Existence of isovolumetric $\mathbb{S}^2$-type stationary surfaces for capillarity functionals

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Abstract. Capillarity functionals are parameter invariant functionals defined on classes of two-dimensional parametric surfaces in $\mathbb{R}^3$ as the sum of the area integral and a non homogenous term of suitable form. Here we consider the case of a class of non homogenous terms vanishing at infinity for which the corresponding capillarity functional has no volume-constrained $\mathbb{S}^2$-type minimal surface. Using variational techniques, we prove existence of extremals characterized as saddle-type critical points.

1. Introduction

Surfaces of constant mean curvature are critical points of the area functional for volume-preserving variations. They constitute a nice model for describing closed capillarity surfaces, i.e., soap bubbles, when the surface energy of the liquid is regarded as isotropic, the liquid is homogeneous and no external force is considered. In this case the surface energy is proportional to the surface area, and soap bubbles correspond to extremal solutions of the isoperimetric problem.

If external forces are taken into account, then the surface energy has to be modified in a suitable way, by considering a generalized area functional

$$(1.1) \qquad A_w(\Sigma) = \int_\Sigma w(p)\, d\Sigma \,,$$

where $w\colon \mathbb{R}^3 \to \mathbb{R}$ is a regular and positive weight.

Functionals of the form (1.1) have been extensively studied from the viewpoint of geometric measure theory (as in [4], for instance). Correspondingly, in the same direction, also isoperimetric problems with weights have been recently studied, in some cases (see [20], [21]).

Here we are interested in investigating some issues about a class of generalized area functionals, from a different perspective, in the frame of differential geometry.

With this approach we are allowed to prescribe the topological type of the surfaces we deal with. In particular, we focus on parametric surfaces of the type of the sphere. This means that we identify surfaces with (the range of) maps from $\mathbb{S}^2$ to $\mathbb{R}^3$. Moreover we consider functionals of the kind

$$F(u) = \int_{\mathbb{S}^2} (1 + Q(u) \cdot \nu) \, d\Sigma \,,$$

where ν is the Gauss map, $d\Sigma$ is the area element of $\mathbb{S}^2$ induced by u, and $Q \colon \mathbb{R}^3 \to \mathbb{R}^3$ is a prescribed smooth vector field such that

$$(1.2) \qquad\qquad\qquad\qquad \|Q\|_\infty < 1 \,.$$

These functionals are known as "capillarity functionals" (see [18]) and they can be seen as a correction of the area functional by a non homogeneous and non-isotropic term. The bound (1.2) is a sufficient (and necessary) condition in order that an isoperimetric inequality for capillarity functionals holds true. We are interested in looking for critical points for these kinds of functionals in the Sobolev space $H^1(\mathbb{S}^2, \mathbb{R}^3)$, for volume-preserving variations, assuming that the non homogeneous term vanishes at infinity, namely

$$(1.3) \qquad\qquad\qquad Q(p) \to 0 \quad \text{as } |p| \to \infty \,.$$

Actually, we can state the precise assumptions just on the scalar field $K = \operatorname{div} Q$, because capillarity functionals depend on the vector field Q only by its divergence.

In fact, the datum of our problem is a regular enough, scalar field $K \colon \mathbb{R}^3 \to \mathbb{R}$ satisfying:

(K_1) $\quad \sup_{p \in \mathbb{R}^3} |K(p)p| =: k_0 < 2$ for every $p \in \mathbb{R}^3$.

(K_2) $\quad K(p)p \to 0$ as $|p| \to \infty$.

Then it is possible to construct a vector field $Q_K \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ such that $\operatorname{div} Q_K = K$ on $\mathbb{R}^3$ and satisfying (1.2) and (1.3) which are direct consequences of (K_1) and (K_2), respectively (see Remark 2.7). For this reason, assumptions (K_1) and (K_2) seem to be reasonably natural to deal with situations with non homogeneous terms vanishing at infinity.

In general, even if the non homogeneous term vanishes at infinity, its presence in the capillarity functional has important consequences on the issue of the existence of extremals for the corresponding isoperimetric inequality. In [7] one can find some results concerning both existence and non-existence of critical points corresponding to minima for the isoperimetric problems

$$(1.4) \qquad \begin{aligned} \mathcal{S}_K(t) &:= \inf \left\{ \mathcal{F}_K(u) \mid u \in H^1(\mathbb{S}^2, \mathbb{R}^3), \ \mathcal{V}(u) = t \right\}, \\ \text{where} \quad \mathcal{F}_K(u) &:= \int_{\mathbb{S}^2} (1 + Q_K(u) \cdot \nu) \, d\Sigma \end{aligned}$$

and $\mathcal{V}(u)$ is the algebraic volume functional, defined as the unique continuous extension to $H^1(\mathbb{S}^2, \mathbb{R}^3)$ of the integral functional

$$\mathcal{V}(u) = \frac{1}{3} \int_{\mathbb{S}^2} u \cdot \nu \, d\Sigma \quad \text{for } u \in H^1(\mathbb{S}^2, \mathbb{R}^3) \cap L^\infty.$$

For future convenience, let us state some results proved in [7], about problems (1.4) with $t > 0$.

Theorem 1.1. *Let $K \in C^1(\mathbb{R}^3)$ satisfy (K_1)–(K_2). Moreover assume that*

$$K(p) < 0 \quad at\ some\ p \in \mathbb{R}^3 \tag{1.5}$$

and that the constant k_0 appearing in (K_1) satisfies

$$2^{2/3}(2 + k_0) < (2 - k_0)^2 . \tag{1.6}$$

Then there exists $t_+ > 0$ such that for every $t \in (0, t_+)$ the minimization problem defined by (1.4) admits a minimizer.

The value t_+ can be characterized as follows

$$t_+ := \sup \left\{ t \geq 0 \mid K \leq 0 \text{ and } K \not\equiv 0 \text{ in some ball of radius } \sqrt[3]{3t/4\pi} \right\}.$$

In particular, $t_+ = \infty$ if $K \leq 0$ everywhere (but also if $K \leq 0$ on the tail of an open cone). Other conditions on K, different from (1.6) and regarding the radial oscillation of K are also considered in [7]. Moreover in [7] it is proved that

Theorem 1.2. *Let $K \in C^0(\mathbb{R}^3)$ satisfy (K_1)–(K_2). If*

$$K(p) > 0 \quad for\ every\ p \in \mathbb{R}^3, \tag{1.7}$$

then there exists $\tau > 0$ such that for every $t \in (0, \tau)$ the minimization problem defined by (1.4) has no minimizer. Moreover $S_K(t) = St^{2/3}$ for $t \in (0, \tau)$, where $S = \sqrt[3]{36\pi}$ is the isoperimetric constant.

The present paper is a continuation and a completion of [7]. Here we focus on the issue of existence of critical points in the case of nonexistence of minima.

Theorem 1.3. *Let $K \in C^{1,\alpha}(\mathbb{R}^3)$ satisfy (K_1)–(K_2). Moreover assume (1.7) and that the constant k_0 appearing in (K_1) satisfies*

$$k_0 < 2(2^{1/3} - 1). \tag{1.8}$$

Then there exists a sequence $t_n \to 0^+$ such that the set of constrained critical points of $\mathcal{F}_K$ at volume t_n, denoted $\mathrm{Crit}_{\mathcal{F}_K}(t_n)$, is non empty.

The proof of this result is mainly based on a minimax argument and on degree theory, in the spirit of a procedure introduced in [3] for certain semilinear elliptic equations in $\mathbb{R}^N$.

More precisely, arguing by contradiction, if there are no volume-constrained critical points, we can construct a suitable minimax level c for the functional which lies between two consecutive levels, corresponding to the the energy at infinity, i.e., the area, of one and two identical spheres at fixed volume. On the other hand, if there are no volume-constrained critical points, then constrained Palais–Smale sequences have a limit configuration made by a finite number of spheres, each

one carrying the same energy. This fact comes out by some key results obtained in [11] and [6]. Hence the contradiction follows by proving the existence of volume-constrained Palais–Smale sequences at the minimax level c (see Proposition 4.3).

We stress that the existence of volume-constrained Palais–Smale sequences at the minimax level c is a delicate and rather technical step. In fact, in general, $\mathcal{F}_K$ is not C^1 and not even Gateaux differentiable. To our knowledge, a similar result is only available in the context of minimax levels for the free functionals (see [19]) and only for C^1 functionals. Furthermore, a constrained version of Proposition 4.3 can be obtained through a deformation-lemma argument, but it requires the functional to be of class C^1 and the constraint to be a Finsler manifold of class $C^{1,1}$. Instead our proof is just based on the Ekeland's variational principle (see, e.g., [15]) and fine estimates.

We also point out that capillarity functionals are particularly meaningful because of their connection with the H-bubble problem. In fact, volume-constrained extremals parametrize $\mathbb{S}^2$-type surfaces with volume t and mean curvature $H(p) = \frac{1}{2}(K(p) - \lambda)$, where $K = \operatorname{div} Q$ is prescribed, and λ is a constant corresponding to the Lagrange multiplier due to the constraint. Differently from previous results obtained for the H-bubble problem, the mean curvature is prescribed up to a constant, while in [8], [12] the mean curvature is of the form $H(p) = \frac{1}{2}(K(p) - \lambda_0)$, where λ_0 is a given constant but no information is provided on the volume of those surfaces. In addition, it is important to note that in our paper we just assume (K_1) with (1.8) and (K_2) (see Theorem 5.2), while in [8] and [12], for analogous results one needs more restrictive assumptions, involving the radial derivative of K.

We also point out that even though we obtain an existence result only for a sequence $t_n \to 0^+$, we believe that our result is relevant in view of the techniques applied for the proof. We suspect that other methods, as the finite-dimensional reduction method, could be used to get an existence result for all t in a small interval $(0, \epsilon)$. By the way, this strategy, already employed for the H-bubble problem (see, e.g., [5], [9], [16], [22]), has not been investigated so far for the generalized isoperimetric problem.

A great part of the tools we use in the present paper is contained in [7] and for the sake of convenience we recall them in Section 2. Sections 3, 4 and 5 are devoted, respectively, to the construction of the minimax scheme, to the existence of constrained Palais–Smale sequences and to the proof of Theorem 1.3.

2. Preliminaries

Let us introduce the space

$$\hat{H}^1(\mathbb{R}^2, \mathbb{R}^3) := \left\{ u \in H^1_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^3); \ \int_{\mathbb{R}^2} (|\nabla u|^2 + \mu^2 |u|^2) < \infty \right\},$$

where

$$(2.1) \qquad\qquad \mu(z) = \frac{2}{1 + |z|^2} \quad \text{for } z = (x, y) \in \mathbb{R}^2.$$

For simplicity we will use the notation $\hat{H}^1$ instead of $\hat{H}^1(\mathbb{R}^2, \mathbb{R}^3)$. The space $\hat{H}^1$ is a Hilbert space with inner product

$$\langle u, v \rangle = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v) + \Big(\frac{1}{4\pi} \int_{\mathbb{R}^2} u\,\mu^2 \Big) \cdot \Big(\frac{1}{4\pi} \int_{\mathbb{R}^2} v\,\mu^2 \Big)$$

and is isomorphic to the space $H^1(\mathbb{S}^2, \mathbb{R}^3)$. The isomorphism is given by the correspondence $\hat{H}^1 \ni u \mapsto u \circ \phi \in H^1(\mathbb{S}^2, \mathbb{R}^3)$, where ϕ is the stereographic projection of $\mathbb{S}^2$ onto the compactified plane $\mathbb{R}^2 \cup \{\infty\}$. As usual, we denote $\|u\| = \langle u, u \rangle^{1/2}$.

It is known that $C^\infty(\mathbb{S}^2, \mathbb{R}^3)$ is dense in $H^1(\mathbb{S}^2, \mathbb{R}^3)$ (see, e.g., [2], Chapter 2). As a consequence, $\hat{C}^\infty := \{u \circ \phi^{-1} \mid u \in C^\infty(\mathbb{S}^2, \mathbb{R}^3)\}$ is dense in $\hat{H}^1$. We point out that constant maps belong to $\hat{H}^1$, and we identify them with $\mathbb{R}^3$. Moreover we observe that $p + \hat{H}^1 = \hat{H}^1$ for every $p \in \mathbb{R}^3$.

We recall now some important facts. Some of them are well known and classical. Others, more related to our problem, are discussed in [7]. We refer to that paper for the proofs or for additional, useful bibliography.

Lemma 2.1. *The space $\mathbb{R}^3 + C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ is dense in $\hat{H}^1$. In particular, for every $u \in \hat{H}^1 \cap L^\infty$ there exists a sequence $(u^n) \subset \mathbb{R}^3 + C_c^\infty(\mathbb{R}^2, \mathbb{R}^3)$ such that $u^n \to u$ in $\hat{H}^1$, in L^∞_{loc} and $\|u^n\|_\infty \leq \|u\|_\infty$.*

Set

$$\mathcal{A}(u) := \int_{\mathbb{R}^2} |u_x \wedge u_y|, \quad \mathcal{D}(u) := \frac{1}{2} \int_{\mathbb{R}^2} |\nabla u|^2 \quad (u \in \hat{H}^1)$$

$$\mathcal{V}(u) := \frac{1}{3} \int_{\mathbb{R}^2} u \cdot u_x \wedge u_y \quad (u \in \hat{H}^1 \cap L^\infty).$$

Lemma 2.2. *The functional $\mathcal{V}$ admits a unique analytic extension on $\hat{H}^1$. In particular, for every $u \in \hat{H}^1$,*

$$\mathcal{V}'(u)[\varphi] = \int_{\mathbb{R}^2} \varphi \cdot u_x \wedge u_y \quad \forall \varphi \in \hat{H}^1 \cap L^\infty$$

and there exists a unique $v \in \hat{H}^1 \cap L^\infty$ which is a (weak) solution of

$$\begin{cases} -\Delta v = u_x \wedge u_y & \text{on } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} v\mu^2 = 0. \end{cases}$$

Moreover,

(2.2) $$\|\nabla v\|_2 + \|v\|_\infty \leq C \|\nabla u\|_2^2$$

for a constant C independent of u. In addition, for every $t \neq 0$ the set

(2.3) $$M_t := \{u \in \hat{H}^1 \mid \mathcal{V}(u) = t\}$$

is a smooth manifold and, for any fixed $u \in M_t$, a function $\varphi \in \hat{H}^1$ belongs to the tangent space to M_t at u, denoted $T_u M_t$, if and only if $\mathcal{V}'(u)[\varphi] = 0$.

Remark 2.3. The second part of Lemma 2.2 states that there exists a number $C > 0$ such that $\|\mathcal{V}'(u)\|_{\hat{H}^{-1}} \le C\|\nabla u\|_2^2$ for every $u \in \hat{H}^1$, where $\hat{H}^{-1}$ denotes the dual of $\hat{H}^1$.

Remark 2.4. The mapping $\omega(z) = (\mu x, \mu y, 1 - \mu)$, with μ defined in (2.1), is a conformal parametrization of the unit sphere. Indeed, it is the inverse of the stereographic projection from the North Pole. Moreover $\mathcal{A}(\omega) = \mathcal{D}(\omega) = 4\pi$ and $\mathcal{V}(\omega) = -4\pi/3$. If $\overline{p} \in \mathbb{R}^3$ and $r \in \mathbb{R} \setminus \{0\}$, then $u = \overline{p} + r\omega$ is a parametrization of a sphere centered at $\overline{p}$ and with radius $|r|$, Moreover, $\mathcal{A}(u) = \mathcal{D}(u) = 4\pi r^2$ and $\mathcal{V}(u) = -4\pi r^3/3$.

Lemma 2.5 (Isoperimetric inequality). *One has that*

$$(2.4) \qquad S\,|\mathcal{V}(u)|^{2/3} \le \mathcal{A}(u) \le \mathcal{D}(u) \quad \forall u \in \hat{H}^1,$$

where $S = \sqrt[3]{36\pi}$ is the best constant. Moreover, any extremal function for (2.4) is a conformal parametrization of a round sphere and it has degree 1 or -1 (as a map from $\mathbb{S}^2$ into $\mathbb{S}^2$).

Fixing $K \in C^1(\mathbb{R}^3)$ satisfying (K_1), set

$$(2.5) \qquad m_K(p) := \int_0^1 K(sp)s^2\,ds \quad \text{and} \quad Q_K(p) := m_K(p)p \quad \forall p \in \mathbb{R}^3,$$

and observe that $\operatorname{div} Q_K = K$. Then set

$$\mathcal{Q}(u) := \int_{\mathbb{R}^2} Q_K(u) \cdot u_x \wedge u_y \quad (u \in \hat{H}^1).$$

Remark 2.6. We point out that under the correspondence $u \mapsto u \circ \phi$ it holds that

$$\mathcal{F}_K(u \circ \phi) = \int_{\mathbb{S}^2} \left(1 + Q_K(u \circ \phi) \cdot \nu\right)\,d\Sigma$$
$$= \int_{\mathbb{R}^2} \left(|u_x \wedge u_y| + Q_K(u) \cdot u_x \wedge u_y\right)\,dx\,dy = \mathcal{A}(u) + \mathcal{Q}(u).$$

In view of this equality we can extend $\mathcal{F}_K$ to the class of non immersed surfaces.

Remark 2.7. Using (2.5) one can easily check that Q_k satisfies (1.2). More precisely,

$$(2.6) \qquad \|Q_K\|_\infty \le \frac{k_0}{2} < 1.$$

Moreover, the functional $\mathcal{Q}$ is well defined on $\hat{H}^1$ and

$$(2.7) \qquad |\mathcal{Q}(u)| \le \|Q_K\|_\infty \mathcal{D}(u) \quad \forall u \in \hat{H}^1.$$

One can also check that Q_k satisfies (1.3). Indeed, for $|p| > R$ write

$$Q_K(p) = \frac{\hat{p}}{|p|^2} \int_0^R K(t\hat{p})\,t^2\,dt + \frac{\hat{p}}{|p|^2} \int_R^{|p|} K(t\hat{p})\,t^2\,dt$$

with $\hat{p} = p/|p|$, and use (K_2) to conclude.

The next result collects some useful properties of the functional $\mathcal{Q}$.

Lemma 2.8. *Let $K\colon \mathbb{R}^3 \to \mathbb{R}$ be a bounded continuous function. Then:*

(i) *the functional $\mathcal{Q}$ is continuous in $\hat{H}^1$.*

(ii) *For every $u \in \hat{H}^1$ and $\varphi \in \hat{H}^1 \cap L^\infty$ one has*

$$\mathcal{Q}(u+\varphi) - \mathcal{Q}(u) = \int_{\mathbb{R}^2} \Big(\int_0^1 K(u+r\varphi)\varphi \cdot (u_x + r\varphi_x) \wedge (u_y + r\varphi_y)\, dr \Big)\, dx\, dy.$$

(iii) *The functional $\mathcal{Q}$ admits directional derivatives at every $u \in \hat{H}^1$ along any $\varphi \in \hat{H}^1 \cap L^\infty$, given by*

$$\mathcal{Q}'(u)[\varphi] = \int_{\mathbb{R}^2} K(u)\varphi \cdot u_x \wedge u_y.$$

If in addition $\sup_{p\in\mathbb{R}^3} |K(p)p| < \infty$, then for every $u \in \hat{H}^1$ the mapping $s \mapsto \mathcal{Q}(su)$ is differentiable and

$$\frac{d}{ds}[\mathcal{Q}(su)] = s^2 \int_{\mathbb{R}^2} K(su)u \cdot u_x \wedge u_y.$$

Now we state and prove a technical result which will be useful in the sequel.

Lemma 2.9. *For any $\varphi \in \mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$, the map $u \mapsto \mathcal{E}'(u)[\varphi]$ from $\hat{H}^1$ to $\mathbb{R}$ is continuous.*

Proof. Thanks to Lemma 2.8 (iii), we have that for any $u \in \hat{H}^1$ and $\varphi \in \mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$ the functional $\mathcal{E}$ admits the directional derivative at u along φ and

$$\mathcal{E}'(u)[\varphi] = \int_{\mathbb{R}^2} \nabla u \cdot \nabla \varphi + \int_{\mathbb{R}^2} K(u)\varphi \cdot u_x \wedge u_y.$$

Since $u \mapsto \mathcal{D}'(u)[\varphi]$ is continuous, it suffices to show that this holds also for $u \mapsto \mathcal{Q}'(u)[\varphi]$. Let $(u^n) \subset \hat{H}^1$ be such that $u^n \to u$ in $\hat{H}^1$. Then

$$|\mathcal{Q}'(u^n)[\varphi] - \mathcal{Q}'(u)[\varphi]|$$

$$= \Big| \int_{\mathbb{R}^2} K(u^n)\varphi \cdot u_x^n \wedge u_y^n - \int_{\mathbb{R}^2} K(u)\varphi \cdot u_x \wedge u_y \Big|$$

$$= \Big| \int_{\mathbb{R}^2} K(u^n)\varphi \cdot u_x^n \wedge u_y^n - \int_{\mathbb{R}^2} K(u)\varphi \cdot u_x^n \wedge u_y^n$$

$$\quad + \int_{\mathbb{R}^2} K(u)\varphi \cdot u_x^n \wedge u_y^n - \int_{\mathbb{R}^2} K(u)\varphi \cdot u_x \wedge u_y \Big|$$

$$\leq \underbrace{\int_{\mathbb{R}^2} |K(u^n) - K(u)|\,|\varphi|\,|u_x^n \wedge u_y^n|}_{I_1^n} + \underbrace{\int_{\mathbb{R}^2} |K(u)|\,|\varphi|\,|u_x^n \wedge u_y^n - u_x \wedge u_y|}_{I_2^n}.$$

Since $u^n \to u$ in $\hat{H}^1$, we get that $u_x^n \wedge u_y^n \to u_x \wedge u_y$ in $L^1(\mathbb{R}^2)$ and $u^n \to u$ a.e. in $\mathbb{R}^2$. Moreover, since K is continuous and satisfies (K_1), then K is bounded by

some positive constant C_K. Now assume by contradiction that $I_1^n \not\to 0$ as $n \to \infty$. This means that there exists $\epsilon > 0$ such that $|I_1^{n_k}| > \epsilon$ for some subsequence $n_k \to \infty$. But since $u_x^n \wedge u_y^n \to u_x \wedge u_y$ in $L^1(\mathbb{R}^2)$, there exists a subsequence n_{k_h} and a nonnegative function $g \in L^1(\mathbb{R}^2)$ such that $|u_x^{n_{k_h}} \wedge u_y^{n_{k_h}}| \leq g$ a.e. in $\mathbb{R}^2$. Thus, by the previous considerations and being $\varphi \in \mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$ it holds that

$$|K(u^{n_{k_h}}) - K(u)|\,|\varphi|\,|u_x^{n_{k_h}} \wedge u_y^{n_{k_h}}| \leq 2\,C_K\,|\varphi|_\infty\,g,$$

and by the dominated convergence theorem we obtain that $I_1^{n_{k_h}} \to 0$, contradicting $|I_1^{n_k}| > \epsilon$. As far as concerns I_2^n, it suffices to observe that

$$I_2^n \leq C_K \|\varphi\|_\infty \int_{\mathbb{R}^2} |u_x^n \wedge u_y^n - u_x \wedge u_y| \to 0, \quad \text{as } n \to \infty$$

because $u_x^n \wedge u_y^n \to u_x \wedge u_y$ in $L^1(\mathbb{R}^2)$. The proof is complete. $\qquad\square$

Remark 2.10. Let ω be the mapping introduced in Remark 2.4. Then, for every $\overline{p} \in \mathbb{R}^3$ and $r > 0$, one has that $\mathcal{Q}(\overline{p} + r\omega) = -\int_{B_r(\overline{p})} K(p)\,dp$, whereas if $r < 0$ then $\mathcal{Q}(\overline{p} + r\omega) = \int_{B_{|r|}(\overline{p})} K(p)\,dp$. For a proof of this fact, see Remark 2.3 in [12].

Now we recall some useful results concerning the following volume-constrained minimization problems:

$$(2.8) \qquad S_K(t) := \inf_{u \in M_t} \mathcal{E}(u) \quad \text{where} \quad \mathcal{E}(u) := \mathcal{D}(u) + \mathcal{Q}(u),$$

$t \in \mathbb{R}$ is fixed, and M_t is defined in (2.3). Unless differently specified, we always assume that $K \in C^1(\mathbb{R}^3)$ satisfy (K_1) and (K_2).

We point out that the mapping $t \mapsto S_K(t)$ is well defined on $\mathbb{R}$ and takes positive values for $t \neq 0$, in view of (2.4), (2.6), and (2.7). It will be named the *isovolumetric function.*

Remark 2.11. For $t = 0$ the class M_t contains the constant functions. Since $0 \leq (1 - \|Q_K\|_\infty)\mathcal{D}(u) \leq \mathcal{E}(u)$, we deduce that $S_K(0) = 0$ and minimizers for $S_K(0)$ are exactly the constant functions.

Remark 2.12. When $K = 0$ we have $\mathcal{E} = \mathcal{D}$ and, by (2.4), $S_0(t) = \inf\{\mathcal{D}(u) \mid u \in M_t\} = St^{2/3}$, for any fixed $t \in \mathbb{R}$.

Now we state some properties of the isovolumetric function $S_K(t)$.

Lemma 2.13. *For every $t \in \mathbb{R}$ the following facts hold*:

(i) $S_K(-t) = S_{-K}(t)$;

(ii) $S_K(t) = S_{K(\cdot + p)}(t)$ *for every $p \in \mathbb{R}^3$.*

(iii) $S_K(t) = \inf\{\mathcal{E}(u) \mid u \in C_c^\infty(\mathbb{R}^2, \mathbb{R}^3),\ \mathcal{V}(u) = t\}$.

Lemma 2.14. *For every $t \in \mathbb{R}$ the following facts hold*:

(i) *For every $t \in \mathbb{R}$ one has that $(1 - \|Q_K\|_\infty)St^{2/3} \leq S_K(t) \leq S_0(t) = St^{2/3}$.*

(ii) *For every $t_1, \ldots, t_k \in \mathbb{R}$ one has that $S_K(t_1) + \cdots + S_K(t_k) \geq S_K(t_1 + \cdots + t_k)$.*

Remark 2.15. The value $S_0(t)$ is the infimum for the Dirichlet integral in the class M_t of mappings in $\hat{H}^1$ parametrizing surfaces with volume t. We know that $S_0(t)$ is attained by a conformal parametrization of a round sphere of volume t with arbitrary center (Lemma 2.5). On the other hand, $S_K(t)$ is is the infimum value for the functional $\mathcal{E} = \mathcal{D} + \mathcal{Q}$ in the same class M_t, and $\mathcal{Q}$ has the meaning of K-weighted algebraic volume (see Remark 2.10; see also [10], Section 2.3).

The next result collects some properties about minimizing sequences for the isovolumetric problem defined by (2.8). In particular we have a bound from above and from below on the Dirichlet norm, and we have that every minimizing sequence shadows another minimizing sequence consisting of approximating solutions for some prescribed mean curvature equation.

Lemma 2.16. *Let $t \in \mathbb{R}$ be fixed. Then*:

(i) $\mathcal{D}(u) \geq \frac{S_K(t)}{1+\|Q_K\|_\infty}$ *for every $u \in M_t$.*

(ii) *If $(u^n) \subset M_t$ is a minimizing sequence for $S_K(t)$, then $\limsup \mathcal{D}(u^n) \leq \frac{St^{2/3}}{1-\|Q_K\|_\infty}$.*

(iii) *For every minimizing sequence $(\tilde{u}^n) \subset M_t$ for $S_K(t)$, there exists another minimizing sequence $(u^n) \subset M_t$ such that $\|u^n - \tilde{u}^n\| \to 0$ and with the additional property that*

$$\Delta u^n - K(u^n)u_x^n \wedge u_y^n + \lambda u_x^n \wedge u_y^n \to 0 \quad in \ \hat{H}^{-1} \ (= \ dual \ of \ \hat{H}^1)$$

for some $\lambda \in \mathbb{R}$.

Definition 2.17. Let $H \in C^0(\mathbb{R}^3)$ be a given function. We call $U \in \hat{H}^1$ an H-bubble if it is a nonconstant solution to

(2.9) $$\Delta U = H(U)U_x \wedge U_y \quad \text{on } \mathbb{R}^2$$

in the distributional sense. If H is constant, an H-bubble will be named H-sphere. The system (2.9) is called H-system.

A first useful property of H-bubbles, for a class of mappings H of our interest, is the following.

Lemma 2.18. *Let $H(p) = K(p) - \lambda$ with $\lambda \in \mathbb{R}$ and $K \in C^0(\mathbb{R}^3)$ satisfying (K_1). If $U \in \hat{H}^1$ is an H-bubble, then $U \in L^\infty$, and $\lambda \mathcal{V}(U) > 0$. If, in addition, $K \in C^1(\mathbb{R}^3)$, then U is of class $C^{2,\alpha}$ as a map on $\mathbb{S}^2$.*

The next result is crucial and explains that Palais–Smale sequences for $\mathcal{E}$ constrained to M_t admit a limit configuration made by bubbles. More precisely:

Lemma 2.19 (Decomposition theorem)**.** *Let $K \colon \mathbb{R}^3 \to \mathbb{R}$ be a continuous function satisfying (K_1) and (K_2). If $(u^n) \subset \hat{H}^1$ is a sequence satisfying*

$$\Delta u^n - K(u^n)u_x^n \wedge u_y^n + \lambda u_x^n \wedge u_y^n \to 0 \quad in \ \hat{H}^{-1},$$

for some $\lambda \in \mathbb{R}$ and such that $c_1 \leq \|\nabla u^n\|_2 \leq c_2$ for some $0 < c_1 \leq c_2 < \infty$ and for every n, then there exist a subsequence of (u^n), still denoted (u^n), finitely many $(K-\lambda)$-bubbles U^i ($i \in I$), finitely many $(-\lambda)$-spheres U^j ($j \in J$) such that, as $n \to \infty$,

$$(2.10) \qquad \begin{cases} \mathcal{D}(u^n) \to \sum_{i \in I} \mathcal{D}(U^i) + \sum_{j \in J} \mathcal{D}(U^j), \\ \mathcal{V}(u^n) \to \sum_{i \in I} \mathcal{V}(U^i) + \sum_{j \in J} \mathcal{V}(U^j), \\ \mathcal{Q}(u^n) \to \sum_{i \in I} \mathcal{Q}(U^i), \end{cases}$$

where I or J can be empty but not both. In particular, if $J = \varnothing$ then the subsequence (u^n) is bounded in $\hat{H}^1$.

3. A constrained minimax result

Let us denote by $\mathrm{Crit}_{\mathcal{E}}(t)$ the set of constrained critical points of the functional $\mathcal{E}$ over M_t, in the following sense:

$$\mathrm{Crit}_{\mathcal{E}}(t) = \{u \in M_t \mid \exists \lambda \in \mathbb{R} \text{ s.t. } \mathcal{E}'(u)[\varphi] - \lambda \mathcal{V}'(u)[\varphi] = 0 \ \forall \varphi \in \mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)\}.$$

For any $p \in \mathbb{R}^3$ and $t > 0$ we set

$$(3.1) \qquad s_t := \sqrt[3]{\frac{3t}{4\pi}} \quad \text{and} \quad \omega_{p,t} := s_t(-\omega + p),$$

where ω the map defined in Remark 2.4.

The goal of this section is to prove the following result.

Proposition 3.1. *Let $K \in C^1(\mathbb{R}^3)$ satisfying (K_1)–(K_2) with (1.8), and $K > 0$ on $\mathbb{R}^3$. Assume that*

$$(3.2) \qquad \exists\, t_0 > 0 \ s.t. \ \mathrm{Crit}_{\mathcal{E}}(t) = \varnothing \quad \forall t \in (0, t_0].$$

Then there exists $R > 0$ such that, for every $t \in (0, t_0)$,

$$S_0(t) < \sup_{p \in \partial B_R} \mathcal{E}(\omega_{p,t}) < \inf_{\phi \in \Phi} \sup_{p \in \overline{B_R}} \mathcal{E}(\phi(p)) < 2^{1/3} S_0(t),$$

where $S_0(t) = St^{2/3}$, $\Phi := \{\phi \in C^0(\overline{B_R}, M_t); \ \phi|_{\partial B_R}(p) = \omega_{p,t}\}$, $\omega_{p,t}$ is the function defined in (3.1).

In order to prove Proposition 3.1, we need to introduce a new tool and some preliminary results. Let us fix $t > 0$ and denote by $\mathcal{B}_t \colon \hat{H}^1 \to \mathbb{R}^3$ the vector-valued map defined by

$$\mathcal{B}_t(u) := \frac{1}{8\pi s_t^2} \int_{\mathbb{R}^2} \Pi(u) |\nabla u|^2,$$

where Π is the minimal distance projection of $\mathbb{R}^3$ onto the closed unit ball, namely,

$$\Pi(p) := \begin{cases} p & \text{if } |p| < 1, \\ p/|p| & \text{if } |p| \geq 1. \end{cases}$$

Since $\Pi \circ u$ is bounded for any $u \in \hat{H}^1$, the mapping $\mathcal{B}_t$ is well defined and continuous on $\hat{H}^1$, in particular $\mathcal{B}_t$ is continuous as mapping from M_t to $\mathbb{R}^3$. We also point out that $\mathcal{B}_t$ is conformally invariant.

Proposition 3.2. *Let $K \in C^1(\mathbb{R}^3)$ satisfying (K_1)–(K_2) and assume that $K > 0$ in $\mathbb{R}^3$ and* (3.2). *Then*

$$\forall t \in (0, t_0) \; \exists\, r_{t,K} > 0 \quad s.t. \quad \inf_{\substack{u \in M_t \\ |\mathcal{B}_t(u)| \leq r_{t,K}}} \mathcal{E}(u) > S_0(t).$$

Proof. We argue by contradiction. Assume the thesis is false, then there exist $t \in (0, t_0)$ and a sequence $(u^n) \subset M_t$ such that $\mathcal{E}(u^n) \to S_0(t)$ and $\mathcal{B}_t(u^n) \to 0$. Observe that, thanks to Theorem 1.2, we can assume without loss of generality that $S_K(t) = S_0(t)$. Hence (u^n) is a minimizing sequence for $S_K(t)$ and, by Lemma 2.16, there exists another minimizing sequence $(\tilde{u}^n) \subset M_t$ such that $\|u^n - \tilde{u}^n\| \to 0$ and

$$\Delta u^n - K(u^n) u_x^n \wedge u_y^n + \lambda u_x^n \wedge u_y^n \to 0 \quad \text{in } \hat{H}^{-1}$$

for some $\lambda \in \mathbb{R}$. Now, being $(\mathcal{D}(u^n))$ bounded, by Lemma 2.19, we get that, up to a subsequence (still denoted (u^n)), there exist finitely many $(K - \lambda)$-bubbles U^i $(i \in I)$, finitely many $(-\lambda)$-spheres U^j $(j \in J)$ for which (2.10) holds, and I or J can be empty but not both. Since we are assuming (3.2), it results that $I = \varnothing$ and thus $J \neq \varnothing$. Now we prove that J is a singleton. Assume, by contradiction, that J is not a singleton, in particular, being J finite and denoting by $|J|$ its cardinality, we have $|J| \geq 2$. We set $t_j := \mathcal{V}(U_j)$ for $j \in J$. By Lemma 2.18 one has that $t_j \lambda > 0$ for any $j \in J$. Hence, from (2.10) we get that $\sum_{j \in J} t_j = t$, and $t_j > 0$ for any $j \in J$. We observe that for any $j \in J$ being U_j a $(-\lambda)$-sphere, there exists a positive integer k_j such that

$$(3.3) \qquad 4\pi k_j \lambda^2 = \mathcal{D}(U_j), \quad \tfrac{4}{3}\pi k_j \lambda^3 = t_j.$$

From (3.3) we deduce that $\mathcal{D}(U_j) = S k_j^{1/3} t_j^{2/3}$. Moreover, thanks to (2.10), being $S_K(t) = S_0(t)$ we have $S_0(t) = \sum_{j \in J} \mathcal{D}(U_j)$, and being $k_j \in \mathbb{N}^+$ it holds

$$\left(\sum_{j \in J} t_j \right)^{2/3} = \sum_{j \in J} k_j^{1/3} t_j^{2/3} \geq \sum_{j \in J} t_j^{2/3}.$$

On the other hand, being $t_j > 0$ for all $j \in J$ and $|J| \geq 2$, by a well-known elementary inequality, it also holds

$$\left(\sum_{j \in J} t_j \right)^{2/3} < \sum_{j \in J} t_j^{2/3},$$

which gives a contradiction. Now, being J a singleton, by Theorem 0.1 of [6] and thanks to (3.2), there exists a sequence (g_n) of conformal transformations of $\mathbb{R}^2 \cup \{\infty\}$ into itself such that setting

$$v_n := \tilde{u}^n \circ g_n \quad \text{and} \quad p_n := \frac{1}{4\pi} \int_{\mathbb{R}^2} \mu^2 \, v_n$$

for a subsequence of (v_n) (still denoted by (v_n)), one has $|p_n| \to \infty$ and $v_n - p_n \to U_j$ weakly in $\hat{H}^1$. In particular, being $\mathcal{D}(\tilde{u}^n) \to \mathcal{D}(U_j)$ it holds that $\nabla v_n \to \nabla U_j$ in L^2. Recalling that $\mathcal{B}_t$ is conformally invariant, we get that

$$\mathcal{B}_t(\tilde{u}^n) = \mathcal{B}_t(v_n) = \frac{1}{8\pi s_t^2} \int_{\mathbb{R}^2} \Pi(v_n)|\nabla U_j|^2 + o(1).$$

Since $|p_n| \to \infty$ and $\int_{\mathbb{S}^2} U_j = 0$ we also have that $v_n - p_n \to U_j$ strongly in $\hat{H}^1$. In particular $|v_n| \to \infty$ a.e. in $\mathbb{R}^2$. Being $p_n/|p_n|$ bounded, up to a subsequence, we have $p_n/|p_n| \to p \in \mathbb{S}^2$ and it follows that $\Pi(v_n) \to p$. Hence we obtain that

$$|\mathcal{B}_t(\tilde{u}^n)| = \frac{1}{8\pi s_t^2} \int_{\mathbb{R}^2} |\nabla U_j|^2 + o(1) \geq c > 0,$$

and being $\mathcal{B}_t$ continuous and $\|\tilde{u}^n - u^n\| \to 0$, this contradicts $\mathcal{B}_t(u^n) \to 0$. □

Lemma 3.3. *Let $K \in C^1(\mathbb{R}^3)$ satisfying (K_1), (K_2) and (3.2). Let $t \in (0, t_0)$ and $p \in \mathbb{R}^3$. As $|p| \to \infty$ it holds that*

$$\mathcal{E}(\omega_{p,t}) = S_0(t) + o(1),$$
$$\mathcal{B}_t(\omega_{p,t}) = \frac{p}{|p|} + o(1),$$

where $\omega_{p,t}$ is the function defined in (3.1).

Proof. The first relation follows from the fact that $\mathcal{D}(\omega_{p,t}) = S_0(t)$ and $\mathcal{Q}(\omega_{p,t}) = \int_{B_{s_t}(s_t p)} K(q) \, dq$ (see Remark 2.10). Thanks to assumption (K_1) we get that

$$|\mathcal{Q}(\omega_{p,t})| \leq \int_{B_{s_t}(s_t p)} |K(q)| \, dq \leq k_0 \int_{B_{s_t}(s_t p)} \frac{dq}{|q|}.$$

Recalling that $1/|q|$ is harmonic in $\mathbb{R}^3$ outside the origin, we have

$$\int_{B_{s_t}(s_t p)} \frac{dq}{|q|} = \frac{4\pi s_t^2}{3|p|} \to 0 \quad \text{as } |p| \to \infty.$$

The first relation is then proved. Concerning the second relation, we observe that $|\omega_{p,t}| \geq 1$ on $\mathbb{R}^2$ for $|p|$ large enough. This implies that $\mathcal{B}_t(\omega_{p,t}) = s_t^2 \mathcal{B}_t(-\omega + p)$ and

$$s_t^2 \mathcal{B}_t(-\omega + p) - \frac{p}{|p|} = \frac{1}{8\pi} \int_{\mathbb{R}^2} \left(\frac{\omega + p}{|\omega + p|} - \frac{p}{|p|}\right)|\nabla \omega|^2 \to 0 \quad \text{as } |p| \to \infty,$$

by dominated convergence theorem. The proof is then complete. □

We have now all the tools to prove Proposition 3.1.

Proof of Proposition 3.1. Let $t \in (0, t_0)$, let $r_{t,K} > 0$ be given by Proposition 3.2, and let $\epsilon \in (0, 1)$ be such that $\epsilon < \inf_{u \in M_t, |\mathcal{B}_t(u)| \leq r_{t,K}} \mathcal{E}(u) - S_0(t)$. According to

Lemma 3.3, there exists a sufficiently large number $R > 1$ such that, for all $p \in \mathbb{R}^3$ with $|p| = R$, one has

$$(3.4) \qquad \mathcal{E}(\omega_{p,t}) < S_0(t) + \epsilon, \quad |\mathcal{B}_t(\omega_{p,t})| > 1 - \epsilon, \quad \frac{p}{|p|} \cdot \mathcal{B}_t(\omega_{p,t}) > 1 - \epsilon.$$

Let Φ be as in the statement of Proposition 3.1. Being $K > 0$, and thanks to (3.4), it follows that

$$S_0(t) < \sup_{p \in \partial B_R} \mathcal{E}(\omega_{p,t}) < S_0(t) + \epsilon.$$

Let us set $c := \inf_{\phi \in \Phi} \sup_{p \in \overline{B_R}} \mathcal{E}(\phi(p))$. We want to prove that $c \geq S_0(t) + \epsilon$. To this goal, assume by contradiction that there exists a map $\phi \in \Phi$ such that

$$\sup_{p \in \overline{B_R}} \mathcal{E}(\phi(p)) < S_0(t) + \epsilon.$$

Hence by Proposition 3.2 we have that

$$(3.5) \qquad\qquad |\mathcal{B}_t(\phi(p))| > r_{t,K} \quad \text{for all } p \in \overline{B_R}.$$

Now consider the map $g : \overline{B_R} \to \mathbb{R}^3$ defined by

$$g(p) := \mathcal{B}_t(\phi(p)),$$

and fix a point $p_0 \in \mathbb{R}^3$ with $0 < |p_0| < \min\{r_{t,K}, 1 - \epsilon\}$. We claim that the topological degree $\deg(g, \overline{B_R}, p_0)$ is well defined and $\deg(g, \overline{B_R}, p_0) = 1$. To this purpose, consider the homotopy $h \colon [0,1] \times \overline{B_R} \to \mathbb{R}^3$ defined by

$$h(s,p) := sp + (1-s)\,\mathcal{B}_t(\phi(p)).$$

Assume by contradiction that h is not admissible, then, there exist $\bar{s} \in [0,1]$ and $\bar{p} \in \partial B_R$ such that $h(\bar{s}, \bar{p}) = p_0$, hence by definition of h and thanks to (3.4) we deduce that

$$|p_0| = |\bar{s}\bar{p} + (1-\bar{s})\mathcal{B}_t(\omega_{\bar{p},t})| \geq \bar{s}R + (1-\bar{s})(1-\epsilon) = 1 - \epsilon + \bar{s}(R - 1 + \epsilon) \geq 1 - \epsilon,$$

which gives a contradiction. Hence h is an admissible homotopy between g and the identity map of $\overline{B_R}$, and by well-known properties of the topological degree it holds that $\deg(g, \overline{B_R}, p_0) = 1$. Now, being $\deg(g, \overline{B_R}, p_0) \neq 0$ in particular we deduce that the equation $g(p) = p_0$ has at least a solution $p \in B_R$. Hence $|\mathcal{B}_t(\phi(p))| = |p_0|$ for some $p \in B_R$ but, being $|p_0| < r_{t,K}$ it follows that $|\mathcal{B}_t(\phi(p))| < r_{t,K}$, contradicting (3.5) and hence $c \geq S_0(t) + \epsilon$.

In order to conclude the proof, it remains to check that $c < 2^{1/3} S_0(t)$. To this goal, let us consider the map $p \mapsto \omega_{p,t}$. It is clear that $\omega_{p,t} \in \Phi$. It is known that $\mathcal{D}(\omega_{p,t}) = S_0(t)$, hence in order to complete the proof we need to estimate $\mathcal{Q}(\omega_{p,t})$. By Remark 2.10, we know that $\mathcal{Q}(\omega_{p,t}) = \int_{B_{s_t}(s_t p)} K(q)\,dq$. Thanks to assumption (K_1) and being $K > 0$, it holds that

$$\int_{B_{s_t}(s_t p)} K(q)\,dq \leq k_0 \int_{B_{s_t}(s_t p)} \frac{1}{|q|}\,dq\,.$$

By a suitable change of variable and elementary computations we get that

$$k_0 \int_{B_{s_t}(s_t p)} K(q) \, dq = k_0 s_t^2 \int_{B_1(0)} \frac{1}{|q - p|} \, dq.$$

Let us set consider the function $I \colon \mathbb{R}^3 \to \mathbb{R}$ defined by $p \mapsto \int_{B_1(0)} \frac{1}{|q-p|}$. We observe that $I(p)$ can be explicitly computed, more precisely we have that

$$(3.6) \qquad I(p) = \begin{cases} \frac{2\pi}{3}(3 - |p|^2) & \text{if } |p| \leq 1, \\ \frac{4\pi}{3|p|} & \text{if } |p| > 1. \end{cases}$$

In fact if $|p| > 1$ the integrand function $q \mapsto \frac{1}{|q-p|}$ is harmonic in $B_1(0)$ and thus by the mean value property we get that $I(p) = \frac{4\pi}{3|p|}$.

The case $|p| \leq 1$ is more delicate: first observe that by dominated convergence theorem we get that $I \colon \mathbb{R}^3 \to \mathbb{R}$ is continuous at any $p \in \mathbb{R}^3$, in particular if $p \in \partial B_1(0)$, from the previous case, we deduce that $I(p) = 4\pi/3$. Now let us consider the vector field $E \colon \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$E(p) := \int_{B_1(0)} \frac{p - q}{|p - q|^3} \, dq.$$

Observe that, by definition and by the dominated convergence theorem, I is differentiable and

$$(3.7) \qquad\qquad\qquad E(p) = -\nabla I(p).$$

We also note that E is of the form $E(p) = g(|p|)\frac{p}{|p|}$ when $p \neq 0$, for some function $g \colon \mathbb{R}^+ \to \mathbb{R}$. In fact, fixing $p \neq 0$ and making a change of variable in the integral defining E by any orthogonal matrix $T \in O_3$ such that $T(p) = p$ we get that $E(p) = T(E(p))$, and thus the fact follows from the arbitrariness of T. Moreover, since $T(0) = 0$ for any $T \in O_3$ it holds that $E(0) = 0$. At the end, by a suitable application of the Stokes theorem, we obtain that

$$E(p) = \begin{cases} \frac{4}{3}\pi p & \text{if } |p| \leq 1, \\ \frac{4}{3}\pi p/|p|^3 & \text{if } |p| > 1. \end{cases}$$

Thanks to (3.7) and the previous characterization, by fixing a point $p_0 \in \partial B_1$, and for any path γ joining p and p_0, we get that

$$I(p) - I(p_0) = -\int_\gamma E \cdot d\gamma = -\frac{4}{3}\pi \int_1^{|p|} r \, dr = -\frac{2}{3}\pi \left(|p|^2 - 1\right).$$

Hence, since $I(p_0) = 4\pi/3$, we have

$$I(p) = \frac{4}{3}\pi - \frac{2}{3}\pi \left(|p|^2 - 1\right) = \frac{2}{3}\pi \left(3 - |p|^2\right) = \frac{2}{3}\pi \left(3 - |p|^2\right).$$

Hence, thanks to (3.6) we deduce that

$$\sup_{p \in \mathbb{R}^3} I(p) = 2\pi$$

and

$$k_0 \, s_t^2 \int_{B_1(0)} \frac{1}{|q-p|} \, dq \le k_0 \, s_t^2 \, 2\pi.$$

Now observe that

$$\mathcal{E}(\omega_{p,t}) \le S_0(t) + 2\pi \, k_0 \Big(\frac{3}{4\pi}\Big)^{2/3} t^{2/3} = S \, t^{2/3} + 2k_0 \Big(\frac{9}{16}\pi\Big)^{1/3} t^{2/3}.$$

Now, thanks to the assumption (1.8), by elementary computations it is easy to verify that

$$S \, t^{2/3} + 2k_0 \Big(\frac{9}{16}\pi\Big)^{1/3} t^{2/3} < 2^{1/3} S \, t^{2/3},$$

which implies that $\mathcal{E}(\omega_{p,t}) < 2^{1/3} S t^{2/3}$, and in particular $c < 2^{1/3} S t^{2/3}$ which is the desired relation. The proof is complete. □

4. Constrained Palais–Smale sequences for $\mathcal{E}$ at the minimax level c

In this section we prove that there exists a Palais–Smale sequence constrained to the smooth manifold M_t at a suitable minimax level. Let $t > 0$ and $R > 0$ be fixed, we define

$$(4.1) \qquad c := \inf_{\phi \in \Phi} \sup_{p \in \overline{B_R}} \mathcal{E}(\phi(p)),$$

where $\Phi := \{f \in C^0(\overline{B_R}, M_t); \; f|_{\partial B_R}(p) = \omega_{p,t}\}$, and $\omega_{p,t}$ is the function defined in (3.1). Moreover we define

$$(4.2) \qquad c_0 := \sup_{p \in \partial B_R} \mathcal{E}(\omega_{p,t}).$$

We begin with a preliminary result.

Lemma 4.1. *Let* $t \in \mathbb{R} \setminus \{0\}$ *and let* $u \in M_t$. *It holds that* $T_u M_t \cap (\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3))$ *is dense in* $T_u M_t$.

Proof. Let us fix $t \in \mathbb{R} \setminus \{0\}$. By Lemma 2.2 we know that $M_t \subset \hat{H}^1$ is a smooth manifold of codimension one. Let us fix $u \in M_t$. Then we can write

$$\hat{H}^1 = T_u M_t \oplus \langle h \rangle,$$

where $h \in \hat{H}^1$ is the Riesz representative of $\mathcal{V}'(u)/\|\mathcal{V}'(u)\|^2$ (see Section 6.1 of [1]). We observe that since $\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$ dense in $\hat{H}^1$ (see Lemma 2.1) then there exists $v \in (\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)) \setminus T_u M_t$. Hence we can also write

$$\hat{H}^1 = T_u M_t \oplus \langle v \rangle.$$

Now let us fix $w \in T_u M_t$, then by the density of $\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$ in $\hat{H}^1$ there exists a sequence $(w_n) \subset \mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$ such that $w_n \to w$ in $\hat{H}^1$. Let us set

$$\tilde{w}_n := w_n - \frac{\mathcal{V}'(u)[w_n]}{\mathcal{V}'(u)[v]}\, v.$$

By construction, $\tilde{w}_n \in \mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$ and $\mathcal{V}'(u)[\tilde{w}_n] = 0$, i.e., $\tilde{w}_n \in (\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)) \cap T_u M_t$. Moreover,

$$\|\tilde{w}_n - w\| \leq \|w_n - w\| + \left| \frac{\mathcal{V}'(u)[w_n]}{\mathcal{V}'(u)[v]} \right| \|v\|,$$

and the right-hand side goes to zero as $n \to \infty$ because $w_n \to w$ in $\hat{H}^1$ and $w \in T_u M_t$. The proof is complete. $\qquad\square$

Proposition 4.2. *Let $t \in \mathbb{R}^+$ and $R > 0$ be fixed and let c, c_0 be the numbers defined, respectively, in (4.1), (4.2). If $c > c_0$ then for any sufficiently small $\epsilon > 0$ and for each $f \in \Phi$ such that*

$$(4.3) \qquad\qquad \sup_{p \in \overline{B_R}} \mathcal{E}(f(p)) \leq c + \epsilon$$

there exists $u \in M_t$ such that

$$c - \epsilon \leq \mathcal{E}(u) \leq \sup_{p \in \overline{B_R}} \mathcal{E}(f(p)),$$
$$\|u - f(p)\| \leq \epsilon^{1/2} \quad \forall p \in \overline{B_R},$$
$$|\mathcal{E}'(u)[\varphi]| \leq 2\epsilon^{1/2} \quad \forall \varphi \in T_u M_t \cap (\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)) \ \ with \ \|\varphi\| = 1.$$

Proof. Let ϵ be such that $0 < \epsilon < c - c_0$. Moreover assume that ϵ satisfies

$$(4.4) \qquad\qquad \epsilon^2 \left(\frac{1}{3t} + \frac{2}{9}\, 2^{7/3} \epsilon^2 \right) < 1.$$

A further restriction on the smallness of ϵ will be specified in the sequel of the proof. Let $f \in \Phi$ satisfy (4.3) and define the function $F \colon \Phi \to \mathbb{R}$ by setting

$$F(g) := \sup_{p \in \overline{B_R}} \mathcal{E}(g(p)).$$

In particular, observe that $c = \inf_\Phi F > c_0$. Thanks to Ekeland's variational principle (see, e.g., [15]), there exists $h \in \Phi$ such that

$$(4.5) \qquad\qquad F(h) \leq F(f) \leq c + \epsilon,$$
$$d(h, f) := \sup_{p \in \overline{B_R}} \|h(p) - f(p)\| \leq \epsilon^{1/2},$$
$$F(g) > F(h) - \epsilon^{1/2} d(h, g) \quad \forall g \in \Phi \ with \ g \neq h.$$

In order to reach the conclusion, it suffices to show that for some $p \in \overline{B_R}$ it holds that

$$c - \epsilon \leq \mathcal{E}(h(p)),$$

(4.6) $|\mathcal{E}'(h(p))[\varphi]| \leq 2\epsilon^{1/2} \quad \forall \varphi \in T_{h(p)}M_t \cap (\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3))$ with $\|\varphi\| = 1$.

Notice that (4.6) is equivalent to

$$\mathcal{E}'(h(p))[\varphi] \geq -2\epsilon^{1/2} \quad \forall \varphi \in T_u M_t \cap (\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)) \text{ with } \|\varphi\| = 1.$$

By contradiction, if this does not happen, then, setting

$$P := \{p \in \overline{B_R} \mid c - \epsilon \leq \mathcal{E}(h(p))\},$$

for each $p \in P$ there exists $\delta_p > 0$, $\varphi_p \in T_{h(p)}M_t \cap (\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3))$ with $\|\varphi_p\| = 1$ and an open ball B_p centered at p such that for $q \in B_p$ and $u \in \hat{H}^1$ with $\|u\| \leq \delta_p$, we have

(4.7) $$\mathcal{E}'(h(q) + u)[\varphi_p] < -2\epsilon^{1/2}.$$

We recall that, by Lemma 2.9, for any fixed $\varphi \in \mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$ the map $u \mapsto \mathcal{E}'(u)[\varphi]$ from $\hat{H}^1$ to $\mathbb{R}$ is continuous. Moreover, since $\mathcal{V}'(h(p))[\varphi_p] = 0$, taking a possibily smaller ball B_p and a smaller constant δ_p, if necessary, we can also assume that

(4.8) $$|\mathcal{V}'(h(q) + u)[\varphi_p]| \leq \epsilon^2 \quad \forall q \in B_p, \quad \forall u \in \hat{H}^1 \text{ with } \|u\| \leq \delta_p.$$

By the continuity of $\mathcal{D}$, assumption (K_1) and (4.5), taking a smaller δ_p, we can also assume that

(4.9) $$\mathcal{D}(h(p) + u) \leq C \text{ for } \|u\| \leq \delta_p,$$

where C is some positive constant depending only on k_0 and c. In fact, by assumption (K_1) we have

$$\mathcal{D}(h(p)) \leq \frac{\mathcal{E}(h(p))}{1 - k_0/2} \leq \frac{F(h)}{1 - k_0/2} \leq \frac{c + \epsilon}{1 - k_0/2} < C,$$

for some positive constant C depending only on k_0 and c and then by continuity of $\mathcal{D}$ we get (4.9). Since P is compact there exists a finite subcovering $B_{p_1}, \ldots, B_{p_k}$ of P and we define $\psi_j : P \to [0, 1]$ by

$$\psi_j(p) = \begin{cases} \dfrac{\text{dist}(p, \complement B_{p_j})}{\sum_{i=1}^k \text{dist}(p, \complement B_{p_i})} & \text{if } p \in \bigcup_{i=1}^k B_{p_i}, \\ 0 & \text{if } p \in P \setminus \bigcup_{i=1}^k B_{p_i}. \end{cases}$$

Furthermore, let $\delta := \min\{1/2, t/2, \delta_{p_1}, \ldots, \delta_{p_k}\}$, let $\psi : \overline{B_R} \to [0, 1]$ be a continuous function such that

$$\psi(p) = \begin{cases} 1 & \text{if } c \leq \mathcal{E}(h(p)), \\ 0 & \text{if } \mathcal{E}(h(p)) \leq c - \epsilon, \end{cases}$$

and let $\tau \colon \overline{B_R} \to \mathbb{R}$ and $g \colon \overline{B_R} \to M_t$ be defined by

$$\tau(p) := \Big(\frac{t}{\mathcal{V}\big(h(p) + \delta\,\psi(p)\sum_{j=1}^{k}\psi_j(p)\varphi_{p_j}\big)}\Big)^{1/3},$$

$$g(p) := \tau(p)\Big(h(p) + \delta\,\psi(p)\sum_{j=1}^{k}\psi_j(p)\varphi_{p_j}\Big).$$

It holds that $g \in \Phi$. In fact, since $0 < \epsilon < c - c_0$, when $p \in \partial B_R$ we have

$$\mathcal{E}(h(p)) = \mathcal{E}(s_t(-\omega + p)) \le c_0 < c - \epsilon,$$

and hence $\psi(p) = 0$ which means that $g(p) = h(p) = s_t(-\omega + p)$. We observe that for $p \in P$ the following inequality holds:

$$(4.10)\quad 1 - \frac{1}{3t}\,\delta\,\psi(p)\epsilon^2 - \frac{2}{9}\,2^{7/3}\delta^2\psi^2(p)\epsilon^4 \le \tau(p) \le 1 + \frac{1}{3t}\,\delta\,\psi(p)\epsilon^2 + \frac{2}{9}\,2^{7/3}\delta^2\psi^2(p)\epsilon^4.$$

In fact, by the mean value theorem, we have

$$\mathcal{V}\Big(h(p) + \delta\,\psi(p)\sum_{j=1}^{k}\psi_j(p)\,\varphi_{p_j}\Big)$$

$$= \mathcal{V}(h(p)) + \mathcal{V}'\Big(h(p) + \sigma\,\delta\,\psi(p)\sum_{j=1}^{k}\psi_j(p)\,\varphi_{p_j}\Big)\Big[\delta\,\psi(p)\sum_{j=1}^{k}\psi_j(p)\,\varphi_{p_j}\Big]$$

$$= t + \delta\,\psi(p)\sum_{j=1}^{k}\psi_j(p)\,\mathcal{V}'\Big(h(p) + \sigma\,\delta\,\psi(p)\sum_{j=1}^{k}\psi_j(p)\,\varphi_{p_j}\Big)[\varphi_{p_j}]$$

for some $\sigma \in (0,1)$. Now, thanks to (4.8) and the definition of the functions ψ_j, we see that

$$\Big|\sum_{j=1}^{k}\psi_j(p)\,\mathcal{V}'\Big(h(p) + \sigma\delta\,\psi(p)\sum_{j=1}^{k}\psi_j(p)\varphi_{p_j}\Big)[\varphi_{p_j}]\Big| \le \epsilon^2.$$

In particular we observe that this estimate is uniform with respect to $p \in P$. Hence we deduce that

$$\tau(p) = \Big(\frac{t}{t + \delta\,\psi(p)O(\epsilon^2)}\Big)^{1/3}$$

with $|O(\epsilon^2)| \le \epsilon^2$ and the desired inequality follows by elementary considerations. More precisely, by the Taylor expansion of the function $s \mapsto 1/(1+s)^{1/3}$ we have

$$\tau(p) = 1 - \frac{1}{3t}\,\delta\,\psi(p)\,O(\epsilon^2)$$

$$+ \int_0^1 (1-s)\frac{4}{9}\Big(1 + s\Big(\frac{\delta}{t}\psi(p)O(\epsilon^2)\Big)\Big)^{-7/3}\Big(\frac{\delta}{t}\psi(p)O(\epsilon^2)\Big)^2 ds.$$

Thanks to the choice of δ and being $|O(\epsilon^2)| \le \epsilon^2$ we have $|\frac{\delta}{t}\psi(p)O(\epsilon^2)| \le \frac{1}{2}\epsilon^2 \le \frac{1}{2}$.

Hence, for any $s \in [0,1]$ we have

$$\left(1 + s\left(\frac{\delta}{t}\,\psi(p)\,O(\epsilon^2)\right)\right)^{-7/3} \leq 2^{7/3}$$

and we get

$$\left| \int_0^1 (1-s)\frac{4}{9}\left(1 + s\left(\frac{\delta}{t}\psi(p)O(\epsilon^2)\right)\right)^{-7/3}\left(\frac{\delta}{t}\psi(p)O(\epsilon^2)\right)^2 ds \right| \leq \frac{2}{9}\,2^{7/3}\frac{\delta^2}{t^2}\,\psi^2(p)\epsilon^4.$$

Hence the estimate (4.10) follows immediately. Now, setting

$$\eta(p) := \psi(p)\sum_{j=1}^k \psi_j(p)\varphi_{p_j} \quad \text{and} \quad h^\delta(p) := h(p) + \delta\eta(p),$$

we write

$$\mathcal{E}(g(p)) - \mathcal{E}(h(p)) = \underbrace{\tau(p)^2\left(\mathcal{E}(h^\delta(p)) - \mathcal{E}(h(p))\right)}_{I_1} + \underbrace{\tau(p)^2\mathcal{E}(h(p)) - \mathcal{E}(h(p))}_{I_2}$$

$$+ \underbrace{\mathcal{Q}(\tau(p)(h^\delta(p))) - \mathcal{Q}(h^\delta(p))}_{I_3} + \underbrace{\mathcal{Q}(h^\delta(p)) - \tau(p)^2\mathcal{Q}(h^\delta(p))}_{I_4}.$$

We begin with the term I_1. Recalling that for any fixed $\varphi \in \mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$ the functional $\mathcal{E}$ is differentiable along φ, by the mean value theorem, for any fixed $p \in P$ there exists $\xi \in (0,1)$ such that

$$\mathcal{E}(h^\delta(p)) - \mathcal{E}(h(p)) = \mathcal{E}'\left(h(p) + \xi\,\delta\,\psi(p)\sum_{j=1}^k \psi_j(p)\,\varphi_{p_j}\right)[\delta\,\psi(p)\sum_{j=1}^k \psi_j(p)\,\varphi_{p_j}]$$

$$= \delta\,\psi(p)\sum_{j=1}^k \psi_j(p)\mathcal{E}'\left(h(p) + \xi\,\delta\,\psi(p)\sum_{j=1}^k \psi_j(p)\,\varphi_{p_j}\right)[\varphi_{p_j}]$$

$$(4.11) \qquad = \delta\,\psi(p)\sum_{j=1}^k \psi_j(p)\mathcal{E}'\left(h(p) + \xi\,\tau(p)\delta\,\psi(p)\sum_{j=1}^k \psi_j(p)\,\varphi_{p_j}\right)[\varphi_{p_j}].$$

Now, from (4.7), (4.10) and (4.11) we get that for $p \in P$

$$I_1 \leq -2\tau(p)^2\delta\,\psi(p)\epsilon^{1/2} \leq -2\left(1 - \frac{1}{3t}\,\delta\,\psi(p)\,\epsilon^2 - \frac{1}{9t^2}\,2^{7/3}\,\delta^2\,\psi^2(p)\,\epsilon^4\right)^2\delta\,\psi(p)\epsilon^{1/2}.$$

Regarding the term I_2, thanks to (4.4), (4.5) and (4.10) we have

$$|I_2| = |\tau^2(p) - 1|\,\mathcal{E}(h(p)) \leq 3(c+\epsilon)\left(\frac{1}{3t}\,\delta\,\psi(p)\,\epsilon^2 + \frac{2}{9t^2}\,2^{7/3}\,\delta^2\,\psi^2(p)\,\epsilon^4\right).$$

For I_3, thanks to Lemma 2.8, we have that

$$\mathcal{Q}(\tau(p)(h^\delta(p))) - \mathcal{Q}(h^\delta(p)) = \int_1^{\tau(p)} s^2 \int_{\mathbb{R}^2} K(s(h^\delta(p)))h^\delta(p) \cdot (h^\delta(p))_x \wedge (h^\delta(p))_y.$$

Now, by assumption (K_1) and thanks to (4.4), (4.9), (4.10) we get that

$$\left| \int_1^{\tau(p)} s^2 \int_{\mathbb{R}^2} K(s(h^\delta(p))) h^\delta(p) \cdot (h^\delta(p))_x \wedge (h^\delta(p))_y \right|$$

$$= \left| \int_1^{\tau(p)} s \int_{\mathbb{R}^2} K(s(h^\delta(p))) s h^\delta(p) \cdot (h^\delta(p))_x \wedge (h^\delta(p))_y \right|$$

$$\leq \int_{\min\{1,\tau(p)\}}^{\max\{1,\tau(p)\}} s \int_{\mathbb{R}^2} \left| K(s(h^\delta(p))) s(h^\delta(p)) \right| \left| (h^\delta(p))_x \wedge (h^\delta(p))_y \right|$$

$$\leq k_0 \int_{\mathbb{R}^2} \left| (h^\delta(p))_x \wedge (h^\delta(p))_y \right| \left| \int_1^{\tau(p)} s \, ds \right|$$

$$\leq k_0 \, \mathcal{D}(h^\delta(p)) \, \frac{|\tau(p)^2 - 1|}{2} \leq \frac{3}{2} \, k_0 \, C \left(\frac{1}{3t} \, \delta \, \psi(p) \epsilon^2 + \frac{2}{9t^2} \, 2^{7/3} \, \delta^2 \, \psi^2(p) \, \epsilon^4 \right).$$

As far as concerns I_4, as before, using assumption (K_1) we get that

$$|I_4| = |1 - \tau(p)^2| \, |\mathcal{Q}(h(p) + \delta\eta(p))| \leq |1 - \tau(p)^2| \frac{k_0}{2} \, \mathcal{D}(h^\delta(p))$$

$$\leq \frac{3}{2} \, k_0 \, C \left(\frac{1}{3t} \, \delta \, \psi(p) \, \epsilon^2 + \frac{2}{9t^2} \, 2^{7/3} \, \delta^2 \, \psi^2(p) \, \epsilon^4 \right).$$

Finally, from these estimates we get that for $p \in P$

$$\mathcal{E}(g(p)) - \mathcal{E}(h(p)) \leq -2\delta \, \psi(p) \, \epsilon^{1/2} + C_1 \, \delta \, \psi(p) \, \epsilon^2,$$

where C_1 is a constant depending only on k_0, t and R. Hence choosing at the beginning of the proof $\epsilon > 0$ sufficiently small such that $-2\epsilon^{1/2} + C_1\epsilon^2 < -\epsilon^{1/2}$ we get that

$$\mathcal{E}(g(p)) - \mathcal{E}(h(p)) \leq -\delta \, \psi(p) \, \epsilon^{1/2}.$$

If $p \notin P$ we have that $\psi(p) = 0$ and $\mathcal{E}(g(p)) = \mathcal{E}(h(p))$. If $\bar{p} \in \overline{B_R}$ is such that $\mathcal{E}(g(\bar{p})) = F(g)$, we have

$$\mathcal{E}(h(\bar{p})) \geq \mathcal{E}(g(\bar{p})) \geq c,$$

and hence $\bar{p} \in P$ and $\psi(\bar{p}) = 1$. Thus, we get that

$$\mathcal{E}(g(\bar{p})) - \mathcal{E}(h(\bar{p})) \leq -\delta \, \epsilon^{1/2}$$

and in particular

$$F(g) + \epsilon^{1/2} \, \delta \leq \mathcal{E}(h(\bar{p})) \leq F(h),$$

so that $g \neq h$. But by definition of g we have

$$d(g, h) \leq \delta$$

and hence

$$F(g) + \epsilon^{1/2} \, d(g, h) \leq F(h),$$

which gives a contradiction. The proof is complete. $\qquad \square$

Proposition 4.3. *Let $K \in C^1(\mathbb{R}^3)$ satisfying (K_1) and (K_2), let $t \in \mathbb{R}^+$ and $R > 0$ be fixed, and let c, c_0 be the numbers defined, respectively, in (4.1), (4.2). Assume that $c > c_0$. Then, for every sequence $(f_n) \subset \Phi$ such that $\sup_{p \in \overline{B}_R} \mathcal{E}(f_n(p)) \to c$, there exists another sequence $(u_n) \subset M_t$ such that $\mathcal{E}(u^n) \to c$ and with the additional property that*

$$\Delta u_n - K(u_n)(u_n)_x \wedge (u_n)_y + \lambda (u_n)_x \wedge (u_n)_y \to 0 \quad in \ \hat{H}^{-1}$$

for some $\lambda \in \mathbb{R}$.

Proof. Let $(f_n) \subset \Phi$ be such that $\sup_{p \in \overline{B}_R} \mathcal{E}(f_n(p)) \to c$. Then, according to Proposition 4.2 we find sequences $(\epsilon_n) \subset (0, 1)$, with $\epsilon_n \to 0$, and $(u_n) \subset M_t$ such that

$$c - \epsilon_n \leq \mathcal{E}(u^n) \leq \sup_{p \in \overline{B}_R} \mathcal{E}(f_n(p))$$

$$|\mathcal{E}'(u^n)[\varphi]| \leq 2\, \epsilon_n^{1/2} \|\varphi\| \quad \forall \varphi \in T_{u^n} M_t \cap (\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)).$$

Then, since $(\mathbb{R}^3 + C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)) \cap T_{u^n} M_t$ is dense in $T_{u^n} M_t$ (see Lemma 4.1) we conclude that

$$(4.12) \qquad |\mathcal{E}'(u^n)[\varphi]| \leq 2\, \epsilon_n^{1/2} \|\varphi\| \quad \forall \varphi \in T_{u^n} M_t.$$

Now let $v^n \in \hat{H}^1$ be the Riesz representative of $\mathcal{V}'(u^n)$. Set

$$\lambda_n = \frac{\mathcal{E}'(u^n)[v^n]}{\|v^n\|^2}$$

(notice that λ_n is well defined because $v^n \in L^\infty$, see Lemma 2.2). For every $\varphi \in \hat{H}^1 \cap L^\infty$ the projection of φ on $T_{u^n} M_t$ is given by

$$\tilde{\varphi} = \varphi - \frac{\langle v^n, \varphi \rangle}{\|v^n\|^2}\, v^n$$

and, by (4.12),

$$|\mathcal{E}'(u^n)[\varphi] - \lambda_n \mathcal{V}'(u^n)[\varphi]| = |\mathcal{E}'(u^n)[\tilde{\varphi}]| \leq 2\, \varepsilon_n^{1/2} \|\tilde{\varphi}\| \leq 2\, \varepsilon_n^{1/2} \|\varphi\|,$$

and then, by density, $\mathcal{E}'(u^n) - \lambda_n \mathcal{V}'(u^n) \to 0$ in $\hat{H}^{-1}$. Now we show that the sequence (λ_n) is bounded. First of all we observe that the sequence $(\mathcal{D}(u^n))$ is bounded, because $\mathcal{E}(u^n) \to c$ and by Remark 2.7 we know that $\mathcal{E}$ is coercive with constants depending only on k_0 (see also (4.9)). Thus, by (2.2), we estimate

$$(4.13) \qquad \|\nabla v^n\|_2 + \|v^n\|_\infty \leq C_1 \|\nabla u^n\|_2^2 \leq C_2,$$

for some positive constants C_1, C_2. Then

$$(4.14) \qquad \begin{aligned} |\mathcal{E}'(u^n)[v^n]| &\leq \left| \int_{\mathbb{R}^2} (\nabla u^n \cdot \nabla v^n + K(u^n) v^n \cdot u_x^n \wedge u_y^n) \right| \\ &\leq \|\nabla u^n\|_2 \|\nabla v^n\|_2 + \|K\|_\infty \|v^n\|_\infty \|\nabla u^n\|_2^2 \leq C. \end{aligned}$$

Moreover, keeping into account that $\int_{\mathbb{R}^2} v^n \mu^2 = 0$ and being $\mathcal{D}(u^n)$ bounded, we have that

$$
\begin{aligned}
(4.15) \qquad |3t| = |\mathcal{V}'(u^n)[u^n]| = |\langle v^n, u^n \rangle| &= \left| \int_{\mathbb{R}^2} \nabla v^n \cdot \nabla u^n \right| \\
&\leq \|\nabla v^n\|_2 \|\nabla u^n\|_2 \leq C \|\nabla v^n\|_2 = C \|v^n\|.
\end{aligned}
$$

Then (4.14) and (4.15) imply that (λ_n) is bounded, because $t \neq 0$. Hence, for a subsequence $\lambda_n \to \lambda \in \mathbb{R}$ and since (v^n) is bounded in $\hat{H}^1$ (use (4.13)), we conclude that $\mathcal{E}(u^n) - \lambda \mathcal{V}'(u^n) \to 0$ in $\hat{H}^{-1}$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5. Proof of Theorem 1.3

In view of Remark 2.6, we consider the functional $\mathcal{F}_K(u) = \mathcal{A}(u) + \mathcal{Q}(u)$ on $\hat{H}^1$. Let $t > 0$ and denote by $\mathrm{Crit}_{\mathcal{F}_K}(t)$ the set of constrained critical points of $\mathcal{F}_K$ at volume t, which we define as

$$
\mathrm{Crit}_{\mathcal{F}_K}(t) := \Big\{ u \in M_t \mid u_x \wedge u_y \neq 0 \text{ a.e and there exists } \lambda \in \mathbb{R} \text{ such that}
$$

$$
(5.1) \qquad \frac{d}{ds}\mathcal{F}_K(u + s\varphi)\Big|_{s=0} = \lambda \frac{d}{ds}\mathcal{V}(u + s\varphi)\Big|_{s=0} \quad \forall \varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^3) \Big\}.
$$

We point out that if u is of class C^2 and free of branch points (i.e., u is an immersion) then, since φ has compact support, we have

$$
(5.2) \qquad \frac{d}{ds}\mathcal{A}(u + s\varphi)\Big|_{s=0} = -2 \int_{\mathbb{R}^2} H(u)\nu \cdot \varphi|u_x \wedge u_y|,
$$

where H is the mean curvature of u, and $\nu = (u_x \wedge u_y)/|u_x \wedge u_y|$ is the Gauss map (see [13], Section 2.1, (7) and (8)).

In general, if u is smooth but not immersed then we can consider only variations φ which have compact support in the set of regular points. Nevertheless, if H is a prescribed function of class $C^{1,\alpha}$, then any H-bubble, namely any non constant (weak) solution $u \in \hat{H}^1$ of $\nabla u = 2H(u)u_x \wedge u_y$ on $\mathbb{R}^2$, is in fact smooth, more precisely, of class $C^{3,\alpha}$, in view of well-known results (see [13], Section 5.1, Theorem 1). Hence, the right-hand side of (5.2) can be continuously extended to variations $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$. Therefore we can take (5.2) as a definition of $\frac{d}{ds}\mathcal{A}(u + s\varphi)\big|_{s=0}$ when u is a H-bubble of class $C^{3,\alpha}$ (see also [13], Sect. 5.3).

Before proving Theorem 1.3, we need the following preliminary lemma.

Lemma 5.1. *Let $K \in C^{1,\alpha}(\mathbb{R}^3)$ satisfy (K_1) and (K_2). Then for any fixed $t > 0$ it holds that*

$$
\mathrm{Crit}_{\mathcal{E}}(t) \subset \mathrm{Crit}_{\mathcal{F}_K}(t).
$$

Proof. If $u \in \mathrm{Crit}_{\mathcal{E}}(t)$, then by definition u is a weak solution of

$$
\Delta u = (K(u) - \lambda)u_x \wedge u_y \quad \text{on } \mathbb{R}^2,
$$

for some $\lambda \in \mathbb{R}$ and, by Lemma 2.18, u is of class $C^{2,\alpha}$ as a map on $\mathbb{S}^2$ and satisfies the conformality relations $u_x \cdot u_y = 0 = |u_x|^2 - |u_y|^2$ (see [12], Remark 2.5). Moreover, since we are assuming $K \in C^{1,\alpha}$, by well-known regularity results (see [14], Section 2.3), we get that u is of class $C^{3,\alpha}$. Hence u describes a closed parametric surface of mean curvature $\frac{1}{2}(K(u) - \lambda)$ in the set of regular points. Concerning the set of branch points of u (i.e., points where $\nabla u = 0$), we point out that it is at most finite (see [17] or [13], Sect. 5.1, [14], Sect. 2.10), and in particular it holds that $u_x \wedge u_y \neq 0$ a.e. in $\mathbb{R}^2$. Since u is a $(K - \lambda)$-bubble of class $C^{3,\alpha}$, by (5.2),

$$\frac{d}{ds}\mathcal{A}(u + s\varphi)\Big|_{s=0} = -2\int_{\mathbb{R}^2} \frac{1}{2}(K(u) - \lambda)\nu \cdot \varphi |u_x \wedge u_y|$$

(5.3)
$$= -\int_{\mathbb{R}^2}(K(u) - \lambda)\varphi \cdot u_x \wedge u_y,$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$, where ν is the extension of the Gauss map (see [13], Section 5.1). Now, from (5.3) and Lemma 2.8 we get that, for any $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$,

$$\frac{d}{ds}\mathcal{F}_K(u + s\varphi)\Big|_{s=0} = -\int_{\mathbb{R}^2}(K(u) - \lambda)\varphi \cdot u_x \wedge u_y + \int K(u)\varphi \cdot u_x \wedge u_y.$$

Moreover, by Lemma 2.2, we have

$$\frac{d}{ds}\mathcal{V}(u + s\varphi)\Big|_{s=0} = \int_{\mathbb{R}^2} \varphi \cdot u_x \wedge u_y.$$

Hence, it immediately follows that

$$\frac{d}{ds}\mathcal{F}_K(u + s\varphi)\Big|_{s=0} = \lambda\frac{d}{ds}\mathcal{V}(u + s\varphi)\Big|_{s=0},$$

for any $\varphi \in C_0^\infty(\mathbb{R}^2, \mathbb{R}^3)$, which means that $u \in \mathrm{Crit}_{\mathcal{F}_K}(t)$ (see (5.1)). The proof is complete. $\qquad\square$

Now we can prove Theorem 1.3.

Proof. Assume by contradiction that the thesis is false. Then, by Lemma 5.1, there exists $t_0 \in (0, \bar{t})$ such that

$$\mathrm{Crit}_{\mathcal{E}}(t) = \varnothing \quad \forall t \in (0, t_0].$$

Hence the assumptions of Proposition 3.1 are satisfied, and so there exists $R > 0$ such that

(5.4)
$$S\,t^{2/3} < c_0 < c < 2^{1/3}\,S\,t^{2/3} \quad \forall t \in (0, t_0).$$

By Proposition 4.3, there exists a constrained Palais–Smale sequence $(u^n) \subset M_t$ at level c. Since $\mathcal{D}(u^n)$ is uniformly bounded (see the proof of Proposition 4.3), then, by Lemma 2.19 we deduce that $I = \varnothing$ and $c = \sum_{j \in J}\mathcal{D}(U_j)$.

Now we observe that, up to changing the index set J, we can assume that the coefficients $k_j \in \mathbb{N}^+$ in (3.3) are all identically 1. In fact, for any given $j \in J$,

if $k_j > 1$ then we can split $\mathcal{D}(U_j)$ as the sum of the area of k_j spheres having the same area $4\pi\lambda^2$ and the same volume $\frac{4}{3}\pi\lambda^3$. Hence, up to replacing j with k_j new indexes $\tilde{j}_1, \ldots, \tilde{j}_{k_j}$ and repeating this operation for all $j \in J$ (we recall that J is finite), then, we get a new finite index set $\tilde{J}$ such that all the algebraic multiplicities of the spheres $U_{\tilde{j}}$ are identically 1.

Hence, denoting by $|\tilde{J}|$ the cardinality of $\tilde{J}$, we have

$$c = \sum_{j \in J} \mathcal{D}(U_j) = \sum_{\tilde{j} \in \tilde{J}} \mathcal{D}(U_{\tilde{j}}) = \sum_{\tilde{j} \in \tilde{J}} S\, t_{\tilde{j}}^{2/3} = S\left(\frac{t}{|\tilde{J}|}\right)^{2/3} |\tilde{J}| = S\, |\tilde{J}|^{1/3}\, t^{2/3},$$

but this contradicts (5.4), because $|\tilde{J}|$ is a positive integer. The proof is complete. □

As a consequence of Theorem 1.3, and arguing as in the proof of Theorem 3.15 in [7], we get an existence result for the H-bubble problem.

Theorem 5.2. *Let $K \in C^{1,\alpha}(\mathbb{R}^3)$ satisfy (K_1) with (1.8), (K_2), and assume that $K > 0$ on $\mathbb{R}^3$. Then there exists a sequence $(\lambda_n) \subset \mathbb{R}$ with $|\lambda_n| \to \infty$ such that for every n there exists a $(K - \lambda_n)$-bubble.*

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