) = \frac{d}{d\theta}\big|_{\theta=0} u(t_\theta)$:

$$\|D_\pi\|_{\mathrm{S}^s_{d_\pi}} = \Big(\sum_{j=1}^{d_\pi} |n_j|^s \Big)^{1/s} \leq (d_\pi d_\pi^s)^{1/s} = d_\pi^{1+1/s}.$$

The same estimates as above show that $D \in \mathcal{A}^p(G, d^\alpha)$ for $\alpha \geq 1$ and $D \in (\mathcal{A}^p_\Delta(G, d^\alpha))^*$ for $p \geq 4/(3 + 2\alpha)$.

Now we set $p = 2$, $\alpha = 0$, but deform the constituent Hilbert–Schmidt matrices. Thanks to Theorem 3.7 we have that the derivation above defines an element of $\mathcal{A}^2_{R^q,\Delta}(G)^*$ if the quantities $d_\pi^{-\gamma(q)-1/r(q)'} d_\pi^{1+1/r(q)'}$ are uniformly bounded in π. Since the dimensions are unbounded – see, e.g. [43] –, this requires that $\gamma(q) \geq 1$ which is always satisfied. $\qquad\square$

Remark 3.11. Consider the weakly dimension weight $\omega_{\alpha,\beta}(\pi) = d_\pi^\alpha (1 + \log d_\pi)^\beta$ on $\widehat{G}$, where $\alpha, \beta \geq 0$. It is easy to check that the addition of the weight $\omega_{w^\beta}(\pi) = (1 + \log d_\pi)^\beta$, as defined in Example 2.7, affects neither of the proofs of the prior two theorems, hence affects neither of their outcomes.

We close by addressing a result on operator amenability. Using terminology of [42], we say that G is *tall* if for each fixed d, $\{\pi \in \widehat{G} : d_\pi = d\}$ is finite. For an infinite group this means $\lim_{\pi \to \infty} d_\pi = \infty$. According to Theorem 3.2 in [29], a semisimple compact Lie group is tall. However, there exist tall totally disconnected groups [30].

Theorem 3.12. *Suppose that G is infinite and tall. Then $\mathcal{A}^p(G)$ is not operator amenable for any $p > 1$, and $\mathcal{A}^2_{R^q}(G)$ is not operator amenable for any $1 \leq q \leq \infty$.*

Proof. This uses the idea of Theorem 3.10 in [37]. Appealing to Theorems 3.5 and 3.7, respectively, we obtain duality relations

$$\mathcal{A}^p_\Delta(G) \cong \Big(c_0\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^{-\frac{1}{r(p)'} - \beta(p)} \mathrm{S}^{r(p)'}_{d_\pi} \Big)^*, \quad \mathcal{A}^2_{R^q,\Delta}(G) \cong \Big(c_0\text{-}\bigoplus_{\pi \in \widehat{G}} d_\pi^{-\frac{1}{r(q)'} - \gamma(q)} \mathrm{S}^{r(q)'}_{d_\pi} \Big)^*.$$

We observe, moreover, that $\beta(p) > 0$ if $p > 1$, and $\gamma(q) \geq 1 > 0$ for $1 \leq q \leq \infty$. Hence, if G is tall, each $\lambda(s) = (\pi(s))_{s \in G}$ is an element of this predual of $\mathcal{A}^p_\Delta(G)$, respectively $\mathcal{A}^2_{R^q,\Delta}(G)$; in other words, evaluation characters are weak*-continuous.

Now suppose $\mathcal{A}^p(G)$ is operator amenable. Then the bounded approximate identity for $\mathrm{I}_{\mathcal{A}^p_\Delta}(e)$ (respectively for $\mathrm{I}_{\mathcal{A}^2_{R^p,\Delta}}(e)$) promised by Proposition 3.3 and

Remark 3.4, admits the indicator function $1_{G\setminus\{e\}}$ as a weak*-cluster point, thanks to weak*-continuity of evaluation characters, and to the regularity of $\mathcal{A}^p_\Delta(G)$ (respectively of $\mathcal{A}^2_{R^q,\Delta}(G)$). This forces G to be discrete, hence finite. $\qquad\square$

It is hence obvious that for tall G and any weight ω on $\widehat{G}$, the algebra $\mathcal{A}^p(G,\omega)$ is operator amenable for no $p > 1$. We suspect that for $p > 1$, $\mathcal{A}^p(G)$ is only operator amenable if and only if G admits an open abelian subgroup. We will be able to transport Theorem 3.12 to connected non-abelian groups. See Section 4.1.

For the case of $\mathcal{A}^2_R(G) = \mathrm{A}_\Delta(G)$, we have that this algebra is operator amenable exactly when G is virtually abelian, and operator weakly amenable exactly when the connected component of the identity, G_e, is abelian. See Theorem 4.1 in [22].

3.2. Amenability and weak amenability

We shall make repeated use the observation that if a commutative Banach algebra has within itself a dense image of a (weakly) amenable commutative algebra, then it is (weakly) amenable.

We observe that it is easy to characterize amenability of $\mathcal{A}^p(G,\omega)$ for a weakly dimension weight ω.

Proposition 3.13. *For any $1 \le p \le \infty$ and weakly dimension weight ω, $\mathcal{A}^p(G,\omega)$ is amenable if and only if G admits an open abelian subgroup.*

Proof. If $\mathcal{A}^p(G,\omega)$ – which, as remarked after (2.8), is dense within $\mathrm{A}(G)$ – is amenable, then so too must be $\mathrm{A}(G)$. Hence G contains an open abelian subgroup by Theorem 2.3 in [20] (or see [51]). The converse is immediate from Proposition 2.9 and the fact that $\mathrm{A}(G)$ is amenable in this case (see [36]). $\qquad\square$

The weak amenability of the Banach algebras $\mathcal{A}^p(G, d^\alpha)$ is now straightforward to establish. It is interesting in its own right to observe that our algebras respect quotient subgroups.

Let $\omega_N = \omega|_{\widehat{G/N}\circ q}$ where $q\colon G \to G/N$ is the quotient map. Notice that if ω is bounded away form zero and a weakly dimension weight, then so too must be ω_N.

Lemma 3.14. *Let N be a closed normal subgroup of G. Then the map $u \mapsto T_N(u) = \int_N u(\cdot n)\, dn$ is a completely contractive projection for which we have $T_N(\mathcal{A}^p(G,\omega)) \cong \mathcal{A}^p(G/N,\omega_N)$ completely isometrically. Moreover, if (N_ι) is a decreasing net of subgroups converging to e (i.e., any open neighborhood of the identity contains some N_ι), then $u = \lim_\iota T_{N_\iota} u$ for each u in $\mathcal{A}^p(G,\omega_N)$.*

Proof. The proof of the first statement may be adapted from that of Proposition 4.14 (i) in [38], with obvious changes. In particular we see that

$$\mathcal{A}^p(G/N,\omega_N) = T_N(\mathcal{A}^p(G,\omega)) \cong \ell^1\text{-}\bigoplus_{\pi\in\widehat{G/N}\circ q} d_\pi^{1+1/p'}\omega(\pi)\, \mathrm{S}^p_{d_\pi}.$$

The second statement is well known, see for example the proof of Theorem 3.3 in [20]. Briefly, given u in $\mathcal{A}^p(G,\omega)$, by continuity of translations we can arrange N_ι

so small that $\|u - u(\cdot n)\|_{\mathcal{A}^p(G,\omega)}$ is uniformly small for n in N_ι. Averaging over N_ι does not increase this norm. $\square$

The result (ii), below, improves upon Theorem 3.14 in [37], where operator weak amenability is established.

Proposition 3.15. (i) *For any $1 \le p \le \infty$ and weakly dimension weight ω, the algebra $\mathcal{A}^p(G,\omega)$ is weakly amenable if and only if G_e is abelian.*

(ii) *For any $1 \le p \le \infty$ and any weight ω, if G is totally disconnected, then $\mathcal{A}^p(G,\omega)$ is weakly amenable.*

Proof. (i) If $\mathcal{A}^p(G,\omega)$ – which is dense within $\mathrm{A}(G)$ – is weakly amenable, then so too must be $\mathrm{A}(G)$. Hence G_e is abelian by Theorem 2.1 in [21]. To see the converse we summarize the proof of Theorem 3.3 in [20], with adaptations to our particular setting. As noted there, there exists a decreasing net of subgroups (N_ι) converging to e for which each G/N_ι is a Lie group with open abelian connected component of the identity. By Proposition 2.9 we obtain that each $\mathcal{A}^p(G/N_\iota, \omega_{N_\iota}) = \mathrm{A}(G/N_\iota)$ isomorphically and, as observed above, is amenable. As in Lemma 3.14, $\mathcal{A}^p(G,\omega)$ is an an inductive limit of the algebras $T_N(\mathcal{A}^p(G,\omega)) \cong \mathcal{A}^p(G/N_\iota, \omega_{N_\iota})$. Any derivation $D : \mathcal{A}^p(G,\omega) \to \mathrm{A}(G,\omega)^*$ must vanish on each $T_N(\mathcal{A}^p(G,\omega))$, whence $D = 0$.

(ii) For totally disconnected G and (N_ι) as above, each G/N_ι is a totally disconnected compact Lie group and hence is finite. $\square$

4. Restriction to subgroups

We do not get a general usable restriction theorem, unless the subgroup is essentially a factor in a direct product. This latter fact gives us enough technology to characterize operator amenability of $\mathcal{A}^p(G)$ for connected groups, or products of finite groups. We will turn our focus to the example of a torus on $\mathrm{SU}(2)$, where the restriction algebra seems to be of a much different form. In doing so we gain, to our knowledge, a new class of Banach algebras of Laurent series.

Let us first consider the general situation. We fix a closed subgroup H of G and let $R_H : \mathrm{A}(G) \to \mathrm{A}(H)$ denote the restriction map. Let us consider this map on $\mathrm{Trig}(G)$. In this context, its adjoint is given by

$$R_H^\dagger (T_\sigma)_{\sigma \in \widehat{H}} = \Big(\sum_{\sigma \subset \pi|_H} \sum_{k=1}^{m(\sigma,\pi)} V_{\pi,\sigma}^{(k)} T_\sigma V_{\pi,\sigma,k}^{(k)*} \Big)_{\pi \in \widehat{G}},$$

where each $V_{\pi,\sigma}^{(k)} : \mathcal{H}_\sigma \to \mathcal{H}_\pi$ is an isometry, and $m(\sigma,\pi)$ is the multiplicity of σ in $\pi|_H$. Indeed, this is straightforward to see if $T = \lambda(s) = (\sigma(s))_{\sigma \in \widehat{H}}$, for s in H; and follows from the weak† density of $\mathrm{span}\,\lambda(G)$ in $\mathrm{Trig}(G)^\dagger$, otherwise. By picking a suitable basis for each $\mathcal{H}_\pi$, we may suppress explicit mention of the isometries $V_{\pi,\sigma}^{(k)}$ and write

(4.1) $R_H^\dagger (T_\sigma)_{\sigma \in \widehat{H}} = \big((T_\sigma^{(m(\sigma,\pi))})_{\sigma \subset \pi|_H} \big)_{\pi \in \widehat{G}},$

where $T^{(m)}$ indicates that T is represented with multiplicity m.

For a weight ω we let

$$\mathcal{A}^p_{G,\omega}(H) = R_H(\mathcal{A}^p(G,\omega))$$

be endowed with the quotient operator space structure which makes $R_H \colon \mathcal{A}^p(G,\omega) \to \mathcal{A}^p_{G,\omega}(H)$ a complete quotient map.

Proposition 4.1. *The operator space structures on $\mathcal{A}^p_{G,\omega}(H)^*$ is determined by the completely isometric embedding*

$$T \mapsto \big((T_\sigma)_{\sigma \subset \pi|_H}\big)_{\pi \in \widehat{G}} : \mathcal{A}^p_{G,d^\alpha}(H)^* \to \ell^\infty\text{-}\bigoplus_{\pi \in \widehat{G}} \frac{d_\pi^{-1/p'}}{\omega(\pi)} \Big(\ell^{p'}\text{-}\bigoplus_{\sigma \subset \pi|_H} m(\sigma,\pi)^{1/p'}\, \mathrm{S}^{p'}_{d_\sigma}\Big).$$

Proof. The embedding result on $\mathcal{A}^p_{G,\omega}(H)^*$ is immediate from (4.1) and the weighted analogue of (2.3). □

We have not come up with an illuminating closed-form formula for the norm on $\mathcal{A}^p_{G,\omega}(H)$, for general p. For $p = 1$ we obtain

$$\mathrm{A}_{G,\omega}(H) = \mathrm{A}(H,\omega_G|_H)$$

completely isometrically, where $\omega_G|_H$ is the restricted weight defined in (2.10). See Proposition 3.5 in [37] or Proposition 4.12 in [38]. We shall see that even with trivial weights, this does not hold generally for p-Fourier algebras.

Let us begin by observing the case of restriction to a central subgroup.

Proposition 4.2. *Let Z be a closed central subgroup of G. Then $\mathcal{A}^p_{G,\omega}(Z) = \mathrm{A}(Z,\omega_G|_Z)$, completely isometrically.*

Proof. It is a consequence of Schur's lemma that for π in $\widehat{G}$, there is a character χ in $\widehat{Z}$ for which $\pi|_Z = \chi(\cdot)I_\pi$. In fact $\chi = \frac{1}{d_\pi}\mathrm{Tr} \circ \pi|_Z$. Furthermore, since $R_Z(\mathrm{Trig}(G)) = \mathrm{Trig}(Z)$, each character on Z is attained thusly. Hence Proposition 4.1 yields completely isometric embedding

$$t \mapsto \Big(t_{\frac{1}{d_\pi}\mathrm{Tr}\circ\pi|_Z}\Big)_{\pi \in \widehat{G}} : \mathcal{A}^p_G(Z)^* \to \ell^\infty\text{-}\bigoplus_{\pi \in \widehat{G}} \frac{d_\pi^{-1/p'}}{\omega(\pi)} \big(d_\pi^{1/p'}\, \mathrm{S}^{p'}_1\big) \cong \ell^\infty(\widehat{G},1/\omega).$$

Following through to the range of this map gives us the isometric identification $\mathcal{A}^p_{G,\omega}(Z)^* \cong \ell^\infty(\widehat{Z},(\omega_G|_Z)^{-1})$. □

4.1. Direct products

The situation of direct product groups is very nice, in this setting. The only weighted version we shall use in the sequel is with dimension weights which are easier to work with as they enjoy a certain multiplicativity with Kronecker products. In a direct product group $H \times K$, we may identify $H = H \times \{e\}$ and $K = \{e\} \times K$.

Theorem 4.3. *Let $G = (H \times K)/Z$ where Z is a central subgroup of $H \times K$ which satisfies $H \cap Z = \{e\} = Z \cap K$. Then $\mathcal{A}_G^p(H) = \mathcal{A}^p(H)$, completely isometrically. Furthermore, we have that $\mathcal{A}_G^p(H, d^\alpha) = \mathcal{A}^p(H, d^\alpha)$, completely isometrically if Z is trivial or if K is abelian, and completely isomorphically if Z is finite.*

Proof. We first recall that $\widehat{H \times K} = \widehat{H} \times \widehat{K}$ via Kronecker products. Then $\widehat{G} = \{\sigma \times \tau : \sigma \in \widehat{H}, \tau \in \widehat{K}$ and $Z \subseteq \ker(\sigma \times \tau)\}$. Let $\sigma \times \tau \in \widehat{G}$. Let $p_J : H \times K \to J$ be the projection map, for $J = H, K$, and note that each $p_J(Z)$ is central in J. Hence an application of Schur's lemma tells us that there are characters χ, χ' on Z for which $\sigma \circ p_H|_Z = \chi(\cdot)I_{d_\sigma}$ and $\tau \circ p_K|_Z = \chi'(\cdot)I_{d_\tau}$, while

$$I_{d_\sigma} \otimes I_{d_\tau} = \sigma \times \tau|_Z = \sigma \circ p_H|_Z \otimes \tau \circ p_K|_Z = \chi\chi'(\cdot)I_{d_\sigma} \otimes I_{d_\tau}$$

which means that $\chi' = \bar\chi$.

Thus we see for any σ in $\widehat{H}$, that there is τ in $\widehat{K}$ for which $\sigma \times \tau \in \widehat{G}$ only if

$$(4.2) \qquad\qquad \frac{1}{d_\sigma}\mathrm{Tr} \circ \bar\sigma \circ p_H|_Z = \frac{1}{d_\tau}\mathrm{Tr} \circ \tau \circ p_K|_Z.$$

Furthermore, $\ker(p_H|_Z) = Z \cap K = \{e\} = H \cap Z = \ker(p_K|_Z)$, so $p_H(Z) \cong Z \cong p_K(Z)$. Hence since $R_{p_J(Z)}(\mathrm{Trig}(J)) = \mathrm{Trig}(p_J(Z)) \cong \mathrm{Trig}(Z)$ for $J = H, K$, we have for any σ in $\widehat{H}$, that there exists τ in $\widehat{K}$ for which (4.2) holds. Thus if we let $\widehat{K}_{Z,\sigma} = \{\tau \in \widehat{K} : (4.2)$ holds$\}$ then we have shown that

$$\widehat{G} = \{\sigma \times \tau : \sigma \in \widehat{H} \text{ and } \tau \in \widehat{K}_{Z,\sigma}\}.$$

It is easily checked that the condition above is symmetric in H and K.

Proposition 4.1 now yields completely isometric embedding

$$T \mapsto (T_\sigma^{d_\tau})_{\sigma \times \tau \in \widehat{G}} : \mathcal{A}_{G,d^\alpha}^p(H)^* \to \ell^\infty\text{-}\bigoplus_{\sigma \times \tau \in \widehat{G}} (d_\sigma d_\tau)^{-1/p'-\alpha} d_\tau^{1/p'} \mathrm{S}_{d_\sigma}^{p'}$$

$$(4.3) \qquad\qquad\qquad = \ell^\infty\text{-}\bigoplus_{\sigma \in \widehat{H}} d_\sigma^{-1/p'-\alpha}\left(\ell^\infty\text{-}\bigoplus_{\tau \in \widehat{K}_{Z,\sigma}} d_\tau^{-\alpha} \mathrm{S}_{d_\sigma}^{p'}\right).$$

In the event that $\sup_{\tau \in \widehat{K}_{Z,\sigma}} d_\tau^{-\alpha} = 1$ (which happens if $\alpha = 0$, or if Z is trivial, or if K is abelian), we see that the range of the map (4.3) lands in a completely isometric copy of $\ell^\infty\text{-}\bigoplus_{\sigma \in \widehat{H}} d_\sigma^{-1/p'-\alpha} \mathrm{S}_{d_\sigma}^{p'} \cong \mathcal{A}^p(H, d^\alpha)^*$.

Otherwise, letting $\chi_\sigma = \frac{1}{d_\sigma}\mathrm{Tr} \circ \bar\sigma \circ p_H|_Z$, which is in $\widehat{Z}$, a simple examination of the definition of $\widehat{K}_{Z,\sigma}$ shows that

$$\sup_{\tau \in \widehat{K}_{Z,\sigma}} d_\tau^{-\alpha} = \left(\inf\{d_\tau^\alpha : \tau \in \widehat{K}, \chi_\sigma \subset \tau\}\right)^{-1} = \frac{1}{d_K^\alpha|_{p_K(Z)}(\chi_\sigma)},$$

where we have a slight abuse of notation: since $p_K(Z) \cong Z$, it would be more logical to write $\chi_\sigma \circ p_K^{-1}|_{p_K(Z)}$, instead of χ_σ, above. The range of the map (4.3) then lands in a completely isometric copy of

$$\ell^\infty\text{-}\bigoplus_{\sigma \in \widehat{H}} d_\sigma^{-1/p'-\alpha} \left[d_K^\alpha|_{p_K(Z)}(\chi_\sigma)\right]^{-1} \mathrm{S}_{d_\sigma}^{p'}.$$

Observe that $\omega^\alpha(\sigma) = d_K^\alpha|_{p_K(Z)}(\chi_\sigma)$ defines a weight on $\widehat{H}$ and the space above is completely isometrically isomorphic to $\mathcal{A}^p(H, d^\alpha\omega^\alpha)^*$. In the event that Z is finite, and hence so too is $\widehat{Z}$, there are only finitely many values of $\omega^\alpha(\sigma) = d_K^\alpha|_{p_K(Z)}(\chi_\sigma)$, and $\mathcal{A}^p(H, d^\alpha\omega^\alpha)$ is completely isomorphic to $\mathcal{A}^p(H, d^\alpha)$. $\square$

Example 4.4. Let us consider an example where $\mathcal{A}^p_{G,d^\alpha}(H) = \mathcal{A}^p(H, d^\alpha)$ completely isomorphically, but not isometrically. We have that $U(2) = (\mathbb{T} \times SU(2))/Z$ where $Z = \{\pm(1, I)\}$. Then $\widehat{Z} = \{1, \mathrm{sgn}\}$. Observe that in the notation above $\omega^\alpha(\mathrm{sgn}) = d_{SU(2)}^\alpha|_{p_{SU(2)}(Z)}(\mathrm{sgn}) = 2^\alpha$, since the standard representation π_1 is the lowest dimension representation of $SU(2)$ which "sees" sgn. Further, letting for k in $\mathbb{Z}$, $\sigma_k(z) = z^k$ on $\mathbb{T}$, we see, again in the notation above, that

$$\chi_{\sigma_k} = \begin{cases} 1 & \text{if } 2 \mid k, \\ \mathrm{sgn} & \text{if } 2 \nmid k, \end{cases} \quad \text{hence} \quad \omega^\alpha(\sigma_k) = \begin{cases} 1 & \text{if } 2 \mid k, \\ 2^\alpha & \text{if } 2 \nmid k. \end{cases}$$

Thus $\mathcal{A}^p_{U(2),d^\alpha}(\mathbb{T}) = \mathcal{A}^p(\mathbb{T}, d^\alpha\omega^\alpha) = A(\mathbb{T})$, completely isomorphically, though not isometrically.

Remark 4.5. Proposition 4.1 admits an obvious analogue when we replace $\mathcal{A}^p(G)$ by $\mathcal{A}^2_{R^q}(G, d^\alpha)$; we get a completely isometric embedding

$$\mathcal{A}^2_{R^q}(G, d^\alpha)^* \hookrightarrow \ell^\infty\text{-}\bigoplus_{\pi \in \widehat{G}} d^{-1/2-\alpha}\left(\ell^2\text{-}\bigoplus_{\sigma \subset \pi|_H} m(\sigma, \pi)^{1/2} \, \mathrm{S}^2_{d_\sigma, R^{q'}}\right).$$

Thus, as in Theorem 4.3, in any of the situations that $\alpha = 0$, Z is trivial, or K is abelian, we obtain that

$$\mathcal{A}^2_{R^q, G, d^\alpha}(H) = R_H(\mathcal{A}^2_{R^q}(G, d^\alpha)) = \mathcal{A}^2_{R^q}(H, d^\alpha).$$

With our restriction formula in hand, we can now characterize operator amenability of our algebras for connected Lie groups and of infinite products of finite groups. Let us first obtain a brief quantitative result on finite groups. Any finite-dimensional amenable algebra admits a cluster point of a bounded approximate diagonal, which we simple call a *diagonal*. It is well known that if the algebra is commutative, then the diagonal is unique; see, for example, Proposition 1.1 in [23].

Proposition 4.6. *If G is finite, then the unique diagonal w for $\mathcal{A}^p(G) = \mathcal{A}^2_{R^q}(G)$ has*

$$\|w\|_{\mathcal{A}^p(G)\hat{\otimes}\mathcal{A}^p(G)} = \frac{1}{|G|}\sum_{\pi \in \widehat{G}} d_\pi^{2+\beta(p)} \quad \text{and} \quad \|w\|_{\mathcal{A}^2_{R^q}(G)\hat{\otimes}\mathcal{A}^2_{R^q}(G)} = \frac{1}{|G|}\sum_{\pi \in \widehat{G}} d_\pi^{2+\gamma(q)},$$

where $\beta(p)$ and $\gamma(q)$ are defined in Theorems 3.5 and 3.7, respectively.

Proof. Let $\mathcal{B}$ be either of $\mathcal{A}^p$ or $\mathcal{A}^2_{R^q}$. Let $N : \mathcal{B}_\Delta(G) \to \mathcal{B}(G)\hat{\otimes}\mathcal{B}(G)$ be given by $Nu(s, t) = u(st^{-1})$, which is an isometry by [21], (1.2). We let 1_e denote the indicator function of $\{e\}$. Then $N1_e$ is the the unique diagonal for $\mathcal{B}(G) \otimes \mathcal{B}(G)$ and has norm $\|1_e\|_{\mathcal{B}_\Delta(G)}$. We have that $\widehat{1_e}(\pi) = \frac{1}{|G|}I_{d_\pi}$ for each π, and we note that $\|I_{d_\pi}\|_{\mathrm{S}^{r(s)}_{d_\pi}} = d_\pi^{1/r(s)}$, for $s = p$ or q. We use Theorems 3.5 and 3.7 to finish. $\square$

Theorem 4.7. (i) *Let G be connected $p > 1$ and $1 \leq q \leq \infty$. Then either of $\mathcal{A}^p(G)$ or $\mathcal{A}^2_{R^q}(G)$ is operator amenable if and only if G is abelian.*

(ii) *If G is connected and non-abelian, then $\mathcal{A}^p(G, d^\alpha)$ is not operator weakly amenable for any $p \geq \frac{4}{3+2\alpha}$, and no $\mathcal{A}^2_{R^q}(G)$ is operator weakly amenable.*

(iii) *Let $(G_n)_{n=1}^\infty$ be a sequence of finite groups, $G = \prod_{n=1}^\infty G_n$, $p > 1$ and $1 \leq q \leq \infty$. Then either of $\mathcal{A}^p(G)$ or $\mathcal{A}^2_{R^q}(G)$ is operator amenable if and only if all but finitely G_n are abelian.*

Proof. (i) If G is abelian, then $\mathcal{A}^p(G) = \mathcal{A}^2(G)$ is amenable by Proposition 3.13. If G is not abelian then there exists a non-empty collection of simple, connected, compact, Lie groups $\{S_i\}_{i \in I}$, a connected abelian group T, and a closed central subgroup D of $T \times S$ where $S = \prod_{i \in I} S_i$, for which $G \cong (T \times S)/D$. This is the Levi–Mal'cev theorem; see, for example, Theorem 9.24 in [28]. The third isomorphism theorem tells us that if we let $H = T/T \cap D$, $K = S/D \cap S$ and $Z = D/(T \cap D \times D \cap S)$, then $G = (H \times K)/Z$ where $H \cap Z = \{e\} = Z \cap K$. Furthermore, $K \cong \prod_{i \in I} S_i/Z_i$ where each Z_i is a central subgroup of S_i. Fix i_0 in I. Then, $K_{i_0} = S_{i_0}/Z_{i_0}$ is a simple Lie group. Also, $\mathcal{A}^p(K_{i_0})$ is a complete quotient of $\mathcal{A}^p(G)$ by Theorem 4.3, while $\mathcal{A}^2_{R^q}(K_{i_0})$ is one of $\mathcal{A}^2_{R^q}(G)$, by Remark 4.5. However, neither of $\mathcal{A}^p(K_{i_0})$ nor $\mathcal{A}^2_{R^q}(K_{i_0})$ is operator amenable by Theorem 3.12, and the fact that K_{i_0} is tall (Theorem 3.2 in [29]).

(ii) If $p \geq 4/(3 + 2\alpha)$, then the complete quotient algebra $\mathcal{A}^p(K_{i_0})$, from the proof of (i), above, is not operator weakly amenable, thanks to Theorem 3.10. Likewise for $\mathcal{A}^2_{R^q}(K_{i_0})$.

(iii) If all but finitely many G_n are abelian, then $\mathcal{A}^p(G)$ is amenable by Proposition 3.13. If $\mathcal{A}^p(G)$ is operator amenable, then $\mathcal{A}^p(G) \hat{\otimes} \mathcal{A}^p(G)$ admits a bounded approximate diagonal (w_i). Let $H_n = \prod_{k=1}^n G_k$. Then by Theorem 4.3 each $(R_{H_n} \otimes R_{H_n} w_i)$ is a bounded approximate diagonal for $\mathcal{A}^p(H_n)$, and hence has limit point the unique diagonal w_n for $\mathcal{A}^p(H_n)$. Hence we appeal to Proposition 4.6 to see for any n that

$$\sup_i \|w_i\|_{\mathcal{A}^p(G)\hat{\otimes}\mathcal{A}^p(G)} \geq \sup_i \|R_{H_n} \otimes R_{H_n} w_i\|_{\mathcal{A}^p(H_n)\hat{\otimes}\mathcal{A}^p(H_n)} \geq \|w_n\|_{\mathcal{A}^p(H_n)\hat{\otimes}\mathcal{A}^p(H_n)}$$

$$= \frac{1}{|H_n|} \sum_{\pi \in \widehat{H_n}} d_\pi^{2+\beta(p)} = \frac{1}{|G_1 \times \cdots \times G_n|} \sum_{\sigma_1 \times \cdots \times \sigma_k \in \widehat{G_1} \times \cdots \times \widehat{G_n}} (d_{\sigma_1} \ldots d_{\sigma_n})^{2+\beta(p)}$$

$$= \prod_{k=1}^n \frac{1}{|G_k|} \sum_{\sigma_k \in \widehat{G_k}} d_{\sigma_k}^{2+\beta(p)}.$$

We have that the sequence above diverges as $n \to \infty$, unless the groups G_k are ultimately abelian. Indeed, it is well known that $|G_k| = \sum_{\sigma_k \in \widehat{G_k}} d_{\sigma_k}^2$ and $|\{\sigma \in \widehat{G_k} : d_\sigma = 1\}| = |G_k/G_k'| = |G_k|/|G_k'|$, where G' is the derived subgroup. Hence if G_k is non-abelian we have

$$\frac{1}{|G_k|} \sum_{\sigma \in \widehat{G_k}} d_\sigma^{2+\beta(p)} \geq \frac{1}{|G_k'|} + \frac{2^{\beta(p)}}{|G_k|} \sum_{\sigma \in \widehat{G_k}, d_\sigma \geq 2} d_\sigma^2 = \frac{1}{|G_k'|} + 2^{\beta(p)}\left(1 - \frac{1}{|G_k'|}\right)$$

$$= 2^{\beta(p)} - \frac{2^{\beta(p)} - 1}{|G_k'|} \geq 2^{\beta(p)} - \frac{2^{\beta(p)} - 1}{2} = \frac{2^{\beta(p)} + 1}{2}.$$

Since $\beta(p) > 0$ provided $p > 1$, we get the desired result.

Obvious modifications show the same for $\mathcal{A}^2_{R^q}(G)$. Again we note that $\gamma(q) \geq 1 > 0$, for all $1 \leq q \leq \infty$.   □

If we could learn the structure of $\mathcal{A}^p_G(G_e)$, say, where G_e is the connected component of the identity, then we may be able to assess conditions of operator amenability more generally. Also, the situation for general totally disconnected groups is unknown to us.

4.2. The torus in SU(2)

We now analyze the example of the (unique up to conjugacy) torus $\mathbb{T}$ in SU(2). We will not bother with general weights, but will consider dimension weights. We recall again that $\widehat{\mathrm{SU}}(2) = \{\pi_n : n = 0, 1, 2, \dots\}$ with $d_{\pi_n} = n+1$, and the character theory reveals

$$\pi_n|_{\mathbb{T}} = \mathrm{diag}(\chi_n, \chi_{n-2}, \dots, \chi_{2-n}, \chi_{-n})$$

where $\chi_n(z) = z^n$. Proposition 4.1 immediately yields the following.

Proposition 4.8. *The operator space structure on $\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha}(\mathbb{T})^* \subset \mathbb{C}^{\mathbb{Z}}$ is given by the completely isometric embedding*

$$t \mapsto ((t_{n-2j})_{j=0}^n)_{n=0}^\infty : \mathcal{A}^p_{\mathrm{SU}(2),d^\alpha}(\mathbb{T})^* \to \ell^\infty\text{-}\bigoplus_{n=0}^\infty (n+1)^{-1/p'-\alpha}\,\ell^{p'}_{n+1} = \mathcal{M}^{\alpha,p'}$$

so

$$\|t\|_{\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha}(\mathbb{T})^*} = \sup_{n=0,1,2,\dots} \frac{\left(\sum_{j=0}^n |t_{n-2j}|^{p'}\right)^{1/p'}}{(n+1)^{1/p'+\alpha}}.$$

Remark 4.9. Notice that when $\alpha = 0$, $\mathcal{A}^p_{\mathrm{SU}(2)}(\mathbb{T})^*$ admits the description

$$\left\{t \in \mathbb{C}^{\mathbb{Z}} : \sup_{n=0,1,2,\dots} \left(\frac{1}{n+1}\sum_{j=0}^n |t_{n-2j}|^{p'}\right)^{1/p'} < \infty\right\}.$$

Thus the space is defined in terms of Cesàro summing norm. We suspect such spaces must exist elsewhere in the literature.

Corollary 4.10. *The Beurling algebra $\mathrm{A}(\mathbb{T}, w^{1/p'+\alpha})$, where $w^{1/p'+\alpha}(n) = (1+|n|)^{1/p'+\alpha}$, embeds completely contractively into $\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha}(\mathbb{T})$ with dense range.*

Proof. We simply observe that for t in $\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha}(\mathbb{T})^*$, and each n in $\mathbb{Z}$, we have

$$\|t\|_{\mathrm{A}(\mathbb{T},w^{1/p'+\alpha})^*} = \sup_{n\in\mathbb{Z}} \frac{|t_n|}{(1+|n|)^{1/p'+\alpha}} \leq \|t\|_{\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha}(\mathbb{T})^*}.$$

Hence it follows that the adjoint of the map $\mathrm{A}(\mathbb{T}, w^{1/p'+\alpha}) \to \mathcal{A}^p_{\mathrm{SU}(2),d^\alpha}(\mathbb{T})$ is contractive and injective, so the map itself is contractive with dense range. Moreover, $\mathrm{A}(\mathbb{T}, w^{1/p'+\alpha}) \cong \ell^1(\mathbb{Z}, w^{1/p'+\alpha})$ is a maximal operator space.   □

Remark 4.11. Techniques of the proof of the corollary show that $A_{SU(2),d^\alpha}(\mathbb{T}) = A(\mathbb{T}, w^\alpha)$, completely isomorphically. Hence it follows from (2.2) that for $p \geq 1$, $\mathcal{A}^p_{SU(2),d^\alpha}(\mathbb{T})$ embeds completely contractively into $A(\mathbb{T}, w^\alpha)$. In particular, with completely contractive inclusions we have that

$$A(\mathbb{T}, w^{1/p'+\alpha}) \subseteq \mathcal{A}^p_{SU(2),d^\alpha}(\mathbb{T}) \subseteq A(\mathbb{T}, w^\alpha).$$

We wish to study the operator weak amenability of $\mathcal{A}^p_{SU(2),d^\alpha}(\mathbb{T})$. We first require an analogue of Theorem 3.5. In preparation we require the following, surely well-known, estimate.

Lemma 4.12. *Let* $T : \mathbb{C}^n \to \mathbb{C}^m$ *be linear. Then*

$$\|T\|_{\mathcal{CB}(\ell^p_n, \ell^{p'}_m)} \leq \begin{cases} (nm)^{1/(2p')-1/(2p)} \|T\|_{\mathcal{B}(\ell^2_n, \ell^2_m)} & \text{if } p \geq 2, \\ \|T\|_{\mathcal{B}(\ell^2_n, \ell^2_m)} & \text{if } 1 \leq p < 2. \end{cases}$$

Proof. Since $\mathcal{CB}(\ell^\infty_n, \ell^1_n) \cong \ell^1_n \check{\otimes} \ell^1_n$, we see that $\| \text{id} \|_{\mathcal{CB}(\ell^\infty_n, \ell^1_n)} \leq n$. Then, since $\ell^2_n = [\ell^\infty_n, \ell^1_n]_{1/2}$ we have that $\| \text{id} \|_{\mathcal{CB}(\ell^\infty_n, \ell^2_n)} \leq \| \text{id} \|^{1/2}_{\mathcal{CB}(\ell^\infty_n, \ell^\infty_n)} \| \text{id} \|^{1/2}_{\mathcal{CB}(\ell^\infty_n, \ell^1_n)} \leq n^{1/2}$. Now if $p \geq 2$ we have $\ell^p_n = [\ell^\infty_n, \ell^2_n]_{2/p}$, so

$$\|\text{id}\|_{\mathcal{CB}(\ell^p_n, \ell^2_n)} \leq \|\text{id}\|^{1-2/p}_{\mathcal{CB}(\ell^\infty_n, \ell^2_n)} \|\text{id}\|^{2/p}_{\mathcal{CB}(\ell^2_n, \ell^2_n)} \leq n^{\frac{1}{2}(1-2/p)} = n^{\frac{1}{2p'}-\frac{1}{2p}}.$$

By duality $\|\text{id}\|_{\mathcal{CB}(\ell^2_m, \ell^{p'}_m)} \leq m^{1/(2p')-1/(2p)}$. Finally we have

$$\|T\|_{\mathcal{CB}(\ell^p_n, \ell^{p'}_m)} \leq \|\text{id}\|_{\mathcal{CB}(\ell^2_m, \ell^{p'}_m)} \|T\|_{\mathcal{CB}(\ell^2_n, \ell^2_m)} \|\text{id}\|_{\mathcal{CB}(\ell^p_n, \ell^2_n)} \leq (nm)^{\frac{1}{2p'}-\frac{1}{2p}} \|T\|_{\mathcal{B}(\ell^2_n, \ell^2_m)}$$

where the last inequality is facilitated by the homogeneity of the operator Hilbert space structure.

If $p < 2$, then since $\|\text{id}\|_{\mathcal{CB}(\ell^1_n, \ell^2_n)} = 1$ by virtue of ℓ^1_n being a maximal space, we apply the reasoning above to $\ell^p_n = [\ell^1_n, \ell^2_n]_{2/p'}$, to see that $\|\text{id}\|_{\mathcal{CB}(\ell^p_n, \ell^2_n)} \leq 1$, likewise $\|\text{id}\|_{\mathcal{CB}(\ell^2_m, \ell^{p'}_m)} \leq 1$, and we finish accordingly. □

We let Γ be as in (2.12) with $G = \mathbb{T}$ and then let

$$\mathcal{A}^p_{SU(2),d^\alpha,\Delta}(\mathbb{T}) = \Gamma\left(\mathcal{A}^p_{SU(2),d^\alpha}(\mathbb{T}) \hat{\otimes} \mathcal{A}^p_{SU(2),d^\alpha}(\mathbb{T})\right).$$

Proposition 4.13. *For* $t \in \mathbb{C}^{\mathbb{Z}}$, *we have*

$$\|t\|_{\mathcal{A}^p_{SU(2),d^\alpha,\Delta}(\mathbb{T})^*} \leq \begin{cases} \sup_{n=0,1,2,\ldots} (n+1)^{-1-2\alpha} \max_{k=0,\ldots,n} |t_{-n+2k}| & \text{if } p \geq 2, \\ \sup_{n=0,1,2,\ldots} (n+1)^{-2/p'-2\alpha} \max_{k=0,\ldots,n} |t_{-n+2k}| & \text{if } 1 \leq p < 2. \end{cases}$$

Proof. We let $\mathcal{M}^{\alpha,p'}$ be as in Proposition 4.8. Since $\mathcal{A}^p_{SU(2),d^\alpha,\Delta}(\mathbb{T})$ is a complete quotient of the predual $\mathcal{M}^{\alpha,p'}_*$ which respects the dual pairing (1.1), and since

$$\mathcal{M}^{\alpha,p'}_* \hat{\otimes} \mathcal{M}^{\alpha,p'}_* \cong \ell^1\text{-}\bigoplus_{m,n=0}^{\infty} [(m+1)(n+1)]^{1+1/p'+\alpha} \ell^p_{m+1} \hat{\otimes} \ell^p_{n+1}$$

we see that the space

$$(\mathcal{M}_*^{\alpha,p'} \hat{\otimes} \mathcal{M}_*^{\alpha,p'})^* \cong \ell^\infty\text{-} \bigoplus_{m,n=0}^{\infty} [(m+1)(n+1)]^{-1/p'-\alpha} \ell_{m+1}^{p'} \check{\otimes} \ell_{n+1}^{p'} \cong \mathcal{M}^{\alpha,p'} \bar{\otimes} \mathcal{M}^{\alpha,p'}$$

will contain $\Gamma(\mathcal{A}_{\mathrm{SU}(2),d^\alpha,\Delta}^p(\mathbb{T})^*)$ completely isometrically. In particular, for $t = (t_k)_{k\in\mathbb{Z}}$ in $\mathbb{C}^\mathbb{Z}$, we have $\|t\|_{\mathcal{A}_{\mathrm{SU}(2),d^\alpha,\Delta}^p(\mathbb{T})^*} = \|\Gamma^*(t)\|_{\mathcal{M}^{\alpha,p'} \bar{\otimes} \mathcal{M}^{\alpha,p'}} < \infty$.

Since $\lambda(s) = (s^k)_{k\in\mathbb{Z}}$, for s in $\mathbb{T}$, (3.1) gives us for t in $\mathcal{A}_{\mathrm{SU}(2),d^\alpha,\Delta}^p(\mathbb{T})^*$ that

$$\Gamma^*(t) = (t_k E_k \otimes E_{-k})_{k\in\mathbb{Z}} \subset \mathbb{C}^{\mathbb{Z}\times\mathbb{Z}}$$

where $(E_k)_{k\in\mathbb{Z}}$ is the standard "basis" in $\mathbb{C}^\mathbb{Z}$. The isometric embedding of Proposition 4.8 gives $E_k \mapsto \left(\sum_{j=0}^n \delta_{k,n-2j} e_j\right)_{n=0}^\infty$, where $(e_0,\ldots,e_n)$ is the standard basis for $\ell_{n+1}^{p'}$. Hence we find for t in $\mathcal{A}_{\mathrm{SU}(2),d^\alpha,\Delta}^p(\mathbb{T})^*$ that

$$\Gamma^*(t) \mapsto \left(t_k \left(\sum_{j=0}^m \delta_{k,n-2j} e_j\right) \otimes \left(\sum_{i=0}^n \delta_{-k,m-2i} e_i\right) \right)_{n,m=0}^\infty$$

(4.4)
$$= \left(\sum_{k\in\mathbb{Z}} t_k \varepsilon_{m-k} \otimes \varepsilon_{n+k} \right)_{m,n=0}^\infty \in \mathcal{M}^{\alpha,p'} \bar{\otimes} \mathcal{M}^{\alpha,p'},$$

where

$$\varepsilon_{n\pm k} = \begin{cases} e_{(n\pm k)/2} & \text{if } 2\,|\,n\pm k \text{ and } n \geq |k|, \\ 0 & \text{otherwise.} \end{cases}$$

We now appeal to the fact that $\ell_{m+1}^{p'} \check{\otimes} \ell_{n+1}^{p'} \cong \mathcal{CB}(\ell_{m+1}^p, \ell_{n+1}^{p'})$, and then to Lemma 4.12, to see that the quantity $\|t\|_{\mathcal{A}_{\mathrm{SU}(2),d^\alpha,\Delta}^p(\mathbb{T})^*}$ is dominated in the case $p \geq 2$ by

$$\sup_{\substack{m,n=0,1,2,\ldots \\ 2\,|\,m+n}} [(m+1)(n+1)]^{(-1/p'-\alpha)+(\frac{1}{2p'}-\frac{1}{2p})} \max_{k=0,\ldots,\min\{m,n\}} |t_{-\min\{m,n\}+2k}|,$$

and in the case $p < 2$ by

$$\sup_{\substack{m,n=0,1,2,\ldots \\ 2\,|\,m+n}} [(m+1)(n+1)]^{-1/p'-\alpha} \max_{k=0,\ldots,\min\{m,n\}} |t_{-\min\{m,n\}+2k}|.$$

In either case the supremum is approximated by choices of $m = n$, giving the desired result. $\qquad\square$

We remark that the algebra $\mathcal{A}_{\mathrm{SU}(2),d^\alpha}^p(\mathbb{T})$ may be easily checked to satisfy the assumptions (a)–(e) given at the beginning of Section 3. Our efforts in this section culminate in the following.

Theorem 4.14. *The algebra $\mathcal{A}_{\mathrm{SU}(2),d^\alpha}^p(\mathbb{T})$ is operator weakly amenable if and only if $1 \leq p < \frac{2}{1+2\alpha}$. Furthermore, it is weakly amenable for such p and α.*

Proof. Thanks to Propositions 3.1 and 3.2, we need only to check when $D(f) = \frac{d}{d\theta} f(e^{i\theta})|_{\theta=0}$ defines a bounded element of $\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha,\Delta}(\mathbb{T})^*$. The functional D corresponds to the sequence $d = (k)_{k\in\mathbb{Z}}$ in $\mathbb{C}^{\mathbb{Z}} \cong \mathrm{Trig}(\mathbb{T})^\dagger$. If $p \geq 2$, it is immediate from Proposition 4.13 that $\|d\|_{\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha,\Delta}(\mathbb{T})^*} < \infty$. If $p < 2$, then $\|d\|_{\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha,\Delta}(\mathbb{T})^*} < \infty$ if $(n+1)^{-2/p'-2\alpha}n \leq (n+1)^{2/p-1-2\alpha}$ is uniformly bounded in $n \geq 0$, i.e., if $2/p - 1 - 2\alpha \leq 0$, which means that $p \geq 2/(1+2\alpha)$. In these cases $\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha,\Delta}(\mathbb{T})$ is not operator weakly amenable, hence not weakly amenable.

Thanks to [3], the algebra $\mathrm{A}(\mathbb{T}, w^{1/p'+\alpha}) \cong \ell^1(\mathbb{Z}, w^{1/p'+\alpha})$ is weakly amenable when $1/p' + \alpha < 1/2$, i.e., when $p < 2/(1+2\alpha)$. Hence it follows from Corollary 4.10 that $\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha,\Delta}(\mathbb{T})$ weakly amenable in this case, thus also operator weakly amenable. $\qquad\square$

Let us observe the following trivial consequence of Proposition 4.8, which uses the proof of Theorem 3.9.

Corollary 4.15. *The algebra $\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha}(\mathbb{T})$ admits a bounded point derivation if and only if $\alpha \geq 1$.*

Thus far, we have observed only situations in which $\mathcal{A}^2_{R^q}(G)$ exhibits the same amenability behaviour for all $1 \leq q \leq \infty$. However, the next result will distinguish these from one another.

Proposition 4.16. *For $t \in \mathbb{C}^{\mathbb{Z}}$, we have*

$$\|t\|_{\mathcal{A}^2_{R^q,\mathrm{SU}(2),d^\alpha,\Delta}(\mathbb{T})^*} = \sup_{n=0,1,2,\dots} (n+1)^{-1-2\alpha}\Big(\sum_{k=0}^n |t_{-n+2k}|^{r(q)'} \Big)^{1/r(q)'}.$$

Proof. Let $\mathcal{M}^{\alpha,2}_{q'} = \ell^\infty\text{-}\bigoplus_{n=0}^\infty (n+1)^{-1/2-\alpha} \mathrm{R}^{p'}_{n+1}$, which completely isometrically hosts $\mathcal{A}^2_{R,\mathrm{SU}(2),d^\alpha,\Delta}(\mathbb{T})^*$, by an obvious modification of Proposition 4.8. Then, as in the proof of Proposition 4.13, we are interested in testing the norm of $\Gamma^*(t)$ in

$$\ell^\infty\text{-}\bigoplus_{m,n=0}^\infty [(m+1)(n+1)]^{-1/2-\alpha} \mathrm{R}^{p'}_{m+1} \check{\otimes} \mathrm{R}^{p'}_{n+1} \cong \mathcal{M}^{\alpha,2}_{q'} \bar{\otimes} \mathcal{M}^{\alpha,2}_{q'}.$$

Appealing to the row version of (3.2) we see that $\mathrm{R}^{p'}_{m+1} \check{\otimes} \mathrm{R}^{p'}_{n+1} \cong \mathrm{S}^{r'}_{m+1,n+1}$, where r' is chosen as above. If $2 \mid m+n$, then the anti-diagonal operator $\sum_{k\in\mathbb{Z}} t_k \varepsilon_{m-k} \otimes \varepsilon_{n+k}$, with notation as in the proof of Proposition 4.13, admits norm

$$\Big(\sum_{k=0}^{\min\{m,n\}} |t_{-\min\{m,n\}+2k}|^{r(q)'} \Big)^{1/r(q)'}.$$

The supremum over all m and n is approximated by values where $m = n$. $\qquad\square$

Theorem 4.17. *The algebra $\mathcal{A}^2_{R^q,\mathrm{SU}(2),d^\alpha}(\mathbb{T})$ is operator weakly amenable if and only if $1 \leq \min\{q,q'\} < 2/(4\alpha+1)$.*

Remark 4.18. Notice that the Banach algebra $\mathcal{A}^2_{\mathrm{SU}(2),d^\alpha}(\mathbb{T})$ is never weakly amenable for any $\alpha \geq 0$. Indeed, this would imply that $\mathcal{A}^2_{R^2,\mathrm{SU}(2)}(\mathbb{T})$ is operator weakly

amenable, violating Theorem 4.14. Alternatively, by Remark 4.11, this would imply that $A(\mathbb{T}, w^{1/2})$ is weakly amenable, violating a result of [25].

Proof. In Proposition 4.16 we gained an exact computation. Hence if we examine the derivation D from the proof of Theorem 4.14, and its associated sequence $d = (k)_{k \in \mathbb{Z}}$, we find

$$\|d\|_{\mathcal{A}^2_{R,\mathrm{SU}(2),d^\alpha,\Delta}(\mathbb{T})^*} = \sup_{n=0,1,2,\ldots} (n+1)^{-1-2\alpha} \Big(\sum_{k=0}^n (-n+2k)^{r(q)'} \Big)^{1/r(q)'}.$$

The estimate (3.4) shows that $\big(\sum_{k=0}^n (-n+2k)^{r'} \big)^{1/r'}$ grows as $(n+1)^{1+1/r'}$. Hence $\|d\|_{\mathcal{A}^2_{R,\mathrm{SU}(2),d^\alpha,\Delta}(\mathbb{T})^*} < \infty$ if and only if $1/r(q)' - 2\alpha \le 0$. Since $r(q) = r(q')$, let us suppose that $1 \le q \le 2$. Then we have $1/r(q)' = (2-q)/(2q) \le 2\alpha$ if and only if $q \ge 2/(4\alpha+1)$. This gives the desired result. □

We close this section by addressing operator amenability. We recall, in passing, that it is well known that $A_{\mathrm{SU}(2)}(\mathbb{T}) = A(\mathbb{T})$ is (operator) amenable.

Theorem 4.19. *If $p > 1$, then $\mathcal{A}^p_{\mathrm{SU}(2)}(\mathbb{T})$ is not operator amenable. The algebra $\mathcal{A}^2_{R^q,\mathrm{SU}(2)}(\mathbb{T})$ is never operator amenable for $1 \le q \le \infty$.*

Proof. Let us begin with $\mathcal{A}^p_{\mathrm{SU}(2)}(\mathbb{T})$. In the notation above, we let $\mathcal{M}^{p'} = \mathcal{M}^{0,p'}$ and consider the predual $\mathcal{L}^p = \mathcal{M}^{p'}_*$ which respects the dual pairing (1.1). Let $\mathcal{D} = \Gamma^*(\mathcal{A}^p_{\mathrm{SU}(2),\Delta}(\mathbb{T})^*) \subset \mathcal{M}^{p'} \bar{\otimes} \mathcal{M}^{p'}$. We combine the observations (4.4), the fact that $\ell^{p'}_{m+1} \check{\otimes} \ell^{p'}_{n+1} \cong \mathcal{CB}(\ell^{p'}_{m+1}, \ell^{p'}_{n+1})$ and Lemma 4.12 to see that in the m,nth component of $\mathcal{M}^{p'} \bar{\otimes} \mathcal{M}^{p'}$, each $\Gamma^*(\lambda(s))$ $(s \in \mathbb{T})$ has norm bounded by

$$[(m+1)(n+1)]^{-\frac{1}{p'}+(\frac{1}{2p'}-\frac{1}{2p})} = [(m+1)(n+1)]^{-1/2}$$

for $p > 2$, and by $[(m+1)(n+1)]^{-1/p'}$ for $1 < p < 2$. In other words, $\Gamma^*(\lambda(s)) \in \mathcal{D}_0 = \mathcal{D} \cap \mathcal{C}_0$, where

$$\mathcal{C}_0 = c_0\text{-}\bigoplus_{m,n=0}^\infty [(m+1)(n+1)]^{-1/p'} \ell^{p'}_{m+1} \check{\otimes} \ell^{p'}_{n+1}.$$

Observe that $\mathcal{C}_0{}^* \cong \mathcal{L}^p \hat{\otimes} \mathcal{L}^p$.

Let $\mathcal{K} = \mathcal{D}_\perp$, the pre-annihilator of $\mathcal{D}$ in $\mathcal{L}^p \hat{\otimes} \mathcal{L}^p$. Since $\Gamma^*(\lambda(G))$ is a weak* spanning set for $\mathcal{D}$, we have that $\mathcal{K} = \bigcap_{s \in \mathbb{T}} \ker \Gamma^*(\lambda(s))$. But then $\mathcal{K} = \mathcal{D}_0^\perp$, where $\mathcal{D}_0$ is the closed subspace generated by $\Gamma^*(\lambda(G))$ in $\mathcal{C}_0$. Collecting all of these facts together, we see that $\mathcal{A}^p_{\mathrm{SU}(2),\Delta}(\mathbb{T}) \cong \mathcal{L}^p \hat{\otimes} \mathcal{L}^p / \mathcal{K}$ is the dual of $\mathcal{D}_0$.

Now by Proposition 3.3, if $\mathcal{A}^p_{\mathrm{SU}(2)}(\mathbb{T})$ were operator amenable, then $\mathrm{I}_{\mathcal{A}^p_{\mathrm{SU}(2),\Delta}}(1)$ would admit a bounded approximate identity. However, since each evaluation functional $\Gamma^*(\lambda(s))$ is weak*-continuous, this bounded net would admit the indicator function of $\mathbb{T} \setminus \{1\}$ as a weak*-cluster point, which is absurd.

Now we consider $\mathcal{A}^2_{R^q,\mathrm{SU}(2)}(\mathbb{T})$. Let $\mathcal{M}^2_{q'} = \ell^\infty\text{-}\bigoplus_{n=0}^\infty (n+1)^{-1/2} \mathrm{R}^{q'}_{n+1}$ and $\mathcal{L}^2_q = (\mathcal{M}^2_{q'})_*$ be the predual which respects the dual pairing (1.1). In $\mathcal{M}^2_{q'} \bar{\otimes} \mathcal{M}^2_{q'}$,

the (m, n)-th component of $\Gamma^*(\lambda(s))$ $(s \in \mathbb{T})$ is an anti-diagonal matrix, thanks to (4.4), with entries bounded in modulus by 1. Hence it has norm bounded by

$$[(m+1)(n+1)]^{-1/2}[\min\{m, n\} + 1]^{1/r(q)'} \leq [(m+1)(n+1)]^{-1/2+1/(2r(q)')}$$

since we can identify that component with a weighted version of $S^{r(q)'}_{m+1,n+1}$, by virtue of (3.2). However, we always have that $1/(2r(q)') = |q-2|/(4q) \leq 1/4 < 1/2$.

Just as above, the subspace $\mathcal{K} = \bigcap_{s \in \mathbb{T}} \ker \Gamma^*(\lambda(s)) \subset \mathcal{L}_q^2 \hat{\otimes} \mathcal{L}_q^2$ serves to allow $\mathcal{A}^2_{R^q, \mathrm{SU}(2)}(\mathbb{T})$ to be viewed as the dual of a space containing each $\Gamma^*(\lambda(s))$. We conclude, as above. $\qquad\square$

5. Arens regularity and representability as an operator algebra

We say that a (commutative) Banach algebra $\mathcal{B}$ is *Arens regular* if for any Φ and Ψ in $\mathcal{B}^{**}$, and any bounded nets (u_i) and (v_j) from $\mathcal{B}$ for which $\Phi = \lim_i u_i$ and $\Psi = \lim_j v_j$, we have that both of the weak* iterated limits $\lim_i \lim_j u_i v_j$ and $\lim_j \lim_i u_i v_j$ exist in $\mathcal{B}^{**}$ and coincide. Arens regularity passes to closed subalgebras and isomorphic quotients of $\mathcal{B}$. Arens regularity is a sufficient condition to see that $\mathcal{B}$ is isomorphic to a closed subalgebra of bounded operators on a reflexive Banach space (see [63]). It is a consequence of the main result of [12] that if $\mathcal{B}$ is isomorphic to a closed subalgebra of bounded operator on a superreflexive Banach space, then $\mathcal{B}$ is Arens regular.

We say $\mathcal{B}$ is *representable as an operator algebra*, if it is isomorphic to a closed subalgebra of operators on a Hilbert space. This property implies Arens regularity. The main result of [4] tells us that $\mathcal{B}$ is representable as an operator algebra provided that it admits an operator space structure with respect to which the multiplication on $\mathcal{B} \otimes \mathcal{B}$ extends to a completely bounded map on the Haagerup tensor product $\mathcal{B} \otimes^h \mathcal{B}$. If $\mathcal{B}$ is already equipped with an operator space structure, we will say it is *completely representable as an operator algebra* provided there is a complete isomorphism between $\mathcal{B}$ and an operator algebra; equivalently, if multiplication factors through $\mathcal{B} \otimes^h \mathcal{B}$.

Let us remark that for the weights $w^\alpha(n) = (1 + |n|)^\alpha$ on $\mathbb{Z} \cong \widehat{\mathbb{T}}$, we have that $\mathrm{A}(\mathbb{T}, w^\alpha)$ is Arens regular exactly when $\alpha > 0$ thanks to [9], [62], and is representable as a Q-algebra (a certain type of operator algebra) exactly when $\alpha > 1/2$, thanks to [58]. As a maximal operator space, $\mathrm{A}(\mathbb{T}, w^\alpha)$ is completely representable an operator algebra exactly when $\alpha > 1/2$, thanks to [24]. A study of Arens regularity of algebras $\mathrm{A}(G, \omega)$ is conducted in [37], and of when $\mathrm{A}(G, \omega)$ is completely representable an operator algebra is conducted in [24].

5.1. Arens regularity

Let us begin by showing that for an infinite G, the algebras $\mathcal{A}^p(G)$ are never Arens regular. We require a supporting result which is of independent interest and was observed in [22], Corollary 2.3, in the case $p = 2$.

Proposition 5.1. *Let* $\mathrm{Z}\mathcal{A}^p(G) = \{u \in \mathcal{A}^p(G) : u(tst^{-1}) = u(s) \text{ for all } s, t \text{ in } G\}$. *Then* $\mathrm{Z}\mathcal{A}^p(G) = \mathrm{ZA}(G)$ *completely isometrically for each* $p \geq 1$.

Proof. We note that for each u in $\mathrm{Z}\mathcal{A}^p(G)$ we have $\hat{u}(\pi) = \frac{1}{d_\pi}\big(\int_G u(s)\overline{\chi_\pi(s)}\,ds\big)I_{d_\pi}$. Moreover, for each scalar matrix, $\|\alpha I_d\|_{\mathrm{S}_d^p} = d^{1/p}|\alpha|$. Hence each space $\mathrm{Z}\mathcal{A}^p(G)$ is completely isometrically isomorphic to $\ell^1(\widehat{G})$. $\qquad\qquad\square$

Theorem 5.2. *The algebra* $\mathcal{A}^p(G)$ *is Arens regular only if* G *is finite.*

Proof. If $\mathcal{A}^p(G)$ is Arens regular, then so too must be the subalgebra $\mathrm{ZA}(G)$ and also its ideal $\mathrm{I}_{\mathrm{ZA}}(e) = \{u \in \mathrm{ZA}(G) : u(e) = 0\}$. It is a well-known consequence of the Schur property for $\mathrm{ZA}(G) \cong \ell^1(\widehat{G})$ that it is weakly sequentially complete, hence so too is the co-dimension one space $\mathrm{I}_{\mathrm{ZA}}(e)$. Furthermore, $\mathrm{ZA}(G)$ is densely spanned by elements whose multiplication operators on $\mathrm{ZA}(G)$ are finite rank. Hence $\mathrm{ZA}(G)$, and its ideal $\mathrm{I}_{\mathrm{ZA}}(e)$, are completely continuous Banach algebras.

By Proposition 3.7 in [18] or Theorem 1.5 in [19], $\mathrm{I}_{\mathrm{A}}(e)$ admits a bounded approximate identity (u_α). Let $P \colon \mathrm{A}(G) \to \mathrm{ZA}(G)$ be given by $Pu(s) = \int_G u(tst^{-1})dt$, which is a surjective contraction with expectation property $P(uv) = P(u)v$ for u in $\mathrm{A}(G)$ and v in $\mathrm{ZA}(G)$. Then (Pu_α) is a bounded approximate identity for $\mathrm{I}_{\mathrm{ZA}}(e)$, as is straightforward to check. Being semisimple, weakly sequentially complete, completely continuous and admitting a bounded approximate identity, Corollary 2.4 in [57] tells us that $\mathrm{I}_{\mathrm{ZA}}(e)$ is a reflexive Banach space. Hence (Pu_α) admits a weak cluster point, which, by regularity of $\mathrm{ZA}(G)$ on its spectrum (the space of conjugacy classes of G, see [1], for example), is necessarily the indicator function of $G \setminus \{e\}$. This implies that G is discrete, hence finite. $\qquad\square$

Let us note a condition which implies Arens regularity.

Theorem 5.3. *Suppose the weight* ω *on* $\widehat{G}$ *satisfies*

$$\lim_{\pi\to\infty} \limsup_{\pi'\to\infty} \frac{\max_{\sigma\subset\pi\otimes\pi'}\omega(\sigma)}{\omega(\pi)\omega(\pi')} = 0 \quad and \quad \lim_{\pi'\to\infty} \limsup_{\pi\to\infty} \frac{\max_{\sigma\subset\pi\otimes\pi'}\omega(\sigma)}{\omega(\pi)\omega(\pi')} = 0.$$

Then $\mathcal{A}^p(G, \omega)$ *is Arens regular for any* $p \geq 1$.

Proof. This is an easy modification of the proof of Theorem 3.16 in [37]. Indeed it is easy to verify that

$$\mathcal{A}^p(G,\omega)^{**} \cong \mathcal{A}^p(G,\omega) \bigoplus \Big(c_0\text{-}\bigoplus_{\pi\in\widehat{G}} \frac{d^{-1/p'}}{\omega(\pi)}\,\mathrm{S}_{d_\pi}^{p'}\Big)^\perp.$$

All other aspects of the proof are similarly straightforward to modify. $\qquad\square$

A function $\tau \colon \widehat{G} \to \mathbb{R}^{\geq 0}$ is called *subadditive* if $\tau(\sigma) \leq \tau(\pi) + \tau(\pi')$ whenever $\sigma \subset \pi \otimes \pi'$. Let for $\alpha > 0$, $\omega_\tau^\alpha(\pi) = (1 + \tau(\pi))^\alpha$. It is easy to verify ω_τ is a weight.

Corollary 5.4. *Let* $\tau \colon \widehat{G} \to \mathbb{R}^{\geq 0}$ *be subadditive and satisfy* $\lim_{\pi\to\infty} \tau(\pi) = \infty$. *Then* $\mathcal{A}^p(G, \omega_\tau^\alpha)$ *is Arens regular for any* $p \geq 1$ *and* $\alpha > 0$.

Proof. Note that of $\sigma \subset \pi \otimes \pi'$ then

$$\omega_\tau^\alpha(\sigma) \leq (1 + \tau(\pi) + \tau(\pi'))^\alpha \leq (1 + 2\tau(\pi))^\alpha + (1 + 2\tau(\pi'))^\alpha \leq 2^\alpha[\omega_\tau^\alpha(\pi) + \omega_\tau^\alpha(\pi')].$$

Hence

$$\frac{\max_{\sigma \subset \pi \otimes \pi'} \omega_\tau^\alpha(\sigma)}{\omega_\tau^\alpha(\pi)\,\omega_\tau^\alpha(\pi')} \leq \frac{2^\alpha}{\omega_\tau^\alpha(\pi)} + \frac{2^\alpha}{\omega_\tau^\alpha(\pi')}.$$

Our assumption on τ assures that either iterated limit of Theorem 5.3 is zero. $\square$

Example 5.5. (i) Let $\tau \colon \widehat{G} \to \mathbb{R}^{>0}$ be given by $\tau(\pi) = \log d_\pi$, which is subadditive. The weight ω_τ^α, given here, is thus the weight ω_{w^α} of Example 2.7. If G is tall, then $\lim_{\pi \to \infty} \tau(\pi) = \infty$.

(ii) For an infinite Lie group G, consider the polynomial weights $\omega_S^\alpha(\pi) = (1 + \tau_S(\pi))^\alpha$, introduced in Section 2.2. Since each $S^{\otimes n}$ is finite, $\lim_{\pi \to \infty} \tau_S(\pi) = \infty$.

For the special unitary groups, a special feature of the dual allows us to deal with dimension weights.

Corollary 5.6. *The algebra* $\mathcal{A}^p(\mathrm{SU}(n), d^\alpha)$ *is Arens regular for any* $\alpha > 0$.

Proof. We parameterize $\widehat{\mathrm{SU}}(n)$ by dominant weights: each $\pi = \pi_\lambda$ where $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ with $\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0$ in $\mathbb{Z}$. Then Corollary 1.2 in [8], in whose notation $\lambda_n = \mu_n = 0$, shows that

$$\frac{d_{\pi_\nu}^\alpha}{d_{\pi_\lambda}^\alpha \, d_{\pi_\mu}^\alpha} \leq C^\alpha \Big(\frac{1}{\lambda_1 + 1} + \frac{1}{\mu_1 + 1}\Big)^\alpha \leq (2C)^\alpha \Big(\frac{1}{(\lambda_1 + 1)^\alpha} + \frac{1}{(\mu_1 + 1)^\alpha}\Big)$$

for $\pi_\nu \subset \pi_\lambda \otimes \pi_\mu$, where the constant C depends only on n. Either iterated limit of Theorem 5.3 is zero. (We remark that the choice of generators corresponding to dominant weights $\{(1, 0, \ldots, 0), (1, 1, 0, \ldots, 0), \ldots, (1, \ldots, 1, 0)\}$ is such that $\tau_S(\pi_\lambda) = \lambda_1$ for $\lambda \neq (0, \ldots, 0)$. See [24], (3.8). Hence the present result may also be viewed as following from Corollary 5.4.) $\square$

We say a compact connected group G is *semisimple* if the commutator subgroup $[G, G]$ equals G. For non-semisimple connected groups, we never have Arens regularity with dimension weights. This generalizes Theorem 4.8 in [24].

Theorem 5.7. *Let G be a non-semisimple connected group and ω be a weakly dimension weight. Then $\mathcal{A}^p(G, \omega)$ is not Arens regular.*

Proof. Thanks to 9.19 and 9.24 in [28], $G = (T \times S)/D$ where T is a non-trivial connected abelian group, and $S = \prod_{i \in I} S_i$ is a product of simple connected Lie groups, and D is totally disconnected. Then $N = DS/D$ is a closed normal subgroup of G and the third isomorphism theorem tells us that $G/N \cong (T \times S)/SD \cong T/T \cap D$. Since $T \cap D$ is totally disconnected, we see that G/N is an infinite abelian group. Thus by Lemma 3.14, $\mathcal{A}^p(G/N) = \mathcal{A}^p(G/N, \omega)$ is isomorphic to a closed subalgebra of $\mathcal{A}^p(G, \omega)$. Proposition 2.9 shows that $\mathcal{A}^p(G/N) = \mathrm{A}(G/N) \cong \ell^1(\widehat{G/N})$, which is never Arens regular by [62].

Let us observe that we have an interesting alternate proof if G is Lie. Indeed, in the Levi–Mal'cev decomposition, $G = (T \times S)/D$ above, we may assume again that T is a non-trivial abelian connected Lie group, S is a semisimple Lie group, and $D = T \cap S$, so $T \cap D = \{e\} = D \cap S$ and D is finite. See Theorem 6.15 in [28]. But then Theorem 4.3 shows that $\mathcal{A}^p(T, \omega) = A(T)$ is an isomorphic quotient of $\mathcal{A}^p(G, \omega)$. As above, this quotient algebra cannot be Arens regular. $\qquad\square$

5.2. Representability as an operator algebra

Let us look at some situations in which $\mathcal{A}^p(G, \omega)$ is representable, in fact completely representable, as an operator algebra. We will focus mainly on the polynomial weights ω_S^α, defined in Section 2.2. We do not know if our estimates are sharp in the ranges of p and α.

We will first develop an analogue of the Littlewood multipliers of [24]. We let $(d_\sigma)_{\sigma \in \Sigma}$ be an indexed collection of positive integers. We then define

$$\mathcal{D}^p = \ell^\infty\text{-}\bigoplus_{\sigma \in \Sigma} d_\sigma^{-1/p'} \, \mathrm{S}_{d_\sigma}^{p'}, \quad \mathcal{H}^p = \ell^2\text{-}\bigoplus_{\sigma \in \Sigma} d_\sigma^{1/2-1/p'} \mathrm{S}_{d_\sigma}^2 \quad \text{and} \quad \mathcal{B}^p = \ell^1\text{-}\bigoplus_{\sigma \in \Sigma} d_\sigma^{1+1/p'} \mathrm{S}_{d_\sigma}^p.$$

We observe that $\mathcal{B}^{p*} = \mathcal{D}^p$ with respect to the dual pairing

$$\langle (A_\sigma)_{\sigma \in \Sigma}, (D_\sigma)_{\sigma \in \Sigma} \rangle = \sum_{\sigma \in \Sigma} d_\sigma \mathrm{Tr}(A_\sigma D_\sigma).$$

Furthermore, with respect to this dual pairing we have linear dual

$$\mathcal{H}^{p*} = \ell^2\text{-}\bigoplus_{\sigma \in \Sigma} d_\sigma^{1/2+1/p'} \, \mathrm{S}_{d_\sigma}^2.$$

This being an ℓ^2-direct sum, we have norm $\|X\|_{\mathcal{H}^{p*}} = \left(\sum_{\sigma \in \Sigma} d_\sigma^{1+2/p'} \|X_\sigma\|_{\mathrm{S}_{d_\sigma}^2}^2 \right)^{1/2}$. Of course, each of $\mathcal{H}^p$ and $\mathcal{H}^{p*}$ are also Hilbert spaces with the map $U(H_\sigma)_{\sigma \in \Sigma} = (d_\sigma^{-1} H_\sigma)_{\sigma \in \Sigma}$ serving as a unitary between them. Since $U : \mathcal{H}_C^p \to \mathcal{H}_C^{p*}$ is a complete isometry, we still obtain operator duality $(\mathcal{H}_C^p)^* \cong \mathcal{H}_R^{p*}$, and the same with row and column structures interchanged.

The spaces $\mathcal{D}^p$ and $\mathcal{B}^p$ will have their usual operator spaces structures. However, we write $\mathcal{D}_C^2$, $\mathcal{B}_C^2$ when the component spaces $\mathrm{S}_{d_\sigma}^2$ have column structure; likewise for rows. For any $A = (A_\sigma)_{\sigma \in \Sigma}$ and $B = (B_\sigma)_{\sigma \in \Sigma}$ in $\prod_{\sigma \in \Sigma} \mathrm{M}_n$, we let $AB = (A_\sigma B_\sigma)_{\sigma \in \Sigma}$.

Proposition 5.8. (i) *The formal identities $\mathcal{H}_C^p \hookrightarrow \mathcal{D}^p$ and $\mathcal{H}_R^p \hookrightarrow \mathcal{D}^p$ are normal complete contractions.*

(ii) *Given X in $\mathcal{H}^{p*}$, the maps $D \mapsto DX$ and $D \mapsto XD$ from $\mathcal{D}^p$ into $\mathcal{H}_C^p$, or equivalently into $\mathcal{H}_R^p$, are normal and completely bounded.*

(ii') *Given X in $\mathcal{H}^{2*}$, the maps $D \mapsto DX$ and $D \mapsto XD$ from $\mathcal{D}_C^2$ or from $\mathcal{D}_R^2$ into either of $\mathcal{H}_C^p$ or $\mathcal{H}_R^p$, are normal and completely bounded.*

Proof. (i) It suffices the show that $\mathcal{B}^p \hookrightarrow \mathcal{H}_C^{p*}$ completely contractively, and then the desired inclusion is the adjoint map. The roles of rows and columns may

be interchanged naturally in all of our manipulations. We first recall that the identity $d^{1/2}\mathrm{C}_d \hookrightarrow \mathrm{R}_d$ is a complete contraction. Thus by interpolation we find that $d^{\frac{1}{2p}}\mathrm{C}_d^p \hookrightarrow \mathrm{C}_d$ is a complete contraction. Hence we get a complete contraction

$$d^{1/2}\mathrm{S}_d^p = d^{1/2}\mathrm{C}_d^p \otimes^h \mathrm{R}_d^p = d^{\frac{1}{2p}}\mathrm{C}_d^p \otimes^h d^{\frac{1}{2p'}}\mathrm{C}_d^{p'} \hookrightarrow \mathrm{C}_d \otimes^h \mathrm{C}_d = \mathrm{S}_{d,C}^2.$$

Thus we obtain a complete contraction from each summand $d_\sigma^{1+1/p'}\mathrm{S}_{d_\sigma}^p$ of $\mathcal{B}^p$ into the summand $d_\sigma^{1/2+1/p'}\mathrm{S}_{d_\sigma,C}^2$ of $\mathcal{H}^{p*}$. By the universal property of direct sums, we are done.

(ii) Given X in $\mathcal{H}^{p*}$, the map $D \mapsto DX \colon \mathcal{D}^p \to \mathcal{H}_C^p$ will be the adjoint of the map $H \mapsto XH \colon \mathcal{H}_R^{p*} \to \mathcal{B}^p$, once we establish that the latter is well-defined. To see that this latter map is well-defined, even completely bounded, is sufficient that $X \otimes H \mapsto XH$ extends to a completely bounded map from $\mathcal{H}_C^{p*} \hat{\otimes} \mathcal{H}_R^{p*}$ to $\mathcal{B}^p$. Let us first observe two facts. First, matrix multiplication $\mathrm{S}_{d,C}^2 \hat{\otimes} \mathrm{S}_{d,R}^2 \to \mathrm{S}_d^1$ is completely contractive. Indeed, this is akin to applying trace to the middle factor of $\mathrm{C}_d \hat{\otimes} \mathrm{S}_d^1 \hat{\otimes} \mathrm{R}_d = \mathrm{C}_d \hat{\otimes} \mathrm{C}_d \hat{\otimes} \mathrm{R}_d \hat{\otimes} \mathrm{R}_d$. Second, we see by interpolation that $d^{\frac{1}{2p'}}\mathrm{C}_p \hookrightarrow \mathrm{C}_d^p$ is a complete contraction. Hence, similarly as in (i), above, we see that $d^{1/p'}\mathrm{S}_d^1 \hookrightarrow \mathrm{S}_d^p$ is completely contractive. Now let us proceed to our multiplication computation. Multiplication may be realized as factoring through the following complete contractions:

$$\mathcal{H}_C^{p*} \hat{\otimes} \mathcal{H}_R^{p*} = \mathrm{S}^1(\mathcal{H}^{p*}) \to \ell^1\text{-}\bigoplus_{\sigma \in \Sigma} \mathrm{S}^1(d^{1/2+1/p'}\,\mathrm{S}_{d_\sigma}^2) = \ell^1\text{-}\bigoplus_{\sigma \in \Sigma} d^{1+2/p'}\,\mathrm{S}_{d_\sigma,C}^2 \hat{\otimes} \mathrm{S}_{d_\sigma,R}^2$$

$$\to \ell^1\text{-}\bigoplus_{\sigma \in \Sigma} d^{1+2/p'}\,\mathrm{S}_{d_\sigma}^1 \to \ell^1\text{-}\bigoplus_{\sigma \in \Sigma} d^{1+1/p'}\,\mathrm{S}_{d_\sigma}^p = \mathcal{B}^p,$$

where the map in the first line is block-diagonal compression, and the two maps in the second line are discussed above.

The commutativity of the projective tensor product, and general symmetry of row versus column operations allows us to switch the order of the computations above rather liberally.

(ii') With considerations so far, it is straightforward to see, for example, that $d^{1/2}\mathrm{S}_d^1 \hookrightarrow \mathrm{S}_{d,C}^2$ is completely contractive. We require this fact exactly at the last step of the multiplication computation, above. $\square$

Our special Littlewood type multipliers are the content of the next theorem. The extended (or weak*) Haagerup tensor product $\otimes^{eh}$ is defined in [6], [14]. It is desirable for us specifically because of the completely isometric duality formula $(\mathcal{V} \otimes^h \mathcal{W})^* \cong \mathcal{V}^* \otimes^{eh} \mathcal{W}^*$ for any operator spaces $\mathcal{V}$ and $\mathcal{W}$.

Theorem 5.9. *Fix T in $\mathcal{H}^{p*}$. Then each of the maps given by $A \otimes B \mapsto TA \otimes B$ $A \otimes B \mapsto A \otimes TB$ extends uniquely to normal linear maps from $\mathcal{D}^p \bar{\otimes} \mathcal{D}^p$ to $\mathcal{D}^p \otimes^{eh} \mathcal{D}^p$. In the case $p = 2$, each of these maps extends uniquely to normal linear maps from $\mathcal{D}_E^2 \bar{\otimes} \mathcal{D}_E^2$ to $\mathcal{D}_E^2 \otimes^{eh} \mathcal{D}_E^2$, where $E = C$ or R.*

Proof. Let us first observe that $\mathcal{H}_C^p \bar{\otimes} \mathcal{D}^p = \mathcal{H}_C^p \otimes^{eh} \mathcal{D}^p$, completely isometrically. Indeed, each space is the dual of $\mathcal{H}_R^{p*} \hat{\otimes} \mathcal{B}^p = \mathcal{H}_R^{p*} \otimes^h \mathcal{B}^p$. Then Proposition 5.8 (ii) shows that $A \otimes B \mapsto TA \otimes B$ extends uniquely to a normal linear map from $\mathcal{D}^p \bar{\otimes} \mathcal{D}^p$ into $\mathcal{H}_C^p \bar{\otimes} \mathcal{D}^p = \mathcal{H}_C^p \otimes^{eh} \mathcal{D}^p$. Then Proposition 5.8 (i) shows that the formal identity on elementary tensors, extends uniquely to a normal complete contraction $\mathcal{H}_C^p \otimes^{eh} \mathcal{D}^p \longrightarrow \mathcal{D}^p \otimes^{eh} \mathcal{D}^p$. The composition of these maps yields the desired result. The right handed case case works similarly.

The proof in the case $p = 2$ is identical. Here we use (ii') in place of (ii), from the prior proposition; and the complete contractivity of the map $\mathcal{B}_E^2 \hookrightarrow \mathcal{H}_E^{2*}$, $E = R$ or C, is obvious. At this particular step, however, we may use only rows or columns. $\qquad\square$

Thus, for elements T of $\mathcal{H}^{p*}$, we may think of $T \otimes I$ and $I \otimes T$ in $\prod_{\sigma,\sigma' \in \Sigma \times \Sigma} \mathrm{M}_{d_\sigma} \otimes \mathrm{M}_{d_{\sigma'}}$, as elements which multiply $\mathcal{D}^p \bar{\otimes} \mathcal{D}^p$ into $\mathcal{D}^p \otimes^{eh} \mathcal{D}^p$.

With our new Littlewood type multipliers in hand, we are almost in position to determine some occasions for which $\mathcal{A}^p(G, \omega)$ is an operator algebra. We will work only with connected Lie groups and polynomial weights, as defined at the end of the previous section.

As in [59], 5.6.5, a connected Lie group G admits a system of fundamental weights $\Lambda_1, \dots, \Lambda_{s(G)}$ which are proportional to the simple roots of G, and $\lambda_1, \dots, \lambda_{z(G)}$ which are related to the characters of the connected component of the centre of G. Each π in $\widehat{G}$ corresponds to some $\Lambda_\pi = \sum_{j=1}^{s(G)} m_j \Lambda_j + \sum_{i=1}^{z(G)} n_i \lambda_i$ where each m_j is a non-negative integer and each n_i is an integer. In this notation we let

$$\|\pi\|_p = \Big(\sum_{j=1}^{s(G)} m_j^p + \sum_{i=1}^{z(G)} |n_i|^p \Big)^{1/p}.$$

Clearly $\|\pi\|_\infty \leq \|\pi\|_2 \leq \|\pi\|_1 \leq (s(G) + z(G)) \|\pi\|_\infty$. The following estimate is a refinement of 5.6.7 in [59]. For $r > 2$, it will allow us better estimates by using more refined data about G

Lemma 5.10. *We have for G as above and any positive real number r, that*

$$\sum_{\pi \in \widehat{G}} \frac{d_\pi^r}{(1 + \|\pi\|_1)^{2\alpha}} \quad \text{converges if} \quad \alpha > \frac{r}{4} d(G) - \frac{r-2}{4}(s(G) + z(G)),$$

where $d(G)$ is the dimension of G.

Proof. We simply adapt the proof of 5.6.7 in [59]. There, it is first established that for some constant C we have $d_\pi \leq C \|\pi\|_2^{\frac{1}{2}(d(G) - (s(G) + z(G)))}$, where of course we can replace $\|\cdot\|_2$ with $\|\cdot\|_1$ at the cost of a new constant, which we shall again denote C.

Now we let $\widehat{G}_j = \{\pi \in \widehat{G} : \|\pi\|_\infty = j\}$, for which the cardinality is estimated by

$$|\widehat{G}_j| \leq (s(G) + 2z(G))(2j + 1)^{s(G)+z(G)-1} \leq K(j + 1)^{s(G)+z(G)-1}$$

where $K = (s(G) + 2z(G))2^{s(G)+z(G)-1}$ is independent of j.

Moreover, for π in $\widehat{G}_j$ we use the estimate of the prior paragraph and the fact that $\|\pi\|_1 \leq \|\pi\|_\infty = j$ to see that

$$\frac{d_\pi^r}{(1+\|\pi\|_1)^{2\alpha}} \leq \frac{C}{(1+j)^{2\alpha - \frac{r}{2}d(G) + \frac{r}{2}(s(G)+z(G))}}.$$

Thus we see that

$$\sum_{\pi \in \widehat{G}} \frac{d_\pi^r}{(1+\|\pi\|_1)^{2\alpha}} = \sum_{j=0}^{\infty} \sum_{\pi \in \widehat{G}_j} \frac{d_\pi^r}{(1+\|\pi\|_1)^{2\alpha}}$$

$$\leq \sum_{j=0}^{\infty} K(j+1)^{s(G)+z(G)-1} \frac{C}{(1+j)^{2\alpha - \frac{r}{2}d(G) + \frac{r}{2}(s(G)+z(G))}},$$

which converges provided $2\alpha > \frac{r}{2}d(G) - \left(\frac{r}{2} - 1\right)(s(G) + z(G))$. $\qquad\square$

Let us recall how to relate the quantity $\|\cdot\|_1$ to our polynomial weight, introduced in Section 2.2. As observed in p. 483 and Theorem 5.4 of [38], there are constants c, C for which

(5.1) $$c\,\tau_S(\pi) \leq \|\pi\|_1 \leq C\,\tau_S(\pi).$$

The following generalizes Theorem 4.5 in [24], which deals only with the case $p=1$.

Theorem 5.11. *Let G be a connected Lie group with polynomial weight ω_S^α. If*

$$\alpha > \left(\frac{1}{2} + \frac{1}{2p'}\right) d(G) - \frac{1}{2p'}(s(G) + z(G)),$$

then $\mathcal{A}^p(G, \omega_S^\alpha)$ is completely representable as an operator algebra.

Proof. We will summarize those details of the proof of Theorem 4.5 in [24] which need adapting. We recall that with the coproduct M, and $W = (\omega(\pi)I_{d_\pi})_{\pi \in \widehat{G}}$, we have that $T = M(W)(W^{-1} \otimes W^{-1})$ in $\mathcal{A}^p(G)^* \bar{\otimes} \mathcal{A}^p(G)^*$ satisfies

$$T_{\pi,\pi'} = \left(\left(\frac{1 + \tau_S(\sigma)}{(1 + \tau_S(\pi))(1 + \tau_S(\pi'))}\right)^\alpha I_{d_\sigma}\right)_{\sigma \subset \pi \otimes \pi'}$$

in the sense of the notation (2.5). Further, just as in the proof of Corollary 5.4, we gain an estimate

$$\left(\frac{1 + \tau_S(\sigma)}{(1 + \tau_S(\pi))(1 + \tau_S(\pi'))}\right)^\alpha \leq 2^\alpha \left(\frac{1}{(1 + \tau_S(\pi))^\alpha} + \frac{1}{(1 + \tau_S(\pi'))^\alpha}\right).$$

We then let T_1 and T_2 be given by

$$T_1 = \left(\frac{1}{(1 + \tau_S(\pi))^\alpha} I_{d_\pi}\right)_{\pi \in \widehat{G}} \quad \text{and} \quad T_2 = \left(\frac{1}{(1 + \tau_S(\pi))^\alpha} I_{d_{\pi'}}\right)_{\pi' \in \widehat{G}}.$$

Then in $\prod_{\pi,\pi'\in\widehat{G}\times\widehat{G}} \mathrm{M}_{d_\pi} \otimes \mathrm{M}_{d_{\pi'}}$ we can write

$$T = S(T_1 \otimes I + I \otimes T_2),$$

where S is diagonal on each block $\mathrm{M}_{d_\pi} \otimes \mathrm{M}_{d_{\pi'}}$ with scalars bounded by 2^α, i.e., $S \in \ell^\infty\text{-}\bigoplus_{\pi,\pi'\in\widehat{G}\times\widehat{G}} \mathrm{S}^\infty_{d_\pi} \check\otimes \mathrm{S}^\infty_{d_{\pi'}}$. It is easy to check that such S acts as a multiplier of $\mathcal{A}^p(G)^* \bar\otimes \mathcal{A}^p(G)^*$.

Letting $\mathcal{H}^{p*}$ denote the multiplier space from Theorem 5.9, it now suffices to find conditions for which $\|T_j\|_{\mathcal{H}^{p*}} < \infty$ for $j = 1, 2$. Now we have for $j = 1, 2$ that

$$\|T_j\|^2_{\mathcal{H}^{p*}} \leq (1+C)^{2\alpha} \sum_{\pi\in\widehat{G}} d_\pi^{1+2/p'} \frac{d_\pi}{(1+\|\pi\|_1)^{2\alpha}}.$$

thanks to (5.1). We then appeal to Lemma 5.10 with the value $r = 2 + 2/p'$ to find values α for which the last series converges.

We have seen that $M(W)(W^{-1} \otimes W^{-1})(\mathcal{A}^p(G)^* \bar\otimes \mathcal{A}^p(G)^*) \subset \mathcal{A}^p(G)^* \otimes^{eh} \mathcal{A}^p(G)^*$. By duality, this is sufficient to see that multiplication on $\mathcal{A}^p(G, \omega_S^\alpha)$ factors completely boundedly through the Haagerup tensor product. Then we appeal to the main result of [4]. $\qquad\square$

Corollary 5.12. *Let G be a connected Lie group. If $\alpha > \frac{3}{4}d(G) - \frac{1}{4}(s(G) + z(G))$, then for any $1 \leq q \leq \infty$ we have that $\mathcal{A}^2_{R^q}(G, \omega_S^\alpha)$ is completely representable as an operator algebra.*

Proof. For $q = 1, \infty$, the proof of Theorem 5.11 can be used to establish that for α as above, $\mathcal{A}^2_E(G, \omega_S^\alpha)$ is an operator algebra for $E = R = R^\infty$ or $C = R^1$. We use a similar formula to (2.9) to obtain that

$$\mathcal{A}^2_{R^q}(G, \omega_S^\alpha) = \left[\mathcal{A}^2_R(G, \omega_S^\alpha), \mathcal{A}^2_C(G, \omega_S^\alpha)\right]_{1/q}.$$

By 2.3.7 in [5], this interpolated algebra is completely isomorphic to an operator algebra. $\qquad\square$

Let us close by returning to dimension weights, although only for special unitary groups.

Corollary 5.13. *If $\alpha > \left(\frac{1}{2} + \frac{1}{2p'}\right)(n^2 - 1) - \frac{1}{2p'}(n - 1)$, then $\mathcal{A}^p(\mathrm{SU}(n), d^\alpha)$ is completely representable as an operator algebra.*

Proof. We use notation and comments in the proof of Corollary 5.6. We have that $\lambda_1 = \|\pi_\lambda\|_1$ in this particular case. The main estimate of that proof hence carries over, whence so too does the proof of Theorem 5.11 and the last corollary. $\qquad\square$

We remark that $\mathrm{A}(\mathrm{SU}(n), d^\alpha)$ is known not to be completely isomorphic to an operator algebra when $\alpha \leq 1/2$ ([24], 4.11). With the difficulty of the restriction result to tori, even for $n = 2$, we have no such result for $\mathcal{A}^p(\mathrm{SU}(n), d^\alpha)$ with $p \neq 1$, at present.

6. Summary of results

6.1. The tables

We summarize our results in the tables below. Labels for our first table are: A = amenable, WA = weakly amenable, OA = operator amenable, OWA = operator weakly amenable, PD = admits bounded point derivation, v.a. = virtually abelian, and c.n.a. = connected non-abelian.

In each table, known ranges of the validity of that property will be given. If a quantitative range is known to us to be sharp we shall indicate so with an (s). All results below can be found in the present article, in [37], or references therein.

	A	WA	OA	OWA	PD
$\mathcal{A}^p(G,d^\alpha)$ G v.a.	always	always	always	always	never
$\mathcal{A}^p(G,d^\alpha)$ G_e abelian	G v.a. (s)	always	$p=1$ & $\alpha=0^{(*)}$	always	never
$\mathcal{A}^p(\mathrm{SU}(2),d^\alpha)$	never	never	$p=1$ & $\alpha=0$ (s)	$p<\frac{4}{3+2\alpha}$ (s)	$\alpha\geq 1$ (s)
$\mathcal{A}^p(G,d^\alpha)$ G c.n.a.	never	never	$p=1$ & $\alpha=0$ (s)	no if $p\geq\frac{4}{3+2\alpha}$	$\alpha\geq 1$
$\mathcal{A}^2_{R^q}(G,d^\alpha)$ G c.n.a.	never	never	never	never	$\alpha\geq 1$
$\mathcal{A}^p_{\mathrm{SU}(2),d^\alpha}(\mathbb{T})$	$p=1$ & $\alpha=0$ (s)	$p<\frac{2}{1+2\alpha}$ (s)	$p=1$ & $\alpha=0$ (s)	$p<\frac{2}{1+2\alpha}$ (s)	$\alpha\geq 1$ (s)
$\mathcal{A}^2_{R^q,\mathrm{SU}(2),d^\alpha}(\mathbb{T})$	never	never	never	$q<\frac{2}{1+4\alpha}$ or $q'<\frac{2}{1+4\alpha}$ (s)	$\alpha\geq 1$ (s)

$(*)$ Sharp for tall groups and those groups with direct product factors which are infinite products of non-abelian finite groups.

	Arens regular	completely representable as an operator algebra
$\mathcal{A}^p(G)$	never	never
$\mathcal{A}^p(G,d^\alpha)$, connected non-semisimple	never	never
$\mathcal{A}^p(G,\omega_S^\alpha)$, infinite connected Lie	$\alpha>0$ (s)	$\alpha>\left(\frac12+\frac{1}{2p'}\right)d(G)-\frac{1}{2p'}(s(G)+z(G))$
$\mathcal{A}^2_{R^q}(G,\omega_S^\alpha)$, infinite connected Lie	$\alpha>0$ (s)	$\alpha>\frac34 d(G)-\frac14(s(G)+z(G))$
$\mathcal{A}^p(\mathrm{SU}(n),d^\alpha)$	$\alpha>0$ (s)	$\alpha>\left(\frac12+\frac{1}{2p'}\right)(n^2-1)-\frac{1}{2p'}(n-1)$

6.2. Questions

The following questions were partially addressed, and arose naturally in the course of this investigation.

(a) If G is a disconnected Lie group and ω_S^α a polynomial weight, is $\mathrm{A}_{G,\omega_S^\alpha}(G_e)$ always regular? [If this is shown to be true, we can conclude that $\mathrm{A}(G,\omega_S^\alpha)$ is always regular, and then extend this to $\mathcal{A}^p(G,\omega_S^\alpha)$.]

(b) If $\mathcal{A}^2(G) \cong \mathcal{A}^2(H)$ isometrically, must we have $G \cong H$, topologically?

(c) Under what general conditions is $\mathcal{A}^p(G)$ operator amenable for $p > 1$?

(d) Under what general conditions is $\mathcal{A}^p(G, d^\alpha)$ operator weakly amenable?

(e) Given a connected non-abelian Lie group G, does $p < 4/(3 + 2\alpha)$ imply that $\mathcal{A}^p(G, d^\alpha)$ is operator weakly amenable? How about the case of G being semi-simple?

(f) If G is a tall infinite group, and H is any closed subgroup which meets every conjugacy class of G, is $\mathcal{A}^p_G(H)$ is ever operator amenable for any $p > 1$.

(g) Is there a good general description of $\mathcal{A}^p_{G,d^\alpha}(G_e)$?

(h) What are the sharp bounds for complete representability as an operator algebra of $\mathcal{A}^p(G, \omega_S^\alpha)$ for a connected Lie group G? How about for $G = \mathrm{SU}(n)$? How about sharp bounds for representability as an operator algebra?

Another question which naturally arises from our investigation is the following:

(i) What is the algebraic centre of $\mathcal{A}^p(G)^{**}$? If G is a non-semisimple connected group and ω is a weakly dimension weight, what is the algebraic centre of $\mathcal{A}^p(G, \omega)^{**}$.

The studies [11] on commutative Beurling algebras, and [35] on Fourier algebras for some compact groups, suggest that these will not be straightforward problems.

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