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On the Navier–Stokes equations in scaling-invariant spaces in any dimension

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Abstract. We study the Navier–Stokes equations with a dissipative term that is generalized through a fractional Laplacian in any dimension higher than two. We extend the horizontal Biot–Savart law beyond dimension three. Using the anisotropic Littlewood–Paley theory with which we distinguish the first two directions from the rest, we obtain a blow-up criteria for its solution in norms which are invariant under the rescaling of these equations. The proof goes through for the classical Navier–Stokes equations if dimension is three, four or five. We also give heuristics and partial results toward further improvement.

1. Introduction, statement of main results, heuristics of the proof

1.1. Regularity criteria and the four-dimensional case

Let us denote by $u\colon \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}^N$, and $\pi\colon \mathbb{R}^N \times \mathbb{R}^+ \mapsto \mathbb{R}$ the fluid velocity and pressure fields, respectively. Furthermore we let $\nu > 0$ represent the viscosity coefficient and study the following N-dimensional generalized Navier–Stokes equations (g-NSE):

$$\frac{\partial u}{\partial t} + (u\cdot\nabla)u + \nabla\pi + \nu\Lambda^{2\alpha}u = 0, \tag{1.1a}$$

$$\nabla\cdot u = 0, \quad u(x,0) \triangleq u_0(x). \tag{1.1b}$$

where Λ^r is the fractional Laplacian, with exponent $r \in \mathbb{R}^+$, defined via the Fourier transform so that the Fourier multiplier is $m(\xi) = |\xi|^r$; i.e.,

$$\widehat{\Lambda^r f}(\xi) = |\xi|^r \hat{f}(\xi).$$

Hereafter, without loss of generality we assume $\nu = 1$, refer to the g-NSE with $\alpha = 1$ as the classical NSE, and write ∂_t for $\partial/\partial t$.

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We emphasize that our result includes the case $\alpha = 1$ and hence the classical NSE. The fractional Laplacian could give the impression to be just of purely mathematical interest and not physical, but this is not the case since it arises naturally in the study of equations in fluid mechanics and geophysics such as the two-dimensional surface quasi-geostrophic equations (see e.g. equation (1) of [12]). Moreover, studying the generalized formulation with fractional Laplacian has allowed us to gain deeper understanding of many equations, e.g. the NSE ([31]), the magneto-hydrodynamics (MHD) system ([32]), Boussinesq equations ([20]), Burgers equations ([22]), or the incompressible porous media equation governed by Darcy's law ([7]).

In case $\alpha < 1/2 + N/4$ and $N \geq 3$, it remains unknown whether or not the solution will preserve sufficiently smooth initial regularity, e.g. $u_0 \in H^s(\mathbb{R}^N)$ with $s \geq N/2 + 1 - 2\alpha$. There are various ways to explain why, and we choose to elaborate below on the reason based on the bounded quantity (1.2) and rescaling. It is clear that if $u(x,t)$ and $\pi(x,t)$ satisfy the g-NSE (1.1a)–(1.1b), then $u_\lambda(x,t)$ and $\pi_\lambda(x,t)$ given by

$$u_\lambda(x,t) \triangleq \lambda^{2\alpha-1} u(\lambda x, \lambda^{2\alpha} t) \quad \text{and} \quad \pi_\lambda(x,t) \triangleq \lambda^{4\alpha-2}\pi(\lambda x, \lambda^{2\alpha} t)$$

for any $\lambda \in \mathbb{R}^+$, also satisfy (1.1a)–(1.1b). Now we may take $L^2(\mathbb{R}^N)$-inner products of (1.1a) with u, and use the divergence-free condition from (1.1b) to obtain the following bound on the kinetic energy and the cumulative kinetic energy dissipation for any solution u in a time interval $[0,T]$:

$$\sup_{t\in[0,T]} \|u(t)\|_{L^2}^2 + \int_0^T \|\Lambda^\alpha u\|_{L^2}^2 \, d\tau \leq \|u_0\|_{L^2}^2. \tag{1.2}$$

Using the homogeneous Sobolev embedding $\dot{H}^\alpha(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2\alpha}}(\mathbb{R}^N)$, and assuming $\alpha < N/2$, we see that

$$\|u\|^2_{L^2_T L_x^{\frac{2N}{N-2\alpha}}} = \int_0^T \|u\|^2_{L^{\frac{2N}{N-2\alpha}}} \, d\tau < \infty$$

due to (1.2) if $u_0 \in L^2(\mathbb{R}^N)$. The norm $\|\cdot\|_{L^2_T L_x^{\frac{2N}{N-2\alpha}}}$ is invariant under the rescaling of (1.1a)–(1.1b); i.e.,

$$\int_0^T \|u_\lambda\|^2_{L^{\frac{2N}{N-2\alpha}}} \, d\tau = \int_0^{\lambda^{2\alpha}T} \|u\|^2_{L^{\frac{2N}{N-2\alpha}}} \, d\tau \quad \text{if and only if } \alpha = \frac{1}{2} + \frac{N}{4}.$$

This computation is valid for $N = 2$ and informally it explains how the global well-posedness for the classical NSE, the g-NSE with $\alpha = 1$ so that $\alpha = 1/2 + N/4$, was achieved by Leray in [25].

Experts in this direction of research classify the g-NSE as subcritical, critical and supercritical if

$$\alpha > \frac{1}{2} + \frac{N}{4}, \quad \alpha = \frac{1}{2} + \frac{N}{4} \quad \text{and} \quad \alpha < \frac{1}{2} + \frac{N}{4},$$

respectively. In both the critical and subcritical cases, the positive result toward the global regularity issue is well known (see, e.g., [32]).

In order to solve the system (1.1a)–(1.1b) in dimensions higher than two, various results have appeared, one of the most prominent being the so-called Serrin-type regularity criteria. It states that if $u(x,t)$ is a weak solution to the N-dimensional classical NSE in $[0,T]$ and

$$u \in L_T^r L_x^p, \quad \frac{N}{p} + \frac{2}{r} \leq 1, \ p \geq N, \tag{1.3}$$

then the solution is smooth in $[0,T]$ (see [30], [16] for the case $N = p = 3$, subsequently generalized to higher dimensions in [14]); the case of $\alpha \in [1, 3/2]$, $N = 3$ for the g-NSE (1.1a)–(1.1b) is shown in [41]. We remark that the norm $\|\cdot\|_{L_T^r L_x^p}$ is scaling-invariant for the classical NSE and the g-NSE (1.1a)–(1.1b) when $N/p + 2/r = 1$ and $N/p + 2\alpha/r = 2\alpha - 1$, respectively. We also mention the results of [3] which state that if u is a weak solution to the classical NSE in $[0,T]$ and

$$\nabla u \in L_T^r L_x^p, \quad \frac{N}{p} + \frac{2}{r} \leq 2, \ r \in \left(1, \min\left\{2, \frac{N}{N-2}\right\}\right], \tag{1.4}$$

then u is regular (see [41] for the case of the g-NSE (1.1a)–(1.1b)). Again, we emphasize that the norm $\|\nabla \cdot\|_{L_T^r L_x^p}$ is scaling-invariant precisely when $N/p + 2/r = 2$.

Because $u = (u^1, \ldots, u^N)$ and $\nabla = (\partial_1, \ldots, \partial_N)$, we have N-many and N^2-many conditions in (1.3) and (1.4), respectively. We now mention some results in the effort to reduce the number of such conditions. The authors in [23] proved that if u solves the three-dimensional classical NSE in $[0,T]$ and

$$u^3 \in L_T^r L_x^p, \quad \frac{3}{p} + \frac{2}{r} \leq \frac{5}{8}, \ \frac{54}{23} \leq r \leq \frac{18}{5}, \tag{1.5}$$

then it is smooth up to time T (see also [5]). We emphasize here that no condition is imposed on u^1, u^2; unfortunately, the condition $3/p + 2/r \leq 5/8$ in (1.5) disallows the scaling-invariant level because $5/8 < 1$. There are many other results reducing the number of components, all of which we cannot mention. Interestingly it is not impossible that the scaling-invariant level is actually kept. For example, due to the work of [2], it is well known that the solution to the three-dimensional classical NSE has no blow-up in $[0,T]$ if $\nabla \times u \in L_T^1 L_x^\infty$; we point out that the norm

$$\|\nabla \times \cdot\|_{L_T^r L_x^p}$$

is actually scaling-invariant when $3/p + 2/r = 2$. Subsequently, the authors in [8] showed that if u is a weak solution to the three-dimensional classical NSE in $[0,T]$, and

$$\sum_{k=1}^{2} (\nabla \times u) \cdot e^k \in L_T^r L_x^p, \quad \frac{3}{p} + \frac{2}{r} \leq 2, \ \frac{3}{2} < p < \infty,$$

where e^k is a standard basis element of $\mathbb{R}^3$, then u is a classical solution. Moreover, the authors in [24] obtained the following regularity criterion for the three-dimensional classical NSE:

$$\partial_3 u \in L_T^r L_x^p, \quad \frac{3}{p} + \frac{2}{r} \leq 2, \ 2 \leq r \leq 3, \tag{1.6}$$

for which the endpoint $3/p + 2/r = 2$ allows the scaling-invariance. On the other hand, results such as those of [6], [42] reduced components furthermore from $\partial_3 u$ of (1.6) to $\partial_3 u^3$, but not at the scaling-invariant level. It remains unknown whether the original result of Serrin for the three-dimensional classical NSE in (1.3) may be reduced to u^3 at the scaling-invariant level. Very recently, Chemin and Zhang in [10] showed that if a blow-up occurs at $T^* > 0$, then

$$\int_0^{T^*} \|u^3\|^p_{\dot{H}^{1/2+2/p}} d\tau = +\infty, \quad p \in (4,6), \tag{1.7}$$

where we emphasize that the norm $\|\cdot\|_{L^p_T \dot{H}^{1/2+2/p}}$ is scaling-invariant; later on, the range of p was improved in [11].

We now discuss the difficulty of extending these results to higher dimensions and in particular the mathematical significance of the four-dimensional case for the fluid equations such as (1.1a)–(1.1b). Essentially due to the Sobolev embedding, proofs of classical results on the four-dimensional NSE turn out to be *barely* successful. However, upon trying to extend more recent results on the three-dimensional NSE such as [10] to the four-dimensional case, the ranges of parameters such as p, r in (1.6) and (1.7) and many others in the necessary estimates such as Propositions A.1, A.2, A.3, A.4 and A.5 become empty.

Long ago, it was realized the relevant role of the four-dimensional case. We quote "$m = 4$ is critical case and the proof would not work for $m > 4$" (Kato, Section 4 of [21]). The study of Scheffer in [29] is also focused strictly on the four-dimensional case; how certain methods in partial regularity theory work for the four-dimensional classical NSE and six-dimensional stationary NSE, but not in any higher dimension, is discussed in [13], Remark 1.3, which we also quote: "four is the highest dimension in which we have such condition. In five or higher dimensional, such condition fails. Therefore, we cannot hope the existing methods work in five or higher dimensional case" (see also p. 2212 of [15]).

In fact, to the best of the author's knowledge, in terms of component reduction theory of the Serrin-type regularity criteria, every result that we have mentioned thus far, and all others in the literature, deal with the three-dimensional case. The only exception is the very recent work [39] which in particular derived the following regularity criteria for the four-dimensional classical NSE:

$$u^3, u^4 \in L^r_T L^p_x, \quad \frac{4}{p} + \frac{2}{r} \leq \frac{1}{p} + \frac{1}{2}, \; 6 < p \leq \infty. \tag{1.8}$$

Recall that $H^1(\mathbb{R}^N) \hookrightarrow L^N(\mathbb{R}^N)$ only for $N \leq 4$ but not for $N > 4$; in fact, $\dot{H}^1(\mathbb{R}^4) \hookrightarrow L^4(\mathbb{R}^4)$. This is why somehow the four-dimensional case is at the threshold between dimensions in which $H^1(\mathbb{R}^N)$ leads to bounds on $L^N(\mathbb{R}^N)$ and those in which it does not. The difficulty of obtaining a criteria such as (1.8) is described with much detail in the introduction of [39], Section 1.2 and Remark 2.1 (2), (3) of [40]. Unfortunately this criteria in (1.8) is not at the scaling-invariant level, and it is expected that improving the upper bound of $1/p + 1/2$ in (1.8) back up to 1 is extremely difficult. We conclude by emphasizing that any extension of the component reduction theory of the Serrin-type regularity criteria to dimension higher than four should be harder than to the four dimensional case.

1.2. Statement of main results

Throughout the rest of this manuscript, in any dimension $N \geq 3$, although not physical and arguably not universally acceptable, we shall refer to the first two directions as "horizontal" and the rest of the $(N-2)$-many directions as "vertical". This will make the notations significantly simpler. In particular, we write $x = (x_h, x_v)$, where

$$x_h \triangleq (x_1, x_2, 0, \dots, 0) \quad \text{and} \quad x_v \triangleq (0, 0, x_3, \dots, x_N).$$

Now we shall extend the notions of three-dimensional anisotropic Sobolev and Besov norms (e.g. [10]) to the N-dimensional case which has been studied by many for long time (see, e.g., p. 51 of [18], [19] and [28]). We let $\mathcal{S}$ denote the Schwartz space and $\mathcal{S}'$ its dual.

Definition 1.1. For $(s, s') \in \mathbb{R}^2$, we let $\dot{H}^{s,s'}(\mathbb{R}^N) = \{f \in \mathcal{S}' : \|f\|_{\dot{H}^{s,s'}} < \infty\}$ where for $\xi = (\xi_h, \xi_v), \xi_h = (\xi_1, \xi_2, 0, \dots, 0)$ and $\xi_v = (0, 0, \xi_3, \dots, \xi_N)$,

$$\|f\|_{\dot{H}^{s,s'}}^2 \triangleq \int_{\mathbb{R}^N} \left| |\xi_h|^s \, |\xi_v|^{s'} \, |\hat{f}(\xi)| \right|^2 d\xi < \infty.$$

We denote by $\omega^3 = \partial_1 u^2 - \partial_2 u^1$ and present our main result; concerning the notations of Littlewood–Paley theory within Theorem 1.1, we refer readers to the Section 2.

Theorem 1.1. *Let the dimension $N \in \mathbb{N}, N \geq 4$, and $u_0 \in \dot{H}^{N/2+1-2\alpha}(\mathbb{R}^N)$ where*

$$\alpha \in \left[\frac{N}{6} + \delta, \frac{N}{6} + \frac{2}{3}\right]$$

for any $\delta > 0$. Then there exists $T^* > 0$ such that on $[0, T^*)$, the g-NSE (1.1a)–(1.1b) possesses a unique solution

$$u \in C([0, T^*); H^{N/2+1-2\alpha}(\mathbb{R}^N)) \cap L^2((0, T^*); H^{N/2+1-\alpha}(\mathbb{R}^N)).$$

Moreover, if

$$\sum_{m=3}^{N} \int_0^{T^*} \|u^m\|_{\dot{H}^{N/2+1-2\alpha+2\alpha/p}}^p + \|\Lambda^{N/2-\alpha}\partial_m u^m\|_{\dot{H}^{-\epsilon,\epsilon}}^2 + \left|\|(\|\dot{\Delta}_k^h \Lambda^{N/2-\alpha}\omega^3\|_{L^2})_k\|_{l^1}\right|^2 d\tau$$

$$+ \sup_{t \in [0,T^*]} \|\omega^3(t)\|_{L^2}^{p(3-\frac{2}{p}-\frac{N}{2\alpha})} \int_0^{T^*} \|\Lambda^\alpha \omega^3\|_{L^2}^{p(-2+\frac{2}{p}+\frac{N}{2\alpha})} d\tau = \infty \tag{1.9}$$

for some $\epsilon \in (0, (N-2)/2)$ and p such that

$$\frac{2\alpha}{3\alpha - N/2} < p \begin{cases} < \frac{4\alpha}{4\alpha - N} & \text{if } \alpha > N/4, \\ < +\infty & \text{if } \alpha = N/4, \\ \leq +\infty & \text{if } \alpha < N/4, \end{cases}$$

then $T^* < \infty$.

Considering the range of $\alpha \geq N/6 + \delta$ and $\delta > 0$ within the statement of Theorem 1.1, we see that when the dimension is four or five, we may take $\alpha = 1$. In that respect, we point out the following corollary, that holds for the classical NSE.

Corollary 1.2. *Let the dimension $N = 4$ or 5, and let $u_0 \in \dot{H}^{N/2-1}(\mathbb{R}^N)$. Then there exists $T^* > 0$ such that, on $[0, T^*)$, the classical NSE has a unique solution*

$$u \in C([0,T^*); H^{N/2-1}(\mathbb{R}^N)) \cap L^2((0,T^*); H^{N/2}(\mathbb{R}^N)).$$

Moreover, if

$$\sum_{m=3}^{N} \int_0^{T^*} \|u^m\|^p_{\dot{H}^{N/2-1+2/p}} + \|\Lambda^{N/2-1}\partial_m u^m\|^2_{\dot{H}^{-\epsilon,\epsilon}} + \left|\|(\|\dot{\Delta}_k^h \Lambda^{N/2-1}\omega^3\|_{L^2})_k\|_{l^1}\right|^2 d\tau$$
$$+ \sup_{t\in[0,T^*]} \|\omega^3(t)\|_{L^2}^{p(3-\frac{2}{p}-\frac{N}{2})} \int_0^{T^*} \|\Lambda\omega^3\|_{L^2}^{p(-2+\frac{2}{p}+\frac{N}{2})} d\tau = \infty$$

for some p such that

$$\frac{2}{3-N/2} < p \begin{cases} < +\infty & \text{if } 1 = N/4, \\ \leq +\infty & \text{if } 1 < N/4, \end{cases}$$

then $T^* < \infty$.

Remark 1.1. 1) All the norms in (1.9), specifically

$$\int_0^T \|\cdot\|^p_{\dot{H}^{N/2+1-2\alpha+2\alpha/p}} d\tau, \qquad \sum_{m=3}^{N} \int_0^T \|\Lambda^{N/2-\alpha}\partial_m \cdot\|^2_{\dot{H}^{-\epsilon,\epsilon}} d\tau,$$
$$\int_0^T \left|\|(\|\dot{\Delta}_k^h \Lambda^{N/2-\alpha}(\partial_1((\cdot)\cdot e^2) - \partial_2((\cdot)\cdot e^1))\|_{L^2})_k\|_{l^1}\right|^2 d\tau,$$
$$\sup_{t\in[0,T]} \|\partial_1((\cdot)\cdot e^2) - \partial_2((\cdot)\cdot e^1)\|_{L^2}^{p(3-\frac{2}{p}-\frac{N}{2\alpha})}$$
$$\times \int_0^T \|\Lambda^\alpha(\partial_1((\cdot)\cdot e^2) - \partial_2((\cdot)\cdot e^1))\|_{L^2}^{p(-2+\frac{2}{p}+\frac{N}{2\alpha})} d\tau,$$

are invariant under the rescaling of (1.1a)–(1.1b). Moreover, the proof goes through in the three-dimensional case, but as the results in [10], [11] are better, we chose not to include such results in the statement of Theorem 1.1.

2) We note that in [17], the local well-posedness of the classical NSE in a critical Sobolev space $\dot{H}^{1/2}(\mathbb{R}^3)$ was obtained. In [10] the authors needed an additional condition on the initial data such that $\nabla \times u|_{t=0} \in L^{3/2}(\mathbb{R}^3)$ where $\dot{W}^{1,3/2}(\mathbb{R}^3) \hookrightarrow \dot{H}^{1/2}(\mathbb{R}^3)$ by the homogeneous Sobolev embedding. In contrast, our initial data space $\dot{H}^{N/2+1-2\alpha}(\mathbb{R}^N)$ is the critical Sobolev space and no additional condition is needed.

3) The anisotropic Sobolev space $\dot{H}^{-\epsilon,\epsilon}$, $\epsilon \in (0,(N-2)/2)$, may be considered as an analogue of $\mathcal{H}_\theta \triangleq \dot{H}^{-1/2+\theta,-\theta}$ for $\theta \in (0,1/2)$ in [10].

4) The results of [10], [11] have been generalized to the three-dimensional MHD system in [37], [26] respectively, and some generalization of Theorem 1.1 to the MHD system may be done as well.

1.3. Heuristic toward eliminating the condition of ω^3 and $\partial_m u^m, m = 3, \dots N$ in (1.9)

The purpose of this subsection is to discuss the idea of the proof of Theorem 1.1 and also the difficulty of eliminating the condition of ω^3 and $\partial_m u^m$, $m = 3, \dots, N$, aiming at a complete extension of the result in [10]. Firstly, one of the most crucial ingredients in the work of [10] was the following decomposition: for $f = (f^1, f^2, f^3)$ such that $\nabla \cdot f = 0, \Delta_h \triangleq \sum_{k=1}^2 \partial_k^2$, with $e^3 = (0, 0, 1)$, we may rewrite the horizontal components of f as

$$(f^1, f^2, 0) = (-\partial_2, \partial_1, 0)\Delta_h^{-1}\nabla \times f \cdot e^3 - (\partial_1, \partial_2, 0)\Delta_h^{-1}\partial_3 f^3$$

(see also [38]). In order to even start considering the higher dimensional extension of the result in [10], it seems that one needs to extend this identity and in particular $\nabla \times u \cdot e^3$ to an appropriate analogue in the higher dimensional case. However, a cross product and a curl operator are meaningful at least physically and formally only in $\mathbb{R}^3$. In the higher dimensional case, the curl of a vector field in any dimension is usually defined as an antisymmetric 2-tensor; however, its form seems too complicated for our purpose. Fortunately the following observation, whose proof via Fourier transform is straightforward, appeared in [40].

Lemma 1.3 (Proposition 1.1, [40]). *Suppose* $f = (f^1, \dots, f^N) \in C^\infty(\mathbb{R}^N)$ *such that* $\nabla \cdot f = 0$. *Under the notation of* $f^h \triangleq (f^1, f^2, 0, \dots, 0)$, $\nabla_h \triangleq (\partial_1, \partial_2, 0, \dots, 0)$, $\nabla_h^\perp \triangleq (-\partial_2, \partial_1, 0, \dots, 0)$ *and* $\Delta_h = \sum_{k=1}^2 \partial_k^2$, *we may write*

$$f^h = f^h_{\text{curl}} + f^h_{\text{div}}$$

where

$$f^h_{\text{curl}} \triangleq \nabla_h^\perp \Delta_h^{-1}(\partial_1 f^2 - \partial_2 f^1), \quad f^h_{\text{div}} \triangleq -\nabla_h \Delta_h^{-1} \sum_{k=3}^N \partial_k f^k.$$

Remark 1.2. To the best of the author's knowledge, this identity in the three-dimensional case was first used implicitly within a certain *a priori* estimate in [27] (see (2.1) of [27]). We also remark that there is a discussion of higher dimensional curl operator on p. 8 of [9], although we failed to find any immediate application such as Lemma 1.3.

Now in the three-dimensional case, one can just apply a curl operator on (1.1a) and consider its third component. We cannot readily follow the same approach due to the lack of precise formulation of a higher dimensional curl operator. However, thanks to Lemma 1.3, we do not necessarily have to take a curl operator in higher dimension but only need to estimate $\omega^3 \triangleq \partial_1 u^2 - \partial_2 u^1$. Thus, we apply ∂_1, ∂_2 on the second and first components of (1.1a), respectively. This leads to

$$\partial_t(\partial_1 u^2) + \partial_1((u\cdot\nabla)u^2) + \partial_{12}\pi + \Lambda^{2\alpha}\partial_1 u^2 = 0,$$
$$\partial_t(\partial_2 u^1) + \partial_2((u\cdot\nabla)u^1) + \partial_{21}\pi + \Lambda^{2\alpha}\partial_2 u^1 = 0.$$

In the three-dimensional case, $\nabla\pi$ disappears because $\nabla\times(\nabla f) = 0$ for all scalar-valued function f. In the higher dimensional case, we may make use of the fact that $\partial_{12}\pi = \partial_{21}\pi$ and deduce

$$\partial_t\omega^3 + \partial_1((u\cdot\nabla)u^2) - \partial_2((u\cdot\nabla)u^1) + \Lambda^{2\alpha}\omega^3 = 0, \tag{1.10}$$

where $\omega^3 = \partial_1 u^2 - \partial_2 u^1$. Moreover, we can rewrite

$$\begin{aligned}
&\partial_2((u\cdot\nabla)u^1) - \partial_1((u\cdot\nabla)u^2)\\
&= (\partial_2 u\cdot\nabla)u^1 - (\partial_1 u\cdot\nabla)u^2 - (u\cdot\nabla)\,\omega^3\\
&= \partial_2 u^1\partial_1 u^1 + \partial_2 u^2\partial_2 u^1 + \sum_{k=3}^N \partial_2 u^k\partial_k u^1\\
&\quad - \partial_1 u^1\partial_1 u^2 - \partial_1 u^2\partial_2 u^2 - \sum_{k=3}^N \partial_1 u^k\partial_k u^2 - (u\cdot\nabla)\,\omega^3\\
&= -\partial_1 u^1\omega^3 - \partial_2 u^2\omega^3 + \Big(\sum_{k=3}^N \partial_2 u^k\partial_k u^1 - \partial_1 u^k\partial_k u^2\Big) - (u\cdot\nabla)\,\omega^3\\
&= \Big(\sum_{k=3}^N \partial_k u^k\Big)\omega^3 + \Big(\sum_{k=3}^N \partial_2 u^k\partial_k u^1 - \partial_1 u^k\partial_k u^2\Big) - (u\cdot\nabla)\,\omega^3
\end{aligned} \tag{1.11}$$

where we used divergence-free condition (1.1b). Applying (1.11) to (1.10) leads to

$$\partial_t\omega^3 + (u\cdot\nabla)\,\omega^3 - \Lambda^{2\alpha}\omega^3 = (\sum_{k=3}^N \partial_k u^k)\,\omega^3 + \sum_{k=3}^N \partial_2 u^k\partial_k u^1 - \partial_1 u^k\partial_k u^2. \tag{1.12}$$

As we will see, e.g. in (4.3), (4.4), in addition to the bound on ω^3, we will also wish to obtain a bound on $\partial_k u^k, k = 3,\dots,N$. Now it is well known that applying divergence operator on (1.1a) leads to $\pi = (-\Delta)^{-1}\sum_{k,m=1}^N \partial_k u^m\partial_m u^k$ due to the divergence-free property (1.1b). Thus, we may write, from the g-NSE (1.1a),

$$\partial_t u^l + (u\cdot\nabla)u^l + \Lambda^{2\alpha}u^l = -\partial_l(-\Delta)^{-1}\sum_{k,m=1}^N \partial_k u^m\,\partial_m u^k \tag{1.13}$$

for $l\in\{3,\dots,N\}$. Applying ∂_l to (1.13) leads to

$$\begin{aligned}
&\partial_t\partial_l u^l + \Lambda^{2\alpha}\partial_l u^l\\
&\qquad = -(\partial_l u\cdot\nabla)u^l - (u\cdot\nabla)\partial_l u^l - \partial_l^2(-\Delta)^{-1}\sum_{k,m=1}^N \partial_k u^m\,\partial_m u^k.
\end{aligned} \tag{1.14}$$

We note that if one tries to estimate $\partial_k u^k, k = 3, \dots, N$, one sees immediately that in contrast to the three-dimensional case, it will be necessary to do additional estimates of the mixed terms, e.g. $\partial_3 u^4$ and $\partial_4 u^3$ in the four-dimensional case.

Now heuristically we can write the right-hand side of (1.12) as

$$\sum_{k=3}^{N} \partial_k u^k \omega^3 + \partial_2 u^k \partial_k (u^1_{\text{curl}} + u^1_{\text{div}}) - \partial_1 u^k \partial_k (u^2_{\text{curl}} + u^2_{\text{div}}) \tag{1.15}$$

due to Lemma 1.3. As $u^1_{\text{curl}}, u^2_{\text{curl}}$ consist of ω^3, we see that $\partial_k u^k \omega^3, \partial_2 u^k \partial_k u^1_{\text{curl}}$ and $\partial_1 u^k \partial_k u^2_{\text{curl}}$ in (1.15) are linear in ω^3 while because $u^1_{\text{div}}, u^2_{\text{div}}$ do not consist of ω^3, we may consider $\partial_2 u^k \partial_k u^1_{\text{div}}, \partial_1 u^k \partial_k u^2_{\text{div}}$ to be just forcing terms in the time evolution equation (1.12) of ω^3. On the other hand, in (1.14),

$$\begin{aligned} & -\partial_l^2 (-\Delta)^{-1} \sum_{k,m=1}^{N} \partial_k u^m \partial_m u^k \\ & = -\partial_l^2 (-\Delta)^{-1} \Big[\sum_{k,m=1}^{2} \partial_k u^m \partial_m u^k + \sum_{k=1}^{2} \sum_{m=3}^{N} \partial_k u^m \partial_m u^k + \sum_{k=3}^{N} \sum_{m=1}^{N} \partial_k u^m \partial_m u^k \Big] \end{aligned} \tag{1.16}$$

where due to Lemma 1.3 we have

$$\sum_{k,m=1}^{2} \partial_k u^m \, \partial_m u^k = \sum_{k,m=1}^{2} \partial_k (u^m_{\text{curl}} + u^m_{\text{div}}) \, \partial_m (u^k_{\text{curl}} + u^k_{\text{div}}).$$

Because u^m_{curl} and u^k_{curl} consist of ω^3, the quadratic terms here turn out to be $\partial_k u^m_{\text{curl}} \partial_m u^k_{\text{curl}}$. This heuristic gives the impression that perhaps the estimate involving ω^3 is somewhat easier than that involving $\partial_k u^k, k = 3, \dots, N$. We explain in the Appendix that even in the four-dimensional case, the estimate involving the former appears to be very difficult. We conclude this discussion by noting that if an estimate of $\|\omega^3(t)\|_{L^2}$ may be completed, it would immediately allow us to eliminate the condition on

$$\sup_{t \in [0, T^*]} \|\omega^3(t)\|_{L^2}^{p(3 - \frac{2}{p} - \frac{N}{2\alpha})} \int_0^{T^*} \|\Lambda^\alpha \omega^3\|_{L^2}^{p(-2 + \frac{2}{p} + \frac{N}{2\alpha})} d\tau$$

in (1.9) using Hölder's inequality.

The structure of the proof of Theorem 1.1 is as follows. Firstly, by Theorem 6.2 in [34] (see also [33]), the existence and uniqueness of the local solution

$$u \in C([0,T); \dot{H}^{N/2+1-2\alpha}(\mathbb{R}^N)) \cap L^2((0,T); \dot{H}^{N/2+1-\alpha}(\mathbb{R}^N))$$

follows. We will show that the converse of (1.9) implies

$$\sum_{k,l=1}^{N} \int_0^T \|\partial_l u^k\|_{\mathcal{B}_{p_{k,l}}}^{p_{k,l}} < \infty$$

in Section 4, which in turn implies that u remains in the regularity class of

$$C([0,T+\delta];H^{N/2+1-2\alpha}(\mathbb{R}^N))\cap L^2((0,T+\delta];H^{N/2+1-\alpha}(\mathbb{R}^N))$$

for some $\delta>0$ by Proposition 3.1 and standard continuation of local theory argument.

2. Preliminaries

2.1. Definitions, notations and past results

We write $A\lesssim_{a,b}B$ and $A\approx_{a,b}B$ when there exists a constant $c\geq 0$ of no significant dependence except on a,b such that $A\leq cB, A=cB$, respectively. For simplicity, we often write $\int f$ to denote $\int_{\mathbb{R}^N} f dx$. Recall the mixed Lebesgue spaces from [4], and notice that its order matters; i.e.,

$$\big\|\|f(\cdot,x_2)\|_{L^{p_1}(X_1,\mu_1)}\big\|_{L^{p_2}(X_2,\mu_2)}\leq\big\|\|f(x_1,\cdot)\|_{L^{p_2}(X_2,\mu_2)}\big\|_{L^{p_1}(X_1,\mu_1)}$$

for any two measure spaces $(X_1,\mu_1),(X_2,\mu_2)$ with $1\leq p_1\leq p_2\leq\infty$. Let us also recall the Littlewood–Paley decomposition. We let χ and ϕ be smooth functions such that

$$\begin{aligned}&\text{supp }\phi\subset\{\zeta\in\mathbb{R}:\tfrac{3}{4}\leq|\zeta|\leq\tfrac{8}{3}\}, &&\sum_{j\in\mathbb{Z}}\phi(2^{-j}\zeta)=1,\\&\text{supp }\chi\subset\{\zeta\in\mathbb{R}:|\zeta|\leq\tfrac{4}{3}\}, &&\chi(\zeta)+\sum_{j\geq 0}\phi(2^{-j}\zeta)=1.\end{aligned}$$

We denote the classical homogeneous and nonhomogeneous Littlewood–Paley operators: for $\xi=(\xi_h,\xi_v)\in\mathbb{R}^N$,

$$\dot{\Delta}_j f\triangleq\mathcal{F}^{-1}(\phi(2^{-j}|\xi|)\hat{f}),\qquad\dot{S}_j f\triangleq\mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\hat{f}),\tag{2.1}$$

and

$$\Delta_j f\triangleq\begin{cases}0 & \text{if } j\leq -2,\\ \mathcal{F}^{-1}(\chi(|\xi|)\hat{f}) & \text{if } j=-1,\\ \dot{\Delta}_j f & \text{if } j\geq 0,\end{cases}\tag{2.2}$$

and similarly, in the anisotropic case:

$$\dot{\Delta}_k^h f\triangleq\mathcal{F}^{-1}(\phi(2^{-k}|\xi_h|)\hat{f}),\quad\dot{S}_k^h f\triangleq\mathcal{F}^{-1}(\chi(2^{-k}|\xi_h|)\hat{f}),\tag{2.3}$$

$$\dot{\Delta}_l^v f\triangleq\mathcal{F}^{-1}(\phi(2^{-l}|\xi_v|)\hat{f}),\quad\dot{S}_l^v f\triangleq\mathcal{F}^{-1}(\chi(2^{-l}|\xi_v|)\hat{f}),\tag{2.4}$$

with

$$\Delta_k^h f\triangleq\begin{cases}0 & \text{if } k\leq -2,\\ \mathcal{F}^{-1}(\chi(|\xi_h|)\hat{f}) & \text{if } k=-1,\\ \dot{\Delta}_k^h f & \text{if } k\geq 0,\end{cases}\qquad\Delta_l^v f\triangleq\begin{cases}0 & \text{if } l\leq -2,\\ \mathcal{F}^{-1}(\chi(|\xi_v|)\hat{f}) & \text{if } l=-1,\\ \dot{\Delta}_l^v f & \text{if } l\geq 0.\end{cases}\tag{2.5}$$

We define $\mathcal{S}'_h$ to be the subspace of $\mathcal{S}'$ such that every $f \in \mathcal{S}'_h$ satisfies $\lim_{j\to-\infty} \|\dot{S}_j f\|_{L^\infty} = 0$.

Definition 2.1. For $p, q \in [1,\infty]$, $s \in \mathbb{R}$, $s < N/p$ ($s = N/p$ if $q = 1$), we define the Besov spaces $\dot{B}^s_{p,q}(\mathbb{R}^N) \triangleq \{f \in \mathcal{S}'_h : \|f\|_{\dot{B}^s_{p,q}} < \infty\}$, where

$$\|f\|_{\dot{B}^s_{p,q}} \triangleq \big\|(2^{js}\|\dot{\Delta}_j f\|_{L^p})_j\big\|_{l^q(\mathbb{Z})}.$$

Moreover, for $p \in (1,\infty)$ we shall use the notations $\mathcal{B}_p \triangleq \dot{B}^{-2\alpha+2\alpha/p}_{\infty,\infty}$.

We define the anisotropic Besov spaces $(\dot{B}^{s_1}_{p,q_1})_h(\dot{B}^{s_2}_{p,q_2})_v$ as the space of distributions in $\mathcal{S}'_h$ endowed with the norm

$$\|f\|_{(\dot{B}^{s_1}_{p,q_1})_h(\dot{B}^{s_2}_{p,q_2})_v} \triangleq \Big(\sum_{k\in\mathbb{Z}} 2^{q_1 k s_1}\Big(\sum_{l\in\mathbb{Z}} 2^{q_2 l s_2}\|\dot{\Delta}^h_k \dot{\Delta}^v_l f\|^{q_2}_{L^p}\Big)^{q_1/q_2}\Big)^{1/q_1}. \tag{2.6}$$

It is well known that $\dot{B}^s_{2,2} = \dot{H}^s$ (cf. [1]). Moreover, the special case of the anisotropic Besov spaces recovers the anisotropic Sobolev spaces:

$$(\dot{B}^{s_1}_{p,q_1})_h(\dot{B}^{s_2}_{p,q_2})_v|_{p=q_1=q_2=2} = \dot{H}^{s_1,s_2}.$$

We recall the important Bony's para-product decomposition:

$$fg = T(f,g) + T(g,f) + R(f,g), \tag{2.7}$$

where

$$T(f,g) \triangleq \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} f \dot{\Delta}_j g, \quad R(f,g) \triangleq \sum_{j\in\mathbb{Z}} \dot{\Delta}_j f \tilde{\dot{\Delta}}_j g \quad \text{and} \quad \tilde{\dot{\Delta}}_j \triangleq \sum_{l=j-1}^{j+1} \dot{\Delta}_l \tag{2.8}$$

(see e.g. [9]). We also recall the useful anisotropic Bernstein's inequalities.

Lemma 2.1. *Let $\mathcal{B}_h$ (respectively $\mathcal{B}_v$) a ball in $\mathbb{R}^2_h$ (resp. $\mathbb{R}^{N-2}_v$) and $\mathcal{C}_h$ (resp. $\mathcal{C}_v$) a ring in $\mathbb{R}^2_h$ (resp. $\mathbb{R}^{N-2}_v$), $\nabla_h = (\partial_1, \partial_2, 0, \ldots, 0), \nabla_v = (0, 0, \partial_3, \ldots, \partial_N)$. Moreover, let $1 \le p_2 \le p_1 \le \infty, 1 \le q_2 \le q_1 \le \infty$. Then*

$$\begin{aligned}
\|\nabla^\alpha_h f\|_{L^{p_1}_h(L^{q_1}_v)} &\lesssim 2^{k(|\alpha|+2(1/p_2-1/p_1))}\|f\|_{L^{p_2}_h(L^{q_1}_v)} && \text{if } \operatorname{supp}\hat{f} \subset 2^k\mathcal{B}_h,\\
\|\nabla^\beta_v f\|_{L^{p_1}_h(L^{q_1}_v)} &\lesssim 2^{l(|\beta|+(N-2)(1/q_2-1/q_1))}\|f\|_{L^{p_1}_h(L^{q_2}_v)} && \text{if } \operatorname{supp}\hat{f} \subset 2^l\mathcal{B}_v,\\
\|f\|_{L^{p_1}_h(L^{q_1}_v)} &\lesssim 2^{-kM}\sup_{|\alpha|=M}\|\nabla^\alpha_h f\|_{L^{p_1}_h(L^{q_1}_v)} && \text{if } \operatorname{supp}\hat{f} \subset 2^k\mathcal{C}_h,\\
\|f\|_{L^{p_1}_h(L^{q_1}_v)} &\lesssim 2^{-lM}\sup_{|\beta|=M}\|\nabla^\beta_v f\|_{L^{p_1}_h(L^{q_1}_v)} && \text{if } \operatorname{supp}\hat{f} \subset 2^l\mathcal{C}_v.
\end{aligned}$$

Remark 2.1. Let us remark on the complexity of the anisotropic Littlewood–Paley theory. Although we have

$$\|\dot{\Delta}_j \partial_2 f\|_{L^p} \le \|\dot{\Delta}_j \nabla f\|_{L^p} \lesssim 2^j\, \|\dot{\Delta}_j f\|_{L^p}$$

by Bernstein's inequality, bounding ∂_2 by ∇ seems not optimal. Hence, one might choose to estimate by

$$\|\dot{\Delta}_j \partial_2 f\|_{L^p} = \|\dot{\Delta}_j \sum_{k\in\mathbb{Z}} \dot{\Delta}_k^h \partial_2 f\|_{L^p} \lesssim \sum_{k\in\mathbb{Z}} \|\dot{\Delta}_j \dot{\Delta}_k^h \partial_2 f\|_{L^p} \lesssim \sum_{k\in\mathbb{Z}} 2^k \|\dot{\Delta}_j \dot{\Delta}_k^h f\|_{L^p}.$$

The difficulty here is now the bi-infinite sum over k, which leads to anisotropic Besov spaces, from which going back to the classical Besov spaces $\dot{B}^s_{p,q}$ requires several conditions; this is well described in the hypothesis of Propositions A.1, A.2. Indeed, if $p = p_1 = p_2$, then $\|f\|_{L_h^{p_1}(L_v^{p_2})} = \|f\|_{L^p}$; however, even if $q_1 = q_2$,

$$\|f\|_{(\dot{B}^{s_1}_{p,q_1})_h(\dot{B}^{s_2}_{p,q_2})} \neq \left(\sum_{j\in\mathbb{Z}} \left|2^{(s_1+s_2)j}\|\dot{\Delta}_j f\|_{L^p}\right|^q\right)^{1/q} = \|f\|_{\dot{B}^{s_1+s_2}_{p,q}}.$$

Finally, for convenience, let us recall the following general partial sum formula, as we will rely on it frequently.

Lemma 2.2. *For any $r \in \mathbb{C}\setminus\{1\}$, not necessarily requiring that $|r| < 1$, $n, m \in \mathbb{Z}$,*

$$\sum_{j=n}^{m} r^j = \frac{r^{m+1} - r^n}{r-1}.$$

3. Preliminary blow-up criterion

We now prove a preliminary blow-up criterion, which is sort of an extension of Lemma 8.1 in [10] and Proposition 4.1 in [37]. In particular, for the sake of generality, we prove the version for $\dot{H}^{N/2+1-2\alpha}(\mathbb{R}^N)$ instead of $\dot{H}^1(\mathbb{R}^3)$.

Proposition 3.1. *Suppose $N \in \mathbb{N}$, $N \geq 3$, $\alpha \in [N/6+\delta, N/6+2/3]$, where $\delta > 0$ and u is a smooth solution for the g-NSE (1.1a)–(1.1b) over time interval $[0,T)$ starting from $u_0 \in \dot{H}^{N/2+1-2\alpha}(\mathbb{R}^N)$. Then*

$$\sup_{t\in[0,T)} \|u(t)\|^2_{\dot{H}^{N/2+1-2\alpha}(\mathbb{R}^N)} + \int_0^T \|\Lambda^\alpha u\|^2_{\dot{H}^{N/2+1-2\alpha}(\mathbb{R}^N)} d\tau \lesssim e^{\sum_{k,l=1}^N \int_0^T \|\partial_l u^k\|^{p_{k,l}}_{\mathcal{B}_{p_{k,l}}} d\tau} \|u_0\|^2_{\dot{H}^{N/2+1-2\alpha}(\mathbb{R}^N)},$$

where $\mathcal{B}_{p_{k,l}} = \dot{B}^{-2\alpha+2\alpha/p_{k,l}}_{\infty,\infty}$, $p_{k,l} \in (1,\infty)$.

Remark 3.1. In contrast to the typical blow-up criterion, the key feature of this estimate is that it allows us to choose different $p_{k,l}, k, l \in \{1,\ldots,N\}$. We will see that this is crucial in (4.2), (4.3), and (4.9).

Proof. We apply $\dot{\Delta}_j$ on the g-NSE (1.1a) and take L^2-inner products with $\dot{\Delta}_j u$ to obtain

$$\frac{1}{2}\partial_t \|\dot{\Delta}_j u\|^2_{L^2} + \|\dot{\Delta}_j \Lambda^\alpha u\|^2_{L^2} = -\int \dot{\Delta}_j((u\cdot\nabla)u)\cdot\dot{\Delta}_j u. \tag{3.1}$$

We multiply (3.1) by $2^{2j(\frac{N}{2}+1-2\alpha)}$ and sum over $j \in \mathbb{Z}$ to compute

$$\frac{1}{2}\,\partial_t\|u\|^2_{\dot{H}^{N/2+1-2\alpha}} + \|\Lambda^\alpha u\|^2_{\dot{H}^{N/2+1-2\alpha}} = -((u\cdot\nabla)u|u)_{\dot{H}^{N/2+1-2\alpha}}\,, \tag{3.2}$$

where we may use Bony's paraproducts (2.7) to write

$$\begin{aligned}|(u^l\partial_l u^k|u^k)_{\dot{H}^{N/2+1-2\alpha}}| \le{}& \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)}|(\dot{\Delta}_j T(u^l,\partial_l u^k)|\dot{\Delta}_j u^k)|\\ &+\sum_j 2^{2j(\frac{N}{2}+1-2\alpha)}|(\dot{\Delta}_j T(\partial_l u^k,u^l)|\dot{\Delta}_j u^k)|\\ &+\sum_j 2^{2j(\frac{N}{2}+1-2\alpha)}|(\dot{\Delta}_j R(u^l,\partial_l u^k)|\dot{\Delta}_j u^k)| \triangleq I_1+I_2+I_3.\end{aligned} \tag{3.3}$$

We start with

$$\begin{aligned}I_1 &= \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)}|(\dot{\Delta}_j(\sum_{j':|j-j'|\le 4}\dot{S}_{j'-1}u^l\dot{\Delta}_{j'}\partial_l u^k)|\dot{\Delta}_j u^k)|\\ &\le \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)}\Big|\int \dot{S}_{j-1}u^l\dot{\Delta}_j\partial_l u^k\dot{\Delta}_j u^k\Big|\\ &\quad + 2^{2j(\frac{N}{2}+1-2\alpha)}\Big|\int\sum_{j':|j-j'|\le 4}[\dot{\Delta}_j,\dot{S}_{j'-1}u^l]\,\dot{\Delta}_{j'}\partial_l u^k\dot{\Delta}_j u^k\Big|\\ &\quad + 2^{2j(\frac{N}{2}+1-2\alpha)}\Big|\int\sum_{j':|j-j'|\le 4}(\dot{S}_{j'-1}u^l-\dot{S}_{j-1}u^l)\dot{\Delta}_j\dot{\Delta}_{j'}\partial_l u^k\dot{\Delta}_j u^k\Big|\\ &\triangleq I_{1,1}+I_{1,2}+I_{1,3}\end{aligned} \tag{3.4}$$

(cf. (2.16), (2.17) in [35]) from (3.3) where we used (2.8) and that we may write $\dot{\Delta}_j\partial_l u^k = \sum_{j':|j-j'|\le 4}\dot{\Delta}_j\dot{\Delta}_{j'}\partial_l u^k$. Firstly, due to the divergence-free property of u, we have from (3.4),

$$I_{1,1} = \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)}\Big|\int \frac{1}{2}\dot{S}_{j-1}u^l\partial_l(\dot{\Delta}_j u^k)^2\Big| = 0. \tag{3.5}$$

Next, from (3.4),

$$\begin{aligned}I_{1,2} &\le \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)}\sum_{j':|j-j'|\le 4}\|[\dot{\Delta}_j,\dot{S}_{j'-1}u^l]\dot{\Delta}_{j'}\partial_l u^k\|_{L^2}\|\dot{\Delta}_j u^k\|_{L^2}\\ &\lesssim \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)}2^{-j}\sum_{j':|j-j'|\le 4}\|\nabla\dot{S}_{j'-1}u^l\|_{L^\infty}\,\|\dot{\Delta}_{j'}\partial_l u^k\|_{L^2}\,\|\dot{\Delta}_j u^k\|_{L^2}\\ &\lesssim \sum_j\sum_{l'=1}^N 2^{j(N+1-4\alpha)}\|\partial_{l'}\dot{S}_{j-1}u^l\|_{L^\infty}\,\|\dot{\Delta}_j\partial_l u^k\|_{L^2}\,\|\dot{\Delta}_j u^k\|_{L^2}\end{aligned}$$

where we used Hölder's inequality, the well-known commutator estimate (Lemma 2.1 in [35], and also Lemma 2.97 in [1]) and that we may assume $j = j'$ when $|j - j'| \leq 4$ by modifying constants appropriately. Furthermore, we bound

$$\begin{aligned}
I_{1,2} &\lesssim \sum_{l'=1}^{N} \sum_{j} 2^{j(N+1-2\alpha-2\alpha/p_{l,l'})} \sum_{j' \leq j-2} 2^{(j'-j)(2\alpha-2\alpha/p_{l,l'})} 2^{j'(-2\alpha+2\alpha/p_{l,l'})} \\
&\quad \times \|\dot{\Delta}_{j'} \partial_{l'} u^l\|_{L^\infty} \|\dot{\Delta}_j \partial_l u^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} \\
&\lesssim \sum_{l'=1}^{N} \sum_{j} 2^{j(N+2-2\alpha-2\alpha/p_{l,l'})} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \|\dot{\Delta}_j u^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} \\
&\approx \sum_{l'=1}^{N} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \sum_{j} (2^{j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j u^k\|_{L^2})^{2/p_{l,l'}} \\
&\quad \times (2^{j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j \Lambda^\alpha u^k\|_{L^2})^{2(1-1/p_{l,l'})} \\
&\lesssim \sum_{l'=1}^{N} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \|u\|_{\dot{H}^{N/2+1-2\alpha}}^{2/p_{l,l'}} \|\Lambda^\alpha u\|_{\dot{H}^{N/2+1-2\alpha}}^{2(1-1/p_{l,l'})} \qquad (3.6)
\end{aligned}$$

where we have used (2.1), Young's inequality for convolution, the hypothesis that $p_{k,l} \in (1, \infty)$ so that $2\alpha - 2\alpha/p_{l,l'} > 0$, and Hölder's inequality. Next, from (3.4),

$$\begin{aligned}
I_{1,3} &\leq \sum_{j} 2^{2j(\frac{N}{2}+1-2\alpha)} \sum_{j':|j-j'|\leq 4} \|(\dot{S}_{j'-1}u^l - \dot{S}_{j-1}u^l)\dot{\Delta}_j \dot{\Delta}_{j'} \partial_l u^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} \\
&\lesssim \sum_{j} 2^{2j(\frac{N}{2}+1-2\alpha)} \\
&\quad \times \sum_{j':|j-j'|\leq 4} \sum_{j'':j''\in[j-2,j'-2]} \|\dot{\Delta}_{j''} u^l\|_{L^\infty} \|\dot{\Delta}_j \dot{\Delta}_{j'} \partial_l u^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2}
\end{aligned}$$

by Hölder's inequality. Now we may follow the approach of [10] to write

$$\dot{\Delta}_{j''} u^l \approx \sum_{l'=1}^{N} \tilde{\dot{\Delta}}_{j''}^{l'} \dot{\Delta}_{j''} u^l \approx \sum_{l'=1}^{N} 2^{-j''} \tilde{\dot{\Delta}}_{j''}^{l'} \dot{\Delta}_{j''} \partial_{l'} u^l$$

(see (8.3) in [10] for the definition of $\tilde{\dot{\Delta}}_{j''}^{l'}$), and continue our estimate as

$$\begin{aligned}
I_{1,3} &\lesssim \sum_{j} 2^{2j(\frac{N}{2}+1-2\alpha)} \sum_{j':|j-j'|\leq 4} \sum_{j'':j''\in[j-2,j'-2]} \\
&\quad \times \|\sum_{l'=1}^{N} 2^{-j''} \tilde{\dot{\Delta}}_{j''}^{l'} \dot{\Delta}_{j''} \partial_{l'} u^l\|_{L^\infty} \|\dot{\Delta}_j \dot{\Delta}_{j'} \partial_l u^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} \\
&\lesssim \sum_{j} 2^{2j(\frac{N}{2}+1-2\alpha)} \sum_{l'=1}^{N} 2^{-j} \|\dot{\Delta}_j \partial_{l'} u^l\|_{L^\infty} \|\dot{\Delta}_j \partial_l u^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2}
\end{aligned}$$

$$\lesssim \sum_{l'=1}^{N} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \sum_j (2^{j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j u^k\|_{L^2})^{2/p_{l,l'}}$$
$$\times (2^{j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j \Lambda^\alpha u^k\|_{L^2})^{2(1-1/p_{l,l'})}$$
$$\lesssim \sum_{l'=1}^{N} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \|u\|_{\dot{H}^{N/2+1-2\alpha}}^{2/p_{l,l'}} \|\Lambda^\alpha u\|_{\dot{H}^{N/2+1-2\alpha}}^{2(1-1/p_{l,l'})}, \tag{3.7}$$

where we used that we may assume $j = j'$ when $|j-j'| \le 4$ modifying constants appropriately, Plancherel's theorem and Hölder's inequality. Thus, appealing to (3.4), and using the estimates (3.5), (3.6), and (3.7), we obtain

$$I_1 \le I_{1,1} + I_{1,2} + I_{1,3} \lesssim \sum_{l'=1}^{N} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \|u\|_{\dot{H}^{N/2+1-2\alpha}}^{2/p_{l,l'}} \|\Lambda^\alpha u\|_{\dot{H}^{N/2+1-2\alpha}}^{2(1-1/p_{l,l'})}. \tag{3.8}$$

Next, from (3.3),

$$I_2 \le \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j \sum_{j'} \dot{S}_{j'-1}\partial_l u^k \dot{\Delta}_{j'} u^l\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2}$$
$$\lesssim \sum_j \sum_{j':|j-j'|\le 4} 2^{2j(\frac{N}{2}+1-2\alpha)} \|\dot{S}_{j'-1}\partial_l u^k\|_{L^\infty} \|\dot{\Delta}_{j'} u^l\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2}$$
$$\lesssim \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \|\dot{S}_{j-1}\partial_l u^k\|_{L^\infty} \|\dot{\Delta}_j u^l\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2}$$

where we used (2.8), Hölder's inequality and that we may assume $j' = j$ when $|j - j'| \le 4$ by modifying constants. We furthermore continue to bound by

$$I_2 \lesssim \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \sum_{j':j'\le j-2} \|\dot{\Delta}_{j'}\partial_l u^k\|_{L^\infty} \|\dot{\Delta}_j u^l\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2}$$
$$\approx \sum_j 2^{j(N+2-2\alpha-\frac{2\alpha}{p_{k,l}})} \sum_{j':j'\le j-2} 2^{(j'-j)(2\alpha-\frac{2\alpha}{p_{k,l}})} 2^{j'(-2\alpha+\frac{2\alpha}{p_{k,l}})}$$
$$\times \|\dot{\Delta}_{j'}\partial_l u^k\|_{L^\infty} \|\dot{\Delta}_j u^l\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2}$$
$$\lesssim \sum_j 2^{j(N+2-2\alpha-\frac{2\alpha}{p_{k,l}})} \|\partial_l u^k\|_{\mathcal{B}_{p_{k,l}}} \|\dot{\Delta}_j u^l\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2}$$
$$\approx \|\partial_l u^k\|_{\mathcal{B}_{p_{k,l}}} \sum_j (2^{j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j u\|_{L^2})^{\frac{2}{p_{k,l}}} (2^{j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j \Lambda^\alpha u\|_{L^2})^{2(1-\frac{1}{p_{k,l}})}$$
$$\lesssim \|\partial_l u^k\|_{\mathcal{B}_{p_{k,l}}} \|u\|_{\dot{H}^{N/2+1-2\alpha}}^{\frac{2}{p_{k,l}}} \|\Lambda^\alpha u\|_{\dot{H}^{N/2+1-2\alpha}}^{2(1-\frac{1}{p_{k,l}})} \tag{3.9}$$

by Young's inequality for convolution, the hypothesis that $p_{k,l} > 1$, Plancherel's theorem, and Hölder's inequality. Finally, from (3.3) we may rewrite

$$I_3 = \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} |(\dot{\Delta}_j \sum_{j':j'\ge j-\delta} \partial_l(\dot{\Delta}_{j'} u^l \tilde{\dot{\Delta}}_{j'} u^k) | \dot{\Delta}_j u^k)| \tag{3.10}$$

for some $\delta \in \mathbb{Z}^+$ due to (2.8) and (1.1b). Now we write

$$\dot{\Delta}_{j'} u^l \approx \sum_{l'=1}^{N} \tilde{\dot{\Delta}}_{j'}^{l'} \dot{\Delta}_{j'} u^l \approx \sum_{l'=1}^{N} 2^{-j'} \tilde{\dot{\Delta}}_{j'}^{l'} \dot{\Delta}_{j'} \partial_{l'} u^l,$$

which is needed to gain a factor of $2^{j-j'}$ as we shall subsequently see. We compute

(3.11)

$$\begin{aligned} I_3 &\approx \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} |(\dot{\Delta}_j \sum_{j':j'\geq j-\delta} \partial_l (\sum_{l'=1}^{N} 2^{-j'} \tilde{\dot{\Delta}}_{j'}^{l'} \dot{\Delta}_{j'} \partial_{l'} u^l \tilde{\dot{\Delta}}_{j'} u^k) |\dot{\Delta}_j u^k)| \\ &\lesssim \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \sum_{j':j'\geq j-\delta} \sum_{l'=1}^{N} 2^{j-j'} \|\tilde{\dot{\Delta}}_{j'}^{l'} \dot{\Delta}_{j'} \partial_{l'} u^l\|_{L^\infty} \|\tilde{\dot{\Delta}}_{j'} u^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} \\ &\lesssim \sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \sum_{j':j'\geq j-\delta} \sum_{l'=1}^{N} 2^{j-j'} 2^{j'(2\alpha - \frac{2\alpha}{p_{l,l'}})} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \|\tilde{\dot{\Delta}}_{j'} u^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} \end{aligned}$$

where we have used Hölder's inequality, Bernstein' inequality, Young's inequality for convolution, and Plancherel's theorem. The powers must be distributed differently from the previous terms such as (3.6), (3.7) and (3.9) here. We compute

(3.12)

$$\begin{aligned} &\sum_j 2^{2j(\frac{N}{2}+1-2\alpha)} \sum_{j':j'\geq j-\delta} \sum_{l'=1}^{N} 2^{j-j'} 2^{j'(2\alpha-\frac{2\alpha}{p_{l,l'}})} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \|\tilde{\dot{\Delta}}_{j'} u^k\|_{L^2} \|\dot{\Delta}_j u^k\|_{L^2} \\ &\approx \sum_{l'=1}^{N} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \sum_j \sum_{j':j'\geq j-\delta} 2^{(j-j')(\frac{N}{2}+2-3\alpha+\frac{\alpha}{p_{l,l'}})} \\ &\quad \times (2^{j'(\frac{N}{2}+1-2\alpha)} \|\tilde{\dot{\Delta}}_{j'} u\|_{L^2})^{1/p_{l,l'}} (2^{j'(\frac{N}{2}+1-2\alpha)} \|\tilde{\dot{\Delta}}_{j'} \Lambda^\alpha u\|_{L^2})^{1-1/p_{l,l'}} \\ &\quad \times (2^{j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j u\|_{L^2})^{1/p_{l,l'}} (2^{j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j \Lambda^\alpha u\|_{L^2})^{1-1/p_{l,l'}} \\ &\lesssim \sum_{l'=1}^{N} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \\ &\quad \times \left\| \left((2^{j(\frac{N}{2}+1-2\alpha)} \|\tilde{\dot{\Delta}}_j u\|_{L^2})^{1/p_{l,l'}} (2^{j(\frac{N}{2}+1-2\alpha)} \|\tilde{\dot{\Delta}}_j \Lambda^\alpha u\|_{L^2})^{1-1/p_{l,l'}} \right)_j \right\|_{l^2} \\ &\quad \times \left\| \left((2^{j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j u\|_{L^2})^{\frac{1}{p_{l,l'}}} (2^{j(\frac{N}{2}+1-2\alpha)} \|\dot{\Delta}_j \Lambda^\alpha u\|_{L^2})^{1-1/p_{l,l'}} \right)_j \right\|_{l^2} \\ &\lesssim \sum_{l'=1}^{N} \|\partial_{l'} u^l\|_{\mathcal{B}_{p_{l,l'}}} \|u\|_{\dot{H}^{N/2+1-2\alpha}}^{2/p_{l,l'}} \|\Lambda^\alpha u\|_{\dot{H}^{N/2+1-2\alpha}}^{2(1-1/p_{l,l'})} \end{aligned}$$

where we have used Hölder's inequality, Young's inequality for convolution and that $N/2 + 2 - 3\alpha + \alpha/p_{l,l'} > 0$ due to the range of α. Thus, in consideration of (3.8), (3.9), (3.10), (3.11), (3.12) in (3.3), we have shown

$$\begin{aligned}(3.13)\qquad &\frac{1}{2}\partial_t\|u\|^2_{\dot H^{N/2+1-2\alpha}}+\|\Lambda^\alpha u\|^2_{\dot H^{N/2+1-2\alpha}}\\ &\lesssim\sum_{k,l=1}^N\|\partial_l u^k\|_{\mathcal{B}_{p_{k,l}}}\|u\|^{2/p_{k,l}}_{\dot H^{N/2+1-2\alpha}}\|\Lambda^\alpha u\|^{2(1-1/p_{k,l})}_{\dot H^{N/2+1-2\alpha}}\\ &\leq\frac{1}{2}\|\Lambda^\alpha u\|^2_{\dot H^{N/2+1-2\alpha}}+c\sum_{k,l=1}^N\|\partial_l u^k\|^{p_{k,l}}_{\mathcal{B}_{p_{k,l}}}\|u\|^2_{\dot H^{N/2+1-2\alpha}}\end{aligned}$$

by Young's inequality. Subtracting $\frac{1}{2}\|\Lambda^\alpha u\|^2_{\dot H^{N/2+1-2\alpha}}$ from both sides and applying Gronwall's inequality, we complete the proof of Proposition 3.1. □

4. Verifying the blow-up criteria

Due to Proposition 3.1, it suffices to show $\sum_{k,l=1}^N\int_0^T\|\partial_l u^k\|^{p_{k,l}}_{\mathcal{B}_{p_{k,l}}}d\tau\lesssim 1$ for some $p_{k,l}\in(1,\infty)$ assuming

$$\begin{aligned}&\sum_{m=3}^N\int_0^T\|u^m\|^p_{\dot H^{N/2+1-2\alpha+\frac{2\alpha}{p}}}+\|\Lambda^{N/2-\alpha}\partial_m u^m\|^2_{\dot H^{-\epsilon,\epsilon}}+\left|\|(\|\dot\Delta_k^h\Lambda^{N/2-\alpha}\omega^3\|_{L^2})_k\|_{l^1}\right|^2d\tau\\ &\quad+\sup_{t\in[0,T]}\|\omega^3(t)\|^{p(3-\frac{2}{p}-\frac{N}{2\alpha})}_{L^2}\int_0^T\|\Lambda^\alpha\omega^3\|^{p(-2+\frac{2}{p}+\frac{N}{2\alpha})}_{L^2}d\tau\lesssim 1.\end{aligned}$$

Firstly, by Bernstein's inequality we may estimate, for any $k\in\{3,\ldots,N\}$,

$$\begin{aligned}(4.1)\qquad\max_{1\leq l\leq N}\|\partial_l u^k\|_{\mathcal{B}_p}&=\max_{1\leq l\leq N}\|\partial_l u^k\|_{\dot B^{(-2+2/p)\alpha}_{\infty,\infty}}\lesssim\sup_j 2^{j(1-2\alpha+\frac{2\alpha}{p})}\|\dot\Delta_j u^k\|_{L^\infty}\\ &\lesssim\sup_j 2^{j(\frac{N}{2}+1-2\alpha+\frac{2\alpha}{p})}\|\dot\Delta_j u^k\|_{L^2}\lesssim\|u^k\|_{\dot H^{N/2+1-2\alpha+2\alpha/p}}\end{aligned}$$

as $l^2\subset l^\infty$, and thus

$$(4.2)\qquad\max_{1\leq l\leq N}\int_0^T\sum_{k=3}^N\|\partial_l u^k\|^p_{\mathcal{B}_p}d\tau\lesssim\int_0^T\sum_{k=3}^N\|u^k\|^p_{\dot H^{N/2+1-2\alpha+2\alpha/p}}d\tau.$$

Next,

$$(4.3)\ \int_0^T\|\nabla_h u^h\|^p_{\mathcal{B}_p}d\tau\lesssim\int_0^T\|\nabla_h(\nabla_h^\perp\Delta_h^{-1}\omega^3)\|^p_{\mathcal{B}_p}d\tau+\sum_{k=3}^N\int_0^T\|\nabla_h(\nabla_h\Delta_h^{-1}\partial_k u^k)\|^p_{\mathcal{B}_p}d\tau$$

by Lemma 1.3.

Now we compute, for M to be chosen shortly,

$$\begin{aligned}\|a\|_{\mathcal{B}_p} &\leq \|a\|_{\dot{B}^{(-2+2/p)\alpha}_{\infty,1}} \\ &\lesssim \sum_{j\leq M} 2^{j((-2+\frac{2}{p})\alpha+\frac{N}{2})}\|\dot{\Delta}_j a\|_{L^2} + \sum_{j>M} 2^{j((-2+\frac{2}{p})\alpha-\alpha+\frac{N}{2})}\|\dot{\Delta}_j\Lambda^\alpha a\|_{L^2} \\ &\lesssim \sum_{j\leq M} 2^{j(-2\alpha+\frac{2\alpha}{p}+\frac{N}{2})}\|a\|_{L^2} + \sum_{j>M} 2^{j(-3\alpha+\frac{2\alpha}{p}+\frac{N}{2})}\|\Lambda^\alpha a\|_{L^2} \\ &\lesssim 2^{M(-2\alpha+\frac{2\alpha}{p}+\frac{N}{2})}\|a\|_{L^2} + 2^{M(-3\alpha+\frac{2\alpha}{p}+\frac{N}{2})}\|\Lambda^\alpha a\|_{L^2}\end{aligned} \tag{4.4}$$

where we used that $l^1 \subset l^\infty$, Bernstein's inequality, that $l^2 \subset l^\infty$, that

$$-3\alpha + \frac{2\alpha}{p} + \frac{N}{2} < 0, \quad -2\alpha + \frac{2\alpha}{p} + \frac{N}{2} > 0$$

and Lemma 2.2. Choosing M so that $2^M = \left(\|\Lambda^\alpha a\|_{L^2}/\|a\|_{L^2}\right)^{1/\alpha}$ in (4.4) leads to

$$\|a\|_{\mathcal{B}_p} \lesssim \|\Lambda^\alpha a\|_{L^2}^{-2+\frac{2}{p}+\frac{N}{2\alpha}} \|a\|_{L^2}^{3-\frac{2}{p}-\frac{N}{2\alpha}} \tag{4.5}$$

Thus, applying (4.5) in (4.3), we deduce

$$\begin{aligned}\int_0^T \|\nabla_h(\nabla_h^\perp\Delta_h^{-1}\omega^3)\|_{\mathcal{B}_p}^p d\tau &\lesssim \int_0^T \|\omega^3\|_{L^2}^{p(3-\frac{2}{p}-\frac{N}{2\alpha})}\|\Lambda^\alpha\omega^3\|_{L^2}^{p(-2+\frac{2}{p}+\frac{N}{2\alpha})} d\tau \\ &\lesssim \sup_{t\in[0,T]} \|\omega^3(t)\|_{L^2}^{p(3-\frac{2}{p}-\frac{N}{2\alpha})} \int_0^T \|\Lambda^\alpha\omega^3\|_{L^2}^{p(-2+\frac{2}{p}+\frac{N}{2\alpha})} d\tau \lesssim 1.\end{aligned} \tag{4.6}$$

On the other hand,

$$\begin{aligned}&\sum_{k=3}^N \int_0^T \|\nabla_h(\nabla_h\Delta_h^{-1}\partial_k u^k)\|_{\mathcal{B}_p}^p d\tau \\ &\quad= \sum_{k=3}^N \int_0^T \left|\sup_j 2^{j(-2+\frac{2}{p})\alpha}\|\dot{\Delta}_j\nabla_h\nabla_h\Delta_h^{-1}\partial_k u^k\|_{L^\infty}\right|^p d\tau \\ &\quad\lesssim \sum_{k=3}^N \int_0^T \left|\sup_j 2^{j((-2+\frac{2}{p})\alpha+N(\frac{1}{2}-\frac{1}{\infty})+1)}\|\dot{\Delta}_j\nabla_h\nabla_h\Delta_h^{-1}u^k\|_{L^2}\right|^p d\tau \\ &\quad\lesssim \sum_{k=3}^N \int_0^T \|u^k\|_{\dot{H}^{N/2+1-2\alpha+2\alpha/p}}^p d\tau \lesssim 1\end{aligned} \tag{4.7}$$

by Bernstein's inequality and the fact that

$$\|\nabla_h\nabla_h\Delta_h^{-1}f\|_{L^2} = \left\|\|\nabla_h\nabla_h\Delta_h^{-1}f\|_{L^2_h}\right\|_{L^2_v} \lesssim \left\|\|f\|_{L^2_h}\right\|_{L^2_v} \approx \|f\|_{L^2},$$

due to the continuity of the Riesz transform in $\mathbb{R}^2$. From (4.6) and (4.7) applied to (4.5), we obtain

$$\int_0^T \|\nabla_h u^h\|_{\mathcal{B}_p}^p d\tau \lesssim 1. \tag{4.8}$$

Finally,

$$\begin{aligned}(4.9)\qquad &\sum_{m=3}^{N}\int_0^T \|\partial_m u^h\|_{\mathcal{B}_2}^2\,d\tau\\ &\lesssim \sum_{m=3}^{N}\int_0^T \|\partial_m \nabla_h^{\perp}\Delta_h^{-1}\omega^3\|_{\mathcal{B}_2}^2 + \Big\|\partial_m \nabla_h \Delta_h^{-1}\Big(\sum_{l=3}^{N}\partial_l u^l\Big)\Big\|_{\mathcal{B}_2}^2\,d\tau\end{aligned}$$

by Lemma 1.3. This is in some sense the most difficult term because $\partial_m\nabla_h^{\perp}\Delta_h^{-1}$, $\partial_m\nabla_h\Delta_h^{-1}$, $m = 3, \dots, N$, are no longer just horizontal Riesz transforms. We may estimate, for some $\delta \in \mathbb{Z}_+$,

$$\begin{aligned}(4.10)\qquad &\sum_{m=3}^{N}\int_0^T \|\partial_m \nabla_h^{\perp}\Delta_h^{-1}\omega^3\|_{\mathcal{B}_2}^2\,d\tau\\ &\lesssim \sum_{m=3}^{N}\int_0^T \Big|\sup_j 2^{j(-\alpha)}\sum_{k\le j+\delta, l\le j+\delta}\|\dot\Delta_j\dot\Delta_k^h\dot\Delta_l^v\partial_m\nabla_h^{\perp}\Delta_h^{-1}\omega^3\|_{L^\infty}\Big|^2 d\tau\\ &\lesssim \sum_{m=3}^{N}\int_0^T \Big|\sup_j 2^{j(-\alpha)}\sum_{k\le j+\delta}\sum_{l\le j+\delta}2^{l(\frac{N-2}{2})}\|\dot\Delta_j\dot\Delta_k^h\dot\Delta_l^v\partial_m\omega^3\|_{L^2}\Big|^2 d\tau,\end{aligned}$$

where we used Bernstein's inequality and the Plancherel theorem. We now continue this bound as

$$\begin{aligned}&\sum_{m=3}^{N}\int_0^T \Big|\sup_j 2^{j(-\alpha)}\sum_{k\le j+\delta}\Big(\sum_{l\le j+\delta}2^{l(N-2)}\Big)^{1/2}\|(\|\dot\Delta_k^h\dot\Delta_j\dot\Delta_l^v\partial_m\omega^3\|_{L^2})_l\|_{l^2}\Big|^2 d\tau\\ &\lesssim \sum_{m=3}^{N}\int_0^T \Big|\sup_j 2^{j(-\alpha+(\frac{N-2}{2}))}\sum_{k\le j+\delta}\|\dot\Delta_k^h\dot\Delta_j\partial_m\omega^3\|_{L^2}\Big|^2 d\tau\\ &\lesssim \int_0^T \Big|\sup_j \sum_{k\le j+\delta}\|\dot\Delta_k^h\dot\Delta_j\Lambda^{-\alpha+\frac{N}{2}}\omega^3\|_{L^2}\Big|^2 d\tau\\ (4.11)\qquad &\lesssim \int_0^T \Big|\sup_j \sum_{k\le j+\delta}\|\dot\Delta_k^h\Lambda^{-\alpha+\frac{N}{2}}\omega^3\|_{L^2}\Big|^2 d\tau \lesssim 1\end{aligned}$$

by Hölder's inequality, Lemma 2.2, the fact that

$$\begin{aligned}\|(\|\dot\Delta_l^v f\|_{L^2})_l\|_{l^2} &= \Big(\sum_l \int_{\mathbb{R}^{N-2}}\int_{\mathbb{R}^2}|\dot\Delta_l^v f|^2\,dx_h\,dx_v\Big)^{1/2}\\ &= \Big(\int_{\mathbb{R}^2}\sum_l\|\dot\Delta_l^v f\|_{L_v^2}^2\,dx_h\Big)^{1/2} = \big\|\|f\|_{L_v^2}\big\|_{L_h^2} = \|f\|_{L^2},\end{aligned}$$

Plancherel's theorem, and the uniform bound of $\dot\Delta_j$ in $L^p(\mathbb{R}^N)$ for all $p \in [1, \infty]$.

On the very last term, it becomes crucial that we have $\dot{H}^{-\epsilon,\epsilon}$, $\epsilon > 0$. Firstly, we estimate, for some $\delta \in \mathbb{N}$,

$$\begin{aligned}(4.12)\quad &\sum_{i,m=3}^{N} \int_0^T \|\partial_m \nabla_h \Delta_h^{-1} \partial_i u^i\|_{\mathcal{B}_2}^2 \, d\tau \\ &\lesssim \sum_{i=3}^{N} \int_0^T \Big| \sup_j 2^{j(-\alpha)} \sum_{k,l \in [j-\delta, j+\delta]} \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \nabla \nabla_h \Delta_h^{-1} \partial_i u^i\|_{L^\infty} \Big|^2 d\tau\end{aligned}$$

and continue as

$$\begin{aligned}&\sum_{i=3}^{N} \int_0^T \Big| \sup_j 2^{j(-\alpha)} \sum_{k,l \in [j-\delta, j+\delta]} \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \nabla \nabla_h \Delta_h^{-1} \partial_i u^i\|_{L^\infty} \Big|^2 d\tau \\ (4.13)\quad &\lesssim \sum_{i=3}^{N} \int_0^T \Big| \sup_j 2^{j(1-N/2)} \sum_{k,l \in [j-\delta, j+\delta]} 2^{k\epsilon}\, 2^{l(\frac{N-2}{2}-\epsilon)}\, 2^{-k\epsilon}\, 2^{l\epsilon} \\ &\qquad \times \|\dot{\Delta}_j \dot{\Delta}_k^h \dot{\Delta}_l^v \Lambda^{N/2-\alpha} \partial_i u^i\|_{L^2} \Big|^2 d\tau \\ &\lesssim \sum_{i=3}^{N} \int_0^T \Big| \sup_j 2^{j(1-N/2)} \Big(\sum_{k \le j+\delta, l \le j+\delta} 2^{2k\epsilon}\, 2^{2l(\frac{N-2}{2}-\epsilon)} \Big)^{1/2} \\ &\qquad \times \Big(\sum_{k,l \in \mathbb{Z}} |2^{-k\epsilon}\, 2^{l\epsilon}\, \|\dot{\Delta}_k^h \dot{\Delta}_l^v \Lambda^{N/2-\alpha} \partial_i u^i\|_{L^2}|^2 \Big)^{1/2} \Big|^2\end{aligned}$$

by Bernstein's inequalities, Plancherel's theorem, Hölder's inequality and uniform bound of $\dot{\Delta}_j$ in $L^2(\mathbb{R}^N)$. Now we use the fact that $\epsilon \in (0, (N-2)/2)$, Lemma 2.2 to continue to bound from (4.13) as

$$\begin{aligned}(4.14)\quad &\sum_{i=3}^{N} \int_0^T \Big| \sup_j 2^{j(1-N/2)} \Big(\sum_{k \le j+\delta, l \le j+\delta} 2^{2k\epsilon} 2^{2l(\frac{N-2}{2}-\epsilon)} \Big)^{1/2} \\ &\qquad \times \Big(\sum_{k,l \in \mathbb{Z}} |2^{-k\epsilon}\, 2^{l\epsilon}\, \|\dot{\Delta}_k^h \dot{\Delta}_l^v \Lambda^{N/2-\alpha} \partial_i u^i\|_{L^2}|^2 \Big)^{1/2} \Big|^2 \\ &\lesssim \sum_{i=3}^{N} \int_0^T \|\Lambda^{N/2-\alpha} \partial_i u^i\|_{\dot{H}^{-\epsilon,\epsilon}}^2 \, d\tau \lesssim 1.\end{aligned}$$

By (4.12), (4.13), (4.14), we conclude $\sum_{i,m=3}^{N} \int_0^T \|\partial_i \nabla_h \Delta_h^{-1} \partial_m u^m\|_{\mathcal{B}_2}^2 d\tau \lesssim 1$ and hence together with (4.10), (4.11), we obtain

$$\sum_{m=3}^{N} \int_0^T \|\partial_m u^h\|_{\mathcal{B}_2}^2 \, d\tau \lesssim 1.$$

This, along with (4.2) and (4.8) concludes the proof of Theorem 1.1.

A. Appendix

The purpose of this Appendix is to announce some estimates that we were able to prove, and sketch their proofs briefly. We hope that they will help to reduce the conditions in (1.9) to just $\sum_{k=3}^{N}\int_0^T \|u^k\|^p_{\dot{H}^{3-2\alpha+2\alpha/p}}d\tau$. We believe that the four-dimensional case is the easiest among any other higher dimensions beyond three to accomplish this improvement.

A.1. Several estimates

The following extensions of Lemmas 4.2 and 4.3 in [10] may be proven.

Proposition A.1. *Let $s \in \mathbb{R}^+, p, q \in [1,\infty]$ and $p \geq q$. Then*

$$\|f\|_{L^p_h((\dot{B}^s_{p,q})_v)} \lesssim \|f\|_{\dot{B}^s_{p,q}}. \tag{A.1}$$

Proof. Firstly for any $l \in \mathbb{Z}$, there exists $N_0 \in \mathbb{Z}$ such that

$$2^{ls}\|\dot{\Delta}^v_l f\|_{L^p} \lesssim 2^{ls}\sum_{j\in\mathbb{Z}:l-N_0\leq j}\|\dot{\Delta}^v_l\dot{\Delta}_j f\|_{L^p} \lesssim \sum_{j\in\mathbb{Z}:l-j\leq N_0} 2^{(l-j)s}2^{js}\|\dot{\Delta}_j f\|_{L^p}. \tag{A.2}$$

Therefore,

$$\big\|(2^{ls}\|\dot{\Delta}^v_l f\|_{L^p})_l\big\|_{l^q(\mathbb{Z})} \lesssim \Big\|\Big(\sum_{j\in\mathbb{Z}:l-j\leq N_0} 2^{(l-j)s}2^{js}\|\dot{\Delta}_j f\|_{L^p}\Big)_l\Big\|_{l^q(\mathbb{Z})} \lesssim \|f\|_{\dot{B}^s_{p,q}} \tag{A.3}$$

by (A.2), Young's inequality for convolution and that $s > 0$ by hypothesis. Hence,

$$\|f\|_{L^p_h((\dot{B}^s_{p,q})_v)} \lesssim \big\|\big(2^{ls}\,\big\|\|\dot{\Delta}^v_l f\|_{L^p_v}\big\|_{L^p_h}\big)\big\|_{l^q(\mathbb{Z})} \lesssim \|f\|_{\dot{B}^s_{p,q}} \tag{A.4}$$

by Minkowski's inequality for integrals and (A.3). This completes the proof. □

Proposition A.2. *For all $s > 0, \beta \in (0,s)$ and $p, q \in [1,\infty]$,*

$$\|f\|_{(\dot{B}^{s-\beta}_{p,q})_h(\dot{B}^{\beta}_{p,1})_v} \lesssim \|f\|_{\dot{B}^s_{p,q}}. \tag{A.5}$$

Proof. We denote by $V_k \triangleq \sum_{l\in\mathbb{Z}} 2^{l\beta}\|\dot{\Delta}^h_k\dot{\Delta}^v_l f\|_{L^p}$ and see that if

$$V_k \lesssim c_k\, 2^{-k(s-\beta)}\,\|f\|_{\dot{B}^s_{p,q}}$$

for some $(c_k)_k \in l^q(\mathbb{Z})$, then it would imply

$$\sum_{l\in\mathbb{Z}} 2^{k(s-\beta)}\,2^{l\beta}\,\|\dot{\Delta}^h_k\dot{\Delta}^v_l f\|_{L^p} \lesssim c_k\,\|f\|_{\dot{B}^s_{p,q}} \tag{A.6}$$

and hence

$$\|f\|_{(\dot{B}^{s-\beta}_{p,q})_h(\dot{B}^{\beta}_{p,q})_v} = \Big\|\,2^{k(s-\beta)}\sum_{l\in\mathbb{Z}} 2^{l\beta}\,\|\dot{\Delta}^h_k\dot{\Delta}^v_l f\|_{L^p}\Big\|_{l^q(\mathbb{Z})} \lesssim \|f\|_{\dot{B}^s_{p,q}} \tag{A.7}$$

by (A.6), completing the proof of Proposition A.2. Next, we would like to show $V_k \lesssim c_k 2^{-k(s-\beta)} \|f\|_{\dot{B}^s_{p,q}}$ for some $(c_k)_k \in l^q(\mathbb{Z})$. Firstly, $\dot{\Delta}^v_l$ is uniformly bounded operators on $L^p(\mathbb{R}^4), p \in [1,\infty]$. Thus, for any fixed $k \in \mathbb{Z}$,

$$\begin{aligned}
&2^{k(s-\beta)} V_k \\
&\lesssim 2^{k(s-\beta)} \sum_{l\in\mathbb{Z}:l\le k} 2^{l\beta} \|\dot{\Delta}^h_k f\|_{L^p} + 2^{k(s-\beta)} \sum_{l\in\mathbb{Z}:l>k} 2^{l\beta} \Big\| \sum_{j\in\mathbb{Z}} \dot{\Delta}_j \dot{\Delta}^h_k \dot{\Delta}^v_l f \Big\|_{L^p} \\
&\lesssim 2^{k(s-\beta)} \|\dot{\Delta}^h_k f\|_{L^p} \sum_{l\in\mathbb{Z}:l\le k} 2^{l\beta} + 2^{k(s-\beta)} \sum_{l\in\mathbb{Z}:l>k} \sum_{j\in\mathbb{Z}:|j-l|\le N_0} 2^{l\beta} \|\dot{\Delta}_j \dot{\Delta}^h_k \dot{\Delta}^v_l f\|_{L^p} \\
&\lesssim \sum_{j\in\mathbb{Z}:j\ge k-N_1} 2^{ks} \|\dot{\Delta}_j \dot{\Delta}^h_k f\|_{L^p} + 2^{k(s-\beta)} \sum_{j\in\mathbb{Z}:j\ge k-N_0} 2^{-j(s-\beta)}\, 2^{js}\, \|\dot{\Delta}_j f\|_{L^p} \\
&\lesssim \sum_{j\in\mathbb{Z}:j\ge k-N_1} 2^{(k-j)s}\, 2^{js}\, \|\dot{\Delta}_j f\|_{L^p} + \sum_{j\in\mathbb{Z}:j\ge k-N_0} 2^{(k-j)(s-\beta)} 2^{js} \|\dot{\Delta}_j f\|_{L^p} \\
&\approx c_k \, \|f\|_{\dot{B}^s_{p,q}}
\end{aligned} \tag{A.8}$$

for all q, where we used Lemma 2.2, the Littlewood–Paley decomposition and the hypothesis that $s > \beta$. □

The following could be considered an extension of Proposition 4.1 in [10].

Proposition A.3. *Let $f = (f^1, f^2, f^3, f^4) \in C_0^\infty(\mathbb{R}^4)$ satisfy $\nabla \cdot f = 0, g = \partial_1 f^2 - \partial_2 f^1$, and $\epsilon > 0$. Furthermore, suppose $s_1 > 1$ and $s_2 \in (\epsilon, s_1 - 1)$ satisfy $s_1 + s_2 < 1 + \alpha$. Then*

$$\begin{aligned}
&\|f^h\|_{(\dot{B}^{s_1}_{2,1})_h (\dot{B}^{s_2}_{2,1})_v} \\
&\lesssim \|g\|_{L^2}^{1-\frac{s_1+s_2-1}{\alpha}} \|\Lambda^\alpha g\|_{L^2}^{\frac{s_1+s_2-1}{\alpha}} + \sum_{m=3}^4 \|\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}^{1-\frac{s_1+s_2-1}{\alpha}} \|\Lambda^\alpha \partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}^{\frac{s_1+s_2-1}{\alpha}}.
\end{aligned} \tag{A.9}$$

Remark A.1. Notice that, in contrast to Proposition 4.1 in [10], we have to assume $s_1 > 1$. We explain why along the proof.

Proof. Firstly,

$$\|f^h\|_{(\dot{B}^{s_1}_{2,1})_h (\dot{B}^{s_2}_{2,1})_v} \lesssim \|g\|_{(\dot{B}^{s_1-1}_{2,1})_h (\dot{B}^{s_2}_{2,1})_v} + \sum_{m=3}^4 \|\partial_m f^m\|_{(\dot{B}^{s_1-1}_{2,1})_h (\dot{B}^{s_2}_{2,1})_v} \tag{A.10}$$

by Lemma 1.3. Now we work on

$$\begin{aligned}
\|g\|_{(\dot{B}^{s_1-1}_{2,1})_h (\dot{B}^{s_2}_{2,1})_v} &\lesssim \|g\|_{\dot{B}^{s_1+s_2-1}_{2,1}} \\
&\approx \sum_{j\le M} 2^{j(s_1+s_2-1)} \|\dot{\Delta}_j g\|_{L^2} + \sum_{j>M} 2^{j(s_1+s_2-1)} \|\dot{\Delta}_j g\|_{L^2} \\
&\lesssim 2^{M(s_1+s_2-1)} \|g\|_{L^2} + 2^{M(s_1+s_2-1-\alpha)} \|\Lambda^\alpha g\|_{L^2}
\end{aligned} \tag{A.11}$$

for $M > 0$ to be determined shortly, by Proposition A.2, Hölder's inequality, Bernstein's inequality and the fact that $s_1 + s_2 - 1 > 0, s_1 + s_2 - 1 - \alpha < 0$.

We highlight here that the issue when $s_1 = 1$ is that we cannot apply Proposition A.2 because $s_1 - 1 = 0 \not> 0$. In [10] the authors are able to dispose of this issue because they can take Bernstein's inequality to go down from $(\dot{B}_{2,1})_h$ to $(\dot{B}^{2(5/9-1/2)}_{9/5,1}) = (\dot{B}^{1/9}_{9/5,1})$ which is not an option in our case because $\dot{W}^{p,1}(\mathbb{R}^4)$, $p < 2$, is not enough regularity for initial data to be even locally well-posed. However, this estimate at $s_1 = 1$ seems necessary in order to obtain the estimates on $\|\omega^3\|_{L^2}$.

Now we choose M such that $2^M = \big(\|\Lambda^\alpha g\|_{L^2}/\|g\|_{L^2}\big)^{1/\alpha}$ so that

$$\|g\|_{(\dot{B}^{s_1-1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} \lesssim \|g\|_{L^2}^{1-\frac{s_1+s_2-1}{\alpha}} \|\Lambda^\alpha g\|_{L^2}^{\frac{s_1+s_2-1}{\alpha}}. \tag{A.12}$$

Next, we write

$$\begin{aligned}
&\|\partial_m f^m\|_{(\dot{B}^{s_1-1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} \\
&= \sum_{k,l:k\le l} 2^{k(s_1-1)}2^{ls_2}\|\dot{\Delta}^h_k\dot{\Delta}^v_l\partial_m f^m\|_{L^2} + \sum_{k,l:k>l} 2^{k(s_1-1)}2^{ls_2}\|\dot{\Delta}^h_k\dot{\Delta}^v_l\partial_m f^m\|_{L^2} \\
&\triangleq H_L(\partial_m f^m) + V_L(\partial_m f^m).
\end{aligned} \tag{A.13}$$

We estimate, for $M > 0$ to be determined shortly,

$$\begin{aligned}
H_L(\partial_m f^m) \lesssim &\sum_{k,l:k\le l\le M} 2^{k(s_1-1+\epsilon)}2^{l(s_2-\epsilon)}2^{-k\epsilon}2^{l\epsilon}\|\dot{\Delta}^h_k\dot{\Delta}^v_l\partial_m f^m\|_{L^2} \\
&+ \sum_{k,l:k\le l,l>M} 2^{k(s_1-1+\epsilon)}2^{l(s_2-\alpha-\epsilon)}2^{-k\epsilon}2^{l\epsilon}\|\dot{\Delta}^h_k\dot{\Delta}^v_l\Lambda^\alpha\partial_m f^m\|_{L^2}
\end{aligned} \tag{A.14}$$

by Bernstein's inequality and the Plancherel theorem. Now we use the fact that $l^2 \subset l^\infty$, that $s_1 - 1 + \epsilon > 0$, and Lemma 2.2 to continue to bound by

$$H_L(\partial_m f^m) \lesssim \|\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}2^{M(s_1+s_2-1)} + \|\Lambda^\alpha\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}2^{M(s_1+s_2-1-\alpha)}. \tag{A.15}$$

We choose M such that that $2^M = \big(\frac{\|\Lambda^\alpha\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}}{\|\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}}\big)^{1/\alpha}$ and hence

$$H_L(\partial_m f^m) \lesssim \|\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}^{1-\frac{s_1+s_2-1}{\alpha}}\|\Lambda^\alpha\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}^{\frac{s_1+s_2-1}{\alpha}}. \tag{A.16}$$

Next,

$$\begin{aligned}
V_L(\partial_m f^m) \lesssim &\sum_{k,l:l<k\le M} 2^{k(s_1-1+\epsilon)}2^{l(s_2-\epsilon)}2^{-k\epsilon}2^{l\epsilon}\|\dot{\Delta}^h_k\dot{\Delta}^v_l\partial_m f^m\|_{L^2} \\
&+ \sum_{k,l:l<k,k>M} 2^{k(s_1-1+\epsilon-\alpha)}2^{l(s_2-\epsilon)}2^{-k\epsilon}2^{l\epsilon}\|\dot{\Delta}^h_k\dot{\Delta}^v_l\Lambda^\alpha\partial_m f^m\|_{L^2}
\end{aligned} \tag{A.17}$$

by Bernstein's inequality and the Plancherel theorem. Again we use the fact that $l^2 \subset l^\infty$, that $s_2 > \epsilon$, and Lemma 2.2, to continue to bound by

$$V_L(\partial_m f^m) \lesssim \|\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}2^{M(s_1+s_2-1)} + \|\Lambda^\alpha\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}2^{M(s_1+s_2-1-\alpha)}. \tag{A.18}$$

Now we choose M such that $2^M = \left(\frac{\|\Lambda^\alpha \partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}}{\|\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}}\right)^{1/\alpha}$ so that

$$V_L(\partial_m f^m) \lesssim \|\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}^{1-\frac{s_1+s_2-1}{\alpha}} \|\Lambda^\alpha \partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}^{\frac{s_1+s_2-1}{\alpha}}. \tag{A.19}$$

Therefore, we have shown by (A.10), (A.12), (A.13), (A.16), and (A.19),

$$\begin{aligned} &\|f^h\|_{(\dot{B}^{s_1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} \\ &\lesssim \|g\|_{L^2}^{1-\frac{s_1+s_2-1}{\alpha}} \|\Lambda^\alpha g\|_{L^2}^{\frac{s_1+s_2-1}{\alpha}} + \sum_{m=3}^{4} \|\partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}^{1-\frac{s_1+s_2-1}{\alpha}} \|\Lambda^\alpha \partial_m f^m\|_{\dot{H}^{-\epsilon,\epsilon}}^{\frac{s_1+s_2-1}{\alpha}}. \end{aligned} \tag{A.20}$$

This completes the proof of Proposition A.3. □

We may also extend the inequalities (94), (95) of [37] to the four-dimensional case as follows.

Proposition A.4. *Let $N = 4$. For $s_1 \le 1$, $s_2 \le 1$, $s_2 + 2\alpha/p - \theta > 0$, $s_1 + 2 - 2\alpha + \theta > 0$, and $\theta \in (2\alpha/p - 1, 2\alpha - 1)$, it holds that for $f, g \in C_0^\infty(\mathbb{R}^4)$,*

$$\|fg\|_{\dot{H}^{s_1+1-2\alpha+\theta, s_2+2\alpha/p-1-\theta}} \lesssim \|f\|_{(\dot{B}^{s_1}_{2,1})_h(\dot{B}^{s_2}_{2,1})_v} \|g\|_{\dot{H}^{2-2\alpha+\theta, 2\alpha/p-\theta}}. \tag{A.21}$$

The proof is standard (see e.g. [37]) and hence we omit it here for brevity. Finally, we would like to point out the product law in anisotropic spaces (cf. Lemma 4.5 in [10]). The proof is well known (see e.g. [19], [36], [37]):

Proposition A.5. *Let $N = 4$, $q \ge 1, p_1 \ge p_2 \ge 1, 1/p_1 + 1/p_2 = 1$, $s_1 < 2/p_1$, $s_2 < 2/p_2$ (resp. $s_1 \le 2/p_1$, $s_2 \le 2/p_2$ if $q = 1$), $s_1 + s_2 > 0$, $\sigma_1 < 2/p_1$, $\sigma_2 < 2/p_2$ (resp. $\sigma_1 \le 2/p_1$, $\sigma_2 \le 2/p_2$ if $q = 1$), and $\sigma_1 + \sigma_2 > 0$. If $f, g \in C_0^\infty(\mathbb{R}^4)$, then*

$$\|fg\|_{(\dot{B}^{s_1+s_2-2/p_2}_{p_1,q})_h(\dot{B}^{\sigma_1+\sigma_2-2/p_2}_{p_1,q})_v} \lesssim \|f\|_{(\dot{B}^{s_1}_{p_1,q})_h(\dot{B}^{\sigma_1}_{p_1,q})_v} \|g\|_{(\dot{B}^{s_2}_{p_2,q})_h(\dot{B}^{\sigma_2}_{p_2,q})_v}. \tag{A.22}$$

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