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Topological recursion, topological quantum field theory and Gromov–Witten invariants of BG

Daniel Hernández Serrano

Abstract. The purpose of this paper is to give a decorated version of the Eynard–Orantin topological recursion using a 2D Topological Quantum Field Theory. We define a kernel for a 2D TQFT and use an algebraic reformulation of a topological recursion to define how to decorate a standard topological recursion by a 2D TQFT. The A-model side enumerative problem consists of counting cell graphs where in addition vertices are decorated by elements in a Frobenius algebra, and which are a decorated version of the generalized Catalan numbers. We show that the function that counts these decorated graphs, which is a decoration of the counting function of the generalized Catalan numbers by a Frobenius algebra, satisfies a topological recursion with respect to the edge-contraction axioms. The path we follow to pass from the A-model side to the remodeled B-model side is to use a discrete Laplace transform as a mirror symmetry map. We show that a decorated version by a 2D TQFT of the Eynard–Orantin differentials satisfies a decorated version of the Eynard–Orantin recursion formula. We illustrate these results using a toy model for the theory arising from the orbifold cohomology of the classifying space of a finite group. In this example, the graphs are orbifold cell graphs (graphs drawn on an orbifold punctured Riemann surface) defined out of the moduli space $\overline{\mathcal{M}}_{g,n}(BG)$ of stable morphisms from twisted curves to the classifying space of a finite group G. In particular we show that the cotangent class intersection numbers on the moduli space $\overline{\mathcal{M}}_{g,n}(BG)$ satisfy a decorated Eynard–Orantin topological recursion and we derive an orbifold DVV equation as a consequence of it. This proves from a different perspective the known result which states that the ψ-class intersection numbers on $\overline{\mathcal{M}}_{g,n}(BG)$ satisfy the Virasoro constraint condition.

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1. Introduction

The recent formalism of the topological recursion given by Eynard–Orantin in [14] is currently a rich and powerful theory to interconnect different areas of mathematics and physics. Many of the uses of their recursion formula are based on the remodeling conjecture of [6], which proposes this theory as a tool to compute the open Gromov–Witten invariants of a Calabi–Yau threefold when using its mirror curve as the spectral curve of Eynard–Orantin. The ideas of [13], [27], [7], [25] and [24] have contributed to solving many of its applications and to accept the Laplace transform as a mirror symmetry map, in the sense that the Laplace transform of many enumerative problems on the A-model side satisfies the Eynard–Orantin recursion on the B-model side for a particular choice of the spectral curve. Many examples have been proven before the conjecture itself had been solved in [17] and [16], such as counting lattice points of $\mathcal{M}_{g,n}$ [7], [28], [29] and [24], single and orbifold Hurwitz numbers [13], [27] and [5], the Weil–Petersson volume of $\overline{\mathcal{M}}_{g,n}$ [15], [25], [20], [21] and [22], the generalised Catalan numbers [11], the stationary Gromov–Witten theory of $\mathbb{P}^1$ [12] or the case of topological vertex [8] and [30] among others.

Recently, a new set of axioms for a 2D TQFT are given in [10] and proved to be equivalent to the standard TQFT rules. One of the key points of this approach is that they transform the classic TQFT rules into new rules which reflects a reduction by 1 of the topological quantity $2g-2+n$, which is one of the conditions needed for a topological recursion to be satisfied. Thus, it is natural to wonder if a 2D TQFT can be included into the topological recursion formalism of Eynard–Orantin. The answer proposed in this paper consists of providing an algebraic reformulation of the Eynard–Orantin topological recursion decorated by a 2D TQFT and to prove that the Laplace transform of a decorated generalization of the Catalan numbers by a Frobenius algebra satisfies this new decorated topological recursion. A toy model for the theory will also be provided by giving an example based on the orbifold cohomology of the classifying space of a finite group as a Frobenius algebra.

The paper is organized as follows. In section 2 we review the definitions of Frobenius algebra, 2D TQFT and ECA axioms. Section 3 is devoted to providing a decorated version of a topological recursion by a 2D TQFT, where kernel and cokernel operators for a 2D TQFT will be defined by using the product and coproduct of a finite dimensional commutative Frobenius algebra. An algebraic reformulation of a topological recursion will be introduced to extend this operators and define a decoration of a standard topological recursion by a 2D TQFT. In section 4 it is shown that a decorated generalization of the Catalan numbers satisfies a topological recursion. The A-model side enumerative problem consists of counting cell graphs which, in addition, vertices are decorated by elements in a Frobenius algebra, which are a decorated version of the generalized Catalan numbers of [25] and [11] by a 2D TQFT. ECA axioms of [10] allow us to show that the function which counts these decorated graphs satisfies the same type of recursion of [7]. The Laplace transform of this recursion is the decorated generalization of the Eynard–Orantin topological recursion by a 2D TQFT proposed in the previous section. Section 5 relates these results to the Gromov–Witten invariants of the classifying space BG of a finite group G. We give an example arising from the orbifold cohomology of BG, where the decorated cell graphs are graphs drawn on an orbifold punctured Riemann surface defined out of the moduli space $\overline{\mathcal{M}}_{g,n}(BG)$ of stable morphisms from twisted curves to the classifying space of a finite group G, and which are given as an orbifold generalization of Grothendieck dessins d'enfants. We generalize the lattice point counting of [24] to this orbifold setting and by taking the Laplace transform of the resulting recursion equation we show that a decorated topological recursion by the 2D TQFT given by the orbifold cohomology of BG as Frobenius algebra is satisfied. This provides us with an orbifold DVV equation, which shows from a different perspective the main result of [18]: the ψ-class intersection numbers on $\overline{\mathcal{M}}_{g,n}(BG)$ satisfy the Virasoro constraint condition. We conclude in section 6 with the proof of Theorem 4.4.

2. Frobenius algebras, 2D TQFT and ECA axioms

In this section we review some definitions. We suggest the readers follow [19] for the notion of Frobenius algebra and its relation with 2-dimensional topological quantum field theory (2D TQFT), [4] for the mathematical definition of TQFT and [10] for the edge-contraction axioms on cell graphs.

Let A be a commutative Frobenius algebra over a field K and let us denote:

- The product:

$$\begin{aligned} m\colon A\otimes A &\to A \\ (u,v) &\mapsto m(u,v)=u\cdot v\,. \end{aligned}$$

- The non-degenerate symmetric bilinear form:

$$\begin{aligned} \eta\colon A\otimes A &\to K \\ (u,v) &\mapsto \eta(u,v)\,. \end{aligned}$$

- The Frobenius form:

$$\begin{aligned} \epsilon\colon A &\to K \\ u &\mapsto \epsilon(u):=\eta(1,u)\,. \end{aligned}$$

There exists a unique coassociative coproduct $\delta\colon A\to A\otimes A$ whose counit is the Frobenius form $\epsilon\colon A\to K$ and which satisfies the Frobenius relation

$$\delta\circ m=(\delta\otimes 1)\circ(1\otimes m).$$

In order to define the coproduct, let us introduce the three-point function

$$\begin{aligned} \phi\colon A\otimes A\otimes A &\to K \\ (u,v,w) &\mapsto \phi(u,v,w):=\eta(u\cdot v,w)=\eta(u,v\cdot w). \end{aligned}$$

Using the standard formula $u = \sum_{a,b} \eta(u, e_a)\, \eta^{ab} e_b$, where $\{e_1, \ldots, e_s\}$ is a K-basis of A, we can then write

$$u \cdot v = \sum_{a,b} \phi(u, v, e_a)\, \eta^{ab} e_b \quad \text{and} \quad \delta(v) = \sum_{i,j,a,b} \phi(v, e_i, e_j)\, \eta^{ia}\, \eta^{jb} e_a \otimes e_b\,,$$

where $\eta_{ij} := \eta(e_i, e_j)$ and $\eta = (\eta_{ij})_{i,j}$ is the associated symmetric matrix, whose inverse is denoted $\eta^{-1} = (\eta^{ij})_{i,j}$.

The last interesting operator (since the genus of a Riemann surface shall be codified on it) is the handle operator

$$h\colon A \xrightarrow{\delta} A \otimes A \xrightarrow{m} A\,. \tag{2.1}$$

The image of $1 \in A$ is the Euler element $\mathbf{e} = (m \circ \delta)(1)$.

Definition 2.1 ([4], [19])**.** A 2D TQFT is a rule F which associates to each closed oriented 1-manifold Σ a vector space $A = F(\Sigma)$, and to each oriented cobordism $M\colon \Sigma_1 \mapsto \Sigma_2$ associates a linear map $F(M)\colon F(\Sigma_1) \to F(\Sigma_2)$. This rule must satisfy:

- Two equivalent cobordisms must have the same image.
- The cylinder cobordism from Σ to itself must be sent to the identity map of $F(\Sigma)$.
- Given a decomposition $M = M'M''$, then $F(M)$ is the composition of the linear maps $F(M')$ and $F(M'')$.
- Disjoint union goes to tensor product, for 1-manifolds and also for cobordisms.
- The empty manifold must be sent to the ground field K.
- Takes the symmetry to the symmetry.

Let $\Sigma_{g,n}$ be an oriented surface of type (g, n) with labeled boundary components by indices $1, \ldots, n$. Let $A = F(S^1)$ and $\Omega_{g,n} := F(\Sigma_{g,n})\colon A^{\otimes n} \to K$ the associated multilinear map. We denote the associated 2D TQFT to a Frobenius algebra A by the tuple $(A, \eta, \{\Omega_{g,n} \in A^{\otimes n *}\})$.

Remark 2.2. Let $\overline{\mathcal{M}}_{g,n}$ be the moduli space of stable genus g curves with n marked points. A TQFT can be thought of as a cohomological field theory which takes values in $H^0(\overline{\mathcal{M}}_{g,n}, K) = K$.

In [10] a new set of rules for a 2D TQFT given in terms of edge-contraction operations on cell-graphs have been proven to be equivalent to the standard set of axioms for a 2D TQFT. For the sake of completeness, these set of axioms will be included.

Let $\Gamma_{g,n}$ be the set of connected cell graphs of type (g, n) with labeled vertices. Recall that a cell graph of type (g, n) is a 1-skeleton of a cell-decomposition of a connected compact oriented topological surface of genus g with n labeled 0-cells, where a 0-cell is called a vertex, a 1-cell an edge and a 2-cell a face (see [11] for details). For each cell graph $\gamma \in \Gamma_{g,n}$, let

$$\begin{aligned}\Omega(\gamma)\colon A^{\otimes n} &\to \mathbb{C}\\ v_1 \otimes \cdots \otimes v_n &\mapsto \Omega(\gamma)(v_1, \ldots, v_n)\end{aligned}$$

be a multilinear map which decorates the i-th vertex of γ with an element $v_i \in A$.

Definition 2.3 (The edge-contraction axioms (ECA), see Definition 4.4 of [10]).

- **ECA 0**: For the cell graph consisting of only one vertex without any edge, $\gamma_0 \in \Gamma_{0,1}$, we define

$$\Omega(\gamma_0)(v) = \epsilon(v), \quad v \in A. \tag{2.2}$$

- **ECA 1**: Suppose there is an edge E connecting the i-th vertex and the j-th vertex for $i < j$ in $\gamma \in \Gamma_{g,n}$. Let $\gamma' \in \Gamma_{g,n-1}$ denote the cell graph obtained by contracting E. Then

$$\Omega(\gamma)(v_1, \ldots, v_n) = \Omega(\gamma')(v_1, \ldots, v_{i-1}, v_i \cdot v_j, v_{i+1}, \ldots, \widehat{v_j}, \ldots, v_n), \tag{2.3}$$

Here $\widehat{v_j}$ means we omit the j-th variable v_j at the j-th vertex of γ.

- **ECA 2**: Suppose there is a loop L attached at the i-th vertex of $\gamma \in \Gamma_{g,n}$. Let γ' denote the possibly disconnected graph obtained by contracting L and separating the vertex to two distinct vertices labeled by i and i'. We assign an ordering $i-1 < i < i' < i+1$.

If γ' is connected, then it is in $\Gamma_{g-1,n+1}$. We then impose

$$\Omega(\gamma)(v_1, \ldots, v_n) = \Omega(\gamma')(v_1, \ldots, v_{i-1}, \delta(v_i), v_{i+1}, \ldots, v_n), \tag{2.4}$$

where the outcome of the comultiplication $\delta(v_i)$ is placed in the i-th and i'-th slots.

If γ' is disconnected, then write $\gamma' = (\gamma_1, \gamma_2) \in \Gamma_{g_1,|I|+1} \times \Gamma_{g_2,|J|+1}$, where

$$\begin{cases} g = g_1 + g_2, \\ I \sqcup J = \{1, \ldots, \widehat{i}, \ldots, n\}. \end{cases} \tag{2.5}$$

Here, vertices labeled by I belong to the connected component of genus g_1, and those labeled by J on the other component. Let (I_-, i, I_+) (resp. (J_-, i, J_+)) be reordering of $I \sqcup \{i\}$ (resp. $J \sqcup \{i\}$) in the increasing order. We impose

$$\begin{aligned}&\Omega(\gamma)(v_1, \ldots, v_n)\\ &\qquad = \sum_{a,b,k,\ell} \eta(v_i, e_k e_\ell)\, \eta^{ka}\, \eta^{\ell b}\, \Omega(\gamma_1)(v_{I_-}, e_a, v_{I_+})\, \Omega(\gamma_2)(v_{J_-}, e_b, v_{J_+}).\end{aligned} \tag{2.6}$$

Theorem 3.8 and Corollary 4.8 of [10] prove that given a Frobenius algebra A, the standard axioms of 2D TQFT and the ECA axioms are equivalent. Moreover they have:

$$\Omega_{g,n}(v_1, \ldots, v_n) = \epsilon(v_1 \cdots v_n \cdot \mathbf{e}^g) = \Omega(\gamma)(v_1, \ldots, v_n)\,,$$

where $\mathbf{e}^g$ is the g-th power of the Euler element.

3. Twisted topological recursion by a 2D TQFT

3.1. Topological recursion

We suggest the readers look at the topological recursion of Eynard–Orantin in [14], and to the special case of genus 0 spectral curve to the mathematical definition given in [11]. In this subsection an algebraic reformulation is given, which will be used to define how to decorate a topological recursion by a 2D TQFT.

Let Σ be a spectral curve and let us denote $V = H^0\big(\Sigma, K_\Sigma(*R)\big)$ the space of meromorphic differentials on Σ (where R is the set of ramification points of the spectral curve) and allow us to write $V^n = \mathrm{Sym}^n H^0\big(\Sigma, K_\Sigma(*R)\big)$.

Definition 3.1. We define a topological recursion *kernel operator* as the following map:

$$
\begin{aligned}
K\colon V\otimes V &\to V\\
(f_0,f_1) &\mapsto K(f_0,f_1)\,,
\end{aligned}
$$

which can naturally be extended to

$$
\begin{aligned}
K\colon V\otimes V\otimes V^{n-1} &\to V\otimes V^{n-1}\\
(f_0,f_1,f_2,\ldots,f_n) &\mapsto \big(K(f_0,f_1),f_2,\ldots,f_n\big)\,,\\
K\colon V\otimes V^{|I|}\otimes V\otimes V^{|J|} &\to V\otimes V^{|I\sqcup J|}\\
(f_0,f_I,f_1,f_J) &\mapsto \big(K(f_0,f_1),f_I,f_J\big)\,.
\end{aligned}
$$

Definition 3.2. Let (g,n) be a pair in the stable range, that is, $g\geq 0$, $n\geq 1$ subject to the condition $2g-2+n>0$. Given $W_{0,2}\in H^0\big(\Sigma, K_\Sigma^{\otimes 2}(2\Delta)\big)$, where Δ denotes the diagonal of $\Sigma\times\Sigma$, the meromorphic differentials $W_{g,n}\in V^n$ are said to satisfy a topological recursion (TR) with respect to the spectral curve Σ and kernel K if:

$$
W_{g,n} = K(W_{g-1,n+1}) + \frac{1}{2}\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\ldots,n\}}}^{\text{no }(0,1)} K(W_{g_1,|I|+1},W_{g_2,|J|+1})\,.
$$

This is called a Eynard–Orantin topological recursion (EO TR) when a explicit form of a EO kernel is chosen.

Remark 3.3. The unstable differential $W_{0,1}\in H^0\big(\Sigma,K_\Sigma\big)$ is also defined for a topological recursion by using the spectral curve, see [14] and [11] for details.

3.2. The kernel and the cokernel operators in a TQFT

In this subsection "kernel and cokernel operators" are defined in a 2D TQFT by using the coproduct and the product of the Frobenius algebra. This will be a useful tool later on to intrinsically define a 2D TQFT-decorated topological recursion.

Let $(A,\eta,\Omega_{g,n}\in A^{\otimes n*})$ be a 2D TQFT, let $A\xrightarrow{\delta} A\otimes A$ be the coproduct in the Frobenius algebra and consider its dual map $\delta^*\colon A^*\otimes A^*\to A^*$.

Definition 3.4. We define the *kernel operator* as $\delta^*\colon A^*\otimes A^*\to A^*$, which can be naturally extended to

$$\begin{aligned}\delta^*\colon\ & A^*\otimes A^*\otimes (A^*)^{n-1}\otimes H^0(\overline{\mathcal{M}}_{g-1,n+1})\to A^*\otimes (A^*)^{n-1}\otimes H^\bullet(\overline{\mathcal{M}}_{g,n})\,,\\ \delta^*\colon\ & A^*\otimes A^*\otimes (A^*)^{|I|}\otimes (A^*)^{|J|}\otimes H^0(\overline{\mathcal{M}}_{g_1,|I|+1})\otimes H^0(\overline{\mathcal{M}}_{g_2,|J|+1})\\ &\qquad\qquad\qquad\qquad\qquad\to A^*\otimes (A^*)^{|I\sqcup J|}\otimes H^0(\overline{\mathcal{M}}_{g,n})\,,\end{aligned}$$

and which will be still denoted by δ^*.

Remark 3.5. Even if $H^0(\overline{\mathcal{M}}_{g,n},K)=K$, we want to still leave it in the definition in order to keep track of the topological type. Since a 2D TQFT is the degree 0 part of the cohomology of a cohomological field theory, it would be an interesting question to generalize this definition for a CohFT.

In this fashion we have that the equation

$$\delta^*(\Omega_{g-1,n+1})=\Omega_{g,n} \tag{3.1}$$

is equivalent to **ECA 2** axiom of equation (2.4),

$$\Omega_{g,n}(v_1,\dots,v_n)=\Omega_{g-1,n+1}(\delta(v_1),v_{[n]\setminus\{1\}})\,.$$

Similarly,

$$\delta^*(\Omega_{g_1,|I|+1},\Omega_{g_2,|J|+1})=\Omega_{g,n} \tag{3.2}$$

produces **ECA 2** axiom of equation (2.6):

$$\begin{aligned}&\Omega_{g,n}\,(v_1,\dots,v_n)\\ &\quad=\sum_{a,b,k,\ell}\phi(v_i,e_k,e_\ell)\,\eta^{ka}\,\eta^{\ell b}\,\Omega_{g_1,|I|+1}(v_{I_-},e_a,v_{I_+})\Omega_{g_2,|J|+1}(v_{J_-},e_b,v_{J_+})\,.\end{aligned}$$

In an analogous way, we can start with the product $m\colon A\otimes A\to A$ and define a cokernel operator.

Definition 3.6. We define the *cokernel operator* as $m^*\colon A^*\to A^*\otimes A^*$, naturally extended to

$$m^*\colon A^*\otimes (A^*)^{n-2}\otimes H^0(\overline{\mathcal{M}}_{g,n-1})\to A^*\otimes A^*\otimes (A^*)^{n-2}\otimes H^0(\overline{\mathcal{M}}_{g,n})\,.$$

We have that

$$m^*(\Omega_{g,n-1})=\Omega_{g,n} \tag{3.3}$$

is equivalent **ECA 1** axiom of equation (2.3):

$$\Omega_{g,n}(v_1,\dots,v_n)=\Omega_{g,n-1}(v_1\cdot v_j,v_{[n]\setminus\{1,j\}})\,.$$

Finally, this section is completed by pointing out the relation between the kernel and the cokernel operators. In the Frobenius algebra A we have the following identity:

$$m = (1 \times \eta) \circ (\delta \times 1)$$
$$m\colon A \otimes A \xrightarrow{\delta\times 1} A \otimes A \otimes A \xrightarrow{1\times\eta} A.$$

Let us define the $(0,2)$ unstable number by using the pairing η:

$$\Omega_{0,2}(v_1, v_2) := \eta(v_1, v_2)\,,$$

which give us

$$m_*(\Omega_{g,n-1}) = \delta^*(\Omega_{g,n-1}, \Omega_{0,2})\,. \tag{3.4}$$

Therefore, and once the unstable $(0,2)$ case is consistently defined in the theory, we could just use the "kernel operator".

3.3. Twisting a topological recursion by a 2D TQFT

Definition 3.7. We define the *decorated kernel*, K_δ, as the following product of the TR kernel of Definition 3.1 with the kernel of Definition 3.4:

$$\begin{aligned} K_\delta\colon (V\otimes A^*)\otimes(V\otimes A^*) &\xrightarrow{K\times\delta^*} (V\otimes A^*) \\ \big((f_0,\Omega_i),(f_1,\Omega_j)\big) &\mapsto \big(K(f_0,f_1),\delta^*(\Omega_i,\Omega_j)\big)\,. \end{aligned}$$

We extend the decorated kernel to

$$\begin{aligned} &(V\otimes A^*)\otimes(V\otimes A^*)\otimes(V^{n-1}\otimes(A^*)^{n-1}) \xrightarrow{K_\delta} (V\otimes A^*)\otimes(V^{n-1}\otimes(A^*)^{n-1}), \\ &(V\otimes A^*)\otimes(V\otimes A^*)\otimes(V^{|I|}\otimes(A^*)^{|I|})\otimes(V^{|J|}\otimes(A^*)^{|J|}) \\ &\qquad\qquad\qquad\qquad \xrightarrow{K_\delta} (V\otimes A^*)\otimes(V^{n-1}\otimes(A^*)^{n-1}). \end{aligned}$$

Definition 3.8. Let (g,n) be a stable pair. We *define* the *decorated-meromorphic differentials* as elements $\mathcal{W}_{g,n} \in V^n \otimes (A^*)^n$.

Definition 3.9. Let (g,n) be a pair in the stable range. The decorated meromorphic differentials $\mathcal{W}_{g,n} \in (V\otimes A^*)^n$ are said to satisfy a topological recursion with respect to the spectral curve Σ and decorated kernel $K_\delta = K\times\delta^*$ if

$$\mathcal{W}_{g,n} = K_\delta(\mathcal{W}_{g-1,n+1}) + \frac{1}{2}\sum_{\substack{g_1+g_2=g \\ I\sqcup J=\{2,\dots,n\}}}^{\text{no }(0,1)} K_\delta(\mathcal{W}_{g_1,|I|+1}, \mathcal{W}_{g_2,|J|+1})\,.$$

This will be called a Eynard–Orantin decorated topological recursion when a explicit form of a EO kernel for K is chosen.

Let us point out that there is no $(0,1)$ terms on the last summand, but there are $(0,2)$ appearing as

$$K_\delta(\mathcal{W}_{g,n-1}, \mathcal{W}_{0,2}) .$$

Using the cokernel m^* of Definition 3.6, we can define a *decorated cokernel* by $K_* := K \times m^*$. Thus, by equation (3.4), and once we define $\mathcal{W}_{0,2}(f_1, f_2; v_1, v_2) := W_{0,2}(f_1, f_2)\Omega_{0,2}(v_1, v_2)$, we have that

$$K_*(\mathcal{W}_{g,n-1}) = K_\delta(\mathcal{W}_{g,n-1}, \mathcal{W}_{0,2}) ,$$

and thus, the TR could be also written as

$$\mathcal{W}_{g,n} = K_*(\mathcal{W}_{g,n-1}) + K_\delta(\mathcal{W}_{g-1,n+1}) + \frac{1}{2} \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\dots,n\}}}^{\text{stable}} K_\delta(\mathcal{W}_{g_1,|I|+1}, \mathcal{W}_{g_2,|J|+1}) .$$

Morover, once we define the unstable differential

$$\mathcal{W}_{0,2}(f_1, f_2; v_1, v_2) := W_{0,2}(f_1, f_2)\, \Omega_{0,2}(v_1, v_2),$$

and since topological recursion is a reduction by 1 of the topological quantity $2g - 2 + n$, using the equations (3.1) and (3.2) we have that $\mathcal{W}_{g,n} = W_{g,n}\Omega_{g,n}$.

Remark 3.10. These definitions could be extended to cohomological field theories and related with [3] and [12]. Properties and details will be studied elsewhere.

4. Twisted topological recursion for decorated Catalan numbers

4.1. Background: Dessins d'enfants

A *dessins d'enfant* of type (g, n) is a topological graph drawn on a genus g connected smooth algebraic curve C which is defined as the inverse image $b^{-1}([0, 1])$ of the closed interval $[0, 1] \subset \mathbb{P}^1$ by a clean Belyi map $b : C \longrightarrow \mathbb{P}^1$ (a meromorphic morphism ramified at three points $\{0, 1, \infty\}$ such that n is the number of poles of b without counting the multiplicity, the ramification type of b above $1 \in \mathbb{P}^1$ is $(2, 2, \dots, 2)$). They are a special kind of metric ribbon graphs and the enumeration of clean Belyi morphism is equivalent to the enumeration of certain ribbon graphs.

A *dessin* is defined as the dual graph $\gamma = b^{-1}([1, i\infty])$, where $[1, i\infty] = \{1 + iy \mid 0 \le y \le \infty\} \subset \mathbb{P}^1$. It has n labeled vertices and it is a connected cell graph. The number of dessins with the automorphism factor is defined by

$$D_{g,n}(\mu_1, \dots, \mu_n) = \sum_{\substack{\gamma \text{ dessin of} \\ \text{type } (g,n)}} \frac{1}{|\mathrm{Aut}_D(\gamma)|}, \tag{4.1}$$

where $(\mu_1, \dots, \mu_n)$ are the prescribed degrees of the n labeled vertices and $\mathrm{Aut}_D(\gamma)$ is the automorphism of γ preserving each vertex point-wise (see [23] and [11] for details).

The *generalized Catalan numbers* of type (g, n) are defined in [26] by

$$C_{g,n}(\mu_1, \ldots, \mu_n) = \mu_1 \cdots \mu_n \, D_{g,n}(\mu_1, \ldots, \mu_n)\,,$$

and they count the dual graphs defined before where, moreover, an outgoing arrow is placed on one of its incident half-edges to the vertices (this is done in order to kill the automorphisms). They are called arrowed cell graphs in [10].

4.2. Counting decorated dessins

We are interested in counting dessins where vertices are decorated by elements in a Frobenius algebra A. We shall refer to them as *decorated dessins*.

Given a Frobenius algebra A, let $(A, \eta, \Omega_{g,n})$ be its associated 2D TQFT. Let us recall that $\Omega_{g,n}(v_1, \ldots, v_n) = \Omega(\gamma)(v_1, \ldots, v_n)$ and by Theorem 4.7 of [10] we have that $\Omega(\gamma)(v_1, \ldots, v_n)$ is graph independent, that is to say, the decorating procedure is graph independent. Let us consider the multilinear maps

$$D_{g,n}(\mu_1, \ldots, \mu_n) \cdot \Omega_{g,n} \colon A^{\otimes n} \to K$$

which to each tuple $(v_1, \ldots, v_n) \in A^{\otimes n}$ associates the number

$$\mathcal{D}_{g,n}(\mu_1, \ldots, \mu_n; v_1, \ldots, v_n) = D_{g,n}(\mu_1, \ldots, \mu_n) \cdot \Omega_{g,n}(v_1, \ldots, v_n)$$

of *decorated dessins* of type (g, n) with prescribed vertices degree profile $(\mu_1, ..., \mu_n)$ decorated by $(v_1, \ldots, v_n)$ (see also Remark 5.4 bellow for a geometric justification). For the unstable case $(0, 2)$, we define

$$\mathcal{D}_{0,2}(\mu_1, \mu_2; v_1, v_2) := D_{0,2}(\mu_1, \mu_2) \cdot \Omega_{0,2}(v_1, v_2), \tag{4.2}$$

where $\Omega_{0,2}(v_1, v_2) := \eta(v_1, v_2)$ and $D_{0,2}(\mu_1, \mu_2)$ is given in Proposition 3.1 of [11].

Proposition 4.1. *The number of decorated dessins satisfies the following recursion equation:*

$$\begin{aligned} &\mathcal{D}_{g,n}(\mu_1, \ldots, \mu_n; v_1, \ldots, v_n) \\ &= \sum_{j=2}^{n} \mu_j \, \mathcal{D}_{g,n-1}(\mu_1 + \mu_j - 2, \mu_2, \ldots, \widehat{\mu_j}, \ldots, \mu_n; v_1 \cdot v_j, v_2, \ldots, \widehat{v_j}, \ldots, v_n) \\ &\quad + \sum_{\alpha+\beta=\mu_1-2} \mathcal{D}_{g-1,n+1}(\alpha, \beta, \mu_2, \ldots, \mu_n; \delta(v_1), v_2, \ldots, v_n) \\ &\quad + \sum_{\alpha+\beta=\mu_1-2} \sum_{\substack{g_1+g_2=g \\ I \sqcup J=\{2,\ldots,n\}}} \delta^*\big(\mathcal{D}_{g_1,|I|+1}(\alpha, \mu_I; \_, v_I), \mathcal{D}_{g_2,|J|+1}(\beta, \mu_J; \_, v_J)\big)(v_1), \end{aligned} \tag{4.3}$$

where δ^* is given by Definition 3.4:

$$\begin{aligned} \delta^*\big(\mathcal{D}_{g_1,|I|+1}(\alpha, \mu_I; \_, v_I), \mathcal{D}_{g_2,|J|+1}(\beta, \mu_J; \_, v_J)\big)(v_1) &= \sum_{k,\ell,a,b} \phi(v_1, e_k, e_\ell)\, \eta^{ka} \eta^{\ell b} \\ \times \big(D_{g_1,|I|+1}(\alpha, \mu_I) \cdot \Omega_{g_1,|I|+1}(e_a, v_I)\big) &\big(D_{g_2,|J|+1}(\beta, \mu_J) \cdot \Omega_{g_2,|J|+1}(e_b, v_J)\big)\,. \end{aligned}$$

Proof. It follows from Theorem 3.3 of [11] by applying the ECA axioms described in terms of the kernel and cokernel operators in a 2D TQFT (see subsection 3.2). When we contract an edge which connects the vertex 1 and the vertex $j > 1$ we need to apply ECA 1 axiom of equation (3.3); if we contract an edge which forms a loop attached to vertex 1, we need to apply ECA 2 axiom of equation (3.1) if the resulting dessin is connected, and ECA 2 axiom of equation (3.2) if the resulting dessin is the disjoint union of two dessins. $\square$

Let $C(\mu_1, \ldots, \mu_n) \cdot \Omega_{g,n} \colon A^{\otimes n} \to K$ be the function which for each tuple $(v_1, \ldots, v_n) \in A^{\otimes n}$ produces the number

$$\mathcal{C}_{g,n}(\mu_1, \ldots, \mu_n; v_1, \ldots, v_n) = C(\mu_1, \ldots, \mu_n) \cdot \Omega_{g,n}(v_1, \ldots, v_n)$$

of decorated arrowed cell graphs. We shall refer to it as *decorated generalized Catalan numbers.*

Corollary 4.2. *The decorated generalized Catalan numbers satisfies the following recursion equation:*

$$\begin{aligned}
&\mathcal{C}_{g,n}(\mu_1, \ldots, \mu_n; v_1, \ldots, v_n) \\
&= \sum_{j=2}^{n} \mu_j \mathcal{C}_{g,n-1}(\mu_1 + \mu_j - 2, \mu_2, \ldots, \widehat{\mu_j}, \ldots, \mu_n; v_1 \cdot v_j, v_2, \ldots, \widehat{v_j}, \ldots, v_n) \\
&\quad + \sum_{\alpha+\beta=\mu_1-2} \mathcal{C}_{g-1,n+1}(\alpha, \beta, \mu_2, \ldots, \mu_n; \delta(v_1), v_2, \ldots, v_n) \\
&\quad + \sum_{\alpha+\beta=\mu_1-2} \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,\ldots,n\}}} \delta^*\big(\mathcal{C}_{g_1,|I|+1}(\alpha, \mu_I; _, v_I), \mathcal{C}_{g_2,|J|+1}(\beta, \mu_J; _, v_J)\big)(v_1).
\end{aligned} \tag{4.4}$$

4.3. Twisted topological recursion for decorated generalized Catalan numbers

We will apply the Laplace transform to the equation (4.3) using the method of [11] and [25] in order to proof that the decorated differentials of Definition 3.8 satisfy the decorated topological recursion of Definition 3.9.

Let $\mu = (\mu_1, \ldots, \mu_n)$ and let $F^D_{g,n}(t_1, \ldots, t_n) = \sum_{\mu \in \mathbb{Z}^n_+} D_{g,n}(\mu) e^{-(w_1\mu_1 + \cdots + w_n\mu_n)}$ be the Laplace transform of the Catalan numbers, where the relation between coordinates are:

$$z_j = \frac{t_j + 1}{t_j - 1}; \quad e^{w_j} = \frac{t_j + 1}{t_j - 1} + \frac{t_j - 1}{t_j + 1}.$$

The Laplace transform of the decorated Catalan numbers is given by

$$\begin{aligned}
\mathcal{F}_{g,n}(t_1, \ldots, t_n; v_1, \ldots, v_n) &= \sum_{\mu \in \mathbb{Z}^n_+} \mathcal{D}_{g,n}(\mu; v_1, \ldots, v_n)\, e^{-(w_1\mu_1 + \cdots + w_n\mu_n)} \\
&= F^D_{g,n}(t_1, \ldots, t_n)\, \Omega_{g,n}(v_1, \ldots, v_n)\,.
\end{aligned}$$

Let $W^D_{g,n}(t_1,\dots,t_n) = dt_1\cdots dt_n F^D_{g,n}(t_1,\dots,t_n)$ be the differential forms of [11] and let x_j be variables defined by $x_j = e^{w_j}$ and write

$$W^D_{g,n}(t_1,\dots,t_n) = w^D_{g,n}(t_1,\dots,t_n)\,dt_1\cdots dt_n = w_{g,n}(x_1,\dots,x_n)\,dx_1\cdots dx_n\,.$$

We have

$$\mathfrak{w}_{g,n}(x_1,\dots,x_n;v_1,\dots,v_n) = w_{g,n}(x_1,\dots,x_n)\,\Omega_{g,n}(v_1,\dots,v_n)\,,$$

and the decorated meromorphic differentials

$$\begin{aligned}\mathcal{W}_{g,n}(t_1,\dots,t_n;v_1,\dots,v_n) &= dt_1\cdots dt_n\,\mathcal{F}_{g,n}(t_1,\dots,t_n;v_1,\dots,v_n)\\ &= W^D_{g,n}(t_1,\dots,t_n)\,\Omega_{g,n}(v_1,\dots,v_n)\,.\end{aligned}$$

Proposition 4.3. *The Laplace transform of the recursion formula (4.3) is the following ECA based differential recursion:*

$$\begin{aligned}&-x_1\,\mathfrak{w}_{g,n}(x_1,\dots,x_n;v_1,\dots,v_n)\\ &= \sum_{j=2}^{n}\frac{\partial}{\partial x_j}\Big(\frac{1}{x_j-x_1}(\mathfrak{w}_{g,n-1}(x_2,\dots,x_n;v_1\cdot v_j,v_{[n]\setminus\{1,j\}})\\ &\qquad\qquad - \mathfrak{w}_{g,n-1}(x_{[n]\setminus\{j\}};v_1\cdot v_j,v_{[n]\setminus\{1,j\}}))\Big)\\ &\quad+\mathfrak{w}_{g-1,n+1}(x_1,x_1,x_{[n]\setminus\{1\}};\delta(v_1),v_{[n]\setminus\{1\}})\\ &\quad+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}\delta^*\big(\mathfrak{w}_{g_1,|I|+1}(x_1,x_I;-,v_I),\mathfrak{w}_{g_2,|J|+1}(x_1,x_J;-,v_J)\big)(v_1).\end{aligned}\tag{4.5}$$

Following the computations for the unstable case $(0,2)$ of [11], we define

$$\begin{aligned}\mathcal{W}_{0,2}(t_1,t_2;v_1,v_2) &= W^D_{0,2}(t_1,t_2)\Omega_{0,2}(v_1,v_2) = d_1d_2F^D_{0,2}(t_1,t_2)\Omega_{0,2}(v_1,v_2)\\ &= \Big(\frac{dt_1\cdot dt_2}{(t_1-t_2)^2}-\frac{dx_1\cdot dx_2}{(x_1-x_2)^2}\Big)\,\Omega_{0,2}(v_1,v_2) = \Omega_{0,2}(v_1,v_2)\frac{dt_1\cdot dt_2}{(t_1+t_2)^2}.\end{aligned}\tag{4.6}$$

Theorem 4.4. *The decorated differential forms*

$$\mathcal{W}_{g,n}(t_1,\dots,t_n;v_1,\dots,v_n) = d_1\cdots d_n\,\mathcal{F}_{g,n}(t_1,\dots,t_n;v_1,\dots,v_n)\tag{4.7}$$

satisfy the Eynard–Orantin decorated topological recursion

$$\begin{aligned}&\mathcal{W}_{g,n}(t_1,t_2,\dots,t_n;v_1,\dots,v_n)\\ &= \frac{1}{2\pi i}\int_{\phi}K^D(t,t_1)\Big[\mathcal{W}_{g-1,n+1}(t,-t,t_2,\dots,t_n;\delta(v_1),v_{[n]\setminus\{1\}})\\ &\quad+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,3,\dots,n\}}}^{\text{No }(0,1)\text{ terms}}\delta^*\big(\mathcal{W}_{g_1,|I|+1}(t,t_I;-,v_I),\mathcal{W}_{g_2,|J|+1}(-t,t_J;-,v_J)\big)(v_1)\Big]\end{aligned}\tag{4.8}$$

with respect to the spectral curve

$$\begin{cases} x = z + \frac{1}{z}, \\ y = -z, \end{cases} \tag{4.9}$$

and the recursion kernel

(4.10)

$$K^D(t,t_1) = \frac{1}{2}\,\frac{\int_t^{-t} W_{0,2}^D(\cdot,t_1)}{W_{0,1}^D(-t) - W_{0,1}^D(t)} = \frac{1}{2}\left(\frac{1}{t+t_1} + \frac{1}{t-t_1}\right)\frac{1}{32}\,\frac{(t^2-1)^3}{t^2}\,\frac{1}{dt}\,dt_1.$$

The integration is taken with respect to a contour ϕ in the complex t-plane consisting of two concentric circles centered around 0*, with a positively oriented small inner circle of radius ϵ and a large negatively oriented circle of radius $1/\epsilon$. This annulus should enclose all values of $\pm t_i$, $i = 1, \ldots, n$.*

Proof. Given in the appendix. □

Remark 4.5. Equation (4.8) is the decorated topological recursion of Definition 3.9 with respect to the spectral curve $\{x = z + 1/z, y = -z\}$ of [11] and the decorated kernel $K_\delta = K \times \delta^*$, where the explicit Eynard–Orintin kernel of equation (4.10) has been chosen. This shows that the decorated Eynard–Orantin differentials satisfies a decorated Eynard–Orantin topological recursion which splits as the product of the EO TR of [11] and a 2D TQFT $(A, \eta, \Omega_{g,n})$.

5. Gromov–Witten theory of BG and orbifold DVV equation

5.1. Background

Following [1], [2], [9] and [18], the moduli stack $\overline{\mathcal{M}}_{g,n}(BG)$ of stable maps from n-pointed twisted curves of genus g to BG is a smooth, proper, Deligne–Mumford stack of dimension $3g-3+n$. The forgetful morphism

$$\varphi : \overline{\mathcal{M}}_{g,n}(BG) \longrightarrow \overline{\mathcal{M}}_{g,n} \tag{5.1}$$

is generically finite. In particular, its restriction to the smooth locus,

$$\varphi : \mathcal{M}_{g,n}(BG) \longrightarrow \mathcal{M}_{g,n}, \tag{5.2}$$

is a finite morphism with the fiber

$$\mathrm{Hom}\big(\pi_1(C \setminus \{p_1, \ldots, p_n\}), G\big) /\!\!/ G \tag{5.3}$$

at each $[C, \{p_1, \ldots, p_n\}] \in \mathcal{M}_{g,n}$, where the G action on the space of homomorphisms is via conjugation action.

Let IBG be the inertia stack, which decomposes as

$$IBG = \coprod_{[\![r]\!]} BG_{[\![r]\!]} = \coprod_{[\![r]\!]} [pt/C(r)]\,,$$

where $C(r)$ denotes the centralizer of $r \in G$ and the evaluation morphisms

$$ev_i : \overline{\mathcal{M}}_{g,n}(BG) \to IBG$$

allows to see that the stack $\overline{\mathcal{M}}_{g,n}(BG)$ breaks up as the disjoint union of open and closed substacks:

$$\overline{\mathcal{M}}_{g,n}(BG) = \coprod_{(\llbracket r_1 \rrbracket, \dots, \llbracket r_n \rrbracket)} \overline{\mathcal{M}}_{g,n}(BG, \llbracket r_1 \rrbracket, \dots, \llbracket r_n \rrbracket),$$

where $\overline{\mathcal{M}}_{g,n}(BG, \llbracket r_1 \rrbracket, \dots, \llbracket r_n \rrbracket) = ev_1^{-1}(BG_{\llbracket r_1 \rrbracket}) \cap \dots \cap ev_n^{-1}(BG_{\llbracket r_n \rrbracket})$ and $\llbracket r_i \rrbracket$ denotes the conjugacy class of the element r_i in G. The map

$$\mathcal{M}_{g,n}(BG, \llbracket r_1 \rrbracket, \dots, \llbracket r_n \rrbracket) \to \mathcal{M}_{g,n}$$

is a finite morphism of degree

$$\Omega^G_{g,n}(\mathbf{r}) = \frac{|\mathcal{X}^G_g(\mathbf{r})|}{|G|}\,, \tag{5.4}$$

where $\mathbf{r} = (\llbracket r_1 \rrbracket, \dots, \llbracket r_n \rrbracket)$ and

$\mathcal{X}^G_g(\mathbf{r})$
$:= \{(\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \sigma_1, \dots, \sigma_n) | \prod_{i=1}^g [\alpha_i, \beta_i] = \prod_{j=1}^n \sigma_j,\ \sigma_j \in \llbracket r_j \rrbracket \text{ for all } j\}.$

Let

$$A := H^*_{\text{orb}}(BG, \mathbb{C}) := H^*(IBG, \mathbb{C}) = \bigoplus_{\llbracket r \rrbracket} \mathbb{C}$$

be the orbifold cohomology of BG as a vector space and, for each conjugacy class $\llbracket r \rrbracket$ in G, let $e_{\llbracket r \rrbracket}$ denote a $\mathbb{C}$-basis of A. It is know that A is a Frobenius algebra isomorphic to the center of the group algebra of G where the non-degenerated bilinear form $\eta\colon A \otimes A \to \mathbb{C}$ is given by

$$\eta_{ij} := \eta(e_{\llbracket r_i \rrbracket}, e_{\llbracket r_j \rrbracket}) = \frac{1}{|C(r_i)|}\, \delta_{\llbracket r_i \rrbracket \llbracket r_j^{-1} \rrbracket}\,, \tag{5.5}$$

and the multiplication (orbifold product) $m\colon A \otimes A \to A$ is given by

$$m(e_{\llbracket r_i \rrbracket}, e_{\llbracket r_j \rrbracket}) = e_{\llbracket r_i \rrbracket} e_{\llbracket r_j \rrbracket} = \sum_{\substack{\sigma_i, \sigma_j \\ \sigma_i \in \llbracket r_i \rrbracket \\ \sigma_j \in \llbracket r_j \rrbracket}} \frac{|C(\sigma_i \sigma_j)|}{|G|} e_{\llbracket \sigma_i \sigma_j \rrbracket}\,. \tag{5.6}$$

In [18] it is proven that the collection

$$\begin{aligned} \Omega^G_{g,n}\colon A^{\otimes n} &\to H^*(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) \\ e_{\llbracket r_1 \rrbracket} \otimes \dots \otimes e_{\llbracket r_n \rrbracket} &\mapsto \Omega^G_{g,n}(e_{\llbracket r_1 \rrbracket} \otimes \dots \otimes e_{\llbracket r_n \rrbracket}) = \varphi_*\big(ev_1^*(e_{\llbracket r_1 \rrbracket}) \dots ev_n^*(e_{\llbracket r_n \rrbracket})\big) \\ &= \Omega^G_{g,n}(\mathbf{r}) \end{aligned}$$

is a cohomological field theory. In fact it is a 2D TQFT since it takes values in $H^0(\overline{\mathcal{M}}_{g,n}, \mathbb{C}) = \mathbb{C}$.

5.2. Orbifold generalized Catalan numbers and decorated topological recursion

The 3-point function $\phi\colon A \otimes A \otimes A \to \mathbb{C}$ is defined by:

$$\begin{aligned}\phi(e_{[\![r_1]\!]}, e_{[\![r_2]\!]}, e_{[\![r_3]\!]}) &:= \eta(e_{[\![r_1]\!]}e_{[\![r_2]\!]}, e_{[\![r_3]\!]}) = \eta(e_{[\![r_1]\!]}, e_{[\![r_2]\!]}e_{[\![r_3]\!]}) \\ &= \epsilon(e_{[\![r_1]\!]}e_{[\![r_2]\!]}e_{[\![r_3]\!]})\,,\end{aligned} \tag{5.7}$$

where $\epsilon\colon A \to \mathbb{C}$ is the counit, and Proposition 3.1 of [18] shows that

$$\phi(e_{[\![r_1]\!]}, e_{[\![r_2]\!]}, e_{[\![r_3]\!]}) = \Omega^G_{0,3}(e_{[\![r_1]\!]}, e_{[\![r_2]\!]}, e_{[\![r_3]\!]})\,. \tag{5.8}$$

Using CohFT 3 axiom of Definition 3.1 of [10] and the expression for $v \in A$,

$$v = \sum_{[\![a]\!],[\![b]\!]} \eta(v, e_{[\![a]\!]})\, \eta^{[\![a]\!][\![b]\!]}\, e_{[\![b]\!]},$$

we can see that the genus 0 values of the collection $\{\Omega^G_{g,n}\}$ are given by

$$\Omega^G_{0,n}(v_{[\![r_1]\!]}, \ldots, v_{[\![r_n]\!]}) = \epsilon(v_{[\![r_1]\!]} \cdots v_{[\![r_n]\!]})\,.$$

Thus, it follows from Theorem 3.8 of [10] that

$$\Omega^G_{g,n}(v_{[\![r_1]\!]}, \ldots, v_{[\![r_n]\!]}) = \epsilon(v_{[\![r_1]\!]} \cdots v_{[\![r_n]\!]}\mathbf{e}^g)\,,$$

where $\mathbf{e}^g$ denotes the g-th power of the Euler element.

Let $\Gamma_{g,n}$ be the set of connected cell graphs of type (g, n) with labeled vertices, and for each cell graph $\gamma \in \Gamma_{g,n}$ let

$$\begin{aligned}\Omega(\gamma)\colon A^{\otimes n} &\to \mathbb{C} \\ v_{[\![r_1]\!]} \otimes \cdots \otimes v_{[\![r_n]\!]} &\mapsto \Omega(\gamma)(v_{[\![r_1]\!]}, \ldots, v_{[\![r_n]\!]})\end{aligned}$$

be an n-variable function which assigns $v_{[\![r_i]\!]} \in A$ to the i-th vertex of γ.

Remark 5.1. This decorating function consists of keeping track of the orbifold information at each marked orbifold point of the twisted curve which maps to BG.

Provided that we define

$$\Omega^G_{0,1}(v) := \epsilon(v)\,, \tag{5.9}$$

$$\Omega^G_{0,2}(v_{[\![r_1]\!]}, v_{[\![r_2]\!]}) := \eta(v_{[\![r_1]\!]}, v_{[\![r_2]\!]})\,, \tag{5.10}$$

and using Theorem 4.7 and Corollary 4.8 of [10] we have:

Proposition 5.2. *For each cell graph $\gamma \in \Gamma_{g,n}$, define*

$$\Omega^G_{g,n}(v_{[\![r_1]\!]}, \ldots, v_{[\![r_n]\!]}) = \Omega(\gamma)(v_{[\![r_1]\!]}, \ldots, v_{[\![r_n]\!]})\,.$$

Then the collection $\{\Omega^G_{g,n}\}$ satisfies the edge-contraction axioms of Definition 2.3.

As a consequence, $\Omega^G_{g,n}(v_{[\![r_1]\!]}, \ldots, v_{[\![r_n]\!]})$ is symmetric with respect to permutation indices and $\{\Omega^G_{g,n}\}$ is the 2D TQFT associated with the Frobenius algebra $A = H^*_{\mathrm{orb}}(BG)$.

Remark 5.3. Let us recall that in terms of the three-point function defined in equation (5.8), the multiplication and the comultiplication in A can be written as

$$v_{[\![r_1]\!]}v_{[\![r_2]\!]} = \sum_{[\![a]\!],[\![b]\!]} \Omega^G_{0,3}(v_{[\![r_1]\!]}, v_{[\![r_2]\!]}, e_{[\![a]\!]})\, \eta^{ab} e_{[\![b]\!]}\,, \tag{5.11}$$

$$\delta(v) = \sum_{[\![r_i]\!],[\![r_j]\!],[\![a]\!],[\![b]\!]} \Omega^G_{0,3}(v, e_{[\![r_i]\!]}, e_{[\![r_j]\!]})\, \eta^{ia}\eta^{jb} e_{[\![a]\!]} \otimes e_{[\![b]\!]}\,. \tag{5.12}$$

So that the product and coproduct of the Frobenius algebra can be thought of as the two kinds of orbifold pair of pants of Figure 5.1.

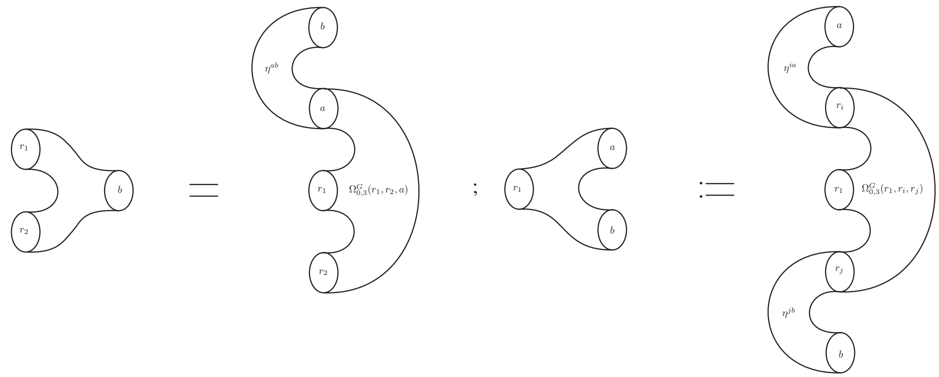

FIGURE 5.1. Orbifold pair of pants as product and coproduct in A, written in terms of the three-point function.

In this fashion, the ECA implies that $\Omega^G_{g,n}(v_{[\![r_1]\!]}, \dots, v_{[\![r_n]\!]})$ satisfy the following relations:

$$\Omega^G_{g,n}(v_{[\![r_1]\!]}, \dots, v_{[\![r_n]\!]}) = \sum_{[\![a]\!],[\![b]\!]} \Omega^G_{0,3}(v_{[\![r_1]\!]}, v_{[\![r_j]\!]}, e_{[\![a]\!]})\, \eta^{ab} \Omega^G_{g,n-1}(e_{[\![b]\!]}, \mathbf{v}_{[n]\setminus\{1,j\}}),$$

$$\begin{aligned}\Omega^G_{g,n}&(v_{[\![r_1]\!]}, \dots, v_{[\![r_n]\!]})\\ &= \sum_{[\![r_i]\!],[\![r_j]\!],[\![a]\!],[\![b]\!]} \Omega^G_{0,3}(v_{[\![r_1]\!]}, e_{[\![r_i]\!]}, e_{[\![r_j]\!]})\, \eta^{ia}\eta^{jb} \Omega^G_{g-1,n+1}(e_{[\![a]\!]}, e_{[\![b]\!]}, \mathbf{v}_{[n]\setminus\{1\}})\,,\end{aligned}$$

$$\begin{aligned}\Omega^G_{g,n}&(v_{[\![r_1]\!]}, \dots, v_{[\![r_n]\!]})\\ &= \sum_{[\![r_i]\!],[\![r_j]\!],[\![a]\!],[\![b]\!]} \Omega^G_{0,3}(v_{[\![r_1]\!]}, e_{[\![r_i]\!]}, e_{[\![r_j]\!]})\, \eta^{ia}\eta^{jb} \Omega^G_{g_1,|I|+1}(e_{[\![a]\!]}, \mathbf{v}_I)\Omega^G_{g_2,|J|+1}(e_{[\![a]\!]}, \mathbf{v}_J),\end{aligned}$$

where $g = g_1 + g_2$ and $I \coprod J = \{2, \dots, n\}$.

These relations are reflected by Figure 5.2 as a cutting off a pair of pants from an n-punctured orbifold surface.

For the $(g, n) = (1, 1)$ case, let us define the $(1, 1)$-operator $e\colon A \to \mathbb{C}$ as the following composite:

$$e\colon A \xrightarrow{h} A \xrightarrow{\epsilon} \mathbb{C}\,, \tag{5.13}$$

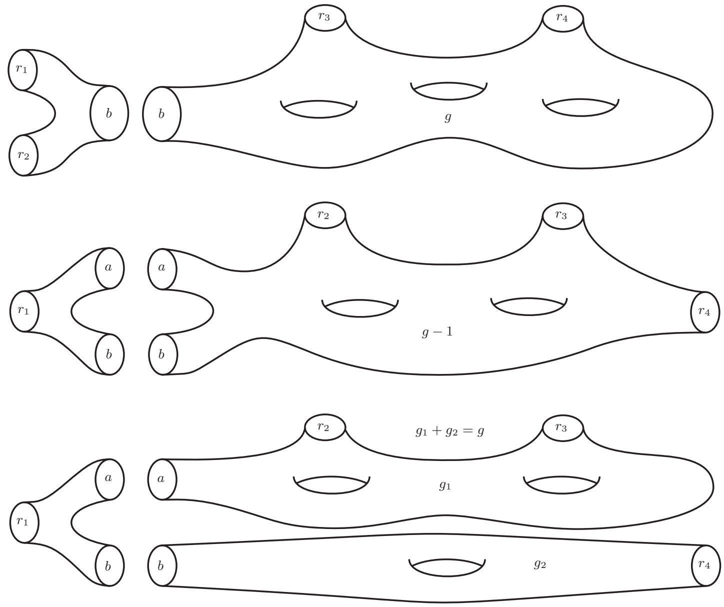

FIGURE 5.2. Cutting off a pair of pants from an n-punctured orbifold surface. Orbifold generalization of Figure 2.1 of [25].

where $h\colon A \to A$ is the handle operator of equation (2.1). The $(1,1)$-operator has one input "decorated marking" and no output markings and we have:

$$\begin{aligned} &\Omega^G_{1,1}(v_{[\![r_1]\!]}) \\ &= \sum_{[\![r_i]\!],[\![r_j]\!],[\![a]\!],[\![b]\!]} \Omega^G_{0,3}(v_{[\![r_1]\!]}, e_{[\![r_i]\!]}, e_{[\![r_j]\!]})\, \eta^{ia}\eta^{jb}\Omega^G_{0,2}(e_{[\![a]\!]}, e_{[\![b]\!]}) = \Omega^G_{0,2}\big(\delta(v_{[\![r_1]\!]})\big). \end{aligned} \tag{5.14}$$

As in section 4.2, let us denote by $\mathcal{C}_{g,n}(\mu_1,\ldots,\mu_n; v_{[\![r_1]\!]},\ldots,v_{[\![r_n]\!]})$ the number of decorated arrowed cell graphs of labeled vertices of degrees $(\mu_1,\ldots,\mu_n)$, where in this particular case we call them *orbifold generalized Catalan numbers*. They satisfy the equation of Corollary 4.2.

Applying the Laplace transform method to this equation we have by Theorem 4.4 that the decorated meromorphic differentials

$$\mathcal{W}_{g,n}(z_1,\ldots,z_n; v_{[\![r_1]\!]},\ldots,v_{[\![r_n]\!]}) = W^D_{g,n}(z_1,\ldots,z_n)\Omega^G_{g,n}(v_{[\![r_1]\!]},\ldots,v_{[\![r_n]\!]})$$

are a solution of a decorated topological recursion (Definition 3.9) which splits as the product of the usual topological recursion given in [11] and the 2D TQFT $(A,\eta,\{\Omega^G_{g,n}\})$ given by the orbifold cohomology of BG as the Frobenius algebra A.

Remark 5.4. The orbifold generalized Catalan numbers counts a special type of orbifold arrowed dessins. Notice that if we denote by D the associated G-cover to a point $C \to BG$ in $\overline{\mathcal{M}}_{g,n}(BG)$, in order to define a Belyi map for D compatible with the action of G one has to have a commutative diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d, "G-equiv"'] & C \simeq D/G \arrow[d, "b"] \\ \mathbb{P}^1 \arrow[r] & \mathbb{P}^1 \simeq \mathbb{P}^1/G \end{tikzcd}$$

and since the G-action on $\mathbb{P}^1$ has to fix the points $0, 1$ and ∞, then G is acting trivially on $\mathbb{P}^1$ and therefore the quotient stack $[\mathbb{P}^1/G]$ is isomorphic to $\mathbb{P}^1 \times BG$. Thus, we define a clean G-Belyi morphism of type (g, n) as a pair

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "b"'] & BG \\ \mathbb{P}^1 & \end{tikzcd}$$

where $b\colon C \to \mathbb{P}^1$ is a clean Belyi morphism and $f\colon C \to BG$ is a point in $\mathcal{M}_{g,n}(BG)$. An orbifold G-*dessin* is a representable map $b^{-1}([1, i\infty]) \to BG$. From this point of view, let us justify the counting of section 4.2. If we denote $\tilde{\gamma}_{\bar{v}}$ the associated G-cover to a map $\gamma \to \mathcal{B}G$ where $\bar{v} = (v_{[\![r_1]\!]}, \dots, v_{[\![r_n]\!]})$ (and $v_{[\![r_i]\!]} \in A$), we have:

$$\sum_{\tilde{\gamma}_{\bar{v}}} \frac{1}{|\mathrm{Aut}^G(\tilde{\gamma}_{\bar{v}})|} = \sum_{\gamma} \sum_{\tilde{\gamma}_{\bar{v}} \to \gamma} \frac{1}{|\mathrm{Aut}(\gamma)|} \frac{|\mathrm{Aut}(\gamma)|}{|\mathrm{Aut}^G(\tilde{\gamma}_{\bar{v}})|} = \sum_{\gamma} \frac{1}{|\mathrm{Aut}(\gamma)|} \Omega^G_{g,n}(\bar{v}).$$

5.3. Intersection numbers of $\overline{\mathcal{M}}_{g,n}(BG)$ and orbifold DVV equation

Let us now briefly comment how to relate these results with the Gromov–Witten intersection numbers of $\overline{\mathcal{M}}_{g,n}(BG)$.

The *n-point correlators* are defined in the usual way:

$$\langle \tau_{k_1}(e_{[\![r_1]\!]}) \dots \tau_{k_n}(e_{[\![r_n]\!]}) \rangle^G_{g,n} := \int_{\overline{\mathcal{M}}_{g,n}(BG)} \prod_{i=1}^{n} \bar{\psi}_i^{k_i} \, e\, v_i^*(e_{[\![r_i]\!]}) \,, \tag{5.15}$$

where the tautological cotangent classes $\bar{\psi}_i := \varphi^* \psi_i$ are defined by pulling back the standard ψ-classes by the canonical projection $\varphi\colon \overline{\mathcal{M}}_{g,n}(BG) \to \overline{\mathcal{M}}_{g,n}$. In [18] it is proven that

$$\langle \tau_{k_1}(e_{[\![r_1]\!]}) \dots \tau_{k_n}(e_{[\![r_n]\!]}) \rangle^G_{g,n} = \Omega^G_g(\mathbf{r}) \langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} \tag{5.16}$$

where $\langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n}$ are the standard n-point correlators for $\overline{\mathcal{M}}_{g,n}$.

Remark 5.5. In [18] it is shown that the intersection numbers on $\overline{\mathcal{M}}_{g,n}(BG)$ satisfy the Virasoro constrain condition. They use a change of basis on the Frobenius algebra to show that the result consist of h copies of Witten–Kontsevich theory of a point, where h is the number of conjugacy classes of G (recall that BG is just h copies of a point). Even though the results of [11] can be straightforward generalized to get h copies of a Eynard–Orantin topological recursion, we can use the approach of the previous sections: even if the intersection numbers on $\overline{\mathcal{M}}_{g,n}(BG)$ are just a scalar multiple of the intersection numbers on $\overline{\mathcal{M}}_{g,n}$, inside the structure there is a new theory of a decorated topological recursion by a 2D TQFT which can produce an orbifold generalization of the DVV equation instead of h copies of the standard DVV equation.

Let $N_{g,n}(\mu_1,\dots,\mu_n)$ be the lattice point counting function of [24] and [11]. By Theorem 1.3 of [24], the leading terms of the Laplace transform, $F^L_{g,n}(t_1,\dots,t_n)$, of the number $N_{g,n}(\mu_1,\dots,\mu_n)$ of ribbon graphs of integer edge lengths $(\mu_1,\dots,\mu_n)$ with a cilium placed on a labeled face, form a homogeneous polynomial of degree $3(2g-2+n)$ given by

$$F^K_{g,n}(t_1,\dots,t_n) = \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{k_1+\cdots+k_n\\=3g-3+n}} \langle \tau_{k_1}\dots\tau_{k_n}\rangle_{g,n} \prod_{j=1}^{n}(2k_j-1)!! \left(\frac{t_j}{2}\right)^{2k_j+1}. \tag{5.17}$$

If we denote by $\mathcal{N}_{g,n}(\mu_1,\dots,\mu_n;v_{[\![r_1]\!]},\dots,v_{[\![r_n]\!]})$ the number of ribbon graphs of integer edge lengths $(\mu_1,\dots,\mu_n)$ with a cilium placed on a labeled face and where faces are also decorated by $(v_{[\![r_1]\!]},\dots,v_{[\![r_n]\!]})\in A^{\otimes n}$, then this number is just the product

$$N_{g,n}(\mu_1,\dots,\mu_n)\,\Omega^G_{g,n}(v_{[\![r_1]\!]},\dots,v_{[\![r_n]\!]})\,,$$

and the multilinear map

$$\mathcal{N}_{g,n}(\mu_1,\dots,\mu_n) = N_{g,n}(\mu_1,\dots,\mu_n)\cdot\Omega^G_{g,n}\colon A^{\otimes n}\to\mathbb{C}$$

satisfies the following ECA based equation:

$$\begin{aligned}
&\mu_1\mathcal{N}_{g,n}(\mu_1,\dots,\mu_n;v_{[\![r_1]\!]},\dots,v_{[\![r_n]\!]})\\
&= \frac{1}{2}\sum_{j=2}^{n}\Bigg[\sum_{q=0}^{\mu_1+\mu_j} q(\mu_1+\mu_j-q)\mathcal{N}_{g,n-1}(q,\mu_{[n]\setminus\{1,j\}};v_{[\![r_1]\!]}v_{[\![r_j]\!]},v_{[\![r_2]\!]},\dots,\widehat{v_{[\![r_j]\!]}},\dots,v_{[\![r_n]\!]})\\
&+H(\mu_1-\mu_j)\sum_{q=0}^{\mu_1-\mu_j} q(\mu_1-\mu_j-q)\mathcal{N}_{g,n-1}(q,\mu_{[n]\setminus\{1,j\}};v_{[\![r_1]\!]}v_{[\![r_j]\!]},v_{[\![r_2]\!]},\dots,\widehat{v_{[\![r_j]\!]}},\dots,v_{[\![r_n]\!]})\\
&-H(\mu_j-\mu_1)\\
&\quad\cdot\sum_{q=0}^{\mu_j-\mu_1} q(\mu_j-\mu_1-q)\mathcal{N}_{g,n-1}(q,\mu_{[n]\setminus\{1,j\}};v_{[\![r_1]\!]}v_{[\![r_j]\!]},v_{[\![r_2]\!]},\dots,\widehat{v_{[\![r_j]\!]}},\dots,v_{[\![r_n]\!]})\Bigg]\\
&+\frac{1}{2}\sum_{0\le q_1+q_2\le\mu_1} q_1q_2(\mu_1-q_1-q_2)\Bigg[\mathcal{N}_{g-1,n+1}(q_1,q_2,\mu_{[n]\setminus\{1\}};\delta(v_{[\![r_1]\!]}),v_{[\![r_2]\!]},\dots,v_{[\![r_n]\!]})\\
&\qquad+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=[n]\setminus\{1\}}}^{\text{stable}} \delta^*\big(\mathcal{N}_{g_1,|I|+1}(q_1,\mu_I;_,v_{[\![I]\!]})\otimes\mathcal{N}_{g_2,|J|+1}(q_2,\mu_J;_,v_{[\![J]\!]})\big)(v_{[\![r_1]\!]})\Bigg],
\end{aligned}\tag{5.18}$$

where $H(x)$ is the Heaviside function,

$$H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

This generalizes Theorem 3.3 of [7]. The leading terms of the Laplace transform of

$$\mathcal{N}_{g,n}(\mu_1, \dots, \mu_n; v_{[\![r_1]\!]}, \dots, v_{[\![r_n]\!]})$$

have then the shape

$$\begin{aligned} &\mathcal{F}^K_{g,n}(t_1, \dots, t_n; v_{[\![r_1]\!]}, \dots, v_{[\![r_n]\!]}) \\ &\quad = \frac{(-1)^n}{2^{2g-2+n}} \sum_{\substack{k_1+\dots+k_n \\ =3g-3+n}} \langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} \, \Omega^G_{g,n}(v_{[\![r_1]\!]}, \dots, v_{[\![r_n]\!]}) \\ &\qquad \cdot \prod_{j=1}^n (2k_j - 1)!! \left(\frac{t_j}{2} \right)^{2k_j+1}, \end{aligned} \tag{5.19}$$

where by equation (5.16) we have

$$\langle \tau_{k_1}(e_{[\![r_1]\!]}) \dots \tau_{k_n}(e_{[\![r_n]\!]}) \rangle^G_{g,n} = \langle \tau_{k_1} \dots \tau_{k_n} \rangle_{g,n} \Omega^G_g(v_{[\![r_1]\!]}, \dots, v_{[\![r_n]\!]}) \,.$$

Applying the Laplace transform method to ECA based equation (5.18) and restricting to the top degree terms, the Frobenius algebra decorated topological recursion produces in this case the following orbifold DVV type equation:

$$\begin{aligned} &\langle \tau_{k_1}(e_{[\![r_1]\!]}) \cdots \tau_{k_n}(e_{[\![r_n]\!]}) \rangle^G_{g,n} \\ &\quad = \sum_{j=2}^n \frac{(2k_1 + 2k_j - 1)!!}{(2k_1 - 1)!!(2k_j - 1)!!} \\ &\qquad\quad \times \langle \tau_{k_1+k_j-1}(e_{[\![r_1]\!]} e_{[\![r_j]\!]}) \tau_{k_2}(e_{[\![r_2]\!]}) \cdots \widehat{\tau_{k_j}(e_{[\![r_j]\!]})} \cdots \tau_{k_n}(e_{[\![r_n]\!]}) \rangle^G_{g,n} \\ &\qquad + \frac{1}{2} \sum_{l+m=k_1-2} \frac{(2l+1)!!(2m+1)!!}{(2k_1+1)!!} \\ &\qquad\quad \times \sum_{[\![r_l]\!], [\![r_m]\!], [\![a]\!], [\![b]\!]} \phi(v_{[\![r_1]\!]}, e_{[\![r_l]\!]}, e_{[\![r_m]\!]}) \, \eta^{ka} \eta^{\ell b} \\ &\qquad\quad \times \Bigg(\langle \tau_l \tau_m \tau_{k_{[n] \setminus \{1\}}} \rangle_{g-1,n+1} \Omega^G_{g-1,n+1}(e_{[\![a]\!]}, e_{[\![b]\!]}, e_{[\![\mathbf{r}_{[n]\setminus\{1\}}]\!]}) \\ &\qquad\qquad + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = [n] \setminus \{1\}}}^{\text{stable}} \langle \tau_l \tau_{k_I} \rangle_{g_1,|I|+1} \Omega^G_{g_1,|I|+1}(e_{[\![a]\!]}, e_{[\![\mathbf{r}_I]\!]}) \langle \tau_m \tau_{k_J} \rangle_{g_2,|J|+1} \\ &\qquad\qquad\qquad \times \Omega^G_{g_2,|J|+1}(e_{[\![b]\!]}, e_{[\![\mathbf{r}_J]\!]}) \Bigg), \end{aligned} \tag{5.20}$$

which we can symbolically write as

$$\begin{aligned}
&\langle \tau_{k_1}(e_{[\![r_1]\!]}) \cdots \tau_{k_n}(e_{[\![r_n]\!]})\rangle^G_{g,n} \\
&\quad = \sum_{j=2}^{n} \frac{(2k_1+2k_j-1)!!}{(2k_1-1)!!(2k_j-1)!!} \\
&\qquad \times \langle \tau_{k_1+k_j-1}(e_{[\![r_1]\!]}e_{[\![r_j]\!]})\tau_{k_2}(e_{[\![r_2]\!]}) \cdots \widehat{\tau_{k_j}(e_{[\![r_j]\!]})} \cdots \tau_{k_n}(e_{[\![r_n]\!]})\rangle^G_{g,n} \\
&\quad + \frac{1}{2} \sum_{l+m=k_1-2} \frac{(2l+1)!!(2m+1)!!}{(2k_1+1)!!} \\
&\qquad \times \Bigg(\delta^*\big(\langle \tau_l \tau_m \tau_{k_{[n]\setminus\{1\}}}(e_{[\![\mathbf{r}_{[n]\setminus\{1\}}]\!]})\rangle^G_{g-1,n+1}\big)(e_{[\![r_1]\!]}) + \\
&\qquad + \sum_{\substack{g_1+g_2=g \\ I\sqcup J=[n]\setminus\{1\}}}^{\text{stable}} \delta^*\big(\langle \tau_l(_)\tau_{k_I}(e_{[\![\mathbf{r}_I]\!]})\rangle^G_{g_1,|I|+1}, \langle \tau_m(_)\tau_{k_J}(e_{[\![\mathbf{r}_J]\!]})\rangle^G_{g_2,|J|+1}\big)(e_{[\![r_1]\!]})\Bigg).
\end{aligned} \tag{5.21}$$

This proves, from a different perspective to that provided in [18], that the intersections numbers on $\overline{\mathcal{M}}_{g,n}(BG)$ satisfy the Virasoro constrain condition.

6. Appendix: proof of Theorem 4.4

In this section the proof of Theorem 4.4 is given in order to clarify how to use the ECA axioms and properly compute the contributions coming from the unstable geometries. We reproduce the proof of Theorem 4.3 of [11] given in the Appendix A of [11] (which states the EO topological recursion satisfied by $W^D_{g,n}(t_1,\dots,t_n)$) and adapt that result for the decorated differentials

$$\mathcal{W}_{g,n}(t_1,\dots,t_n;v_1,\dots,v_n) = W^D_{g,n}(t_1,\dots,t_n)\Omega_{g,n}(v_1,\dots,v_n)\,.$$

Let us separate first the contributions from unstable geometries $(g,n)=(0,1)$ and $(0,2)$ in the last line of equation (4.5). Using equation (2.6) of ECA2 axiom for $g_1=0$ and $I=\emptyset$, or $g_2=0$ and $J=\emptyset$ we have a contribution of

$$\begin{aligned}
&\delta^*\big(\mathfrak{w}_{0,1}(x_1;-), \mathfrak{w}_{g,n}(x_{[n]};-,v_{[n]\setminus\{1\}})\big)(v_1) \\
&\qquad + \delta^*\big(\mathfrak{w}_{g,n}(x_{[n]};-,v_{[n]\setminus\{1\}}), \mathfrak{w}_{0,1}(x_1;-)\big)(v_1) \\
&\qquad\qquad = 2\, w_{0,1}(x_1)\, w_{g,n}(x_1,x_2,\dots,x_n)\, \Omega_{g,n}(v_1,\dots,v_n)\,.
\end{aligned}$$

Similarly, for $g_1=0$ and $I=\{j\}$, or $g_2=0$ and $J=\{j\}$, we have

$$\begin{aligned}
&\sum_{j=2}^{n} \delta^*\big(\mathfrak{w}_{0,2}(x_1,x_j;-,v_j), \mathfrak{w}_{g,n-1}(x_1,x_{[n]\setminus\{1,j\}};-,v_{[n]\setminus\{1,j\}})\big)(v_1) \\
&\qquad = \sum_{j=2}^{n} w_{0,2}(x_1,x_j)\, w_{g,n-1}(x_1,\dots,\widehat{x_j},\dots,x_n)\, \Omega_{g,n}(v_1,\dots,v_n)\,.
\end{aligned}$$

Thus, bearing in mind that

$$\begin{aligned} w_{0,1}(x) &= -\frac{t+1}{t-1}\,, \\ w_{0,2}(x_1,x_2) &= \frac{1}{(t_1+t_2)^2}\,\frac{(t_1^2-1)^2}{8t_1}\,\frac{(t_2^2-1)^2}{8t_2}\,, \\ w_{g,n}(x_1,\dots,x_n) &= (-1)^n w^D_{g,n}(t_1,\dots,t_n)\prod_{i=1}^n \frac{(t_i^2-1)^2}{8t_i}\,, \end{aligned}$$

the differential equation (4.5) is equivalent to

$$\begin{aligned} &2\Big(\frac{t_1^2+1}{t_1^2-1}-\frac{t_1+1}{t_1-1}\Big)\mathfrak{w}_{g,n}(t_1,\dots,t_n;v_1,\dots,v_n) \\ &=\sum_{j=2}^n\Bigg(\frac{(t_1^2-1)^2(t_j^2-1)^2}{16(t_1^2-t_j^2)^2}\,\frac{8t_j}{(t_j^2-1)^2}\,\mathfrak{w}_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n;v_1\cdot v_j,v_{[n]\setminus\{1,j\}}) \\ &\quad+\frac{\partial}{\partial t_j}\Big(\frac{(t_1^2-1)(t_j^2-1)}{4(t_1^2-t_j^2)}\,\frac{8t_1}{(t_1^2-1)^2}\,\frac{(t_j^2-1)^2}{8t_j}\mathfrak{w}_{g,n-1}(t_2,\dots,t_n;v_1\cdot v_j,v_{[n]\setminus\{1,j\}})\Big)\Bigg) \\ &\quad+\frac{(t_1^2-1)^2}{8t_1}\Bigg(\delta^*\big(\mathfrak{w}_{g-1,n+1}(t_1,t_1,t_2,\dots,t_n;-,-,v_{[n]\setminus\{1\}})\big)(v_1) \\ &\quad+\sum^{\text{stable}}_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}\delta^*\big(\mathfrak{w}_{g_1,|I|+1}(t_1,t_I;-,v_I),\mathfrak{w}_{g_2,|J|+1}(t_1,t_J;-,v_J)\big)(v_1)\Bigg) \\ &\quad+2\sum_{j=2}^n\frac{1}{(t_1+t_j)^2}\,\frac{(t_1^2-1)^2}{8t_1}\,w^D_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n)\,\Omega_{g,n}(v_1,\dots,v_n) \\ &=\sum_{j=2}^n\Bigg(\Big(\frac{t_j(t_1^2-1)^2}{2(t_1^2-t_j^2)^2}+\frac{1}{(t_1+t_j)^2}\,\frac{(t_1^2-1)^2}{4t_1}\Big) \\ &\qquad\times\mathfrak{w}_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n;v_1\cdot v_j,v_{[n]\setminus\{1,j\}}) \\ &\quad+\frac{t_1}{t_1^2-1}\,\frac{\partial}{\partial t_j}\Big(\frac{(t_j^2-1)^3}{4t_j(t_1^2-t_j^2)}\,\mathfrak{w}_{g,n-1}(t_2,\dots,t_n;v_1\cdot v_j,v_{[n]\setminus\{1,j\}})\Big)\Bigg) \\ &\quad+\frac{(t_1^2-1)^2}{8t_1}\Bigg(\delta^*\big(\mathfrak{w}_{g-1,n+1}(t_1,t_1,t_2,\dots,t_n;-,-,v_{[n]\setminus\{1\}})\big)(v_1) \\ &\quad+\sum^{\text{stable}}_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}\delta^*\big(\mathfrak{w}_{g_1,|I|+1}(t_1,t_I;-,v_I),\mathfrak{w}_{g_2,|J|+1}(t_1,t_J;-,v_J)\big)(v_1)\Bigg)\,, \end{aligned}$$

where we have used ECA1 equation (2.3) to convert the second and sixth lines into the seventh and eighth lines. Since

$$2\Big(\frac{t_1^2+1}{t_1^2-1}-\frac{t_1+1}{t_1-1}\Big)=-\frac{4t_1}{t_1^2-1}\,,$$

we obtain

$$
\begin{aligned}
&\mathfrak{w}_{g,n}(t_1,\dots,t_n;v_1,\dots,v_n)\\
&\quad= -\sum_{j=2}^{n}\Bigg(\frac{\partial}{\partial t_j}\Big(\frac{(t_j^2-1)^3}{16t_j(t_1^2-t_j^2)}\,\mathfrak{w}_{g,n-1}(t_2,\dots,t_n;v_1\cdot v_j,v_{[n]\setminus\{1,j\}})\Big)\\
&\qquad+\frac{(t_1^2-1)^3}{16t_1^2}\,\frac{t_1^2+t_j^2}{(t_1^2-t_j^2)^2}\mathfrak{w}_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n;v_1\cdot v_j,v_{[n]\setminus\{1,j\}})\Bigg)\\
&\qquad-\frac{(t_1^2-1)^3}{32t_1^2}\Bigg(\delta^*\big(\mathfrak{w}_{g-1,n+1}(t_1,t_1,t_2,\dots,t_n;-,-,v_{[n]\setminus\{1\}})\big)(v_1)\\
&\qquad+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,\dots,n\}}}^{\text{stable}}\delta^*\big(\mathfrak{w}_{g_1,|I|+1}(t_1,t_I;-,v_I),\mathfrak{w}_{g_2,|J|+1}(t_1,t_J;-,v_J)\big)(v_1)\Bigg).
\end{aligned}
\tag{6.1}
$$

We can then compute the integral (notice that we are already separating the unstable $(0,2)$ differential forms and substituting the kernel by its value (4.10)):

$$
\begin{aligned}
&\mathcal{W}_{g,n}(t_1,\dots,t_n;v_1,\dots,v_n)\\
&\quad=-\frac{1}{64}\,\frac{1}{2\pi i}\int_\phi\Big(\frac{1}{t+t_1}+\frac{1}{t-t_1}\Big)\frac{(t^2-1)^3}{t^2}\cdot\frac{1}{dt}\cdot dt_1\\
&\qquad\times\Bigg[\sum_{j=2}^{n}\Bigg(\delta^*\big(\mathcal{W}_{0,2}(t,t_j;-,v_j),\mathcal{W}_{g,n-1}(-t,t_{[n]\setminus\{1,j\}};-,v_{[n]\setminus\{1,j\}})\big)(v_1)\\
&\qquad+\delta^*\big(\mathcal{W}_{0,2}(-t,t_j;-,v_j),\mathcal{W}_{g,n-1}(t,t_{[n]\setminus\{1,j\}};-,v_{[n]\setminus\{1,j\}})\big)(v_1)\Bigg)\\
&\qquad+\delta^*\big(\mathcal{W}_{g-1,n+1}(t,-t,t_2,\dots,t_n;-,-,v_{[n]\setminus\{1\}})\big)(v_1)\\
&\qquad+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=\{2,3,\dots,n\}}}^{\text{stable}}\delta^*\big(\mathcal{W}_{g_1,|I|+1}(t,t_I;-,v_I),\mathcal{W}_{g_2,|J|+1}(-t,t_J;-,v_J)\big)(v_1)\Bigg].
\end{aligned}
\tag{6.2}
$$

Recall that for $2g-2+n>0$, $w^D_{g,n}(t_1,\dots,t_n)$ is a Laurent polynomial in $t_1^2,\dots,t_n^2$. Thus the last two lines of (6.2) are immediately calculated because the integration contour ϕ encloses $\pm t_1$ and contributes residues with the negative sign. The result is exactly the last two lines of (6.1). Similarly, since

$$
\begin{aligned}
&\delta^*\big(\mathcal{W}_{0,2}(t,t_j;-,v_j),\mathcal{W}_{g,n-1}(-t,t_{[n]\setminus\{1,j\}};-,v_{[n]\setminus\{1,j\}})\big)(v_1)\\
&\quad+\delta^*\big(\mathcal{W}_{0,2}(-t,t_j;-,v_j),\mathcal{W}_{g,n-1}(t,t_{[n]\setminus\{1,j\}};-,v_{[n]\setminus\{1,j\}})\big)(v_1)\\
&\qquad=-\Big(\frac{1}{(t+t_j)^2}+\frac{1}{(t-t_j)^2}\Big)\Omega_{g,n}(v_1,\dots,v_n)\\
&\qquad\quad\times w^D_{g,n-1}(t,t_2,\dots,\widehat{t_j},\dots,t_n)\,dt\,dt\,dt_2\cdots\widehat{dt_j}\cdots dt_n,
\end{aligned}
$$

the residues at $\pm t_1$ contributes

$$
-\frac{(t_1^2-1)^3(t_1^2+t_j^2)}{16t_1^2(t_1^2-t_j^2)^2}\,w^D_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n)\,\Omega_{g,n}(v_1,\dots,v_n),
$$

which by ECA1 axiom of equation (2.3) is equal to

$$-\frac{(t_1^2-1)^3(t_1^2+t_j^2)}{16t_1^2(t_1^2-t_j^2)^2}\, w^D_{g,n-1}(t_1,\dots,\widehat{t_j},\dots,t_n)\,\Omega_{g,n-1}(v_1\cdot v_j, v_{[n]\setminus\{1,j\}}).$$

This is the same as the third line of (6.1).

Within the contour γ, there are second order poles at $\pm t_j$ for each $j\geq 2$ which come from $(0,2)$ unstable cases, using ECA 1 and ECA 2 axioms of equations (2.3) and (2.6) respectively, we calculate

$$\begin{aligned}
&\frac{1}{64}\frac{1}{2\pi i}\int_\gamma\left(\frac{1}{t+t_1}+\frac{1}{t-t_1}\right)\frac{(t^2-1)^3}{t^2}\\
&\qquad\times\sum_{j=2}^{n}\Big(\delta^*\big(\mathfrak{w}_{0,2}(t,t_j;-,v_j),\mathfrak{w}_{g,n-1}(-t,t_{[n]\setminus\{1,j\}};-,v_{[n]\setminus\{1,j\}})\big)(v_1)\\
&\qquad+\delta^*\big(\mathfrak{w}_{0,2}(-t,t_j;-,v_j),\mathfrak{w}_{g,n-1}(t,t_{[n]\setminus\{1,j\}};-,v_{[n]\setminus\{1,j\}})\big)(v_1)\Big)\\
&=-\frac{1}{32}\frac{\partial}{\partial t_j}\Big(\Big(\frac{1}{t_j+t_1}+\frac{1}{t_j-t_1}\Big)\frac{(t_j^2-1)^3}{t_j^2}w^D_{g,n-1}(t_j,t_2,\dots,\widehat{t_j},\dots,t_n)\,\Omega_{g,n}(v_1,\dots,v_n)\Big)\\
&=-\frac{1}{16}\frac{\partial}{\partial t_j}\Big(\frac{1}{t_j^2-t_1^2}\,\frac{(t_j^2-1)^3}{t_j}\,w^D_{g,n-1}(t_j,t_2,\dots,\widehat{t_j},\dots,t_n)\,\Omega_{g,n-1}(v_1\cdot v_j,v_{[n]\setminus\{1,j\}})\Big).
\end{aligned}$$

This gives the second line of (6.1). We have thus completed the proof of Theorem 4.4.

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Daniel Hernández Serrano: Department of Mathematics and IUFFYM, University of Salamanca, Salamanca 37008, Spain.
E-mail: dani@usal.es

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