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Hörmander type theorem on bi-parameter Hardy spaces for bi-parameter Fourier multipliers with optimal smoothness

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Abstract. The main purpose of this paper is to establish, using the bi-parameter Littlewood–Paley–Stein theory (in particular, the bi-parameter Littlewood–Paley–Stein square functions), a Hörmander–Mihlin type theorem for the following bi-parameter Fourier multipliers on bi-parameter Hardy spaces $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ $(0<p\le 1)$ with optimal smoothness:

$$T_m f(x_1,x_2)=\frac{1}{(2\pi)^{n_1+n_2}}\int_{\mathbb{R}^n\times\mathbb{R}^m} m(\xi,\eta)\hat{f}(\xi,\eta)e^{2\pi(x_1\xi+x_2\eta)}\,d\xi\,d\eta.$$

One of our results (Theorem 1.7) is the following: assume that $m(\xi,\eta)$ is a function on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_1}$ satisfying

$$\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{W^{(s_1,s_2)}}<\infty,$$

with $s_1>n_1(1/p-1/2)$, $s_2>n_2(1/p-1/2)$. Then T_m is bounded from $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for all $0<p\le 1$, and

$$\|T_m\|_{H^p\to H^p}\lesssim\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{W^{(s_1,s_2)}}.$$

Moreover, the smoothness assumption on s_1 and s_2 is optimal. Here, $m_{j,k}(\xi,\eta)=m(2^j\xi,2^k\eta)\Psi(\xi)\Psi(\eta)$, where $\Psi(\xi)$ and $\Psi(\eta)$ are suitable cut-off functions on $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$, respectively, and $W^{(s_1,s_2)}$ is a two-parameter Sobolev space on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. We also establish that under the same smoothness assumption on the multiplier m, $\|T_m\|_{H^p\to L^p}\lesssim\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{W^{(s_1,s_2)}}$ and $\|T_m\|_{\mathrm{CMO}_p\to\mathrm{CMO}_p}\lesssim\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{W^{(s_1,s_2)}}$ for all $0<p\le 1$. Moreover, $\|T_m\|_{L^p\to L^p}\lesssim\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{W^{(s_1,s_2)}}$ for all $1<p<\infty$ under the assumption $s_1>n_1/2$ and $s_2>n_2/2$.

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1. Introduction

The aim of this paper is to consider the minimal smoothness condition on the bi-parameter Fourier multipliers to guarantee their boundedness on the bi-parameter Hardy spaces. This is a bi-parameter version of the well-known Hörmander–Mihlin type multiplier theorem on one-parameter Hardy spaces due to Calderón and Torchinsky [1].

Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of Schwartz functions, and let $\mathcal{S}'(\mathbb{R}^d)$ denote the class of tempered distributions. The Fourier transform $\hat{f}$ and the inverse Fourier transform $\check{f}$ of $f \in \mathcal{S}(\mathbb{R}^d)$ are defined by

$$\mathcal{F}f(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\, dx\,, \quad \mathcal{F}^{-1}f(\xi) = \check{f}(x) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{ix\cdot\xi} f(\xi)\, d\xi.$$

We first recall the following Mihlin theorem in the linear case [32]. We use $[\alpha]$ to denote the largest integer not exceeding the real number α.

Theorem 1.1. *If a multiplier $m \in C^{[n/2]+1}(\mathbb{R}^n \backslash \{0\})$ satisfies the following condition*:

$$|\partial^\alpha m(\xi)| \le C_\alpha\, |\xi|^{-|\alpha|}, \quad \textit{for all } |\alpha| \le [n/2]+1,$$

then the Fourier multiplier operator $m(D)f = \mathcal{F}^{-1}[m\hat{f}]$ defined with the symbol $m(\xi)$ is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

On the other hand, Hörmander [24] reformulated and improved Mihlin's theorem using the Sobolev regularity of the multiplier. To describe Hörmander's theorem, we let $\Psi \in \mathcal{S}(\mathbb{R}^d)$ be a Schwartz function in $\mathbb{R}^d$ (with d changing from time to time as needed) satisfying

$$\text{(1.1)} \quad \operatorname{supp}\Psi \subset \{\xi \in \mathbb{R}^d : 1/2 \le |\xi| \le 2\}, \quad \sum_{j\in\mathbb{Z}} \Psi(\xi/2^j) = 1, \quad \text{for all } \xi \in \mathbb{R}^d\backslash\{0\}.$$

For $s \in \mathbb{R}$, the Sobolev space $W^s(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{W^s} \triangleq \|(I-\triangle)^{s/2} f\|_{L^2} < \infty,$$

where $(I-\triangle)^{s/2}f = \mathcal{F}^{-1}[(1+|\xi|^2)^{s/2}\hat{f}(\xi)]$ and $\xi \in \mathbb{R}^n$. Then the Hörmander multiplier theorem says:

Theorem 1.2. *If $m \in L^\infty(\mathbb{R}^n)$ satisfies*

$$\sup_{j\in\mathbb{Z}} \|m(2^j\cdot)\Psi\|_{W^s(\mathbb{R}^n)} < \infty \quad \textit{for all } s > \frac{n}{2},$$

where Ψ is the same as in (1.1) when $d = n$ and $W^s(\mathbb{R}^n)$ is the Sobolev space, then the Fourier multiplier operator $m(D)$ defined with the symbol m is bounded from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$.

Clearly, Hörmander's theorem is stronger than Mihlin's one and the number $n/2$ cannot be improved in Hörmander's theorem.

In order to study the boundedness of Fourier multipliers with optimal smoothness on Hardy spaces $H^p(\mathbb{R}^n)$ for $0 < p \le 1$, Calderón and Torchinsky [1] set up the following Hörmander's multiplier theorem in Hardy spaces.

Theorem 1.3. *If $m \in L^\infty(\mathbb{R}^n)$ satisfies*

$$\sup_{j\in\mathbb{Z}} \|m(2^j\cdot)\Psi\|_{W^s(\mathbb{R}^n)} < \infty \qquad \text{for all } s > \frac{n}{p} - \frac{n}{2},$$

where Ψ is the same as in (1.1) when $d = n$ and $W^s(\mathbb{R}^n)$ is the Sobolev space, then the Fourier multiplier operator $m(D)$ defined with the symbol m is bounded from $H^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ for all $0 < p \le 1$.

Before we proceed further, we give a brief introduction on the theory of multi-parameter singular integrals and Hardy spaces. Multi-parameter structures play an important role in harmonic analysis. On the one hand, the Calderón–Zygmund operators are extension of the classical Hilbert transform and can be regarded as centering around singular integrals associated with the Hardy–Littlewood maximal operator M that commutes with the usual dilations on $\mathbb{R}^n$, $\delta \cdot x = (\delta x_1, \ldots, \delta x_n)$ for $\delta > 0$. On the other hand, *multi-parameter* Calderón–Zygmund operators are singular integral operators that are extension of the double Hilbert transform and are associated with the *strong* maximal function M_S that commutes with the multi-parameter dilations on $\mathbb{R}^n$, $\delta \cdot x = (\delta_1 x_1, \ldots, \delta_n x_n)$ for $\delta = (\delta_1, \ldots, \delta_n) \in \mathbb{R}^n_+$, [26].

For Calderón–Zygmund theory in this multi-parameter setting, we are interested in considering operators of the form $Tf = K * f$, where K is homogeneous in the sense of $\delta_1 \ldots \delta_n K(\delta \cdot x) = K(x)$, or more generally, $K(x)$ satisfies certain differential inequalities and cancellation conditions such that the kernels $\delta_1 \ldots \delta_n K(\delta \cdot x)$ also satisfy the same bounds. These operators and their non-convolution type analogues have been studied extensively in the literature. The L^p $(1 < p < \infty)$ boundedness of such operators of convolution type was established by R. Fefferman and E. Stein [15]. To study the endpoint estimates, the multi-parameter Hardy spaces introduced by Gundy–Stein ([16]) have been further investigated by R. Fefferman ([13]), Chang and R. Fefferman ([4], [6]). The non-convolution type multi-parameter singular integral operators were studied by Journé ([27], [28]). We also refer the reader to the more recent work [10], [11], [12], [31] on boundedness on multi-parameter Triebel–Lizorkin and Besov spaces for Fourier multipliers and singular integral operators, L^p estimates for multi-parameter Fourier integral operators [22], [23] and L^p estimates by Street and Stein on multi-parameter singular Radon transforms [36], [37], [38].

However, as far as the endpoint theory for $p = 1$ and $p = \infty$ in the multi-parameter setting is considered, it is well-known that there is a basic obstacle to both the multi-parameter Hardy space and the multi-parameter BMO space theory. The role of cubes in the classical atomic decomposition of one-parameter Hardy spaces $H^p(\mathbb{R}^n)$ is replaced by arbitrary open sets of finite measure in the multi-parameter Hardy space $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. This makes the multi-parameter

Hardy space theory more difficult. Motivated by a counterexample of L. Carleson [3], the multi-parameter $\mathrm{BMO}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and Hardy space $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ theory was developed by Chang and R. Fefferman ([4], [6], [5]). Due to the complicated nature of atoms in multi-parameter Hardy spaces, it was a difficult task to establish boundedness of singular integral operators from multi-parameter Hardy spaces H^p to H^p or from H^p to L^p. R. Fefferman discovered a criterion for the H^p to L^p boundedness of a Calderón–Zygmund operator T obtained by restricting the action of T to *rectangle* atoms and applying a geometric lemma due to Journé (see Journé [27], [28], [29]). However, this beautiful theorem is restricted to two parameters only as observed by Journé and cannot be applied to the case of three or more parameters [27], [28]. Subsequently, the H^p to L^p boundedness for Journé's class of singular integral operators with arbitrary number of parameters was established by J. Pipher [35] by considering directly the action of the operator on (non-rectangle) atoms. More recently, the boundedness criterion on multiparameter Hardy spaces for Journé's class of singular integral operators with arbitrary number of parameters were given in [18], [19].

We are now ready to review the early works on multi-parameter Fourier multipliers in the literature. We refer the reader to definitions of multi-parameter Hardy spaces $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to Section 2.

R. Fefferman and K. C. Lin [14] extended the Fourier multiplier theorems in one-parameter setting to product Hardy spaces $H^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and proved the following.

Theorem 1.4. *Let $k=[n_1/2]+1$, $l=[n_2/2]+1$. Suppose $m\in C^k(\mathbb{R}^{n_1})\times C^l(\mathbb{R}^{n_2})$ and*

$$\int_{r_1<|\xi|\le 2r_1}\int_{r_2<|\eta|\le 2r_2}|\partial_\xi^\alpha\partial_\eta^\beta m(\xi,\eta)|^2\,d\xi\,d\eta\le C\,{r_1}^{-2|\alpha|+n_1}\,{r_2}^{-2|\beta|+n_2},$$

where $|\alpha|\le k,|\beta|\le l$. Then the multiplier operator T_m maps $H^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ boundedly to $L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for $1\le q\le 2$.

Lung-Kee Chen ([9]) extended the above multiplier theorem to product Hardy spaces $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for $0<p\le 1$ under stronger hypothesis and proved the following.

Theorem 1.5. *Let $k=[n_1(1/p-1/2)]+1$, $l=[n_2(1/p-1/2)]+1$, $0<p\le 1$. Suppose $m\in C^k(\mathbb{R}^{n_1})\times C^l(\mathbb{R}^{n_2})$ and*

$$|\partial_\xi^\alpha\partial_\eta^\beta m(\xi,\eta)|\le C\,|\xi|^{-|\alpha|}\,|\eta|^{-|\beta|}\quad \textit{for all } |\alpha|\le k,|\beta|\le l.$$

Then the multiplier operator T_m maps $H^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ boundedly to $L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for $p\le q\le 2$.

Viet-Le Hung extended the above result in [9] under weaker condition on the multiplier m and proved the following.

Theorem 1.6. *Let m be a bounded function in $C^k(\mathbb{R}^{n_1})\times C^l(\mathbb{R}^{n_2})$, where*

$$k=[n_1(1/p-1/2)]+1,\quad l=[n_2(1/p-1/2)]+1,\quad 0<p\le 1.$$

Suppose that

$$\int_{\Delta_i\times\Delta_j}\sum_{|\alpha|\le k}\sum_{|\beta|\le l}|2^{i|\alpha|}\,2^{j|\beta|}\,\partial_\xi^\alpha\partial_\eta^\beta(\xi^u\eta^v m(\xi,\eta))|^2 d\xi\,d\eta\le C\,2^{n_1 i}\,2^{n_2 j}\,2^{2i|u|}\,2^{2j|v|},$$

where $\Delta_i=\{\xi\in\mathbb{R}^{n_1}:2^i\le|\xi|\le 2^{i+1}\}$, *and a similar definition for* Δ_j,

$$\sup_{\xi\in\mathbb{R}^{n_1}}\left\{\int_{\Delta_j}\sum_{|\beta|\le l}|2^{j|\beta|}\partial_\eta^\beta(\eta^v m(\xi,\eta))|^2|\,d\eta\right\}\le C\,2^{n_2 j}\,2^{2j|v|},$$

and

$$\sup_{\eta\in\mathbb{R}^{n_2}}\left\{\int_{\Delta_i}\sum_{|\beta|\le k}|2^{i|\beta|}\partial_\xi^\alpha(\xi^u m(\xi,\eta))|^2|\,d\xi\right\}\le C\,2^{n_1 i}\,2^{2i|v|}.$$

Then the Fourier multiplier T_m *maps* $H^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ *boundedly to* $L^q(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ *for* $p\le q\le 2$.

As we have observed in the above theorems, the $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ boundedness has been established in [14], [9], [25] under the smoothness assumption on the bi-parameter Fourier multiplier, roughly speaking, of order $k=[n_1(1/p-1/2)]+1$ in the first parameter and $l=[n_2(1/p-1/2)]+1$ in the second parameter.

One of the main goals of this paper is to extend Calderón and Torchinsky's Hörmander–Mihlin type multiplier theorem [1] to the setting of product Hardy spaces and improve those bi-parameter multiplier theorems in [14], [9], [25] by proving that the bi-parameter multiplier operator is bounded from the bi-parameter Hardy spaces $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ for all $0<p\le 1$. It is known that $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\subsetneq L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and $||f||_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\le C||f||_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$ for all $0<p\le 1$ (see [17], [20]), thus our theorem indeed sharpens those results on $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ in the literature. Moreover, our theorem is optimal as far as the smoothness of the multiplier is concerned.

To describe our theorem, we introduce the two-parameter Sobolev spaces. For $s_1,s_2\in\mathbb{R}$, the two-parameter Sobolev space $W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is defined to be the class of all $f\in\mathcal{S}'(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ such that

$$\|f\|_{W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}=\|(I-\triangle)^{s_1/2,s_2/2}f\|_{L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}<\infty,$$

where $(I-\triangle)^{s_1/2,s_2/2}f=\mathcal{F}^{-1}[(1+|\xi|^2)^{s_1/2}(1+|\eta|^2)^{s_2/2}\hat{f}(\xi,\eta)]$ and $\xi\in\mathbb{R}^{n_1}$ and $\eta\in\mathbb{R}^{n_2}$.

Our first result is the following.

Theorem 1.7. *Assume that* $m(\xi,\eta)$ *is a function on* $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ *satisfying*

$$\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{W^{(s_1,s_2)}}<\infty$$

with $s_1>n_1(1/p-1/2)$, $s_2>n_2(1/p-1/2)$. *Then* T_m *is bounded from* $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ *to* $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ *for all* $0<p\le 1$ *and*

$$\|T_m\|_{H^p\to H^p}\lesssim\sup_{j,k\in\mathbb{Z}}\|m_{j,k}\|_{W^{(s_1,s_2)}}.$$

Moreover, the smoothness assumption on s_1 and s_2 is optimal in the sense that there exists a bi-parameter multiplier m with smoothness with $s_1 \le n_1(1/p - 1/2)$ and $s_2 \le n_2(1/p - 1/2)$ such that T_m is not bounded on $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

In the above theorem, and below,

$$m_{j,k}(\xi, \eta) = m(2^j \xi, 2^k \eta)\, \Psi(\xi)\, \Psi(\eta), \tag{1.2}$$

where $\Psi(\xi)$ is the same as in (1.1) with $d = n_1$ and $\Psi(\eta)$ is the same as in (1.1) with $d = n_2$.

Since $H^p \to H^p$ boundedness implies $H^p \to L^p$ boundedness, we in fact derive the same conclusion as those by Lung-Kee Chen and Viet-Le Hung but under weaker conditions on the multiplier m. We state them as follows.

Theorem 1.8. *Assume that $m(\xi, \eta)$ is a function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ satisfying*

$$\sup_{j,k \in \mathbb{Z}} \|m_{j,k}\|_{W^{(s_1,s_2)}} < \infty$$

with $s_1 > n_1(1/p - 1/2)$, $s_2 > n_2(1/p - 1/2)$. Then T_m is bounded from $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for all $0 < p \le 1$. Moreover,

$$\|T_m\|_{H^p \to L^p} \lesssim \sup_{j,k \in \mathbb{Z}} \|m_{j,k}\|_{W^{(s_1,s_2)}}\,.$$

Moreover, by interpolation and duality argument of the multi-parameter multiplier operators (see [5]), for $1 < p < \infty$ we have:

Theorem 1.9. *Assume that $m(\xi, \eta)$ is a function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ satisfying*

$$\sup_{j,k \in \mathbb{Z}} \|m_{j,k}\|_{W^{(s_1,s_2)}} < \infty$$

with $s_1 > n_1/2$, $s_2 > n_2/2$. Then T_m is bounded from $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for all $1 < p < \infty$. Moreover,

$$\|T_m\|_{L^p \to L^p} \lesssim \sup_{j,k \in \mathbb{Z}} \|m_{j,k}\|_{W^{(s_1,s_2)}}.$$

By duality of the product $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ and $\mathrm{CMO}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ (see [30] and Section 2 in this paper) and the $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ boundedness of T_m, we have:

Theorem 1.10. *Assume that $m(\xi, \eta)$ is a function on $\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$ satisfying*

$$\sup_{j,k \in \mathbb{Z}} \|m_{j,k}\|_{W^{(s_1,s_2)}} < \infty$$

with$s_1 > n_1(1/p - 1/2)$, $s_2 > n_2(1/p - 1/2)$ and $0 < p \le 1$. Then T_m is bounded from $\mathrm{CMO}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $\mathrm{CMO}^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Moreover,

$$\|T_m\|_{\mathrm{CMO}^p \to \mathrm{CMO}^p} \lesssim \sup_{j,k \in \mathbb{Z}} \|m_{j,k}\|_{W^{(s_1,s_2)}}.$$

In the case of $p = 1$, we derive the boundedness of T_m on the bi-parameter $\mathrm{BMO}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ under the assumption that the multiplier m satisfies the minimal smoothness $s_1 > n_1/2$ and $s_2 > n_2/2$.

We end this introduction with the following remarks. In order to establish our Theorem 1.7, we will need to show that Fefferman's criterion (see Theorems 3.1 and 3.2) is satisfied on rectangle atoms a for the Littlewood–Paley–Stein square function $T_m^*(a)$ for $T_m(a)$. It is a very delicate issue to show that Fefferman's criterion holds under the *minimal smoothness condition* on the multiplier m. A careful and rather complicated analysis is required to accomplish this. More precisely, for instance, the proof of our Theorem 1.7 can be reduced into proving that the operator T_m^* satisfies Fefferman's criterion on rectangle atom a, where $T_m^*(a)$ is defined as the bi-parameter Littlewood–Paley–Stein function of the $T_m(a)$ defined by

$$T_m^*(a)(x,y) = \Big(\sum_{j,k} |\psi_{j,k} * T_m(a)|^2(x,y)\Big)^{1/2}.$$

(See Section 2 for more details of definitions of the bi-parameter Littlewood–Paley–Stein square function). The detailed proof is presented in Section 3.

Finally, we mention that the results of this paper on bi-parameter Fourier multipliers have been extended to the case of arbitrary number of parameters in [7].

The organization of this paper is as follows. In Section 2 we recall some preliminary facts and give some relevant definitions. In Section 3, we prove Theorem 1.7. Then Theorems 1.8, 1.9 and 1.10 follow. In Section 4, we show the smoothness in our Theorem 1.7 is optimal.

2. Preliminary results

Theorem 2.1 ([15]). *Let $1 < p < \infty$, and let $\Psi_1 \in \mathcal{S}(R^{n_1}), \Psi_2 \in \mathcal{S}(R^{n_2})$ be such that* $\operatorname{supp}\psi_1 \subset \{\xi \in \mathbb{R}^{\mathrm{n}_1} : 1/\mathrm{a} \le |\xi| \le \mathrm{a}\}$ *for some $a > 1$,* $\operatorname{supp}\psi_2 \subset \{\eta \in \mathbb{R}^{\mathrm{n}_2} : 1/\mathrm{b} \le |\eta| \le \mathrm{b}\}$ *for some $b > 1$. Then there exists a constant $C > 0$ such that*

$$\Big\|\Big\{\sum_{j,k\in\mathbb{Z}} |\Psi_1(D/2^j)\Psi_2(D/2^k)f|^2\Big\}^{1/2}\Big\|_{L^p} \le C\,\|f\|_{L^p} \quad \textit{for all } f \in L^p(\mathbb{R}^{n_1+n_2}),$$

where

$$[\Psi_1(D/2^j)\Psi_2(D/2^k)f](\xi_1,\xi_2) = \mathcal{F}^{-1}\left[\Psi_1(\cdot/2^j)\Psi_2(\cdot/2^k)f(\cdot,\cdot)\right](\xi_1,\xi_2).$$

Moreover, if $\sum_{j\in\mathbb{Z}} \Psi_i(\xi_i/2^j) = 1$ for all $\xi_i \neq 0$, *for $i = 1, 2$, then*

$$\Big\|\Big\{\sum_{j,k\in\mathbb{Z}} |\Psi_1(D/2^j)\Psi_2(D/2^k)f|^2\Big\}^{1/2}\Big\|_{L^p} \approx \|f\|_{L^p} \quad \text{for all } f \in L^p(\mathbb{R}^{n_1+n_2}).$$

Now we recall the definitions of product Hardy spaces $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ and $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$-$(p,r)$-atoms, where $2\le r<\infty$ and $0<p\le 1$. Denote by $\mathcal{S}_\infty'(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ the functions $f\in\mathcal{S}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ such that for every i, $1\le i\le 2$,

$$\int_{\mathbb{R}^{n_1}} f(x_1,x_2)\,x_1^{\alpha_1}\,dx_1=0 \quad \text{for any } |\alpha_1|\ge 0,$$
$$\int_{\mathbb{R}^{n_2}} f(x_1,x_2)\,x_2^{\alpha_2}\,dx_2=0 \quad \text{for any } |\alpha_2|\ge 0.$$

The product Littlewood–Paley square function of $f\in\mathcal{S}'(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is defined by

$$\mathcal{G}(f)(x_1,x_2)=\Big\{\sum_{j,k\in\mathbb{Z}}|\Psi_1(D/2^j)\,\Psi_2(D/2^k)f|^2\Big\}^{1/2}.$$

For $0<p\le 1$, the product Hardy space on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ can be defined by

$$H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})=\{f\in\mathcal{S}_\infty'(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}):\mathcal{G}(f)\in L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})\},$$

with the norm $\|f\|_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}\approx\|\mathcal{G}(f)\|_{L^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$ ([20]).

By this definition, the proof of our main theorem (Theorem 1.7) can be reduced into proving that T^* maps $H^p\to L^p$, where T^* is defined as the bi-parameter Littlewood–Paley function of Tf defined by

$$T^*(f)(x,y)=\Big(\sum_{j,k}|\psi_{j,k}*Tf|^2(x,y)\Big)^{1/2}.$$

Next we introduce the definition of atoms in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ which provide a powerful tool in proving the boundedness of singular integrals on $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ ([4], [22]) Let $r\ge 2$. A function a defined in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ is called an $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is called an $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$-$(p,r)$-atom if a is supported in an open set $\Omega\subseteq\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with finite measure and satisfies the following conditions:

- $\|a\|_{L^r_w}\le|\Omega|^{1/r-1/p}$;
- a can be decomposed as $a(x_1,x_2)=\sum_{R\in\mathcal{M}(\Omega)}a_R(x_1,x_2)$, where a_R are supported on $2R=2(I_1\times I_2)$, I_i are dyadic cubes in $\mathbb{R}^{n_i}$, $i=1,2$, and $\mathcal{M}(\Omega)$ is the collection of all dyadic rectangles contain in Ω which are maximal in all directions x_1 and x_2. Moreover,

$$\Big\{\sum_{R\in\mathcal{M}(\Omega)}\|a_R\|_{L^r}^r\Big\}^{1/r}\le|\Omega|^{1/r-1/p};$$

- $\int_{2I_1}a_R(x_1,x_2)\,x_1^\alpha\,dx_1=0$ for all $x_2\in\mathbb{R}^{n_2}$ and $0\le|\alpha|\le N_{p,n_1}$,
  $\int_{2I_2}a_R(x_1,x_2)\,x_2^\alpha\,dx_2=0$ for all $x_1\in\mathbb{R}^{n_1}$ and $0\le|\alpha|\le N_{p,n_2}$.

where N_{p,n_i} is a large integer depending on p and n_i.

The dual space of weighted multi-parameter Hardy spaces $H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is introduced in [30]. We only consider the nonweighted case here. It is the so-called Carleson measure space $\mathrm{CMO}^p=\mathrm{CMO}^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. We refer to [30] for more details.

Definition 2.1. For $0<p\le 1$, we call $f\in \mathrm{CMO}^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ if $f\in (\mathcal{S}_\infty)'(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ with the finite norm defined by

$$\sup_{\Omega}\left\{\frac{1}{|\Omega|^{2/p-1}}\sum_{j,k\in\mathbb{Z}}\sum_{I_1\times I_2}|\Psi_1(D/2^j)\,\Psi_2(D/2^k)\,f(2^{-jl_1},2^{-kl_2})|^2\times|I_1\times I_2|\right\}^{1/2}$$

for all open sets Ω in $\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$ with finite measures, here I_1 are dyadic cubes in $\mathbb{R}^{n_1}$ with the side length 2^{-j} and the left lower corners of I_1 is $2^{-j}l_1, l_1\in\mathbb{Z}^{n_1}$ and I_2 are dyadic cubes in $\mathbb{R}^{n_2}$ with the side length 2^{-k} and the left lower corners of I_2 is $2^{-k}l_2, l_2\in\mathbb{Z}^{n_2}$.

We use $\mathrm{BMO}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ to denote $\mathrm{CMO}^1(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. From [30], we know the definition of the space CMO^p is independent of choice of functions Ψ_1 and Ψ_2, Thus the space $\mathrm{CMO}^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is well defined. Then the authors in [30] set up the following.

Theorem 2.2. *For $0<p\le 1$,*

$$(H^p)^*(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})=\mathrm{CMO}^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}).$$

To be precise, if $g\in\mathrm{CMO}^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, the map l_g given by $l_g(f)=\langle f,g\rangle$, defined initially for $f\in\mathcal{S}_\infty$, extends to a continuous linear functional on H^p with $\|l_g\|\approx\|g\|_{\mathrm{CMO}^p}$.

Conversely, for every $l\in(H^p)^*(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ there exists some $g\in\mathrm{CMO}^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ so that $l=l_g$. In particular, $(H^1)^*(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})=\mathrm{BMO}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$.

We now state some lemmas which will be needed in the sequel. The proofs of these lemmas are easy.

Lemma 2.1. *([34]) Let $2\le q<\infty$, $r>0$ and $s\ge 0$. Then there exists a constant $C>0$ depending on r such that*

$$\|\hat{f}\|_{L^q(w_{sq})}\le C\,\|f\|_{W^s}$$

for all $f\in W^s(\mathbb{R}^n)$ with $\mathrm{supp} f\subset\{|x|\le r\}$, where $w_{sq}=(1+|x|^2)^{sq/2}$ is a weight function.

Proposition 2.1. *If $s_j>n/2$ for $1\le j\le 2$, then $W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$ is an algebra under pointwise multiplication.*

Lemma 2.2. *Assume $s>n/2$ and $\max\{1,n/s\}<q<2$ and suppose $m\in W^s(R^n)$, $s>0$ and $\mathrm{supp}\, m\subset\{|\xi|\le 2/t\}$. Then there exists a constant $C>0$ depending only on N,n,s and q such that, for all $f\in\mathcal{S}(\mathbb{R}^n)$,*

$$|T_m(f)(x)|\le C\,\|m(\cdot/t)\|_{W^s(\mathbb{R}^n)}\,M(|f|^q)(x)^{1/q}$$

for all $x\in\mathbb{R}^n$, where M is the Hardy–Littlewood maximal function of f.

Lemma 2.3. *Let* $s_1, s_2 \in \mathbb{R}$ *and let* $\Psi_1 \in \mathcal{S}(\mathbb{R}^{n_1})$ *and* $\Psi_2 \in \mathcal{S}(\mathbb{R}^{n_2})$ *be such that* $\operatorname{supp}\Psi_1, \operatorname{supp}\Psi_2$ *are compact and none of them contains the origin. Assume that* $\Phi \in C^\infty(\mathbb{R}^{n_1}\setminus\{0\}\times\mathbb{R}^{n_2}\setminus\{0\})$ *satisfies*

$$|\partial_\xi^\alpha \partial_\eta^\beta \Phi(\xi,\eta)| \le C_{\alpha,\beta}\, |\xi|^{-|\alpha|}|\eta|^{-|\beta|}$$

for all $\alpha, \beta \in \mathbb{N}_0^n$. *Then there exists a constant* $C > 0$ *such that*

$$\sup_{t,s>0} \|m(t\xi, s\eta)\,\Phi(t\xi,s\eta)\,\Psi_1(\xi)\,\Psi_2(\eta)\|_{W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \le C \sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$

for all m *satisfying* $\sup_{j,k\in\mathbb{Z}} \|m_{j,k}\|_{W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} < \infty$, *where* $m_{j,k}$ *is defined by* (1.2).

Proof. We mimic the proof of Lemma (3.4) in [8]. For simplicity, we use W^{s_1,s_2} to denote $W^{(s_1,s_2)}(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$. First, we assume that $\operatorname{supp}\Psi_1 \subset \{1/2^{j_0} \le |\xi| \le 2^{j_0}\}$ and $\operatorname{supp}\Psi_2 \subset \{1/2^{k_0} \le |\eta| \le 2^{k_0}\}$ for some $j_0, k_0 \in \mathbb{N}$. Given $t, s > 0$, take $j, k \in \mathbb{Z}$ satisfying $2^{j-1} \le t \le 2^j$, $2^{k-1} \le s \le 2^k$. Then, since $1 < 2^j/t \le 2, 1 < 2^k/s \le 2$, by change of variables,

$$\|m(t\cdot, s\cdot)\Phi(t\cdot,s\cdot)\,\Psi_1(\cdot)\,\Psi_2(\cdot)\|_{W^{s_1,s_2}} \lesssim \|m(2^j\cdot, 2^k\cdot)\Phi(2^j\cdot,2^k\cdot)\Psi_1(2^jt^{-1}\cdot)\Psi_2(2^ks^{-1}\cdot)\|_{W^{s_1,s_2}}\,.$$

Let $\Psi^1(\xi), \Psi^2(\eta)$ be as in (1.1) with $d = n_1$ and $d = n_2$ respectively. Using $\operatorname{supp}\Psi_1(2^jt^{-1}\cdot) \subset \{1/2^{j_0+1} \le |\xi| \le 2^{j_0}\}$ and $\operatorname{supp}\Psi_2(2^ks^{-1}\cdot) \subset \{1/2^{k_0+1} \le |\eta| \le 2^{k_0}\}$, we have

$$\begin{aligned}
&\|m(2^j\cdot, 2^k\cdot)\Phi(2^j\cdot,2^k\cdot)\Psi_1(2^jt^{-1}\cdot)\Psi_2(2^ks^{-1}\cdot)\|_{W^{s_1,s_2}}\\
&\le C \sum_{j_1=-(j_0+1)}^{j_0}\ \sum_{k_1=-(k_0+1)}^{k_0} \Big\|m(2^j\cdot,2^k\cdot)\,\Phi(2^j\cdot,2^k\cdot)\,\Psi_1(2^jt^{-1}\cdot)\\
&\qquad\qquad \times\ \Psi_2(2^ks^{-1}\cdot)\,\Psi(\cdot/2^{j_1})\,\Psi(\cdot/2^{k_1})\Big\|_{W^{s_1,s_2}}\\
&\le C \sum_{j_1=-(j_0+1)}^{j_0}\ \sum_{k_1=-(k_0+1)}^{k_0} \Big\|m(2^j\cdot,2^k\cdot)\,\Psi(\cdot/2^{j_1})\,\Psi(\cdot/2^{k_1})\Big\|_{W^{s_1,s_2}}\\
&\qquad\qquad \times \Big\|\Phi(2^j\cdot,2^k\cdot)\,\Psi_1(2^jt^{-1}\cdot)\,\Psi_2(2^ks^{-1}\cdot)\Big\|_{W^{s_1,s_2}}\\
&\le C \sum_{j_1=-(j_0+1)}^{j_0}\ \sum_{k_1=-(k_0+1)}^{k_0} \|m(2^{j+j_1}\cdot, 2^{k+k_1}\cdot)\Psi(\cdot)\Psi(\cdot)\|_{W^{s_1,s_2}}\|\Phi(t\cdot,s\cdot)\Psi_1\Psi_2\|_{W^{s_1,s_2}}\\
&\le C\Big(\sup_{j,k\in\mathbb{Z}} \|m(2^{j+j_1}\cdot, 2^{k+k_1}\cdot)\Psi\Psi\|_{W^{s_1,s_2}}\Big)\Big(\sup_{j,s>0}\|\Phi(t\cdot,s\cdot)\Psi_1\Psi_2\|_{W^{s_1,s_2}}\Big)
\end{aligned}$$

Obviously, $\sup_{j,s>0}\|\Phi(t\cdot,s\cdot)\Psi_1\Psi_2\|_{W^{s_1,s_2}} < \infty$.

The proof is then complete. □

3. The proof of Theorem 1.7

In this section, we use the notations $A \approx B$ to denote $C^{-1}B \le A \le CB$ for some absolute constant $C \ge 1$ and $A \lesssim B$ to denote $A \le CB$ for some absolute constant $C > 0$.

We first recall a boundedness criterion due to R. Fefferman [13].

Theorem 3.1. *Suppose that T is a bounded linear operator from $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Let $\tilde{R}_r$ denote the r fold enlargement of R and ${}^c\tilde{R}_r$ denote its complement. Suppose further that if a is an $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ rectangle atom $(0 < p \le 1)$ supported on R, we have*

$$\iint_{{}^c\tilde{R}_r} |T(a)(x,y)|^p \, dx\, dy \lesssim B\, r^{-\delta} \quad \text{for all } r \ge 2$$

and some fixed $\delta > 0$. Then T is a bounded operator from $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Moreover,

$$\|T\|_{H^p \to L^p} \lesssim (\|T\|_{L^2 \to L^2} + B).$$

If we replace Tf by T^*f and use the Fefferman criterion, then we can obtain:

Theorem 3.2. *Suppose that T is a bounded linear operator from $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $L^2(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Suppose further that if a is an $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ rectangle atom $(0 < p \le 1)$ supported on R, we have*

$$\iint_{{}^c\tilde{R}_r} |T^*(a)(x,y)|^p \, dx\, dy \lesssim B\, r^{-\delta} \quad \text{for all } r \ge 2$$

and some fixed $\delta > 0$. Then T is a bounded operator from $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ to $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Moreover,

$$\|T\|_{H^p \to H^p} \lesssim (\|T\|_{L^2 \to L^2} + B).$$

Therefore, to establish the main theorem (Theorem 1.7), we only need to prove: if a is an $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ rectangle atom $(0 < p \le 1)$ supported on R, we have

$$\iint_{{}^c\tilde{R}_r} |T^*_m(a)(x,y)|^p \, dx\, dy \lesssim B\, r^{-\delta} \quad \text{for all } r \ge 2$$

and some fixed $\delta > 0$, where

$$T^*_m(f)(x,y) = \Big(\sum_{j,k} |\psi_{j,k} * T_m f|^2(x,y) \Big)^{1/2}.$$

Since T_m is a convolution operator, we have

$$\|T_m\|_{L^2 \to L^2} \lesssim \|m\|_{L^\infty}$$

By the Sobolev embedding theorem, we have

$$\|m\|_{L^\infty} \lesssim \sup_{j,k} \|m_{j,k}\|_{W^{(s_1,s_2)}}$$

when $s_1 > n_1/2, s_2 > n_2/2$

Thus, to prove $T_m : H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}) \longrightarrow H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})$, by a translation it only needs to show the estimate

$$\|T_m^*(a)(x,y)\|^p_{L^p({}^c\tilde{R}_r)} \lesssim \Big(\sup_{j,k}\|m_{j,k}\|_{W^{s_1,s_2}}\Big) r^{-\delta} \tag{3.1}$$

for all $s_1 > n_1/2 - p/2$, $s_2 > n_2/2 - p/2$, where a is a rectangle atom supported in R, where $R = I \times J$ is centered at (0,0).

By Sobolev's embedding theorem, it is sufficient to consider the case

$$n_i(1/p - 1/2) < s_1 < [n_i(1/p-1)] + n_i/2 + 1, \quad \text{for } 1 \le i \le 2.$$

Define

$$K_{j,k}(x,y) = \mathcal{F}^{-1}[m(\cdot,\cdot)\Psi(\cdot/2^j)\Psi(\cdot/2^k)](x,y).$$

If we write $\tilde{K}_{j,k} = \mathcal{F}^{-1}[m_{j,k}]$, then $K_{j,k}(x,y) = 2^{jn_1+kn_2}\tilde{K}_{j,k}(2^jx, 2^ky)$, where $x, y \in \mathbb{R}^{n_1}\times\mathbb{R}^{n_2}$. Then

$$T_m^*(a)(x,y) = \Big(\sum_{j,k}|T_{m(\cdot,\cdot)\Psi(\cdot/2^j)\Psi(\cdot/2^k)}(a)|^2(x,y)\Big)^{1/2}$$
$$= \Big(\sum_{j,k}\Big|\int_{\mathbb{R}^{2n}} K_{j,k}(x-x_1, y-y_1)a(x_1,y_1)dx_1\,dy_1\Big|^2\Big)^{1/2} \triangleq \Big(\sum_{j,k}|F_{j,k}(x,y)|^2\Big)^{1/2}.$$

We decompose the integral domain ${}^c\tilde{R}_r$ into three parts:

$$\begin{aligned} {}^c\tilde{R}_r^1 &= \{(\xi,\eta)|\xi \in {}^c\tilde{I}_r, \eta \in \tilde{J}_r\},\\ {}^c\tilde{R}_r^2 &= \{(\xi,\eta)|\xi \in \tilde{I}_r, \eta \in {}^c\tilde{J}_r\},\\ \text{and } {}^c\tilde{R}_r^3 &= {}^c\tilde{R}_r \backslash ({}^c\tilde{R}_r^1 \cup {}^c\tilde{R}_r^2). \end{aligned}$$

By the subadditivity of the p-th power of the L^p-norm, $p \le 1$ and Hölder's inequality, to prove (3.1), we estimate

$$\begin{aligned} \int_{{}^c\tilde{R}_r^3} |T_m^*(a)(x,y)|^p\,dx\,dy &\le \sum_{j,k\in\mathbb{Z}} \int_{{}^c\tilde{R}_r^3} |F_{j,k}(x,y)|^p\,dx\,dy\\ &\le \sum_{j,k\in\mathbb{Z}} \Big(\int_{{}^c\tilde{R}_r^3} |x|^{-s_1(\frac{2p}{2-p})}|y|^{-s_2(\frac{2p}{2-p})}\,dx\,dy\Big)^{1-p/2}\\ &\qquad \times \Big(\int_{{}^c\tilde{R}_r^3} |x_1|^{2s_1}|x_2|^{2s_2}|F_{j,k}(x_1,x_2)|^2\,dx\,dy\Big)^{p/2}\\ &\lesssim \sum_{j,k\in\mathbb{Z}} \{(r^{-s_1+n_1/p-n_1/2} + r^{-s_2+n_2/p-n_2/2})|I|^{-s_1/n_1+1/p-1/2}\\ &\qquad \times |J|^{-s_2/n_2+1/p-1/2}\,\||x|^{s_1}|y|^{s_2}F_{j,k}\|_{L^2({}^c\tilde{R}_r^3)}\}^p, \end{aligned}$$

where we used the condition $s_1 > n_1/p - n_1/2$ and $s_2 > n_2/p - n_2/2$ to obtain the last inequality.

We note

$$\begin{aligned}
F_{j,k}(x,y) &= \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \Big(K_{k,j}(x-x_1,y-y_1) - \sum_{|\alpha|\le L_1-1} \frac{1}{\alpha!}(-x_1)^\alpha\, \partial_1^\alpha K_{j,k}(x,y-y_1)\Big) \\
&\qquad \times a(x_1,y_1)\,dx_1\,dy_1 \\
&= L_1 \sum_{|\alpha|=L_1} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \int_{0<t<1} \frac{(-x_1)^\alpha}{\alpha!}(1-t)^{L_1-1} \\
&\qquad \times \partial_1^\alpha K_{j,k}(x-tx_1,y-y_1)\,a(x_1,y_1)\,dt\,dx_1\,dy_1 \\
&= L_1 \sum_{|\alpha|=L_1} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \int_{0<t<1} \frac{(-x_1)^\alpha}{\alpha!}(1-t)^{L_1-1}\Big(\partial_1^\alpha K_{j,k}(x-tx_1,y-y_1) \\
&\qquad - \sum_{|\beta|\le L_2-1} \frac{(-y_1)^\beta}{\beta!}\partial_3^\beta\, \partial_1^\alpha K_{j,k}(x-tx_1,y-y_1)\Big)\,a(x_1,y_1)\,dt\,dx_1\,dy_1 \\
&= L_1L_2 \sum_{|\alpha|=L_1}\sum_{|\beta|=L_2} \int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}} \int_{I\times J}\int_{0<t<1}\int_{0<s<1} \frac{(-x_1)^\alpha}{\alpha!}\frac{(-y_1)^\beta}{\beta!} \\
&\quad \times (1-t)^{L_1-1}(1-s)^{L_2-1}\,\partial_1^\alpha\partial_2^\beta K_{j,k}(x-tx_1,y-sy_1)a(y_1,y_2)\,ds\,dt\,dx_1\,dy_1,
\end{aligned}$$

where L_1 and L_2 are integers satisfying

$$0\le L_1\le [n_1(1/p-1/2)], \quad 0\le L_2\le [n_2(1/p-1/2)].$$

Thus we have

$$\begin{aligned}
&|F_{j,k}(x,y)| \\
&\ \lesssim |I|^{L_1/n_1}|J|^{L_2/n_2} \\
&\quad \times \sum_{|\alpha|=L_1}\sum_{|\beta|=L_2}\Big(\int_{I\times J}\int_{0<t<1}\int_{0<s<1} |\partial_1^\alpha\partial_2^\beta K_{j,k}(x-tx_1,y-sy_1)|^2 ds\,dt\,dx_1\,dy_1\Big)^{1/2} \\
&\quad \times \Big(\int_{I\times J}|a(x_1,y_1)|^2 dx_1\,dy_1\Big)^{1/2} \\
&\ \lesssim |I|^{1/2-1/p+L_1/n_1}|J|^{1/2-1/p+L_2/n_2} \\
&\quad \times \sum_{|\alpha|=L_1}\sum_{|\beta|=L_2}\Big(\int_{I\times J}\int_{0<t<1}\int_{0<s<1} |\partial_1^\alpha\partial_2^\beta K_{j,k}(x-tx_1,y-sy_1)|^2 ds\,dt\,dx_1\,dy_1\Big)^{1/2}.
\end{aligned}$$

Next we estimate the term

$$\big\|\,|x|^{s_1}\,|y|^{s_2}\,F_{j,k}(x,y)\big\|_{L^2({}^c\tilde{R}_r^3)}.$$

Since it is easy to see $|x-tx_1|\approx|x|$, $|y-sy_1|\approx|y|$.

Hence, we have

$$\Big\||x|^{s_1}|y|^{s_2}\Big(\int_{I\times J}\int_{0<t<1}\int_{0<s<1}|\partial_1^\alpha\partial_2^\beta K_{j,k}(x-tx_1,y-sy_1)|^2 ds\,dt\,dx_1\,dy_1\Big)^{1/2}\Big\|_{L^2({}^c\tilde{R}_r^3)}$$
$$=\Big(\int_{I\times J}\int_{0<t<1}\int_{0<s<1}\int_{{}^c\tilde{R}_r^3}||x-tx_1|^{s_1}|y-sy_1|^{s_2}\partial_1^\alpha\partial_2^\beta$$
$$\times K_{j,k}(x-tx_1,y-sy_1)|^2\,dx\,dy\,ds\,dt\,dx_1\,dy_1\Big)^{1/2}$$
$$\leq\Big(\int_{I\times J}\int_{0<t<1}\int_{0<s<1}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\big||x|^{s_1}|y|^{s_2}\partial_1^\alpha\partial_2^\beta K_{j,k}(x,y)\big|^2\,dx\,dy\,ds\,dt\,dx_1\,dy_1\Big)^{1/2}$$
$$=|I|^{1/2}\,|J|^{1/2}\,\big\||x|^{s_1}\,|y|^{s_2}\,\partial_1^\alpha\partial_2^\beta K_{j,k}(x,y)\big\|_{L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}.$$

Since $K_{j,k}(x,y)=2^{jn_1+kn_2}\tilde{K}_{j,k}(2^jx,2^ky)$, the last term above can be written as

$$2^{j(-s_1+n_1+|\alpha|)}\,2^{k(-s_2+n_2+|\beta|)}|I|^{1/2}|J|^{1/2}$$
$$\times\big\||2^jx|^{s_1}|2^ky|^{s_2}\partial_1^\alpha\partial_2^\beta\tilde{K}_{j,k}(2^jx,2^ky)\big\|_{L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$
$$=2^{j(-s_1+n_1/2+L_1)}2^{k(-s_2+n_2/2+L_2)}|I|^{1/2}|J|^{1/2}$$
$$\times\big\||x|^{s_1}|y|^{s_2}\partial_1^\alpha\partial_2^\beta\tilde{K}_{j,k}(x,y)\big\|_{L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$
$$\lesssim 2^{j(-s_1+n_1/2+L_1)}\,2^{k(-s_2+n_2/2+L_2)}\,|I|^{1/2}\,|J|^{1/2}$$
$$\times\big\|\langle x\rangle^{s_1}\langle y\rangle^{s_2}\partial_1^\alpha\partial_2^\beta\tilde{K}_{j,k}(x,y)\big\|_{L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$
$$=2^{j(-s_1+n_1/2+L_1)}\,2^{k(-s_2+n_2/2+L_2)}\,|I|^{1/2}\,|J|^{1/2}$$
$$\times\big\|\langle x\rangle^{s_1}\langle y\rangle^{s_2}\mathcal{F}^{-1}[\xi^\alpha\eta^\beta m_{j,k}(\xi,\eta)](x,y)\big\|_{L^2(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})}$$
$$\lesssim 2^{j(-s_1+n_1/2+L_1)}\,2^{k(-s_2+n_2/2+L_2)}\,|I|^{1/2}\,|J|^{1/2}\,\|m_{j,k}\|_{W^{s_1,s_2}}.$$

Thus we have

$$\Big\||x|^{s_1}|y|^{s_2}\int_{I\times J}\int_{0<t,s<1}|\partial_1^\alpha\partial_2^\beta K_{j,k}(x-tx_1,y-sy_1)a(x_1,y_1)|dt\,ds\,dx_1\,dy_1\Big\|_{L^2({}^c\tilde{R}_r^3)}$$
$$\lesssim 2^{j(-s_1+n_1/2+L_1)}\,2^{k(-s_2+n_2/2+L_2)}\,|I|^{1/2}\,|J|^{1/2}\,\|m_{j,k}\|_{W^{s_1,s_2}}.$$

Thus, we obtain

$$(r^{-s_1+n_1/p-n_1/2}+r^{-s_2+n_2/p-n_2/2})\,|I|^{-s_1/n_1+1/p-1/2}$$
$$\times|J|^{-s_2/n_2+1/p-1/2}\big\||x|^{s_1}|y|^{s_2}F_{j,k}(x,y)\big\|_{L^2({}^c\tilde{R}_r^3)}$$
$$\lesssim r^{-\delta}\,2^{j(-s_1+n_1/2+L_1)}\,2^{k(-s_2+n_2/2+L_2)}\,|I|^{-s_1/n_1+L_1/n_1+1/2}$$
$$\times|J|^{-s_2/n_2+L_2/n_2+1/2}\,\|m_{j,k}\|_{W^{s_1,s_2}},$$

where $\delta=\min\{s_1-n_1/p+n_1/2,s_2-n_2/p+n_2/2\}$.

For each rectangle $I\times J$, there exist integers j_0,k_0 such that $|I|^{1/n_1}\approx 2^{-j_0}$,

$|J|^{1/n_2} \approx 2^{-k_0}$. Therefore

$$\|T_m^*(a)\|^p_{L^p({}^c\tilde{R}_r^3)} \le \sum_{j,k} \|F_{j,k}\|^p_{L^p({}^c\tilde{R}_r^3)} = \sum_{j\ge j_0}\sum_{k\ge k_0} + \sum_{j\ge j_0}\sum_{k<k_0} + \sum_{j<j_0}\sum_{k\ge k_0} + \sum_{j<j_0}\sum_{k<k_0} \{\|F_{k,j}\|^p_{L^p({}^c\tilde{R}_r^3)}\}.$$

First we consider the sums $\sum_{j\ge j_0}\sum_{k\ge k_0}$ and pick $L_1 = 0$, $L_2 = 0$. Since $s_1 > n_1/p - n_1/2$, $s_2 > n_2/p - n_2/2$ and $0 < p \le 1$, we have

$$\begin{aligned}\sum_{j\ge j_0}\sum_{k\ge k_0} |F_{j,k}\|^p_{L^p({}^c\tilde{R}_r^3)} &\lesssim r^{-p\delta} \sum_{j\ge j_0} 2^{pj(-s_1+n_1/2)} |I|^{p(-s_1/n_1+1/2)} \\ &\qquad \times \sum_{k\ge k_0} 2^{kp(-s_2+n_2/2)} |J|^{p(-s_2/n_2+1/2)} \|m_{j,k}\|_{W^{(s_1,s_2)}} \\ &\approx r^{-p\delta} \sum_{j\ge j_0} 2^{p(j-j_0)(-s_1+n_1/2)} \sum_{k\ge k_0} 2^{p(k-k_0)(-s_2+m/2)} \|m_{j,k}\|_{W^{(s_1,s_2)}} \\ &\lesssim r^{-p\delta} \sup_{j,k} \|m_{j,k}\|_{W^{(s_1,s_2)}}.\end{aligned}$$

We pick $L_1 = 0$, $L_2 = [n_2/p - n_2] + 1$ for the sums $\sum_{j\ge j_0}\sum_{k<k_0}$. Then

$$\begin{aligned}&\sum_{j\ge j_0}\sum_{k\ge k_0} \|F_{j,k}\|^p_{L^p({}^c\tilde{R}_r^3)} \\ &\quad \lesssim r^{-p\delta} \sum_{j\ge j_0} 2^{pj(-s_1+n_1/2)} |I|^{p(-s_1/n_1+1/2)} \|m_{j,k}\|_{W^{(s_1,s_2)}} \\ &\qquad \cdot \sum_{k<k_0} 2^{pk(-s_2+n_2/2+[n_2/p-n_2])} |J|^{p(-s_2/n_2+1/2+[n_2/p-n_2]/n_2)} \|m_{j,k}\|_{W^{(s_1,s_2)}} \\ &\quad \approx r^{-p\delta} \sum_{j\ge j_0} 2^{p(j-j_0)(-s_1+n_1/2)} \sum_{k<k_0} 2^{p(k_0-k)(s_2-n_2/2-[n_2/p-n_2]-1)} \|m_{j,k}\|_{W^{(s_1,s_2)}} \\ &\quad \lesssim r^{-p\delta} \sup_{j,k} \|m_{j,k}\|_{W^{(s_1,s_2)}}.\end{aligned}$$

By picking $L_1 = [n_1/p - n_1] + 1$, $L_2 = 0$ for the sums $\sum_{j\le j_0}\sum_{k>k_0}$ and $L_1 = [n_1/p - n_1] + 1$, $L_2 = [n_2/p - n_2] + 1$ for the sums $\sum_{j\le j_0}\sum_{k\le k_0}$, we can reach the same conclusion.

Since $0 < p \le 1$ and by Hölder's inequality, we have

$$\begin{aligned}&\int_{{}^c\tilde{R}_r^1} |T_m^*(a)(x,y)|^p \,dx\,dy \le \sum_{j\in\mathbb{Z}} \int_{{}^c\tilde{R}_r^1} \Big(\sum_k |F_{j,k}|^2(x,y)\Big)^{p/2} dx\,dy \\ &\le \sum_{j\in\mathbb{Z}} \Big(\int_{{}^c\tilde{R}_r^1} |x|^{-s_1(\frac{2p}{2-p})} \,dx\,dy\Big)^{1-p/2} \Big(\int_{{}^c\tilde{R}_r^1} |x|^{2s_1} \sum_k |F_{j,k}|^2(x,y)\,dx\,dy\Big)^{p/2} \\ &\lesssim \sum_{j\in\mathbb{Z}} \big\{r^{-s_1+n_1/p-n_1/2} |I|^{-s_1/n_1+1/p-1/2} |J|^{1/p-1/2} \|\{|x|^{s_1} F_{j,k}\}_{l_k^2}\|_{L^2({}^c\tilde{R}_r^1)}\big\}^p,\end{aligned}$$

where the last inequality is obtained by the condition $s_1 > n_1/p - n_1/2$.

Using the same method as above, we can deduce for $F_j(x,y)=\sum_k F_{j,k}(x,y)$ and $K_j(x,y)=\sum_k K_{j,k}(x,y)$ the following estimate:

$$F_j(x,y)$$
$$= L_1 \sum_{|\alpha|=L_1} \int_{I\times J}\int_0^1 \frac{(-x_1)^\alpha}{\alpha!}(1-s)^{L_1-1}\partial_1^\alpha K_j(x-\theta x_1, y-y_1)a(x_1,y_1)d\theta\, dx_1\, dy_1.$$

Note that

$$\begin{aligned}\|\{|x|^{s_1}F_{j,k}\}_{l_k^2}\|_{L^2({}^c\tilde{R}_r^1)} &\le \|\{|x|^{s_1}F_{j,k}\}_{l_k^2}\|_{L^2({}^c\tilde{I}_r\times\mathbb{R}^{n_2})}\\ &\le \sup_{\|\{h_k\}_{l_k^2}\|_{L^2({}^c\tilde{I}_r\times\mathbb{R}^{n_2})}\le 1}\Big|\sum_k\int_{{}^c\tilde{I}_r\times\mathbb{R}^{n_2}}|x|^{s_1}F_{j,k}(x,y)h_k(x,y)\,dx\,dy\Big|.\end{aligned}$$

Fixing h, we have

$$\begin{aligned}&\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}|x|^{s_1}F_{j,k}(x,y)\,h_k(x,y)\,dx\,dy\\ &\quad=\sum_{|\alpha|=L_1}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_1}\times\mathbb{R}^{n_2}}\int_0^1|x|^{s_1}\,\frac{(-x_1)^\alpha}{\alpha!}\,(1-\theta)^{L_1-1}\\ &\qquad\cdot\partial_1^\alpha K_{j,k}(x-\theta x_1,y-y_1)\,a(x_1,y_1)\,d\theta\,dx_1\,dy_1\,h(x,y)\,dx\,dy.\end{aligned}$$

Define

$$T_{x,\theta,x_1,k}(a)(y)=\int_{\mathbb{R}^{n_2}}\partial_1^\alpha K_{j,k}(x-\theta x_1,y-y_1)\,a(x_1,y_1)\,dy_1.$$

Obviously, T_{x_1,θ,y_1,z_1} is a Fourier multiplier operator with symbol

$$m_{x,\theta,x_1,k}(\eta)=\partial_1^\alpha\hat{K}_{j,k}^2(x-\theta x_1,\eta).$$

Denote a_{x_1} and $h_{k,x}$ the cross-section functions by $a_{x_1}(y_1)=a(x_1,y_1)$ and $h_{k,x}(y)=h_k(x,y)$, then we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^{n_2}}T_{x,\theta,x_1,k}(a_{x_1})(y)h_{k,x}(y)dy\Big| &= \Big|\int_{\mathbb{R}^{n_2}}T_{x,\theta,x_1,k}(a_{x_1})(y)\psi_k * h_{k,x}(y)\,dy\Big|\\ &\lesssim \|\mathcal{F}^{-1}m_{x,\theta,x_1,k}\|_{L^{q'}(w_{sq'})}\int_{\mathbb{R}^{n_2}}M(|\psi_k * a_{x_1}|^q)(y)^{1/q}|\psi_k * h_{k,x}(y)|\,dy.\end{aligned}$$

Then, we have

$$\begin{aligned}&\Big|\sum_k\int_{{}^c\tilde{I}_r\times\mathbb{R}^{n_2}}|x|^{s_1}F_{j,k}(x,y)\,h_k(x,y)\,dx\,dy\Big|\\ &\quad\lesssim\sum_{|\alpha|=L_1}|I|^{L_1/n_1}\sum_k\int_0^1\int_{{}^c\tilde{I}_r}\int_{\mathbb{R}^{n_2}}\int_{\mathbb{R}^{n_1}}|x|^{s_1}\|\mathcal{F}^{-1}m_{x,\theta,x_1,k}\|_{L^{q'}(w_{s_2q'})}\\ &\qquad\cdot M(|\psi_k * a_{x_1}|^q)(y)^{1/q}|\psi_k * h_{k,x}(y)|\,dx_1\,dy\,dx\,d\theta.\end{aligned}$$

First, since

$$\|\mathcal{F}^{-1}m_{x,\theta,x_1,k}\|_{L^{q'}(w_{s_2q'})} \leq \|\langle y\rangle^{s_2}\partial_1^{\alpha}K_{j,k}(x-\theta x_1,y)\|_{L^2(\mathbb{R}_y^{n_2})},$$

we have

$$\begin{aligned}
&\int_{{}^c\tilde{I}_r} |x|^{s_1}\|\langle y\rangle^{s}\partial_1^{\alpha}K_{j,k}(x-\theta x_1,y)\|_{L^2(\mathbb{R}_y^{n_2})}|\psi_k * h_{k,x}(y)|dx\\
&\quad\leq \int_{{}^c\tilde{I}_r} |x-\theta x_1|^{s_1}\|\langle y\rangle^{s}\partial_1^{\alpha}K_{j,k}(x-\theta x_1,y)\|_{L^2(\mathbb{R}_y^{n_2})}|\psi_t * h_{k,x}(y)|dx\\
&\quad\leq \big\||x-\theta x_1|^{s_1}\langle y\rangle^{s_2}\partial_1^{\alpha}K_{j,k}(x-\theta x_1,y)\big\|_{L^2(\mathbb{R}^{n_1+n_2})}\|\psi_k * h_{k,x}(y)\|_{L^2(\mathbb{R}_x^{n_1})}\\
&\quad= \big\||x|^{s_1}\langle y\rangle^{s_2}\partial_1^{\alpha}K_{j,k}(x,y)\big\|_{L^2(\mathbb{R}^{n_1+n_2})}\|\psi_t * h_{k,x}(y)\|_{L^2(\mathbb{R}_x^{n_1})}\\
&\quad= 2^{j(-s_1+n_1+|\alpha|)}\big\||2^jx|^{s_1}\langle y\rangle^{s_2}\partial_1^{\alpha}\tilde{K}_{j,k}(2^jx,y)\big\|_{L^2(\mathbb{R}^{n_1+n_2})}\|\psi_k * h_{k,x}(y)\|_{L^2(\mathbb{R}_x^{n_1})}\\
&\quad= 2^{j(-s_1+n_1/2+|L_1|)}\big\||x|^{s_1}\langle y\rangle^{s_2}\partial_1^{\alpha}\tilde{K}_{j,k}(x,y)\big\|_{L^2(\mathbb{R}^{n_1+n_2})}\|\psi_k * h_{k,x}(y)\|_{L^2(\mathbb{R}_x^{n_1})}\\
&\quad\leq 2^{j(-s_1+n_1/2+|L_1|)}\big\|\langle x\rangle^{s_1}\langle y\rangle^{s_2}\partial_1^{\alpha}\tilde{K}_{j,k}(x,y)\big\|_{L^2(\mathbb{R}^{n_1+n_2})}\|\psi_k * h_{k,x}(y)\|_{L^2(\mathbb{R}_x^{n_1})}\\
&\quad= 2^{j(-s_1+n_1/2+|L_1|)}\|\xi^{\alpha}m_{j,k}(\xi,\eta)\|_{W^{(s_1,s_2)}}\|\psi_k * h_{k,x}(y)\|_{L^2(\mathbb{R}_x^{n_1})}\\
&\quad\leq 2^{j(-s_1+n_1/2+|L_1|)}\|m_{j,k}(\xi,\eta)\|_{W^{(s_1,s_2)}}\|\psi_k * h_{k,x}(y)\|_{L^2(\mathbb{R}_x^{n_1})}\\
&\quad\leq 2^{j(-s_1+n_1/2+|L_1|)}\sup_k\|m_{j,k}\|_{W^{(s_1,s_2)}}\|\psi_k * h_{k,x}(y)\|_{L^2(\mathbb{R}_x^{n_1})},
\end{aligned}$$

where in the above string of inequalities we obtain the first and second inequalities by Schwarz's inequality. Since $s_1 > n_1/2$, we have

$$\begin{aligned}
&\Big|\sum_k\int_{{}^c\tilde{I}_r\times\mathbb{R}^{n_2}} |x|^{s_1}F_{j,k}(x,y)h_k(x,y)\,dx\,dy\Big|\\
&\qquad\leq 2^{j(-s_1+n_1/2+|L_1|)}\,|I|^{L_1/n_1}\,\sup_k\|m_{j,k}\|_{W^{(s_1,s_2)}}\\
&\qquad\quad\cdot\sum_k\int_{\mathbb{R}^{n_1+n_2}} M(|\psi_k * a_{x_1}|^q)(y)^{1/q}\|\psi_k * h_{k,x}(y)\|_{L^2(\mathbb{R}_x^{n_1})}\,dy\,dx_1.
\end{aligned}$$

By Schwarz's inequality,

$$\begin{aligned}
&\sum_k\int_{\mathbb{R}^{n_2}} M(|\psi_k * a_{x_1}|^q)(y)^{1/q}\|\psi_k * h_{k,x}(y)\|_{L^2(\mathbb{R}_x^{n_2})}dy\\
&\leq \Big(\sum_k\int_{\mathbb{R}^{n_2}} M(|\psi_k * a_{x_1}|^q)(y)^{2/q}dy\Big)^{1/2}\Big(\sum_k\int_{\mathbb{R}^{n_1}}\int_{\mathbb{R}^{n_2}} |\psi_k * h_{k,x}|^2(y)\,dx\,dy\Big)^{1/2}\\
&\leq \Big(\sum_k\int_{\mathbb{R}^{n_2}} |\psi_k * a_{x_1}|^2(y)dy\Big)^{1/2}\Big(\sum_k\int_{\mathbb{R}^{n_2}}\int_{{}^c\tilde{I}_r} |h_k|^2(x,y)\,dx\,dy\Big)^{1/2}\\
&\leq \|a_{x_1}\|_{L^2(\mathbb{R}^{n_2})}\|\{h_k\}_{l_2}\|_{L^2({}^c\tilde{I}_r\times\mathbb{R}^{n_2})}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\Big|\sum_k \int_{{}^c\tilde{I}_r\times\mathbb{R}^{n_2}} |x|^{s_1}\, F_{j,k}(x,y)h_k(x,y)\,dx\,dy\Big| \\
&\lesssim 2^{j(-s_1+n_1/2+|L_1|)}\,|I|^{\frac{L_1}{n_1}}\sup_k \|m_{j,k}\|_{w^{s_1,s_2}}\int_{\mathbb{R}^{n_1}}\|a_{x_1}\|_{L^2(\mathbb{R}^{n_2})}dx_1\|\{h_k\}_{l_2}\|_{L^2({}^c\tilde{I}_r\times\mathbb{R}^{n_2})} \\
&\lesssim 2^{j(-s_1+n_1/2+|L_1|)}\,|I|^{L_1/n_1+1/2}\,\sup_k\|m_{j,k}\|_{w^{s_1,s_2}}\Big(\int_{I\times J}|a(x_1,y_1)|^2dx_1\,dy_1\Big)^{1/2} \\
&\qquad\cdot\|\{h_k\}_{l_2}\|_{L^2({}^c\tilde{I}_r\times\mathbb{R}^{n_2})} \\
&\lesssim 2^{j(-s_1+n_1/2+|L_1|)}\,|I|^{L_1/n_1+1-1/p}|J|^{1/2-1/p}\sup_k\|m_{j,k}\|_{W^{(s_1,s_2)}}\|\{h_k\}_{l_2}\|_{L^2({}^c\tilde{I}_r\times\mathbb{R}^{n_2})}.
\end{aligned}$$

Thus, we have obtained

$$\begin{aligned}
&\int_{{}^c\tilde{R}^1_r}|T^*_m(a)(x,y)|^p\,dx\,dy \\
&\quad\lesssim\sum_{j\in\mathbb{Z}}\big\{r^{-s_1+n_1/p-n_1/2}|I|^{-s_1/n_1+1/p-1/2}|J|^{1/p-1/2}\|\{|x|^{s_1}F_{j,k}\}_{l^2_k}\|_{L^2({}^c\tilde{R}^1_r)}\big\}^p \\
&\quad\le\sum_{j\in\mathbb{Z}}\big\{r^{-s_1+n_1/p-n_1/2}|I|^{-s_1/n_1+1/p-1/2}|J|^{1/p-1/2}2^{j(-s_1+n_1/2+|L_1|)} \\
&\qquad\qquad\cdot|I|^{L_1/n_1+1-1/p}\,|J|^{1/2-1/p}\,\sup_k\|m_{j,k}\|_{W^{(s_1,s_2)}}\big\}^p \\
&\quad=\sum_{j\in\mathbb{Z}}\big\{r^{-s_1+n_1/p-n_1/2}|I|^{L_1-s_1/n_1+1/2}2^{j(-s_1+n_1/2+|L_1|)}\sup_k\|m_{j,k}\|_{W^{(s_1,s_2)}}\big\}^p \\
&\quad\le\sum_{j\in\mathbb{Z}}\big\{r^{-\delta}|I|^{L_1-s_1/n_1+1/2}2^{j(-s_1+n_1/2+|L_1|)}\sup_k\|m_{j,k}\|_{W^{(s_1,s_2)}}\big\}^p,
\end{aligned}$$

where δ is the same as above.

For each rectangle I, there exists an integer j_0 such that $|I|^{1/n}\approx 2^{-j_0}$. Therefore

$$\begin{aligned}
\|T^*_m(a)\|^p_{L^p({}^c\tilde{R}^1_r)}&\le\sum_{j\in\mathbb{Z}}\big\{r^{-\delta}|I|^{L_1-s_1/n_1+1/2}\,2^{j(-s_1+n_1/2+|L_1|)}\sup_k\|m_{j,k}\|_{W^{(s_1,s_2)}}\big\}^p \\
&=\sum_{j\ge j_0}+\sum_{j<j_0}\big\{r^{-\delta}|I|^{L_1-s_1/n_1+1/2}\,2^{j(-s_1+n_1/2+|L_1|)}\sup_k\|m_{j,k}\|_{W^{(s_1,s_2)}}\big\}^p.
\end{aligned}$$

First we consider the sums $\sum_{j\ge j_0}$ and pick $L_1=0$. Since $s_1>n_1/p-n/2$ and $0<p\le 1$, then we have

$$\begin{aligned}
&\sum_{j\ge j_0}\big\{r^{-\delta}|I|^{-s_1/n_1+1/2}2^{j(-s_1+n_1/2)}\big\}^p \\
&\quad\lesssim r^{-p\delta}\sum_{j\ge j_0}2^{pj(-s_1+n_1/2)}|I|^{p(-s_1/n_1+1/2)}\sup_k\|m_{j,k}\|^p_{W^{(s_1,s_2)}} \\
&\quad\approx r^{-p\delta}\sum_{j\ge j_0}2^{p(j-j_0)(-s_1+n_1/2)}\sup_k\|m_{j,k}\|^p_{W^{(s_1,s_2)}}\lesssim r^{-p\delta}\sup_{j,k}\|m_{j,k}\|^p_{W^{(s_1,s_2)}}.
\end{aligned}$$

Next, we pick $L_1 = [n_1/p - n_1] + 1$ for the sums $\sum_{j<j_0}$. Then

$$\sum_{j<j_0} \{r^{-\delta} 2^{j(-s_1+n_1/2+[n_1/p-n_1])} |I|^{(-s_1/n_1+1/2+[n_1/p-n_1]/n_1)} \sup_k \|m_{j,k}\|_{W^{(s_1,s_2)}}\}^p$$

$$\approx r^{-p\delta} \sum_{j<j_0} 2^{p(j_0-j)(s_1-\frac{n_1}{2}-[\frac{n_1}{p}-n_1]-1)} \sup_{j,k} \|m_{j,k}\|^p_{W^{(s_1,s_2)}} \lesssim r^{-p\delta} \sup_{j,k} \|m_{j,k}\|^p_{W^{(s_1,s_2)}}.$$

Thus, we have completed the proof of the main theorem, i.e., Theorem 1.7.

4. The sharpness of Theorem 1.7

To establish the sharpness of smoothness of the multiplier in Theorem 1.7, we only need to consider the case when $m(\xi,\eta) = m_1(\xi)m_2(\eta)$. Suppose $f(x,y) = f_1(x)f_2(y)$. Then we have

$$T_m(f,g)(x,y) = T_{m_1}(f_1)(x)\, T_{m_2}(f_2)(y)$$

if $f_1 \in H^p(\mathbb{R}^{n_1})$ and $f_2 \in H^p(\mathbb{R}^{n_2})$, by the Littlewood–Paley–Stein characterization of Hardy spaces $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, we can conclude $f(x,y) = f_1(x)f_2(y) \in H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$. Moreover,

$$\|T_m(f)\|_{H^p(\mathbb{R}^{n_1}\times\mathbb{R}^{n_2})} \approx \|T_{m_1}(f_1)\|_{H^p(\mathbb{R}^{n_1})} \|T_{m_2}(f_2)\|_{H^p(\mathbb{R}^{n_2})}$$

By the sharpness of Calderón–Torchinsky's theorem ([33]), we can see that Theorem 1.7 is sharp in the sense that there exists a multiplier m with smoothness of order $s_1 \le n_1/p - n_1/2$ and $s_2 \le n_2/p - n_2/2$ such that T_m is not bounded on Hardy spaces $H^p(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ for $0 < p \le 1$.

Note added in proof. After this paper was accepted for publication, the authors learned from Professor Jill Pipher that the Hörmander type multiplier theorem (Theorem 1.7) has been established by Carbery and Seeger in their earlier paper [2], among other results, using a very different method from ours by considering vector-valued rectangle atoms of multi-parameter Hardy spaces. The authors would like to thank Jill Pipher for bringing this to our attention.

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