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Hodge–Dirac, Hodge-Laplacian and Hodge–Stokes operators in L^p spaces on Lipschitz domains

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Abstract. This paper concerns Hodge–Dirac operators $D_\| = d + \underline{\delta}$ acting in $L^p(\Omega, \Lambda)$ where Ω is a bounded open subset of $\mathbb{R}^n$ satisfying some kind of Lipschitz condition, Λ is the exterior algebra of $\mathbb{R}^n$, d is the exterior derivative acting on the de Rham complex of differential forms on Ω, and $\underline{\delta}$ is the interior derivative with tangential boundary conditions. In $L^2(\Omega, \Lambda)$, $\underline{\delta} = d^*$ and $D_\|$ is self-adjoint, thus having bounded resolvents $\{(I + itD_\|)^{-1}\}_{t \in \mathbb{R}}$ as well as a bounded functional calculus in $L^2(\Omega, \Lambda)$. We investigate the range of values $p_H < p < p^H$ about $p = 2$ for which $D_\|$ has bounded resolvents and a bounded holomorphic functional calculus in $L^p(\Omega, \Lambda)$. On domains which we call very weakly Lipschitz, we show that this is the same range of values as for which $L^p(\Omega, \Lambda)$ has a Hodge (or Helmholz) decomposition, being an open interval that includes 2.

The Hodge-Laplacian $\Delta_\|$ is the square of the Hodge–Dirac operator, i.e., $-\Delta_\| = D_\|{}^2$, so it also has a bounded functional calculus in $L^p(\Omega, \Lambda)$ when $p_H < p < p^H$. But the Stokes operator with Hodge boundary conditions, which is the restriction of $-\Delta_\|$ to the subspace of divergence free vector fields in $L^p(\Omega, \Lambda^1)$ with tangential boundary conditions, has a bounded holomorphic functional calculus for further values of p, namely for $\max\{1, p_{HS}\} < p < p^H$ where p_{HS} is the Sobolev exponent below p_H, given by $1/p_{HS} = 1/p_H + 1/n$, so that $p_{HS} < 2n/(n+2)$. In 3 dimensions, $p_{HS} < 6/5$.

We show also that for bounded strongly Lipschitz domains Ω, $p_H < 2n/(n+1) < 2n/(n-1) < p^H$, in agreement with the known results that $p_H < 4/3 < 4 < p^H$ in dimension 2, and $p_H < 3/2 < 3 < p^H$ in dimension 3. In both dimensions 2 and 3, $p_{HS} < 1$, implying that the Stokes operator has a bounded functional calculus in $L^p(\Omega, \Lambda^1)$ when Ω is strongly Lipschitz and $1 < p < p^H$.

[†]Alan McIntosh passed away shortly after submitting this manuscript. We deeply miss him.

1. Introduction

In this paper, we take a first order approach to developing an L^p theory for the Hodge-Laplacian and the Stokes operator with Hodge boundary conditions, acting on a bounded open subset Ω of $\mathbb{R}^n$. In particular, we give conditions on Ω and p under which these operators have bounded resolvents, generate analytic semigroups, have bounded Riesz transforms, or have bounded holomorphic functional calculi. The first order approach of initially investigating the Hodge–Dirac operator, provides a framework for strengthening known results and obtaining new ones on general classes of domains, in what we believe is a straightforward manner.

In particular we consider the usual strongly Lipschitz and weakly Lipschitz domains (see Section 2.2), but mostly we only need the still weaker concept of a *very weakly Lipschitz domain* Ω, by which we mean that $\Omega = \cup_{j=1}^{M}\Omega_j$ where each Ω_j is a biLipschitz transformation of the unit ball, and $1\!\!1_\Omega = \sum_{j=1}^{M}\chi_j$ for some Lipschitz functions $\chi_j\colon \Omega \to [0,1]$ with sppt $\chi_j \subset \Omega_j$.

When $1 < p < \infty$, we consider the *exterior derivative* $d = \nabla\wedge$ as an unbounded operator in the space $L^p(\Omega, \Lambda)$ with domain $\mathsf{D}^p(d) = \{u \in L^p(\Omega, \Lambda)\,;\, du \in L^p(\Omega, \Lambda)\}$, where $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^n$ is the exterior algebra of $\mathbb{R}^n$ and $L^p(\Omega, \Lambda) = \oplus_{k=0}^{n}L^p(\Omega, \Lambda^k)$ is the space of differential forms on Ω. We shall see that on a very weakly Lipschitz domain Ω, the range $\mathsf{R}^p(d)$ of the exterior derivative is a closed subspace of the null space $\mathsf{N}^p(d)$ with finite codimension. Similar results hold for the *interior derivative* $\delta = -\nabla\lrcorner$.

The duals of the operators d and δ in $L^{p'}(\Omega, \Lambda)$ are denoted by $\underline{\delta}$ and $\underline{d}$, being restrictions of the operators δ and d to smaller domains, namely to the completion of $\mathscr{C}_c^\infty(\Omega)$ in the graph norms. By duality, the range $\mathsf{R}^p(\underline{\delta})$ is a closed subspace of the null space $\mathsf{N}^p(\underline{\delta})$ with finite codimension, and similarly for $\underline{d}$. We remark that when Ω is weakly Lipschitz, so that the unit normal ν is defined a.e. on the boundary $\partial\Omega$, then $\underline{\delta}$ and $\underline{d}$ have domains $\mathsf{D}^p(\underline{\delta}) = \{u \in \mathsf{D}^p(\delta)\,;\, \nu\lrcorner\, u_{|\partial\Omega} = 0\}$ and $\mathsf{D}^p(\underline{d}) = \{u \in \mathsf{D}^p(d)\,;\, \nu\wedge u_{|\partial\Omega} = 0\}$ (called *tangential* and *normal boundary conditions* respectively).

When $p = 2$ and Ω is very weakly Lipschitz, then $\underline{\delta} = d^*$, so the *Hodge–Dirac operator* $D_\parallel = d + \underline{\delta}$ is self-adjoint in $L^2(\Omega, \Lambda)$, and thus has bounded resolvents $\{(\mathrm{I} + itD_\parallel)^{-1}\}_{t\in\mathbb{R}}$ as well as a bounded functional calculus in $L^2(\Omega, \Lambda)$. Moreover there is a *Hodge decomposition*

$$L^2(\Omega, \Lambda) = \mathsf{R}^2(d) \overset{\perp}{\oplus} \mathsf{R}^2(\underline{\delta}) \overset{\perp}{\oplus} \mathsf{N}^2(D_\parallel)\,,$$

where the space of harmonic forms $\mathsf{N}^2(D_\parallel) = \mathsf{N}^2(d) \cap \mathsf{N}^2(\underline{\delta})$ is finite-dimensional (owing to the finite codimension of $\mathsf{R}^2(\underline{\delta})$ in $\mathsf{N}^2(\underline{\delta})$). Similar results hold for $D_\perp = \underline{d} + \delta$.

When Ω is smooth (see, e.g., [26]), then each of these L^2 results has an L^p analogue for all $p \in (1, \infty)$ (provided we drop orthogonality from the definition of the Hodge decomposition). This is known not to be the case on all Lipschitz domains, though typically L^p results do hold for all p sufficiently close to 2 (see, e.g., Theorem 6.1 in [20]). In this paper we prove that the following results hold, provided that Ω is a very weakly Lipschitz domain.

- There exist Hodge exponents p_H, $p^H = p_H{}'$ with $1 \le p_H < 2 < p^H \le \infty$, such that the Hodge decomposition

$$L^p(\Omega, \Lambda) = \mathsf{R}^p(d) \oplus \mathsf{R}^p(\underline{\delta}) \oplus (\mathsf{N}^p(d) \cap \mathsf{N}^p(\underline{\delta}))$$

  holds if and only if $p_H < p < p^H$. Moreover, for p in this range, $D_\| = d + \underline{\delta}$ is a closed operator in $L^p(\Omega, \Lambda)$, and $\mathsf{N}^p(d) \cap \mathsf{N}^p(\underline{\delta}) = \mathsf{N}^p(D_\|) = \mathsf{N}^2(D_\|)$. (Theorem 4.3).

- The Hodge–Dirac operator $D_\|$ is bisectorial with a bounded holomorphic functional calculus in $L^p(\Omega, \Lambda)$ if and only if $p_H < p < p^H$; in particular, for each such p there exists $C_p > 0$ such that $\|(\mathrm{I} + itD_\|)^{-1}u\|_p \le C_p\|u\|_p$ for all $t \in \mathbb{R}$ (Theorem 5.1 (i) and (ii)).

- When $p_H < p < p^H$, the *Hodge-Laplacian* $\Delta_\| = -D_\|{}^2 = -(d\underline{\delta} + \underline{\delta}d)$ is sectorial with a bounded holomorphic functional calculus in $L^p(\Omega, \Lambda)$ and has a bounded Riesz transform in the sense that $\|\sqrt{-\Delta_\|}\, u\|_p \approx \|D_\|u\|_p$; in particular, $\|(\mathrm{I} + t^2\Delta_\|)^{-1}u\|_p \le C_p{}^2\|u\|_p$ for all $t > 0$, and $\Delta_\|$ generates an analytic semigroup in $L^p(\Omega, \Lambda)$ (Corollary 8.1). Let us mention that sectoriality ([24], Theorems 6.1 and 7.1) and boundedness of Riesz transforms ([16], Theorem 5.1) have already been proved in the case of bounded strongly Lipschitz domains.

- If $\max\{1, p_{HS}\} < p < p^H$ (where $p_{HS} = np_H/(n + p_H) < 2n/(n + 2)$), then the operators $f(D_\|)$ in the holomorphic functional calculus of $D_\|$ are bounded on $\mathsf{N}^p(\underline{\delta})$ and on $\mathsf{N}^p(d)$ (Theorem 5.1 (iii)).

- When $\max\{1, p_{HS}\} < p < p^H$, the restriction of the Hodge-Laplacian $\Delta_\|$ to $\mathsf{N}^p(\underline{\delta})$ is sectorial with a bounded holomorphic functional calculus; in particular, the estimate $\|(\mathrm{I} - t^2\Delta_\|)^{-1}u\|_p \le C_p{}^2\|u\|_p$ holds for all $u \in \mathsf{N}^p(\underline{\delta})$ and all $t > 0$, and $\Delta_\|$ generates an analytic semigroup on $\mathsf{N}^p(\underline{\delta})$. The corresponding results also hold on $\mathsf{N}^p(d)$ (Corollary 8.2).

- If Ω is strongly Lipschitz, then $p_H < 2n/(n + 1) < 2n/(n - 1) < p^H$ and $p_{HS} < 2n/(n + 3)$, in particular $\max\{1, p_{HS}\} = 1$ in dimensions 2 and 3. (Theorem 7.1).

The last two points are of particular relevance to the Stokes operator with Hodge boundary conditions, which is the restriction of $-\Delta_\|$ to $\{u \in L^p(\Omega, \Lambda^1)\,;\, \underline{\delta}u = 0\}$. In dimension $n = 3$, the last point shows that the Stokes operator has a bounded holomorphic functional calculus for all $p \in (1, p^H)$ where $p^H > 3$ depends on Ω. This result completes the result stated in Theorem 7.2 of [24], where only sectoriality for $p \in (p_H, p^H)$ has been proved.

A similar lower Hodge exponent arises when considering perturbed Hodge–Dirac operators of the form $D_{\|,B} = d + \underline{\delta}_B = d + B^{-1}\underline{\delta}B$, where $B, B^{-1} \in L^\infty(\Omega, \mathscr{L}(\Lambda))$ with $\mathrm{Re}\,B \ge \kappa\mathrm{I}$, which we shall only do in the case of bounded strongly Lipschitz domains. In this case, all of the above points, except for the final one, hold with $\underline{\delta}$ replaced by $\underline{\delta}_B$, $D_\|$ replaced by $D_{\|,B}$, and $\Delta_\|$ replaced by $\Delta_{\|,B} = -(D_{\|,B})^2$, though of course the Hodge exponents depend on B, with p^H possibly unequal to $p_H{}'$. See Section 6.

Our proofs of the results announced above rely strongly on the potential maps defined in Section 4. Those maps can be of independent interest. They are refined versions of the ones developed in [21] and [12], refined in two ways:

- we can deal here with very weakly Lipschitz domains while [21] and [12] only treat the case of bounded strongly Lipschitz domains;
- we obtain true potentials, in the sense that the maps R, S, T and Q defined in Section 4 are right inverses of d, $\underline{\delta}$, $\underline{d}$ and δ on their ranges.

As a direct consequence, the families of ranges and of null spaces of these operators in L^p, $1 < p < \infty$, form complex interpolation scales (see Corollary 4.2).

In the case of $\mathbb{R}^n$, results in the same spirit (extending the range of p for which a bounded holomorphic functional calculus holds outside the Hodge range) have been recently obtained in [15] and [5]. The methods used there are different, and specific to $\mathbb{R}^n$.

2. Setting

In this section, we specify some concepts used throughout the paper. At all times we are considering functions and operators defined on bounded open subsets Ω of Euclidean space $\mathbb{R}^n$ with dimension $n \geq 2$.

2.1. Notation

Notation 2.1. For $1 \leq p \leq \infty$, we denote by p' the Hölder conjugate exponent, i.e., $1/p + 1/p' = 1$ (with the convention that $1/\infty = 0$), by p_S the lower Sobolev exponent defined by $1/p_S = 1/p + 1/n$, and by p^* the exponent for which $W^{1/p,p}(\mathbb{R}^n) \hookrightarrow L^{p^*}(\mathbb{R}^n)$, i.e., $p^* = np/(n-1)$.

We denote by p^S the Sobolev exponent given by $1/p^S = 1/p - 1/n$ if $1 \leq p < n$, $p^S = \infty$ if $p > n$. If $p = n$, p^S is multivalued, it takes any value in $[p, \infty)$.

Remark 2.2. Note that if $p \in [1, n)$, then $(p^S)' \in (1, n]$ and $(p^S)' = (p')_S$. Note also that if $r \in (1, \infty)$, then $(r^*)' \in (1, n)$ and

$$(2.1) \qquad\qquad \left((r^*)'\right)^S = (r')^*.$$

Notation 2.3. The following sectors in the complex plane will be considered:

$$S_{\mu+}^\circ := \{z \in \mathbb{C} \setminus \{0\};\, |\arg z| < \mu\} \quad \text{and} \quad S_{\mu+} := \overline{S_{\mu+}^\circ} \quad \text{if } \mu \in (0, \pi),$$
$$S_{\mu-}^\circ := -S_{\mu+}^\circ \quad \text{and} \quad S_\mu^\circ := S_{\mu+}^\circ \cup S_{\mu-}^\circ \quad \text{if } \mu \in (0, \pi/2),$$
$$S_\mu := \overline{S_\mu^\circ} \quad \text{if } \mu \in (0, \pi/2) \quad \text{and} \quad S_0 := \mathbb{R} \times \{0\} \subset \mathbb{C}.$$

Notation 2.4. The domain of an (unbounded linear) operator A is denoted by $\mathsf{D}(A)$, its null space by $\mathsf{N}(A)$, its range by $\mathsf{R}(A)$, and its graph by $\mathsf{G}(A)$. When the operator A acts in $L^p(\Omega)$, these are sometimes written as $\mathsf{D}^p(A, \Omega)$, $\mathsf{N}^p(A, \Omega)$, $\mathsf{R}^p(A, \Omega)$, and $\mathsf{G}^p(A, \Omega)$.

Notation 2.5. For $E, F \subset \mathbb{R}^n$ two Borel sets, denote by $\mathrm{dist}\,(E, F)$ the distance between E and F defined by $\mathrm{dist}\,(E, F) = \inf \big\{ |x - y|\,;\, x \in E,\, y \in F \big\}$.

For a distribution f defined on an open subset Ω of $\mathbb{R}^n$, we denote the support of f by $\mathrm{sppt}_\Omega f$, or sometimes just by $\mathrm{sppt}\, f$.

Notation 2.6. We denote by $B(x, r)$ the ball in $\mathbb{R}^n$ with centre $x \in \mathbb{R}^n$ and radius $r > 0$, and set $B_\Omega(x, r) = B(x, r) \cap \Omega$, namely the ball in Ω with the same centre and radius.

2.2. Various types of Lipschitz domains

In the following definitions and properties, we follow the paper [9] by Axelsson (now Rosén) and the first author. By a bounded weakly Lipschitz domain we mean a bounded open set Ω separated from the exterior domain $\mathbb{R}^n \setminus \overline{\Omega}$ by a weakly Lipschitz interface $\Sigma = \partial\Omega = \partial(\mathbb{R}^n \setminus \overline{\Omega})$, defined as follows.

Definition 2.7. Let $\Omega \subset \mathbb{R}^n$ be an open set. A function $f \colon \Omega \to \mathbb{R}^p$ is said to be *uniformly locally Lipschitz* (or *Lipschitz* for short) if there exists $C > 0$ such that for all $x \in \Omega$ there exists $r_x > 0$ such that $|f(y) - f(z)| \le C|y - z|$ for all $y, z \in B_\Omega(x, r_x)$.
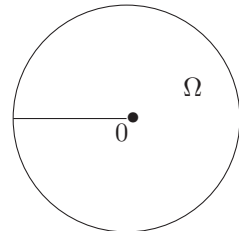
We remark that every such function f is differentiable a.e. with derivatives $\partial_j f \in L^\infty(\Omega, \mathbb{R}^p)$.

Example 2.8. Let

$$\Omega := \big\{ (x, y) \in \mathbb{R}^2\,;\, 0 < x^2 + y^2 < 1,\, |\arg(x, y)| < \pi \big\}$$

as in the picture and define $f \colon \Omega \to \mathbb{R}$ by

$$f(x, y) = (x^2 + y^2)^{1/2} \arg(x, y).$$

Then f is a uniformly locally Lipschitz function in the sense of Definition 2.7, but not globally Lipschitz; i.e., there is no $C > 0$ such that $|f(z) - f(w)| \le C|z - w|$ for all $z, w \in \Omega$.

Definition 2.9. Let $\Omega \subset \mathbb{R}^n$ and let $\rho \colon \Omega \to \rho(\Omega) \subset \mathbb{R}^n$. We say that ρ is a *biLipschitz map* if ρ is a bijective map from Ω to $\rho(\Omega)$ and ρ and ρ^{-1} are both uniformly locally Lipschitz.

Definition 2.10. The interface Σ (between a bounded domain $\Omega \subset \mathbb{R}^n$ and $\mathbb{R}^n \setminus \overline{\Omega}$) is *weakly Lipschitz* if, for all $y \in \Sigma$, there is a neighbourhood $V_y \ni y$ and a global biLipschitz map $\rho_y : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$\Omega \cap V_y = \rho_y\big(\mathbb{R}^{n-1} \times (0, +\infty)\big) \cap V_y, \quad \Sigma \cap V_y = \rho_y\big(\mathbb{R}^{n-1} \times \{0\}\big) \cap V_y,$$
$$\text{and } \big(\mathbb{R}^n \setminus \overline{\Omega}\big) \cap V_y = \rho_y\big(\mathbb{R}^{n-1} \times (-\infty, 0)\big) \cap V_y.$$

In that case, Ω is called (bounded) weakly Lipschitz domain (following [9], Definition 2.1).

A special case of a weakly Lipschitz domain is a strongly Lipschitz domain defined as follows.

Definition 2.11. A (possibly bounded) *strongly Lipschitz domain* is a (possibly bounded) weakly Lipschitz domain such that for all $y \in \Sigma$, there is a neighbourhood $V_y \ni y$ and a global biLipschitz map $\rho_y \colon \mathbb{R}^n \to \mathbb{R}^n$, satisfying the conditions of Definition 2.10, that takes the form

$$\rho_y(x) = E_y\big(x', x_n - g_y(x')\big), \quad x = (x', x_n), \ x' = (x_1, \dots, x_{n-1})$$

where $g_y \colon \mathbb{R}^{n-1} \to \mathbb{R}$ is a Lipschitz function such that $g_y(0) = 0$ and E_y is a Euclidian transformation.

Reasoning as in the proof of Theorem 1.3 in [9], we see that bounded weakly Lipschitz domains have the following property.

Remark 2.12. By Definition 2.10 it follows that there exist biLipschitz maps $\rho_j \colon B \to \rho_j(B) =: \Omega_j \subset \Omega$ ($j = 1, \dots, M$) (where $B = B(0,1)$ denotes the unit ball in $\mathbb{R}^n$) such that $\Omega = \bigcup_{j=1}^{M} \Omega_j$, and there exist Lipschitz functions $\chi_j \colon \Omega \to [0,1]$ such that $\mathrm{sppt}_\Omega \chi_j \subset \Omega_j$ and $\sum_{j=1}^{M} \chi_j = 1$ on Ω.

Furthermore, we may assume that for each $j = 1, \dots, M$, ρ_j extends to a biLipschitz map between slightly larger open sets.

Example 2.13. An important example of a weakly Lipschitz domain that is not strongly Lipschitz is the "two brick" domain in $\mathbb{R}^3$ defined as the interior of

$$\big\{(x,y,z) \in \mathbb{R}^3; 0 \le z \le 1, -2 \le y \le 2, -1 \le x \le 1\big\}$$
$$\cup \big\{(x,y,z) \in \mathbb{R}^3; -1 \le z \le 0, -1 \le y \le 1, -2 \le x \le 2\big\}.$$
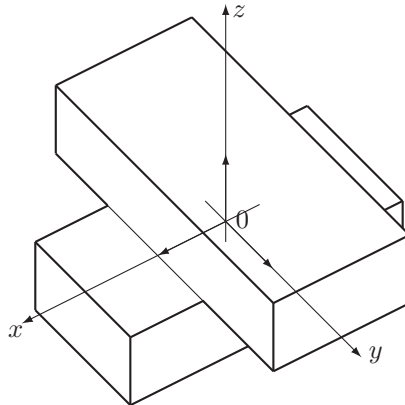
See, e.g., Example 1.5.6 in [6].

A bounded strongly Lipschitz domain is biLipschitz equivalent to a smooth domain in the following sense. The proof of this fact is given in the Appendix A.

Proposition 2.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded strongly Lipschitz domain. Then there exists a biLipschitz map $\phi\colon \mathbb{R}^n \to \mathbb{R}^n$ where $\phi^{-1}(\Omega) = \Omega'$ is a smooth domain in $\mathbb{R}^n$ satisfying $\phi(\mathbb{R}^n \setminus \overline{\Omega'}) = \mathbb{R}^n \setminus \overline{\Omega}$ and $\phi(\partial\Omega') = \partial\Omega$.*

We now take the property of weakly Lipschitz domains spelled out in Remark 2.12 (though without the condition that the biLipschitz maps extend to slightly larger sets) as our definition of very weakly Lipschitz domains, because this is all that is needed in proving many of our results.

Definition 2.15. We call an open set $\Omega \subset \mathbb{R}^n$ a *very weakly Lipschitz domain* provided it satisfies the property (VWL) below:

(VWL)
$$
\begin{aligned}
&\text{there exist } (\rho_j : B \to \Omega_j)_{j=1,\ldots,M} \text{ biLipschitz maps such that} \\
&\Omega = \textstyle\bigcup_{j=1}^M \Omega_j, \text{ and for each } j = 1, \ldots, M, \text{ there exists} \\
&\text{a Lipschitz function } \chi_j \colon \Omega \to [0,1] \text{ such that } \mathrm{sppt}\, \chi_j \subset \Omega_j \\
&\text{and } \textstyle\sum_{j=1}^M \chi_j(x) = 1 \text{ for all } x \in \Omega.
\end{aligned}
$$

Example 2.16. Let us reconsider the domain Ω of Example 2.8. It is not weakly Lipschitz because its boundary does not form an interface between Ω and $\mathbb{R}^n \setminus \overline{\Omega}$. However it is very weakly Lipschitz (with $M = 1$ and $\chi_1 = 1$) as can be shown as follows. Set

$$
\Omega' := \left\{ (x,y) \in \mathbb{R}^2 \,;\, 0 < x^2 + y^2 < 1, |\arg(x,y)| < \pi/2 \right\},
$$

and define $\phi\colon \Omega \to \Omega'$ by $\phi(x,y) := (r\cos(\theta/2), r\sin(\theta/2))$, where $r := (x^2 + y^2)^{1/2}$ and $\theta = \arg(x,y)$.
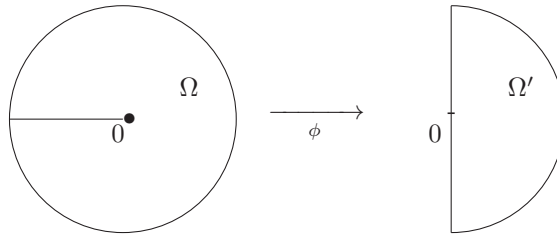
Now ϕ is a biLipschitz map from Ω to Ω' in the sense of Definition 2.9, and Ω' is biLipschitz equivalent to a ball, so that Ω is biLipschitz equivalent to a ball.

2.3. Differential forms

We consider the *exterior derivative* $d := \nabla\wedge = \sum_{j=1}^n \partial_j e_j \wedge$ and the *interior derivative* (or co-derivative) $\delta := -\nabla\lrcorner = -\sum_{j=1}^n \partial_j e_j \lrcorner$ acting on *differential forms* on a domain $\Omega \subset \mathbb{R}^n$, i.e., acting on functions from Ω to the exterior algebra $\Lambda = \Lambda^0 \oplus \Lambda^1 \oplus \cdots \oplus \Lambda^n$ of $\mathbb{R}^n$.

We denote by $\{e_S \,;\, S \subset \{1, \ldots, n\}\}$ the basis for Λ. The space of ℓ-vectors Λ^ℓ is the span of $\{e_S \,;\, |S| = \ell\}$, where

$$
e_S = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_\ell} \quad \text{for} \quad S = \{e_{j_1}, \ldots, e_{j_\ell}\} \quad \text{with} \quad j_1 < j_2 < \cdots < j_\ell.
$$

Remark that Λ^0, the space of complex scalars, is the span of $e_\emptyset$ ($\emptyset$ being the empty set). We set $\Lambda^\ell = \{0\}$ if $\ell < 0$ or $\ell > n$.

On the exterior algebra Λ, the basic operations are

(i) the exterior product $\wedge : \Lambda^k \times \Lambda^\ell \to \Lambda^{k+\ell}$,

(ii) the interior product $\lrcorner : \Lambda^k \times \Lambda^\ell \to \Lambda^{\ell-k}$,

(iii) the Hodge star operator $\star : \Lambda^\ell \to \Lambda^{n-\ell}$,

(iv) the inner product $\langle \cdot, \cdot \rangle : \Lambda^\ell \times \Lambda^\ell \to \mathbb{R}$.

If $a \in \Lambda^1$, $u \in \Lambda^\ell$ and $v \in \Lambda^{\ell+1}$, then

$$\langle a \wedge u, v \rangle = \langle u, a \lrcorner v \rangle.$$

For more details, we refer to, e.g., Section 2 of [9] and Section 2 of [12], noting that both these papers contain some historical background (and being careful that δ has the opposite sign in [9]). In particular, we note the relation between d and δ via the Hodge star operator:

(2.2) $\star \delta u = (-1)^\ell d(\star u)$ and $\star du = (-1)^{\ell-1}\delta(\star u)$ for an ℓ-form u.

The domains of the differential operators d and δ, denoted by $\mathsf{D}(d,\Omega)$ and $\mathsf{D}(\delta,\Omega)$, or more simply $\mathsf{D}(d)$ and $\mathsf{D}(\delta)$, are defined by

$$\mathsf{D}(d) := \big\{ u \in L^2(\Omega, \Lambda); du \in L^2(\Omega, \Lambda) \big\}$$
$$\text{and} \quad \mathsf{D}(\delta) := \big\{ u \in L^2(\Omega, \Lambda); \delta u \in L^2(\Omega, \Lambda) \big\}.$$

Similarly, the L^p versions of these domains read

$$\mathsf{D}^p(d,\Omega) := \big\{ u \in L^p(\Omega, \Lambda); du \in L^p(\Omega, \Lambda) \big\}$$
$$\text{and} \quad \mathsf{D}^p(\delta,\Omega) := \big\{ u \in L^p(\Omega, \Lambda); \delta u \in L^p(\Omega, \Lambda) \big\}.$$

The differential operators d and δ satisfiy $d^2 = d \circ d = 0$ and $\delta^2 = \delta \circ \delta = 0$. We will also consider the adjoints of d and δ in the sense of maximal adjoint operators in a Hilbert space: $\underline{\delta} := d^*$ and $\underline{d} := \delta^*$. They are defined as the closures in $L^2(\Omega, \Lambda)$ of the closable operators $(d^*, \mathscr{C}_c^\infty(\Omega, \Lambda))$ and $(\delta^*, \mathscr{C}_c^\infty(\Omega, \Lambda))$. The next result was proved in [9], Corollary 4.4.

Proposition 2.17. *In the case where Ω is a bounded weakly Lipschitz domain, the operators $d^* = \underline{\delta}$ and $\delta^* = \underline{d}$ have the following representation:*

$\mathsf{D}(\underline{d}, \Omega) = \mathsf{D}(\underline{d}) : = \big\{ u \in L^2(\Omega, \Lambda); d\tilde{u} \in L^2(\mathbb{R}^n, \Lambda) \big\}, \quad \underline{d}u = (d\tilde{u})_{|\Omega} \text{ for } u \in \mathsf{D}(\underline{d}),$
$\mathsf{D}(\underline{\delta}, \Omega) = \mathsf{D}(\underline{\delta}) : = \big\{ u \in L^2(\Omega, \Lambda); \delta\tilde{u} \in L^2(\mathbb{R}^n, \Lambda) \big\}, \quad \underline{\delta}u = (\delta\tilde{u})_{|\Omega} \text{ for } u \in \mathsf{D}(\underline{\delta}),$

where $\tilde{u}$ denotes the zero-extension of u to $\mathbb{R}^n$.

A well-known property of the differential operator d is that it commutes with a change of variables as stated below, see, e.g., Definition 1.2.1 and Proposition 1.2.2 in [6].

Definition 2.18. Let Ω be an open set in $\mathbb{R}^n$ and $\rho\colon \Omega \to \rho(\Omega)$ a biLipschitz transformation. Denote by $J_\rho(y)$ the Jacobian matrix of ρ at a point $y \in \Omega$ and extend it to an isomorphism $J_\rho(y)\colon \Lambda \to \Lambda$ such that $J_\rho(y)(e_0) = e_0$ and

$$J_\rho(y)(e_{i_1} \wedge \cdots \wedge e_{i_k}) = (J_\rho(y)e_{i_1}) \wedge \cdots \wedge (J_\rho(y)e_{i_k}), \quad \{i_1,\ldots,i_k\} \subset \{0,1,\ldots,n\}.$$

To a differential form $u\colon \rho(\Omega) \to \Lambda$ we associate its *pullback* $\rho^*u\colon \Omega \to \Lambda$ and its *push forward* $\rho_*^{-1}u\colon \Omega \to \Lambda$ defined by

$$(\rho^*u)(y) := J_\rho(y)^*\big(u(\rho(y))\big) \quad \text{and} \quad (\rho_*^{-1}u)(y) := J_\rho(y)^{-1}\big(u(\rho(y))\big), \quad y \in B.$$

For convenience, we define the *reduced push forward* of u by

$$\tilde{\rho}_*^{-1}u := \mathrm{Jac}(\rho)\rho_*^{-1}u : \Omega \to \Lambda$$

where $\mathrm{Jac}(\rho)(y)$ denotes the Jacobian determinant of ρ at a point $y \in \Omega$.

Remark 2.19. Note that for all $p \in [1,\infty]$, $\rho^*\colon L^p(\rho(\Omega),\Lambda) \to L^p(\Omega,\Lambda)$ and $(\rho_*)^{-1}\colon L^p(\rho(\Omega),\Lambda) \to L^p(\Omega,\Lambda)$ are bounded with norms controlled by

$$\operatorname*{ess\,sup}_{y\in\Omega} \|J_\rho(y)\|_{\mathscr{L}(\Lambda)} \quad \text{and} \quad \operatorname*{ess\,sup}_{y\in\Omega} \|J_\rho(y)^{-1}\|_{\mathscr{L}(\Lambda)},$$

and hence by the Lipschitz constants of ρ and ρ^{-1}.

Remark 2.20. For ρ as in Definition 2.18 and a differential form $u\colon \rho(\Omega) \to \Lambda$, the following commutation properties hold:

$$(2.3) \qquad\qquad d(\rho^*u) = \rho^*(du) \quad \text{and} \quad \delta(\tilde{\rho}_*^{-1}u) = \tilde{\rho}_*^{-1}(\delta u).$$

In particular, if $u \in \mathsf{D}(d, \rho(\Omega))$, then $\rho^*u \in \mathsf{D}(d,\Omega)$ and if $u \in \mathsf{D}(\delta, \rho(\Omega))$, then $\tilde{\rho}_*^{-1}u \in \mathsf{D}(\delta,\Omega)$.

We also have the following homomorphism properties:

$$\begin{aligned}
\rho^*(u \wedge v) &= \rho^*u \wedge \rho^*v, & \rho_*^{-1}(u \wedge v) &= \rho_*^{-1}u \wedge \rho_*^{-1}v, \\
\rho^*(u \lrcorner\, v) &= \rho_*^{-1}u \lrcorner\, \rho^*v, & \rho_*^{-1}(u \lrcorner\, v) &= \rho^*u \lrcorner\, \rho_*^{-1}v.
\end{aligned}$$

Remark 2.21. By the product rule for the exterior derivative and the interior derivative we have that for all bounded Lipschitz scalar-valued functions η, for all $u \in \mathsf{D}^p(d,\Omega)$ and $v \in \mathsf{D}^p(\underline{\delta},\Omega)$, then $\eta u \in \mathsf{D}^p(d,\Omega)$, $\eta v \in \mathsf{D}^p(\underline{\delta},\Omega)$ with

$$(2.4) \qquad\qquad d(\eta u) = \eta\, du + \nabla\eta \wedge u \quad \text{and} \quad \delta(\eta v) = \eta\, \delta v - \nabla\eta \lrcorner\, v.$$

More generally, for u a bounded Lipschitz ℓ-form, for all $v \in \mathsf{D}^p(d,\Omega)$, it holds

$$(2.5) \qquad\qquad d(u \wedge v) = du \wedge v + (-1)^\ell u \wedge dv,$$

which gives also for all bounded Lipschitz scalar-valued functions η, and for all $u \in \mathsf{D}^p(d,\Omega)$:

$$(2.6) \qquad\qquad d(\nabla\eta \wedge u) = -\nabla\eta \wedge du.$$

2.4. Bisectoriality, sectoriality and functional calculus

Definition 2.22. A closed unbounded operator A on a Banach space X is said to be *bisectorial* of angle $\omega \in [0, \pi/2)$ if the spectrum of A is contained in the double sector S_ω and for all $\theta \in (\omega, \pi/2)$, the following resolvent estimate holds:

$$\sup_{z \in \mathbb{C} \setminus S_\theta} \|(\mathrm{I} + zA)^{-1}\|_{\mathscr{L}(X)} < \infty.$$

Remark 2.23. Let $\mu \in (0, \pi/2)$. Denote by $\Psi(S_\mu^\circ)$ the subspace of continuous functions $f \colon S_\mu \to \mathbb{C}$ holomorphic on S_μ° for which there exists $s > 0$ such that $\sup_{z \in S_\mu^\circ} \big\{ \frac{|z|^s |f(z)|}{1+|z|^{2s}} \big\} < \infty$. Let A be a bisectorial operator of angle $\omega \in [0, \mu)$ on a Banach space X. For all $f \in \Psi(S_\mu^\circ)$, we can define, for $\theta \in (\omega, \mu)$,

$$f(A)u := \frac{1}{2\pi i} \int_{\partial S_\theta^\circ} f(z)(z\mathrm{I} - A)^{-1}\, u \,\mathrm{d}z,$$

where the boundary of the double sector ∂S_θ° is oriented counterclockwise. Note that the integral above converges in norm thanks to the definition of functions belonging to $\Psi(S_\mu^\circ)$ and the estimate on the resolvents of A.

Definition 2.24. Let $0 \leq \omega < \mu < \pi/2$. A bisectorial operator A of angle ω on a Banach space X is said to admit a *bounded S_μ° holomorphic functional calculus* in X if for $\theta \in (\omega, \mu)$ there exists a constant $K_\theta > 0$ such that for all $f \in \Psi(S_\mu^\circ)$, we have that

$$\|f(A)\|_{\mathscr{L}(X)} \leq K_\theta \|f\|_{L^\infty(S_\theta)}.$$

Remark 2.25. Every self-adjoint operator S in a Hilbert space X is bisectorial of angle 0 with resolvent estimate $\sup_{z \in \mathbb{C} \setminus S_\theta} \|(\mathrm{I} + zS)^{-1}\|_{\mathscr{L}(X)} \leq 1/\sin\theta$, and has a bounded holomorphic functional calculus for all $\theta \in (0, \pi/2)$ with $K_\theta = 1$. See, e.g., [19].

The results above can be adapted to the case of sectorial operators suited for second order differential operators.

Definition 2.26. A closed unbounded operator A on a Banach space X is said to be *sectorial* of angle $\omega \in [0, \pi)$ if the spectrum of A is contained in the sector $S_{\omega+}$ and for all $\theta \in (\omega, \pi)$, the following resolvent estimate holds:

$$\sup_{z \in \mathbb{C} \setminus S_{\theta+}} \|(\mathrm{I} + zA)^{-1}\|_{\mathscr{L}(X)} < \infty.$$

Remark 2.27. Let $\mu \in (0, \pi)$. As before, denote by $\Psi(S_{\mu+}^\circ)$ the subspace of continuous functions $f \colon S_{\mu+} \to \mathbb{C}$, holomorphic on $S_{\mu+}^\circ$ for which there exists $s > 0$ such that $\sup_{z \in S_{\mu+}^\circ} \big\{ \frac{|z|^s |f(z)|}{1+|z|^{2s}} \big\} < \infty$. Let A be a sectorial operator of angle $\omega \in [0, \mu)$ on a Banach space X. For all $f \in \Psi(S_{\mu+}^\circ)$, we can define for $\theta \in (\omega, \mu)$,

$$f(A)u := \frac{1}{2\pi i} \int_{\partial S_{\theta+}^\circ} f(z)(z\mathrm{I} - A)^{-1} u \,\mathrm{d}z,$$

where the boundary of the sector $\partial S_{\theta+}^{\circ}$ is oriented counterclockwise. Note that the integral above converges in norm thanks to the definition of functions belonging to $\Psi(S_{\mu+}^{\circ})$ and the estimate on the resolvents of A.

Definition 2.28. Let $0 \leq \omega < \mu < \pi/2$. A sectorial operator A of angle ω on a Banach space X is said to admit a *bounded $S_{\mu+}^{\circ}$ holomorphic functional calculus* in X if for $\theta \in (\omega, \mu)$ there exists a constant $K_{\theta} > 0$ such that for all $f \in \Psi(S_{\mu+}^{\circ})$, we have that
$$\|f(A)\|_{\mathscr{L}(X)} \leq K_{\theta} \|f\|_{L^{\infty}(S_{\theta+})}.$$

Definition 2.29. Let $\Omega \subset \mathbb{R}^n$ be an open set and let $q \in [1, \infty)$. A family of bounded operators $\{R_z, z \in Z\}$ (where $Z \subset \mathbb{C}$) on $L^q(\Omega)$ is said to admit (exponential) *off-diagonal bounds L^q-L^q (of first order)* if there exists $C, c > 0$ such that for all $E, F \subset \mathbb{R}^n$ Borel sets, we have that

$$\left\| \mathbb{1}_E R_z \mathbb{1}_F u \right\|_{L^q(\Omega)} \leq C\, e^{-c\,\mathrm{dist}\,(E,F)/|z|}\, \|u\|_{L^q(\Omega)}, \quad \forall\, z \in Z,\ \forall\, u \in L^q(\Omega).$$

Remark 2.30. If a family of bounded operators $\{R_z, z \in Z\}$ on $L^q(\Omega)$ admits off-diagonal bounds L^q-L^q, then the family of adjoints $\{R_z{}^*, z \in Z\}$ admits off-diagonal bounds $L^{q'}$-$L^{q'}$.

3. Hodge–Dirac operators

Definition 3.1. (i) The *Hodge–Dirac operator* on Ω with *normal boundary conditions* is
$$D_{\perp} := \delta^* + \delta = \underline{d} + \delta.$$
Note that $-\Delta_{\perp} := D_{\perp}{}^2 = \underline{d}\delta + \delta\underline{d}$ is the Hodge-Laplacian with relative (generalised Dirichlet) boundary conditions.

For a scalar function $u\colon \Omega \to \Lambda^0$ we have that $-\Delta_{\perp} u = \delta\underline{d}u = -\Delta_D u$, where Δ_D is the Dirichlet Laplacian.

(ii) The *Hodge–Dirac operator* on Ω with *tangential boundary conditions* is

$$D_{\|} := d + d^* = d + \underline{\delta}.$$

Note that $-\Delta_{\|} := D_{\|}{}^2 = d\underline{\delta} + \underline{\delta}d$ is the Hodge-Laplacian with absolute (generalised Neumann) boundary conditions.

For a scalar function $u\colon \Omega \to \Lambda^0$ we have that $-\Delta_{\|} u = \underline{\delta}du = -\Delta_N u$, where Δ_N is the Neumann Laplacian.

Following [8], Section 4, we have that the operators $D_{\perp}$ and $D_{\|}$ are closed densely defined operators in $L^2(\Omega, \Lambda)$, and that

$$L^2(\Omega, \Lambda) = \overline{\mathsf{R}(d)} \overset{\perp}{\oplus} \overline{\mathsf{R}(\underline{\delta})} \overset{\perp}{\oplus} \mathsf{N}(D_{\|}) = \overline{\mathsf{R}(\delta)} \overset{\perp}{\oplus} \overline{\mathsf{R}(\underline{d})} \overset{\perp}{\oplus} \mathsf{N}(D_{\perp}),$$

where $\mathsf{N}(D_{\|}) = \mathsf{N}(d) \cap \mathsf{N}(\underline{\delta}) = \mathsf{N}(\Delta_{\|})$ and $\mathsf{N}(D_{\perp}) = \mathsf{N}(\delta) \cap \mathsf{N}(\underline{d}) = \mathsf{N}(\Delta_{\perp})$

Remark 3.2. If Ω satisfies (VWL), then it is essentially proved in [9] (see the proof of Theorem 1.3 (i), p. 19–20) that $\mathsf{R}(d)$ and $\mathsf{R}(\underline{\delta})$, as well as $\mathsf{R}(\delta)$ and $\mathsf{R}(\underline{d})$, are closed subspaces of $L^2(\Omega, \Lambda)$ and that $\mathsf{N}(D_\parallel) = \mathsf{N}(\Delta_\parallel)$ and $\mathsf{N}(D_\perp) = \mathsf{N}(\Delta_\perp)$ are finite dimensional. We shall include a proof of these facts in Section 4.

Definition 3.3. The *Hodge decompositions* from the paragraph just before Remark 3.2 are accompanied with the orthogonal projections

$$\mathcal{P}_{\mathsf{R}(d)} : L^2(\Omega, \Lambda) \to \overline{\mathsf{R}(d)}, \quad \mathcal{P}_{\mathsf{R}(\underline{\delta})} : L^2(\Omega, \Lambda) \to \overline{\mathsf{R}(\underline{\delta})}, \quad \mathcal{P}_{\mathsf{N}(D_\parallel)} : L^2(\Omega, \Lambda) \to \mathsf{N}(D_\parallel);$$

$$\mathcal{P}_{\mathsf{R}(\delta)} : L^2(\Omega, \Lambda) \to \overline{\mathsf{R}(\delta)}, \quad \mathcal{P}_{\mathsf{R}(\underline{d})} : L^2(\Omega, \Lambda) \to \overline{\mathsf{R}(\underline{d})}, \quad \mathcal{P}_{\mathsf{N}(D_\perp)} : L^2(\Omega, \Lambda) \to \mathsf{N}(D_\perp).$$

Moreover, noting that $d \colon \mathsf{D}(d) \cap \overline{\mathsf{R}(\underline{\delta})} \to \overline{\mathsf{R}(d)}$ is one-to-one we define

$$\underline{R} : L^2(\Omega, \Lambda) \to \mathsf{R}(\underline{\delta}), \quad \begin{cases} d\underline{R}u = u & \text{if } u \in \overline{\mathsf{R}(d)}, \\ \underline{R}u = 0 & \text{if } u \in \overline{\mathsf{R}(\underline{\delta})} \overset{\perp}{\oplus} \mathsf{N}(D_\parallel). \end{cases}$$

In particular, we have that

$$\mathrm{I} = d\underline{R} + \overline{\underline{R}d} + \mathcal{P}_{\mathsf{N}(D_\parallel)},$$

where $\overline{\underline{R}d}$ denotes the closure in $L^2(\Omega, \Lambda)$ of the operator $\underline{R}d$. Note that $\underline{R}$ is a potential operator, in the sense that, if $u \in \mathsf{R}(d)$ then $u = df$ where $f = \underline{R}u$.

Remark 3.4. If the domain $\Omega \subset \mathbb{R}^n$ is convex or of class $\mathscr{C}^{1,1}$, we have that $\mathsf{D}(D_\perp), \mathsf{D}(D_\parallel) \subset H^1(\Omega, \Lambda)$ (see Theorems 2.9, 2.12, 2.17 in [1] for the proof in dimension $n = 3$, and Theorem 4.10 and Remark 4.11 in [9]). This is however not true in general. If Ω is a strongly Lipschitz domain, then it can be proved that $\mathsf{D}(D_\perp), \mathsf{D}(D_\parallel) \subset H^{1/2}(\Omega, \Lambda)$ as shown in [11] in dimension 3 and in [22], Theorem 11.2, in arbitrary dimension (see also the estimate (7.1) below).

Remark 3.5. At this point we remark that the theory concerning the Hodge–Dirac operator with normal boundary conditions, $D_\perp = \delta^* + \delta = \underline{d} + \delta$, is entirely analogous to the theory concerning the Hodge–Dirac operator with tangential boundary conditions, $D_\parallel = d + d^* = d + \underline{\delta}$. Either the proofs for one can be mimicked for the other, or the results for one can be obtained form the results for the other by the Hodge star operator and appropriate changes of sign. So from now on we will state our results for d, $\underline{\delta}$ and $D_\parallel$, noting here that corresponding results hold for δ, $\underline{d}$ and $D_\perp$.

4. Potential operators on very weakly Lipschitz domains

The unit ball $B = B(0, 1)$ in $\mathbb{R}^n$ is starlike with respect to the ball $\frac{1}{2}B := B(0, 1/2)$. For $p \in (1, \infty)$ and $s \in \mathbb{R}$, let $R_B : W^{s-1,p}(B, \Lambda) \to W^{s,p}(B, \Lambda)$ be a Poincaré-type map (relative to a non negative smooth function $\theta \in \mathscr{D} := \mathscr{C}_c^\infty(B)$ with support in $\frac{1}{2}B$ and $\int \theta = 1$) as defined in [12], Definition 3.1 and (3.9) (building on [21]; see also [10]), in the case of domains which are starlike with respect to a ball.

In those papers a theory of potential operators in Sobolev spaces on strongly Lipschitz domains was developed. Following the notations used in § 2 of [12], we denote by $W^{1,p}_{\overline{\Omega}}(\Lambda)$ the space $\{u \in W^{1,p}(\mathbb{R}^n, \Lambda) : \operatorname{sppt} u \subset \overline{\Omega}\}$. In this section we follow some of the techniques developed there to consider a somewhat different context, namely potential operators mapping $L^p(\Omega, \Lambda)$ to $L^{p^S}(\Omega, \Lambda)$ on very weakly Lipschitz domains.

The operator R_B has the following representation:

$$(4.1) \qquad R_B f_\ell(y) := \int_{\frac{1}{2}B} \theta(a)(y-a) \lrcorner \left(\int_0^1 t^{\ell-1} f_\ell(a + t(y-a)) \, \mathrm{d}t \right) \mathrm{d}a$$

for an ℓ-form f_ℓ ($\ell = 1, \ldots, n$)

($R_B f_0 = 0$), and satisfies

$$(4.2) \qquad R_B d f + d R_B f = f - K_B f \quad \text{where} \quad K_B f = {}_{\mathscr{D}}\langle \theta, f_0 \rangle_{\mathscr{D}'} e_\emptyset$$

for all $f = f_0 + f_1 + \cdots + f_n \in W^{s,p}(B, \Lambda) = W^{s,p}(B, \Lambda^0) \oplus W^{s,p}(B, \Lambda^1) \oplus \cdots \oplus W^{s,p}(B, \Lambda^n)$, where ${}_{\mathscr{D}}\langle \cdot, \cdot \rangle_{\mathscr{D}'}$ denotes the duality pairing between $\mathscr{D}$ and $\mathscr{D}'$. The operator K_B is infinitely smoothing in the sense that for all $f \in (\mathscr{C}_c^\infty(B, \lambda))'$, $K_B f \in \mathscr{C}^\infty(B, \Lambda)$. Moreover, $K_B f = 0$ if $f = dg$ for $g \in \mathsf{D}(d, B)$, which implies that the operator R_B is a true potential for d on B in the sense that for all $p \in (1, \infty)$,

$$(4.3) \qquad \text{if } f \in \mathsf{R}^p(d, B), \quad \text{then } f = d R_B f.$$

The mapping properties of R_B imply in particular that

$$(4.4) \qquad dR_B : L^p(B, \Lambda) \to L^p(B, \Lambda), \quad \forall p \in (1, \infty),$$

so that dR_B is a projection from $L^p(B, \Lambda)$ onto $\mathsf{R}^p(d, B)$.

We also have that for $p \in (1, \infty)$, the adjoint operator of R_B, $R_B{}^*$ maps $L^p(B, \Lambda)$ to $W^{1,p}_{\overline{B}}(\Lambda) \hookrightarrow L^{p^S}(B, \Lambda) \cap \mathsf{D}^p(\underline{\delta}, B)$ where p^S is as in Notation 2.1. As for the adjoint operator of K_B, K_B^* maps in particular $L^p(B, \Lambda)$ to $L^\infty(B, \Lambda) \cap \mathsf{D}^p(\underline{\delta}, B)$.

Therefore, we have that

$$(4.5) \qquad \begin{aligned} R_B &: L^p(B, \Lambda) \to L^{p^S}(B, \Lambda) \cap \mathsf{D}^p(d, B), \\ R_B^* &: L^p(B, \Lambda) \to L^{p^S}(B, \Lambda) \cap \mathsf{D}^p(\underline{\delta}, B) \end{aligned}$$

and

$$(4.6) \qquad \begin{aligned} K_B &: L^p(B, \Lambda) \to L^\infty(B, \Lambda) \cap \mathsf{D}^p(d, B), \\ K_B^* &: L^p(B, \Lambda) \to L^\infty(B, \Lambda) \cap \mathsf{D}^p(\underline{\delta}, B) \end{aligned}$$

are bounded for all $p \in (1, \infty)$. Since the range of K_B is one-dimensional, the operator K_B is compact in $L^p(B, \Lambda)$ for every $p \in (1, \infty)$ (as is K_B^*). Note that the operators R_B and R_B^* are also compact in $L^p(B, \Lambda)$ for every $p \in (1, \infty)$.

Let now $\Omega \subset \mathbb{R}^n$ satisfy property (VWL): $\Omega = \cup_{j=1}^{M} \rho_j(B)$ with $\chi_j \colon \Omega \to [0,1]$ Lipschitz functions such that $\mathrm{sppt}_\Omega \, \chi_j \subset \rho_j(B)$ and $\sum_{j=1}^{M} \chi_j = 1$ on Ω. Following the construction of [12], we define for $u \in L^p(\Omega, \Lambda)$,

$$\tilde{R}_\Omega u = \sum_{j=1}^{M} \chi_j (\rho_j^*)^{-1} R_B(\rho_j^* u).$$

By Remark 2.19, $\tilde{R}_\Omega \colon L^p(\Omega, \Lambda) \to L^{p^S}(\Omega, \Lambda) \cap \mathsf{D}^p(d, B)$ for all $p \in (1, \infty)$. Moreover, for all $u \in \mathsf{D}^p(d, \Omega)$ we have, thanks to the product rule (2.4), the commutation property (2.3) and the relation (4.2) satisfied by R_B, that

$$\begin{aligned}
d\tilde{R}_\Omega u &= \sum_{j=1}^{M} \chi_j \, d\big[(\rho_j^*)^{-1} R_B(\rho_j^* u)\big] + \sum_{j=1}^{M} \nabla \chi_j \wedge \big[(\rho_j^*)^{-1} R_B(\rho_j^* u)\big] \\
&= \sum_{j=1}^{M} \chi_j \big[(\rho_j^*)^{-1} dR_B(\rho_j^* u)\big] + \sum_{j=1}^{M} \nabla \chi_j \wedge \big[(\rho_j^*)^{-1} R_B(\rho_j^* u)\big] \\
&= \sum_{j=1}^{M} \chi_j \big[(\rho_j^*)^{-1}(\mathrm{I} - K_B - R_B d)(\rho_j^* u)\big] + \sum_{j=1}^{M} \nabla \chi_j \wedge \big[(\rho_j^*)^{-1} R_B(\rho_j^* u)\big] \\
&= u - \tilde{R}_\Omega \, du - \tilde{K}_\Omega u
\end{aligned}$$

where

$$\tilde{K}_\Omega u = \sum_{j=1}^{M} \big(\chi_j (\rho_j^*)^{-1} K_B(\rho_j^* u) - \nabla \chi_j \wedge \big[(\rho_j^*)^{-1} R_B(\rho_j^* u)\big]\big).$$

The operator $\tilde{K}_\Omega$ is compact in $L^p(\Omega, \Lambda)$ for all $p \in (1, \infty)$; it is indeed a sum of compositions of bounded operators $(\rho_j^*, (\rho_j^*)^{-1}$ and multiplication with χ_j or $\nabla \chi_j)$ with compact operators $(K_B$ and $R_B)$.

The relation $d\tilde{R}_\Omega + \tilde{R}_\Omega d = \mathrm{I} - \tilde{K}_\Omega$ on $\mathsf{D}^p(d, \Omega)$ implies directly that $\tilde{K}_\Omega$ commutes with d on $\mathsf{D}^p(d, \Omega)$. Moreover, thanks to the mapping properties of R_B and K_B, it is clear that $\tilde{K}_\Omega$ maps $L^q(\Omega, \Lambda)$ to $L^{q^S}(\Omega, \Lambda)$ for all $q \in (1, \infty)$. It is also obvious that $\tilde{K}_\Omega$ maps $L^p(\Omega, \Lambda)$ to $\mathsf{D}^p(d, \Omega)$ thanks to the mapping properties of R_B and K_B, the commutation property (2.3) and the product rules (2.4) and (2.6). Therefore, we see that $\tilde{K}_\Omega{}^n$ maps $L^p(\Omega, \Lambda)$ to $L^\infty(\Omega, \Lambda) \cap \mathsf{D}^p(d, \Omega)$ for all $p > 1$. We define the following operators $\tilde{\tilde{R}}_\Omega$ and $\tilde{\tilde{K}}_\Omega$:

$$\tilde{\tilde{R}}_\Omega := \big(\mathrm{I} + \tilde{K}_\Omega + \tilde{K}_\Omega{}^2 + \cdots + \tilde{K}_\Omega{}^{n-1}\big) \tilde{R}_\Omega \quad \text{and} \quad \tilde{\tilde{K}}_\Omega := \tilde{K}_\Omega{}^n.$$

It follows that $\tilde{\tilde{K}}_\Omega$ is compact in $L^p(\Omega, \Lambda)$ for all $p \in (1, \infty)$ (as a composition of compact operators) and

$$\begin{aligned}
&\tilde{\tilde{R}}_\Omega \colon L^p(\Omega, \Lambda) \to L^{p^S}(\Omega, \Lambda) \cap \mathsf{D}^p(d, \Omega), && \forall\, p \in (1, \infty) \\
&\tilde{\tilde{K}}_\Omega \colon L^p(\Omega, \Lambda) \to L^\infty(\Omega, \Lambda) \cap \mathsf{D}^p(d, \Omega), && \forall\, p \in (1, \infty), \\
&d\tilde{\tilde{R}}_\Omega + \tilde{\tilde{R}}_\Omega d = \mathrm{I} - \tilde{\tilde{K}}_\Omega, && d\tilde{\tilde{K}}_\Omega = \tilde{\tilde{K}}_\Omega d \qquad \text{on } \mathsf{D}^p(d, \Omega).
\end{aligned}$$

Note that $\tilde{R}_\Omega$ is a potential operator modulo compactness, in the sense that, if $u \in \mathsf{R}^p(d, \Omega)$, then $u = df + \tilde{K}_\Omega u$ where $f = \tilde{R}_\Omega u$. It is good enough for most purposes, but it can be improved as follows. Define

$$(4.7) \qquad R_\Omega := \tilde{R}_\Omega + \tilde{K}_\Omega \tilde{R}_\Omega + \tilde{K}_\Omega \underline{R} \tilde{K}_\Omega \quad \text{and} \quad K_\Omega := \tilde{K}_\Omega \mathcal{P}_{\mathsf{N}(D_\parallel)} \tilde{K}_\Omega,$$

where $\underline{R}$ and $\mathcal{P}_{\mathsf{N}(D_\parallel)}$ were defined in Definition 3.3. Then the operators R_Ω and K_Ω satisfy, for every $p \in (1, \infty)$,

$$dR_\Omega u + R_\Omega du = u - K_\Omega u, \quad \text{for all } u \in \mathsf{D}^p(d).$$

By construction, K_Ω is zero on $\mathsf{R}^p(d, \Omega)$; it follows that R_Ω is a true potential operator in the sense that, if $u \in \mathsf{R}^p(d, \Omega)$, then $u = df$ where $f = R_\Omega u$. It is not as natural in $L^2(\Omega, \Lambda)$ as the potential operator $\underline{R}$, but it has the advantage of working for all $p \in (1, \infty)$. (We remark that a similar improvement could be made to the potential operators in strongly Lipschitz domains studied in [12].)

Using the properties of R_B^* and K_B^* and the same construction as above, as well as the Hodge star operator we have similar properties for potential operators (defined below) associated with δ (Q_Ω and L_Ω), $\underline{d}$ ($T_\Omega = Q_\Omega^*$ and L_Ω^*) and $\underline{\delta}$ (S_Ω and K_Ω^*). We define

$$\star Q_\Omega u := (-1)^{\ell-1} R_\Omega(\star u), \quad \star L_\Omega u := K_\Omega(\star u) \quad \text{for an } \ell\text{-form } u;$$
$$T_\Omega u := Q_\Omega^* u;$$
$$\star S_\Omega u := (-1)^{\ell-1} T_\Omega(\star u), \quad \text{for an } \ell\text{-form } u.$$

The properties of the operators R_Ω, S_Ω and K_Ω are summarised in the following proposition. The properties of T_Ω, Q_Ω and L_Ω, can be deduced in a straightforward way.

Proposition 4.1. *Suppose Ω is a very weakly Lipschitz domain. Then the potential operators R_Ω, S_Ω and K_Ω defined above satisfy, for all $p \in (1, \infty)$,*

$$R_\Omega : L^p(\Omega, \Lambda) \to L^{p^S}(\Omega, \Lambda) \cap \mathsf{D}^p(d, \Omega), \quad S_\Omega : L^p(\Omega, \Lambda) \to L^{p^S}(\Omega, \Lambda) \cap \mathsf{D}^p(\underline{\delta}, \Omega),$$

$$K_\Omega : L^p(\Omega, \Lambda) \to L^\infty(\Omega, \Lambda) \cap \mathsf{D}^p(d, \Omega), \quad K_\Omega^* : L^p(\Omega, \Lambda) \to L^\infty(\Omega, \Lambda) \cap \mathsf{D}^p(\underline{\delta}, \Omega),$$

K_Ω, K_Ω^* are compact operators in $L^p(\Omega, \Lambda)$,

$$dR_\Omega + R_\Omega d = \mathrm{I} - K_\Omega, \quad \underline{\delta} S_\Omega + S_\Omega \underline{\delta} = \mathrm{I} - K_\Omega^*,$$

$$dK_\Omega = 0, \quad \underline{\delta} K_\Omega^* = 0 \quad and \quad K_\Omega = 0 \text{ on } \mathsf{R}^p(d, \Omega), \quad K_\Omega^* = 0 \text{ on } \mathsf{R}^p(\underline{\delta}, \Omega),$$

$$dR_\Omega u = u \text{ if } u \in \mathsf{R}^p(d, \Omega), \quad \underline{\delta} S_\Omega u = u \text{ if } u \in \mathsf{R}^p(\underline{\delta}, \Omega).$$

As direct consequence we obtain that dR_Ω, $\underline{\delta} S_\Omega$, $\underline{d} T_\Omega$, and δQ_Ω are projections from $L^p(\Omega, \Lambda)$ onto the ranges of d, $\underline{d}$, δ or $\underline{\delta}$ for all $p \in (1, \infty)$.

Corollary 4.2. *Suppose Ω is a very weakly Lipschitz domain. Then*

(i) *for all $p \in (1, \infty)$, the spaces $\mathsf{R}^p(d, \Omega)$, $\mathsf{R}^p(\underline{d}, \Omega)$, $\mathsf{R}^p(\delta, \Omega)$ and $\mathsf{R}^p(\underline{\delta}, \Omega)$ are closed linear subspaces of $L^p(\Omega, \Lambda)$;*

(ii) *for all $p \in (1, \infty)$, the operators d, $\underline{d}$, δ and $\underline{\delta}$ are closed (unbounded) operators in $L^p(\Omega, \Lambda)$;*

(iii) *there exist finite dimensional subspaces $\mathcal{Z}_d, \mathcal{Z}_\delta, \mathcal{Z}_{\underline{d}}, \mathcal{Z}_{\underline{\delta}} \subset L^\infty(\Omega, \Lambda)$, such that*
$\mathsf{N}^p(d, \Omega) = \mathsf{R}^p(d, \Omega) \oplus \mathcal{Z}_d$, $\mathsf{N}^p(\delta, \Omega) = \mathsf{R}^p(\delta, \Omega) \oplus \mathcal{Z}_\delta$, $\mathsf{N}^p(\underline{d}, \Omega) = \mathsf{R}^p(\underline{d}, \Omega) \oplus \mathcal{Z}_{\underline{d}}$
and $\mathsf{N}^p(\underline{\delta}, \Omega) = \mathsf{R}^p(\underline{\delta}, \Omega) \oplus \mathcal{Z}_{\underline{\delta}}$ for all $p \in (1, \infty)$.

(iv) *The families of spaces $\{\mathsf{R}^p(d, \Omega), 1 < p < \infty\}$, $\{\mathsf{R}^p(\underline{d}, \Omega), 1 < p < \infty\}$, $\{\mathsf{R}^p(\delta, \Omega), 1 < p < \infty\}$ and $\{\mathsf{R}^p(\underline{\delta}, \Omega), 1 < p < \infty\}$ are complex interpolation scales. So too are the families of null spaces.*

(v) *When $1 < p < q < \infty$, then $\mathsf{R}^q(d, \Omega) = \mathsf{R}^p(d, \Omega) \cap L^q(\Omega, \Lambda)$, and similarly for the other range spaces.*

(vi) *When $1 < p < q < \infty$, then $\mathsf{R}^q(d, \Omega)$ is dense in $\mathsf{R}^p(d, \Omega)$, and similarly for the other range spaces.*

Proof. (i) This follows from the fact that the ranges are images of bounded projections.

(ii) The cases of δ, $\underline{d}$ and $\underline{\delta}$ are similar to the case of d. Let $(u_k)_{k \in \mathbb{N}}$ be a sequence in $\mathsf{D}^p(d, \Omega)$ converging to u in $L^p(\Omega, \Lambda)$ such that $(du_k)_{k \in \mathbb{N}}$ converges to v in $L^p(\Omega, \Lambda)$. By (i), $v \in \mathsf{R}^p(d, \Omega)$ (in particular, $v = dR_\Omega v$) and by Proposition 4.1 applied to u_k, we have that $u = dR_\Omega u + R_\Omega v + K_\Omega u$. Therefore $u \in \mathsf{D}^p(d, \Omega)$ satisfies $du = d(dR_\Omega u + R_\Omega v + K_\Omega u) = dR_\Omega v = v$ since $d^2 = 0$ and $dK_\Omega = 0$. This proves that d is a closed operator in $L^p(\Omega, \Lambda)$.

(iii) We just consider the case of d. Let $\mathcal{Z}_d^p = K_\Omega(\mathsf{N}^p(d, \Omega)) \subset L^\infty(\Omega)$. Then $\mathsf{N}^p(d, \Omega) = \mathsf{R}^p(d, \Omega) \oplus \mathcal{Z}_d^p$ with decomposition $u = dR_\Omega u + K_\Omega u$ for all $u \in \mathsf{N}^p(d, \Omega)$. So the spaces in the decomposition are closed, and $\mathcal{Z}_d^p$ is finite dimensional (on account of the compactness of K_Ω). Moreover if $u \in \mathsf{N}^q(d, \Omega)$, then $K_\Omega u = K_\Omega{}^2 u \in K_\Omega(\mathsf{N}^p(d, \Omega)) = \mathcal{Z}_d^p$ so that $\mathcal{Z}_d^q \subset \mathcal{Z}_d^p$, and conversely $\mathcal{Z}_d^p \subset \mathcal{Z}_d^q$, implying that the spaces $\mathcal{Z}_d^p$ are independent of p and can just be named $\mathcal{Z}_d$.

(iv) The spaces $L^p(\Omega, \Lambda)$ interpolate by the complex method, and hence so do their images under bounded projections (see [18], Chap. 1, §14.3; see also Lemma 2.12 in [23]).

(v) If $u \in \mathsf{R}^p(d, \Omega) \cap L^q(\Omega, \Lambda)$, then $u = d(R_\Omega u) \in \mathsf{R}^q(d, \Omega)$.

(vi) $L^q(\Omega, \Lambda)$ is dense in $L^p(\Omega, \Lambda)$, and so then is $dR_\Omega L^q(\Omega, \Lambda)$ dense in the space $dR_\Omega L^p(\Omega, \Lambda)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

We have now essentially proved Remark 3.2. In particular the L^2 range spaces are all closed, and for the space $L^2(\Omega, \Lambda)$, the following decompositions are equally valid:

$$L^2(\Omega, \Lambda) = \mathsf{R}(d) \overset{\perp}{\oplus} \mathsf{R}(\underline{\delta}) \overset{\perp}{\oplus} \mathsf{N}(D_\parallel) = \mathsf{R}(d) \overset{\perp}{\oplus} \mathsf{R}(\underline{\delta}) \oplus \mathcal{Z}_{\underline{\delta}} = \mathsf{R}(\underline{\delta}) \overset{\perp}{\oplus} \mathsf{R}(d) \oplus \mathcal{Z}_d$$

$$= \mathsf{R}(\delta) \overset{\perp}{\oplus} \mathsf{R}(\underline{d}) \overset{\perp}{\oplus} \mathsf{N}(D_\perp) = \mathsf{R}(\delta) \overset{\perp}{\oplus} \mathsf{R}(\underline{d}) \oplus \mathcal{Z}_{\underline{d}} = \mathsf{R}(\underline{d}) \overset{\perp}{\oplus} \mathsf{R}(\delta) \oplus \mathcal{Z}_\delta$$

So the spaces $\mathsf{N}(D_\parallel)$, $\mathcal{Z}_{\underline{\delta}}$ and $\mathcal{Z}_d$ all have the same finite dimension. So do their components of ℓ forms in $L^2(\Omega, \Lambda^\ell)$, which can be identified with the de Rham

cohomology spaces of Ω with tangential (absolute) boundary conditions, and thus have dimensions determined by the global topology of Ω. The spaces $\mathsf{N}(D_\perp)$, $\mathcal{Z}_{\underline{d}}$ and $\mathcal{Z}_\delta$ also have the same finite dimension, as well as their components, which can be identified with the de Rham cohomology spaces of Ω with normal (relative) boundary conditions.

A further important consequence of the existence of these potentials is the fact that the above Hodge decompositions in $L^2(\Omega, \Lambda)$ extend to $L^p(\Omega, \Lambda)$ for p in an interval around 2.

Theorem 4.3. *Let $\Omega \subset \mathbb{R}^n$ be a very weakly Lipschitz domain. There exist Hodge exponents p_H, $p^H = p_H{}'$ with $1 \le p_H < 2 < p^H \le \infty$, such that the Hodge decomposition*

$$(H_p) \qquad L^p(\Omega, \Lambda) = \mathsf{R}^p(d, \Omega) \oplus \mathsf{R}^p(\underline{\delta}, \Omega) \oplus \big(\mathsf{N}^p(d, \Omega) \cap \mathsf{N}^p(\underline{\delta}, \Omega)\big)$$

holds if and only if $p_H < p < p^H$. Moreover, for p in this range, $D_\| = d + \underline{\delta}$ (with $\mathsf{D}^p(D_\|) = \mathsf{D}^p(d) \cap \mathsf{D}^p(\underline{\delta})$) is a closed operator in $L^p(\Omega, \Lambda)$, and $\mathsf{N}^p(d, \Omega) \cap \mathsf{N}^p(\underline{\delta}, \Omega) = \mathsf{N}^p(D_\|) = \mathsf{N}(D_\|)$, so that

$$(4.8) \qquad L^p(\Omega, \Lambda) = \mathsf{R}^p(d, \Omega) \oplus \mathsf{R}^p(\underline{\delta}, \Omega) \oplus \mathsf{N}(D_\|);$$

and also $D_\perp = \delta + \underline{d}$ is a closed operator in $L^p(\Omega, \Lambda)$ with Hodge decomposition

$$(4.9) \qquad L^p(\Omega, \Lambda) = \mathsf{R}^p(\delta, \Omega) \oplus \mathsf{R}^p(\underline{d}, \Omega) \oplus \mathsf{N}(D_\perp).$$

Proof. Let $p \in (1, \infty)$. The decomposition (H_p) holds if and only if

$$(4.10) \qquad L^p(\Omega, \Lambda) = \mathsf{R}^p(d, \Omega) \oplus \mathsf{N}^p(\underline{\delta}, \Omega) \qquad \text{and}$$
$$(4.11) \qquad L^p(\Omega, \Lambda) = \mathsf{N}^p(d, \Omega) \oplus \mathsf{R}^p(\underline{\delta}, \Omega).$$

Now each of these decompositions hold for $p = 2$, and all of the families interpolate with respect to p by the the complex method, so by the properties of interpolation together with Šneĭberg's theorem [27] (see also Theorem 2.7 in [17]), (4.10) holds if and only if p belongs to some open interval $J = (q_\Omega, r_\Omega)$ containing 2, while (4.11) holds if and only if p belongs to another open interval, which, by duality, is $J' = (r_{\Omega'}, q_{\Omega'})$. Therefore (H_p) holds if and only if $p \in J \cap J'$, i.e., $p_H < p < p^H$, where $p_H = \max\{q_\Omega, r_{\Omega'}\}$ and $p^H = \min\{r_\Omega, q_{\Omega'}\} = p_H{}'$.

Once we have the Hodge decomposition (H_p), it is straightforward to verify that $D_\| = d + \underline{\delta}$ is a closed operator in $L^p(\Omega, \Lambda)$ (using the closedness of d and $\underline{\delta}$ proved in Corollary 4.2 (ii)), and that $\mathsf{N}^p(d, \Omega) \cap \mathsf{N}^p(\underline{\delta}, \Omega) = \mathsf{N}^p(D_\|)$.

Moreover, following the reasoning above, we have that $\dim(\mathsf{N}^p(D_\|)) = \dim \mathcal{Z}_d$ which is independent of $p \in (p_H, p^H)$. Now $\mathsf{N}^q(D_\|) \subset \mathsf{N}^p(D_\|)$ when $p_H < p < q < p^H$, and these null spaces all have the same dimension, so they are all equal to $\mathsf{N}(D_\|)$.

The results for $D_\perp = \delta + \underline{d}$ are proved in a similar way, and have the same Hodge exponents by Hodge duality. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We record the following facts about the closed operator $D_\| = d + \underline{\delta}$ (with $\mathsf{D}^p(D_\|) = \mathsf{D}^p(d) \cap \mathsf{D}^p(\underline{\delta})$) in $L^p(\Omega, \Lambda)$.

Proposition 4.4. *If $p_H < p < q < p^H$, then $\mathsf{R}^p(D_\parallel) = \mathsf{R}^p(d) \oplus \mathsf{R}^p(\underline{\delta})$ and $\mathsf{R}^q(D_\parallel) = \mathsf{R}^q(d) \oplus \mathsf{R}^q(\underline{\delta})$ is dense in $\mathsf{R}^p(D_\parallel)$. Moreover, $\mathsf{G}^q(D_\parallel)$ is dense in $\mathsf{G}^p(D_\parallel)$.*

Proof. The density of the ranges follows from Corollary 4.2 (vi). In proving the density of the graphs, we assume that $q \le p^S$. Otherwise we proceed by induction. Let us introduce the potential map $Z \colon L^p(\Omega, \Lambda) \to L^q(\Omega, \Lambda)$ defined by

$$Zv = \mathcal{P}_{\mathsf{R}(d)} S_\Omega \mathcal{P}_{\mathsf{R}(\underline{\delta})} v + \mathcal{P}_{\mathsf{R}(\underline{\delta})} R_\Omega \mathcal{P}_{\mathsf{R}(d)} v$$

where R_Ω and S_Ω have the properties stated in Proposition 4.1. This is a potential map in the sense that, for all $v \in \mathsf{R}^p(D_\parallel)$,

$$\begin{aligned} D_\parallel Zv &= \underline{\delta} \mathcal{P}_{\mathsf{R}(d)} S_\Omega \mathcal{P}_{\mathsf{R}(\underline{\delta})} v + d \mathcal{P}_{\mathsf{R}(\underline{\delta})} R_\Omega \mathcal{P}_{\mathsf{R}(d)} v \\ &= \underline{\delta} S_\Omega \mathcal{P}_{\mathsf{R}(\underline{\delta})} v + d R_\Omega \mathcal{P}_{\mathsf{R}(d)} v = \mathcal{P}_{\mathsf{R}(\underline{\delta})} v + \mathcal{P}_{\mathsf{R}(d)} v = v \,. \end{aligned}$$

Let $(u, D_\parallel u) \in \mathsf{G}^p(D_\parallel)$. By density of the ranges, there exists a sequence $(w_k)_{k\in\mathbb{N}}$ in $\mathsf{R}^q(D_\parallel)$ such that $w_k \to D_\parallel u$ as $k \to \infty$ in L^p. Let $u_k = Zw_k + (u - ZD_\parallel u)$. Then $(u_k)_{k\in\mathbb{N}}$ is a sequence in $L^q(\Omega, \Lambda)$ (because $Zw_k \in L^q$ and $u - ZD_\parallel u \in \mathsf{N}(D_\parallel) \in L^q$) and $D_\parallel u_k = w_k$, so $(u_k, D_\parallel u_k) = (u_k, w_k) \in \mathsf{G}^q(D_\parallel)$. Also $u_k - u = Z(w_k - D_\parallel u) \to 0$ as $k \to \infty$, so that $(u_k, D_\parallel u_k) \to (u, D_\parallel u)$ as $k \to \infty$ in $L^p \oplus L^p$. We conclude that $\mathsf{G}^q(D_\parallel)$ is dense in $\mathsf{G}^p(D_\parallel)$. $\square$

Remark 4.5. If $\Omega \subset \mathbb{R}^n$ is smooth, it is known that $p_H = 1$ and $p^H = \infty$ (see Theorems 2.4.2 and 2.4.14 in [26]). We will see in Section 7 that if $\Omega \subset \mathbb{R}^n$ is a bounded strongly Lipschitz domain, then $p_H < 2n/(n+1)$ and $p^H > 2n/(n-1)$.

5. Hodge–Dirac operators on very weakly Lipschitz domains

On any $\Omega \subset \mathbb{R}^n$, the Hodge–Dirac operator $D_\parallel = d + \underline{\delta}$ with domain $\mathsf{D}(D_\parallel) = \mathsf{D}(d, \Omega) \cap \mathsf{D}(\underline{\delta}, \Omega)$, as defined in Definition 3.1, is self-adjoint in $L^2(\Omega, \Lambda)$. Therefore, by Remark 2.25, $D_\parallel$ is bisectorial of angle $\omega = 0$ in $L^2(\Omega, \Lambda)$, and, for all $\mu \in (0, \pi/2)$, $D_\parallel$ admits a bounded S_μ° holomorphic functional calculus in $L^2(\Omega, \Lambda)$.

Our aim in this section is to extend this result to a range of values of p under the condition that Ω satisfies condition (VWL).

In the case of a strongly Lipschitz domain, it has been proved in Theorem 7.1 of [24] that the semigroup generated by the Hodge-Laplacian $\Delta_\parallel = -D_\parallel{}^2$ in $L^2(\Omega, \Lambda)$ extends to an analytic semigroup in $L^p(\Omega, \Lambda)$ if $p_H < p < p^H$. Moreover, the Riesz transforms $d/\sqrt{-\Delta_\parallel}$ and $\underline{\delta}/\sqrt{-\Delta_\parallel}$ are bounded in $L^p(\Omega, \Lambda)$ for $p_H < p < p^H$ as proved in Theorem 1.1 of [16].

Recall that the results presented here for $D_\parallel$ are equally valid for $D_\perp$ (see Remark 3.5).

Theorem 5.1. *Suppose Ω is a very weakly Lipschitz domain, $1 < p < \infty$, and $D_\parallel = d + \underline{\delta}$ is the Hodge–Dirac operator in $L^p(\Omega, \Lambda)$ with domain $\mathsf{D}^p(D_\parallel) = \mathsf{D}^p(d, \Omega) \cap \mathsf{D}^p(\underline{\delta}, \Omega)$.*

(i) *If $p_H < p < p^H$, then the operator $D_\parallel = d + \underline{\delta}$ is bisectorial of angle $\omega = 0$ in $L^p(\Omega, \Lambda)$, and for all $\mu \in (0, \pi/2)$, $D_\parallel$ admits a bounded S_μ° holomorphic functional calculus in $L^p(\Omega, \Lambda)$.*

(ii) *Conversely, if, for some $p \in (1, \infty)$, the operator $D_\parallel$ is bisectorial with a bounded holomorphic functional calculus in $L^p(\Omega, \Lambda)$, then $p_H < p < p^H$.*

(iii) *Moreover, for all $r \in (\max\{1, p_{HS}\}, p^H)$ (recall that $p_S := np/(n+p)$) and all $\theta \in (0, \pi/2)$, there exists $C_{r,\theta} > 0$ such that*

$$(5.1) \qquad (\mathrm{I} + zD_\parallel)^{-1} : \begin{cases} \mathsf{R}^r(d, \Omega) \\ \mathsf{R}^r(\underline{\delta}, \Omega) \end{cases} \longrightarrow L^r(\Omega, \Lambda) \quad \forall z \in \mathbb{C} \setminus S_\theta,$$

with the estimates

$$(5.2) \quad \sup_{z \in \mathbb{C} \setminus S_\theta} \|(\mathrm{I} + zD_\parallel)^{-1} u\|_r \le C_{r,\theta} \|u\|_r \quad \forall u \in \mathsf{R}^r(d, \Omega) \text{ and } \forall u \in \mathsf{R}^r(\underline{\delta}, \Omega).$$

For all $\mu \in (0, \pi/2)$, there exists a constant $K_{r,\mu}$ such that for all $f \in \Psi(S_\mu^\circ)$,

$$(5.3) \qquad\qquad f(D_\parallel) : \begin{cases} \mathsf{R}^r(d, \Omega) \\ \mathsf{R}^r(\underline{\delta}, \Omega) \end{cases} \longrightarrow L^r(\Omega, \Lambda),$$

with the estimates

$$(5.4) \quad \|f(D_\parallel) u\|_r \le K_{r,\mu} \|f\|_{L^\infty(S_\mu^\circ)} \|u\|_r, \quad \forall u \in \mathsf{R}^r(d, \Omega) \text{ and } \forall u \in \mathsf{R}^r(\underline{\delta}, \Omega).$$

The proof of this result is iterative. In the iteration arguments, we will apply the following two intermediate results. The heart of the extrapolation method is deferred to Section 9.

Proposition 5.2. *Suppose Ω is a very weakly Lipschitz domain, that $p_H < q < p^H$, and that $D_\parallel$ is bisectorial of angle $\omega \ge 0$ in $L^q(\Omega, \Lambda)$. Suppose $\omega < \mu < \pi/2$ and $\max\{1, q_S\} < p < q$.*

(i) *The family of resolvents*

$$(5.5) \qquad\qquad \left\{ (\mathrm{I} + zD_\parallel)^{-1} \,;\, z \in \mathbb{C} \setminus S_\mu \right\}$$

admits off-diagonal bounds L^q-L^q as defined in Definition 2.29. Moreover the following families of operators

$$(5.6) \quad \left\{ zd(\mathrm{I} + zD_\parallel)^{-1} \,;\, z \in \mathbb{C} \setminus S_\mu \right\} \quad \text{and} \quad \left\{ z\underline{\delta}(\mathrm{I} + zD_\parallel)^{-1} \,;\, z \in \mathbb{C} \setminus S_\mu \right\}$$

also admit off-diagonal bounds L^q-L^q, as (by Remark 2.30) do the families of adjoints

$$(5.7) \quad \left\{ z\overline{(\mathrm{I} + zD_\parallel)^{-1}\underline{\delta}} \,;\, z \in \mathbb{C} \setminus S_\mu \right\} \quad \text{and} \quad \left\{ z\overline{(\mathrm{I} + zD_\parallel)^{-1}d} \,;\, z \in \mathbb{C} \setminus S_\mu \right\}.$$

(ii) *Condition (A) of Theorem 9.1 (which we state here for the convenience of the reader):*

X_p is a closed subspace of $L^p(\Omega)$ such that for all $u \in X_p$, there exist $w, v \in L^q(\Omega)$ with $w \in \mathsf{D}^p(B)$, $\|w\|_q, \|v\|_q \lesssim \|u\|_p$ and $u = Bw + v$. Moreover, for each $t \in (0, \operatorname{diam}\Omega]$ there exists a family $\{Q_k^t, k \in \mathbb{Z}^n\}$ of open subsets of Ω with the property that

$$|Q_k^t| \lesssim t^n, \quad \mathbb{1}_\Omega \le \sum_k \mathbb{1}_{Q_k^t} \le N\,\mathbb{1}_\Omega,$$

$$\sup_j \sum_k e^{-\varepsilon \operatorname{dist}(Q_k^t, Q_j^t)/t} = \sup_k \sum_j e^{-\varepsilon \operatorname{dist}(Q_k^t, Q_j^t)/t} \le C_\varepsilon$$

for all $\varepsilon > 0$, where C_ε does not depend on t, and for all $u \in X_p$, there exist $w_k, v_k \in L^q(\Omega)$ such that $w_k \in \mathsf{D}^p(B)$ for all k, and w_k, v_k satisfy

$$\operatorname{sppt} w_k, \operatorname{sppt} v_k \subset Q_k^t, \quad \|w_k\|_q, \|v_k\|_q \lesssim t^{1-n(1/p-1/q)} \|\mathbb{1}_{Q_k^t} u\|_p,$$

$$u = \sum_k \left(Bw_k + \tfrac{1}{t} v_k \right).$$

holds in each of the following cases:

(a) The operator $B = d$ and the subspace $X_p = \mathsf{R}^p(d, \Omega)$,
(b) the operator $B = \underline{\delta}$ and the subspace $X_p = \mathsf{R}^p(\underline{\delta}, \Omega)$.

(iii) There exist constants $M_{p,\mu}$ such that

$$\left\| (I + zD_\parallel)^{-1} u \right\|_p \le M_{p,\mu} \|u\|_p, \quad \forall\, z \in \mathbb{C} \setminus S_\mu,$$
$$\forall\, u \in \mathsf{R}^p(d, \Omega) \cap L^q(\Omega, \Lambda) = \mathsf{R}^q(d, \Omega)$$
$$and \quad \forall\, u \in \mathsf{R}^p(\underline{\delta}, \Omega) \cap L^q(\Omega, \Lambda) = \mathsf{R}^q(\underline{\delta}, \Omega).$$

(iv) If in addition $p > p_H$, then $D_\parallel$ is bisectorial of angle ω in $L^p(\Omega, \Lambda)$.

Proof. (i) The methods used in this proof are inspired by those developed for the proof of Lemma 2.1 in [3]; see also Proposition 5.1 in [2].

We start with the proof of off-diagonal bounds for the families (5.5). Let $\mu \in (\omega, \pi/2)$ and $E, F \subset \mathbb{R}^n$ be Borel sets. Let $z \in \mathbb{C} \setminus S_\mu$ and $t := |z| > 0$. If $\operatorname{dist}(E, F) = 0$, the result is immediate since the resolvent is bounded in $L^q(\Omega, \Lambda)$ by assumption, so suppose that $\operatorname{dist}(E, F) > 0$. Let $M_{q,\mu} := \sup_{z \in \mathbb{C} \setminus S_\mu} \|(I + zD_\parallel)^{-1}\|_{\mathscr{L}(L^q(\Omega, \Lambda))}$. Let ξ be a real-valued function satisfying

(5.8) $\xi \in \operatorname{Lip}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, $\xi = 1$ on E, $\xi = 0$ on F and $\|\nabla\xi\|_\infty \le \dfrac{1}{\operatorname{dist}(E, F)}$,

for example taking $\xi(x) = \min\{\operatorname{dist}(x, F)/\operatorname{dist}(E, F), 1\}$. Let $\alpha > 0$ (which will be determined later) and let $\eta := e^{\alpha\xi}$. Note that $\nabla\eta = \alpha\eta\nabla\xi$.

For each $u \in L^q(\Omega, \Lambda)$, set $v := (I + zD_\parallel)^{-1}(\mathbb{1}_F u) = (I + zD_\parallel)^{-1}(\eta\,\mathbb{1}_F u) \in \mathsf{D}^q(D_\parallel, \Omega)$ (since $\eta = 1$ on F), noting that $\eta v \in \mathsf{D}^q(D_\parallel, \Omega)$ (see Remark 2.21).

Hence we have the following commutator identity:

$$
\begin{aligned}
\eta v &= v + \big[\eta, (\mathrm{I} + zD_\parallel)^{-1}\big](\mathbb{1}_F u) \\
&= v - (\mathrm{I} + zD_\parallel)^{-1}\big[\eta, zD_\parallel\big](\mathrm{I} + zD_\parallel)^{-1}(\mathbb{1}_F u) \\
&= v + z(\mathrm{I} + zD_\parallel)^{-1}\big(\nabla\eta \wedge v - \nabla\eta_\lrcorner v\big) \qquad \text{by (2.4)} \\
&= v + z(\mathrm{I} + zD_\parallel)^{-1}\alpha\big(\nabla\xi \wedge (\eta v) - \nabla\xi_\lrcorner (\eta v)\big).
\end{aligned}
$$

Since $\|v\|_q \leq M_{q,\mu}\|u\|_q$, we have the estimate

$$
\tag{5.9}
\|\eta v\|_q \leq M_{q,\mu}\,\|u\|_q + 2\alpha\, t\, M_{q,\mu}\,\frac{1}{\mathrm{dist}\,(E,F)}\,\|\eta v\|_q.
$$

We now choose $\alpha = \frac{\mathrm{dist}\,(E,F)}{4tM_{q,\mu}}$ and since $\eta = e^\alpha$ on E, (5.9) implies

$$
\tag{5.10}
e^\alpha\|\mathbb{1}_E v\|_q \leq \|\eta v\|_q \leq 2M_{q,\mu}\,\|u\|_q.
$$

Therefore, we have proved off-diagonal bounds (as in Definition 2.29) for $\big\{(\mathrm{I} + zD_\parallel)^{-1}, z \in \mathbb{C}\setminus S_\mu\big\}$ with $C = 2M_{q,\mu}$ and $c = 1/(4M_{q,\mu})$.

We turn now to the proof of off-diagonal bounds for the first family in (5.6), and use the same notation as above for $q, t, \xi, \eta, M_{q,\mu}, u, v$, first noting that

$$
\tag{5.11}
\|zD_\parallel(\mathrm{I} + zD_\parallel)^{-1}u\|_q \leq (1 + M_{q,\mu})\|u\|_q, \quad \forall z \in \mathbb{C}\setminus S_\mu.
$$

Since $q \in (p_H, p^H)$, the Hodge projection $P_{\mathsf{R}^q(d,\Omega)}\colon L^q(\Omega,\Lambda) \to \mathsf{R}^q(d,\Omega)$ is bounded on $L^q(\Omega,\Lambda)$; we denote by M_q its norm. It is straightforward that $P_{\mathsf{R}^q(d,\Omega)}D_\parallel v = dv$ for all $v \in \mathsf{D}^q(D_\parallel,\Omega)$. From (5.11) follows the estimate

$$
\begin{aligned}
\|zd(\mathrm{I} + zD_\parallel)^{-1}u\|_q &\leq M_q\,(1 + M_{q,\mu})\,\|u\|_q \quad \forall z \in \mathbb{C}\setminus S_\mu\ , \quad \text{and therefore} \\
\|zdv\|_q &\leq M_q\,(1 + M_{q,\mu})\,\|u\|_q \quad \forall z \in \mathbb{C}\setminus S_\mu\ .
\end{aligned}
$$

Further,

$$
\begin{aligned}
\eta z\,dv - z\,dv &= \eta z\,dv - z\,d(\eta v) + z\,d(\eta v) - z\,dv \\
&= -\alpha z\nabla\xi \wedge (\eta v) + z\,d(\eta v - v) \\
&= \alpha z\big(-\nabla\xi \wedge (\eta v) + z\,d(\mathrm{I} + zD_\parallel)^{-1}\big(\nabla\xi \wedge (\eta v) - \nabla\xi_\lrcorner (\eta v)\big)\big).
\end{aligned}
$$

This gives the estimate

$$
\|\eta z\,dv\|_q \leq M_q(1 + M_{q,\mu})\|u\|_q + \frac{\alpha\, t}{\mathrm{dist}\,(E,F)}\big(1 + 2M_q(1 + M_{q,\mu})\big)\|\eta v\|_q.
$$

Choosing $\alpha = \frac{\mathrm{dist}\,(E,F)}{4tM_{q,\mu}}$, and using the bound proved in (5.10) for $\|\eta v\|_q$, we obtain

$$
\|\eta z\,dv\|_q \leq \big(1/2 + 2M_q(1 + M_{q,\mu})\big)\|u\|_q
$$

and conclude as before that $\{zd(\mathrm{I} + zD_\parallel)^{-1}, z \in \mathbb{C}\setminus S_\mu\}$ satisfies off-diagonal bounds with $C = 1/2 + 2M_q(1 + M_{q,\mu})$ and $c = 1/(4M_{q,\mu})$.

The proof of the off-diagonal bound for the other family in (5.6) follows the same lines.

(ii) The proofs of points (a) and (b) are similar and rely on the properties of the potentials described in Section 4. We present the proof of point (a), so suppose $B = d$ and $X_p = \mathsf{R}^p(d, \Omega)$.

For the family Q_k^t required in (A) of Theorem 9.1, proceed as follows.

- Suppose $0 < t \leq \operatorname{diam} \Omega$.

- Cover Ω: Let $\underline{Q}_k^t$ $(k \in J)$ be the cubes in $\mathbb{R}^n$ with side-length t and corners at points in $t\mathbb{Z}^n$, which intersect Ω. Let $Q_k^t = 4\underline{Q}_k^t \cap \Omega$. Then $\Omega = \cup Q_k^t$.

- There exist functions $\underline{\eta}_k \in \mathscr{C}_c^1(4\underline{Q}_k^t, [0,1])$ with $\|\nabla \underline{\eta}_k\|_\infty \leq 1/t$ and $\sum \underline{\eta}_k{}^2 = 1$ on Ω (see Remark below). Then $\eta_k := \underline{\eta}_k|_\Omega$ is a Lipschitz function on Ω with values in $[0,1]$, $\operatorname{sppt}(\eta_k) \subset Q_k^t$, $\|\nabla \eta_k\|_\infty \leq 1/t$ and $\sum \eta_k{}^2 = 1$ on Ω.

- $d(\eta_k f) - \eta_k df = (\nabla \eta_k) \wedge f$.

For $u \in \mathsf{R}^p(d, \Omega) \cap L^q(\Omega, \Lambda)$, $u = dR_\Omega u$ (where R_Ω is the potential map defined in Section 4) and we define

$$w_k = \eta_k R_\Omega(\eta_k u) \quad \text{and} \quad v_k = \eta_k R_\Omega(t\nabla\eta_k \wedge u) - t\nabla\eta_k \wedge R_\Omega(\eta_k u) + t\eta_k K_\Omega(\eta_k u),$$

where K_Ω is defined in Section 4. It is clear that $\operatorname{sppt} w_k, \operatorname{sppt} v_k \subset Q_k^t$. Thanks to the relations listed in Proposition 4.1, it is immediate that

$$\eta_k{}^2 u = \eta_k(dR_\Omega + R_\Omega d + K_\Omega)\eta_k u = dw_k + \frac{1}{t}v_k$$

and so

$$u = \sum_k \eta_k{}^2 u = \sum_k \left(dw_k + \tfrac{1}{t}v_k\right).$$

It remains to prove estimates on w_k and v_k. They come from the mapping properties of R_Ω and K_Ω. Denote by $r \in (1, \infty)$ the real number satisfying $1/q = 1/p^S + 1/r$. In other words, r satisfies $1/r = 1/n - (1/p - 1/q)$. We have that

$$\|w_k\|_q \lesssim \|\eta_k\|_r \|R_\Omega(\eta_k u)\|_{p^S} \lesssim |Q_k^t|^{1/r} \|\eta_k u\|_p \lesssim t^{1-n(1/p-1/q)} \|\mathbb{1}_{Q_k^t} u\|_p$$

and similarly

$$\|v_k\|_q \lesssim \|\eta_k\|_r\|R_\Omega(t\nabla\eta_k \wedge u)\|_{p^S} + \|t\nabla\eta_k\|_r\|R_\Omega(\eta_k u)\|_{p^S} + t\|\eta_k\|_q\|K_\Omega(\eta_k u)\|_\infty$$

$$\lesssim t^{1-n(1/p-1/q)}\left(1 + t^{n/p}\right)\|\mathbb{1}_{Q_k^t} u\|_p \leq t^{1-n(1/p-1/q)}\left(1 + \operatorname{diam}(\Omega)^{n/p}\right)\|\mathbb{1}_{Q_k^t} u\|_p,$$

and thus the condition (A) of Theorem 9.1 is satisfied.

(iii) This is now a consequence of Theorem 9.1. By density of $\mathsf{R}^q(d, \Omega)$ in $\mathsf{R}^p(d, \Omega)$ and of $\mathsf{R}^q(\underline{\delta}, \Omega)$ in $\mathsf{R}^p(\underline{\delta}, \Omega)$ (see Corollary 4.2 (vi)), the estimate in (iii) holds for all $u \in \mathsf{R}^p(d, \Omega)$ and for all $u \in \mathsf{R}^p(\underline{\delta}, \Omega)$.

(iv) It is a consequence of (iii) and the Hodge decomposition of $L^p(\Omega, \Lambda)$, that there exist constants $M_{p,\mu}$ such that

$$\left\|(\mathrm{I} + zD_\parallel)^{-1}u\right\|_p \leq M_{p,\mu}\|u\|_p, \quad \forall z \in \mathbb{C} \setminus S_\mu, \ \forall u \in L^q(\Omega, \Lambda).$$

By the density of $\mathsf{G}^q(D_\parallel)$ in $\mathsf{G}^p(D_\parallel)$ (Proposition 4.4) it then follows that the L^p operator $(\mathrm{I} + zD_\parallel)$ is invertible in $L^p(\Omega, \Lambda)$, with

$$\left\|(\mathrm{I} + zD_\parallel)^{-1}u\right\|_p \leq M_{p,\mu}\|u\|_p, \quad \forall z \in \mathbb{C} \setminus S_\mu, \ \forall u \in L^p(\Omega, \Lambda). \qquad \square$$

Remark 5.3. A partition of unity associated with a family of cubes of length 1 $\{C_k, k \in \mathbb{Z}^n\}$ usually has the form $\sum \chi_k = 1$ where $0 \leq \chi_k \leq 1$ and $\chi_k = 1$ on C_k. The functions $\chi_k \in \mathscr{C}_c^\infty(2C_k)$ are obtained with the help of (translation, dilation, product of) bump functions such as

$$b(x_j) = \begin{cases} 0 & \text{if } x_j \leq -2, \\ \left(1 + \exp(\frac{1}{x_j+2} + \frac{1}{x_j-1})\right)^{-1} & \text{if } -2 < x_j < -1, \\ 1 & \text{if } -1 \leq x_j \leq 1, \qquad 1 \leq j \leq n. \\ \left(1 + \exp(\frac{1}{2-x_j} - \frac{1}{x_j-1})\right)^{-1} & \text{if } 1 < x_j < 2, \\ 0 & \text{if } x_j \geq 2 \end{cases}$$

Using $\sqrt{b}$ instead of b in the construction of χ_k to obtain functions η_k, we obtain the desired form of the partition of unity $\sum \eta_k^2 = 1$.

Proposition 5.4. *Suppose that, in addition to the hypotheses of Proposition 5.2, $D_\parallel$ has a bounded holomorphic functional calculus in $L^q(\Omega, \Lambda)$ with $p_H < q < p^H$ and $\max\{1, q_S\} < p < q$.*

Then condition (B) of Theorem 9.2 (stated below)

> X_p *is a closed subspace of* $L^p(\Omega)$ *such that there is a Calderón–Zygmund type decomposition: for all* $\alpha > 0$ *and all* $u \in X_p$ *there exist functions* $g, w_k, v_k \in L^q(\Omega)$, $t_k > 0$ *and cubes* $Q_k = Q(x_k, t_k) \subset \mathbb{R}^n$ *of center* x_k *and sidelength* t_k *such that*
>
> $$\|g\|_p \lesssim \|u\|_p, \quad \|g\|_\infty \leq \alpha, \ \mathbb{1}_\Omega \leq \sum_k \mathbb{1}_{Q_k} \leq N \, \mathbb{1}_\Omega,$$
>
> $$\|\mathbb{1}_{Q_k \cap \Omega} u\|_p \lesssim \alpha |Q_k|^{1/p}, \ \sum_k |Q_k| \lesssim \frac{1}{\alpha^p} \|u\|_p^p,$$
>
> $$\mathrm{sppt}\, w_k, \mathrm{sppt}\, v_k \subset Q_k \cap \Omega,$$
>
> $$w_k \in \mathsf{D}_{L^p}(B), \quad \|w_k\|_q, \|v_k\|_q \lesssim t_k^{1-n(1/p-1/q)} \|\mathbb{1}_{Q_k \cap \Omega} u\|_p,$$
>
> *and* $\quad u = g + \sum_k \left(Bw_k + \frac{1}{t_k}v_k\right).$

holds in each of the following cases:

1. *the operator* $B = d$ *and the subspace* $X_p = \mathsf{R}^p(d, \Omega)$,
2. *the operator* $B = \underline{\delta}$ *and the subspace* $X_p = \mathsf{R}^p(\underline{\delta}, \Omega)$.

Consequently, for each $r \in (p, q)$ there exist constants $\kappa_{r,\mu}$ such that

$$(5.12) \quad \left\| f(D_{\parallel}) u \right\|_r \leq \kappa_{r,\mu} \| f \|_{\infty} \| u \|_r, \quad \forall z \in \mathbb{C} \backslash S_{\mu}, \forall u \in \mathsf{R}^r(d, \Omega) \ \text{and} \ \forall u \in \mathsf{R}^r(\underline{\delta}, \Omega)$$

for all $f \in \Psi(S_{\mu}^{\circ})$.

Proof. Our aim is to prove that the condition (B) of Theorem 9.2 is satisfied in case 1 (case 2 can be treated similarly). Let $\alpha > 0$, $u \in X_p \cap L^q(\Omega, \Lambda)$ and let

$$F := \left\{ x \in \mathbb{R}^n; \left(\mathcal{M}(|\tilde{u}|^p)(x) \right)^{1/p} \leq \alpha \right\}, \quad E_{\alpha} := \mathbb{R}^n \setminus F,$$

where $\mathcal{M}$ denotes the uncentered Hardy–Littlewood maximal operator on $\mathbb{R}^n$, i.e.,

$$\mathcal{M}(f)(x) := \sup_{Q \ni x} \fint_Q |f(y)| \, \mathrm{d}y, \quad x \in \mathbb{R}^n, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n),$$

where the sup is taken over all cubes $Q \subset \mathbb{R}^n$ containing x and $\tilde{u}$ denotes the extension by zero to $\mathbb{R}^n$ of u. Let $Q_k = Q(x_k, t_k)$, $k \in \mathbb{N}$ be the family of cubes relative to F given by Theorem 3 in [28], Chap. I, §3, and denote by $2^j Q_k$ the dilated cube $Q(x_k, 2^j t_k)$. Since $2Q_k \cap F \neq \emptyset$, we have that

$$\int_{Q_k \cap \Omega} |u|^p \, \mathrm{d}x = \int_{Q_k} |\tilde{u}|^p \, \mathrm{d}x \leq |2Q_k| \fint_{2Q_k} |\tilde{u}|^p \, \mathrm{d}x \lesssim \alpha^p \, |Q_k|.$$

Moreover, by the finite overlapping property of the family of Q_k's and the properties of the maximal operator (see, e.g., Chap. I, §1, Theorem 1 in [28]), we have that

$$\sum_k |Q_k| \lesssim \left| \bigcup_k Q_k \right| = |E_{\alpha}| = \left| \left\{ x \in \mathbb{R}^n; \mathcal{M}(|\tilde{u}|^p)(x) > \alpha^p \right\} \right|$$

$$\lesssim \frac{1}{\alpha^p} \, \| |\tilde{u}|^p \|_{L^1(\mathbb{R}^n)} = \frac{1}{\alpha^p} \, \| u \|_p^p.$$

Next, for each $k \in \mathbb{N}$, let $\eta_k \in \mathscr{C}_c^{\infty}(Q_k, [0, 1])$ be such that $\sum_k \eta_k^2 = 1\!\!1_{E_{\alpha}}$ and $\| \nabla \eta_k \|_{\infty} \lesssim 1/t_k$. We define g by $g := 1\!\!1_{\Omega \setminus E_{\alpha}} u$. It is clear that $\| g \|_p \leq \| u \|_p$ and by the Lebesgue differentiation theorem, we have that

$$|g(x)| \leq \alpha \quad \text{for almost all } x \in \Omega.$$

We define next, for the relevant $k \in \mathbb{N}$, i.e., those $k \in \mathbb{N}$ such that $Q_k \cap \Omega \neq \emptyset$ and $t_k \leq \operatorname{diam} \Omega$,

$$w_k := \eta_k R_{\Omega}(\eta_k u) \quad \text{and} \quad v_k := \eta_k R_{\Omega}(t_k \nabla \eta_k \wedge u) - t_k \nabla \eta_k \wedge R_{\Omega}(\eta_k u) - t_k \eta_k K_{\Omega}(\eta_k u).$$

Since η_k is smooth and $R_{\Omega}(\eta_k u) \in \mathsf{R}^p(d, \Omega)$, it follows that $w_k \in \mathsf{R}^p(d, \Omega)$. We have that $\eta_k^2 u = dw_k + \frac{1}{t_k} v_k$, and therefore

$$u = g + \sum_k \left(dw_k + \tfrac{1}{t_k} v_k \right).$$

Moreover, w_k and v_k, $k \in \mathbb{N}$, satisfy the estimate

$$\|w_k\|_q, \|v_k\|_q \lesssim t_k^{1-n(1/p-1/q)} \|\mathbb{1}_{Q_k \cap \Omega} u\|_p,$$

which proves that in our case, the conditions of (B) of Theorem 9.2 are satisfied. Therefore, applying the result of Theorem 9.2, we obtain the following weak L^p-estimate:

$$\|f(D_\parallel)u\|_{p,w} \le K_{p,\mu}\|f\|_\infty\|u\|_p, \quad \forall\, z \in \mathbb{C} \setminus S_\mu,$$
$$\forall\, u \in \mathsf{R}^q(d,\Omega) = \mathsf{R}^p(d,\Omega) \cap L^q(\Omega,\Lambda) \quad \text{and}$$
$$\forall\, u \in \mathsf{R}^q(\underline{\delta},\Omega) = \mathsf{R}^p(\underline{\delta},\Omega) \cap L^q(\Omega,\Lambda).$$

By interpolation between this last result and the fact that $D_\parallel$ has a bounded holomorphic functional calculus in $L^q(\Omega,\Lambda)$, and using the density of $\mathsf{R}^q(d,\Omega)$ in $\mathsf{R}^r(d,\Omega)$ and of $\mathsf{R}^q(\underline{\delta},\Omega)$ in $\mathsf{R}^r(\underline{\delta},\Omega)$ for all $p < r < q$ (see Corollary 4.2 (vi)), we obtain (5.12). $\qquad \square$

Proof of Theorem 5.1. We are now in position to prove our main theorem.

The assertion (iii) is proved by iteration: we start with $q = 2$ and apply Proposition 5.2 (iii) and Proposition 5.4 to obtain (iii) for all $r \in \big(\max\{1, 2_S\}, 2\big]$. We iterate the procedure a times where a is the smallest integer defined by $\frac{2n}{n+2a} < (p_H)_S$ (we can take $a = 1 + E(n/2)$, were $E(s)$ denotes the integer part of a real s) and we obtain (iii) for all $r \in \big(\max\{1, p_{HS}\}, 2\big]$. The range $[2, p^H)$ is obtained by taking adjoints in the interval $(p_H, 2]$.

(iii) $\implies$ (i): For p in the range where (H_p) holds, it is immediate that for all $u \in L^p(\Omega,\Lambda)$, and all $z \in \mathbb{C} \setminus S_\theta$, $\theta \in (0, \pi/2)$,

$$(\mathrm{I} + zD_\parallel)^{-1}u = (\mathrm{I} + zD_\parallel)^{-1}\big(\mathcal{P}_{\mathsf{R}^p(d)}u\big) + (\mathrm{I} + zD_\parallel)^{-1}\big(\mathcal{P}_{\mathsf{R}^p(\underline{\delta})}u\big) + \mathcal{P}_{\mathsf{N}^p(D_\parallel)}u,$$

and therefore, by (5.2),

$$\|(\mathrm{I} + zD_\parallel)^{-1}u\|_p \le C_{p,\theta}\big(\|\mathcal{P}_{\mathsf{R}^p(d)}u\|_p + \|\mathcal{P}_{\mathsf{R}^p(\underline{\delta})}u\|_p\big) + \|\mathcal{P}_{\mathsf{N}^p(D_\parallel)}u\|_p \le (C_{p,\theta}+1)\|u\|_p.$$

Similarly, for all $f \in \Psi(S_\mu^\circ)$, $\mu \in (0, \pi/2)$,

$$f(D_\parallel)u = f(D_\parallel)\big(\mathcal{P}_{\mathsf{R}^p(d)}u\big) + f(D_\parallel)\big(\mathcal{P}_{\mathsf{R}^p(\underline{\delta})}u\big),$$

which gives the estimate

$$\|f(D_\parallel)u\|_p \le K_{p,\mu}\big(\|\mathcal{P}_{\mathsf{R}^p(d)}u\|_p + \|\mathcal{P}_{\mathsf{R}^p(\underline{\delta})}u\|_p\big) \le K_{p,\mu}\|u\|_p$$

thanks to (5.3).

(ii): Assume that p is such that $D_\parallel$ admits a bounded S_μ° holomorphic functional calculus in $L^p(\Omega,\Lambda)$. The fact that $D_\parallel$ is bisectorial in $L^p(\Omega,\Lambda)$ implies that

$$L^p(\Omega,\Lambda) = \overline{\mathsf{R}^p(D_\parallel)} \oplus \mathsf{N}^p(D_\parallel),$$

the projections on each subspace being bounded. See, e.g., Theorem 3.8 in [13]. Then the restriction of $D_\parallel$ in $Y_p := \overline{\mathsf{R}^p(D_\parallel)}$ with domain $\mathsf{D}^p(D_\parallel) \cap Y_p$ is densely defined, one-to-one and admits a bounded S_μ° holomorphic functional calculus in Y_p. Following the idea of [4], §5.3, let sgn be the (bounded) holomorphic function in S_μ° defined by $\mathrm{sgn}(z) = z/\sqrt{z^2}$ where $\sqrt{\cdot}$ is the holomorphic continuation of $(0, +\infty) \ni x \mapsto \sqrt{x}$ to $\mathbb{C} \setminus (-\infty, 0]$. Then we have that $\mathrm{sgn}^2(D_\parallel)u = \mathrm{sgn}(\mathrm{sgn}(D_\parallel)u) = u$ for all $u \in Y_p$. Now, (H_p) is a consequence of $\|D_\parallel u\|_p \approx \|du\|_p + \|\underline{\delta} u\|_p \approx \big\|\sqrt{D_\parallel^2}\, u\big\|_p$. Indeed, assuming these equivalences hold, for all $u \in Y_p$,

$$u = dv + \underline{\delta}w, \quad \text{where} \quad v = \frac{\underline{\delta}}{D_\parallel^2}\, u \quad \text{and} \quad w = \frac{d}{D_\parallel^2}\, u,$$

$$v \in \mathsf{D}^p(d, \Omega),\ \|dv\|_p \lesssim \|u\|_p \quad \text{and} \quad w \in \mathsf{D}^p(\underline{\delta}, \Omega),\ \|\underline{\delta}w\|_p \lesssim \|u\|_p.$$

The equivalence $\|D_\parallel u\|_p \approx \big\|\sqrt{D_\parallel^2}\, u\big\|_p$ comes from the boundedness of the holomorphic functional calculus for $D_\parallel$ in Y_p. To prove $\|D_\parallel u\|_p \approx \|du\|_p + \|\underline{\delta}u\|_p$, it is sufficient to show that $\|du\|_p \lesssim \|D_\parallel u\|_p$ for all $u \in \mathsf{D}^p(D_\parallel)$. Write $u = \sum_{k=0}^n u^k$ where $u^k \in L^p(\Omega, \Lambda^k)$. Then

$$\|du\|_p \approx \sum_{k=0}^n \|(du)^k\|_p = \sum_{\ell=0}^n \|d(u^\ell)\|_p \le \sum_{\ell=0}^n \|D_\parallel(u^\ell)\|_p \approx \sum_{\ell=0}^n \big\|\sqrt{D_\parallel^2}(u^\ell)\big\|_p$$

$$= \sum_{\ell=0}^n \big\|(\sqrt{D_\parallel^2}\, u)^\ell\big\|_p \approx \big\|\sqrt{D_\parallel^2}\, u\big\|_p \approx \|D_\parallel u\|_p.$$

The bound $\sum_{\ell=0}^n \|d(u^\ell)\|_p \le \sum_{\ell=0}^n \|D_\parallel(u^\ell)\|_p$ holds because $d(u^\ell) \in L^p(\Omega, \Lambda^{\ell+1})$ and $\underline{\delta}(u^\ell) \in L^p(\Omega, \Lambda^{\ell-1})$. This proves then that (H_p) holds if p is as in (ii). $\square$

6. Perturbed Hodge–Dirac operators on strongly Lipschitz domains

Let $\Omega \subset \mathbb{R}^n$ be a bounded strongly Lipschitz domain. Let $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ a biLipschitz map as in Proposition 2.14 for which $\Omega' = \phi^{-1}(\Omega)$ is a smooth domain. The following result is the perturbed version of Theorem 5.1 in the case of bounded strongly Lipschitz domains.

Theorem 6.1. *Let $B \in L^\infty(\Omega, \mathscr{L}(\Lambda))$ such that $\Re e B \ge \kappa I$ ($\kappa > 0$) and $B(x)$ is invertible for almost all $x \in \Omega$. We assume moreover that $B^{-1} \colon \Omega \to \mathscr{L}(\Lambda)$ defined by $B^{-1}(x) := (B(x))^{-1}$ belongs to $L^\infty(\Omega, \mathscr{L}(\Lambda))$. Let $D_{\parallel,B}$ be the (unbounded) operator defined on $L^2(\Omega, \Lambda)$ by*

$$D_{\parallel,B} = d + \underline{\delta}_B = d + B^{-1}\underline{\delta}B \quad \mathsf{D}(D_{\parallel,B}) = \mathsf{D}(d) \cap \mathsf{D}(\underline{\delta}B).$$

Then there exist $\omega_B \in [0, \pi/2)$ and $\varepsilon_B, \tilde{\varepsilon}_B > 0$ such that for all $\theta \in (\omega_B, \pi/2)$ and

all $p \in \big(\max\{1, (2 - \tilde{\varepsilon}_B)_S\}, 2 + \varepsilon_B\big)$, *there exists* $C_{p,\theta} > 0$ *such that*

$$(6.1) \qquad (I + zD_{\|,B})^{-1} : \begin{cases} \mathsf{R}^p(d, \Omega) \\ \mathsf{R}^p(\underline{\delta}_B, \Omega) \end{cases} \longrightarrow L^p(\Omega, \Lambda), \quad \forall z \in \mathbb{C} \setminus S_\theta,$$

with the estimates

$$(6.2) \quad \sup_{z \in \mathbb{C} \setminus S_\theta} \|(I + zD_{\|,B})^{-1} u\|_p \leq C_{p,\theta} \|u\|_p, \quad \forall u \in \mathsf{R}^p(d, \Omega) \ and \ \forall u \in \mathsf{R}^p(\underline{\delta}_B, \Omega).$$

For all $\mu \in (0, \pi/2)$, *there exists a constant* $K_{p,\mu}$ *such that for all* $f \in \Psi(S_\mu^\circ)$,

$$(6.3) \qquad f(D_{\|,B}) : \begin{cases} \mathsf{R}^p(d, \Omega) \\ \mathsf{R}^p(\underline{\delta}_B, \Omega) \end{cases} \longrightarrow L^p(\Omega, \Lambda),$$

with the estimates

$$(6.4) \ \|f(D_{\|,B}) u\|_p \leq K_{p,\mu} \|f\|_{L^\infty(S_\mu^\circ)} \|u\|_p, \quad \forall u \in \mathsf{R}^p(d, \Omega) \ \ and \ \ \forall u \in \mathsf{R}^p(\underline{\delta}_B, \Omega).$$

Proof. The proof follows the lines of the proof of Theorem 5.1. To prove that $D_{\|,B}$ admits a bounded holomorphic functional calculus in $L^2(\Omega, \Lambda)$, we use the characterization of Theorem 2 in [7] after transformation of the problem in the smooth domain Ω' from Proposition 2.14: $\Omega = \phi(\Omega')$ where $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ is a biLipschitz map. The triplet $(d, \tilde{B}^{-1}, \tilde{B})$ with $\tilde{B} = (\tilde{\phi}_*)^{-1} B(\phi^*)^{-1}$ satisfies the conditions $(H1) - (H8)$ of [7] (the condition $(H8)$ is satisfied thanks to the embedding of $\mathsf{D}(d + \underline{\delta}, \Omega')$ into $H^1(\Omega', \Lambda)$ since Ω' is smooth: see Remark 3.4). We conclude then that the operator $d + \underline{\delta}_{\tilde{B}}$ admits a bounded holomorphic functional calculus in $L^2(\Omega', \Lambda)$. Therefore, $d + \underline{\delta}_B$ admits a bounded holomorphic functional calculus in $L^2(\Omega, \Lambda)$. By the same arguments as in the proof of Theorem 5.1 (ii), we see that the Hodge decomposition

$$L^p(\Omega, \Lambda) = \mathsf{R}^p(d, \Omega) \oplus \mathsf{R}^p(\underline{\delta}_B, \Omega) \oplus \mathsf{N}^p(d + \underline{\delta}_B, \Omega)$$

holds for $p = 2$. Let $\tilde{\varepsilon}_B, \varepsilon_B > 0$ such that the above Hodge decomposition holds for all $p \in \big(2 - \tilde{\varepsilon}_B, 2 + \varepsilon_B\big)$. Next, instead of potentials R_Ω and S_Ω, we use R_Ω and $B^{-1} S_\Omega B$ which have the same mapping properties as R_Ω and S_Ω listed in Proposition 4.1. This gives the result in the range $\big(\max\{1, (2 - \tilde{\varepsilon}_B)_S\}, 2\big]$. To obtain the range $[2, 2 + \varepsilon_B)$, we proceed by duality, using $\underline{\delta} + B^* d(B^*)^{-1}$ the adjoint of $D_{\|,B}$ and the potential maps S_Ω and $B^* R_\Omega(B^*)^{-1}$ instead of R_Ω and S_Ω. □

7. Estimates of the Hodge exponents on strongly Lipschitz domains

In this section, we focus on the case of bounded strongly Lipschitz domains. We start with a result which gives good integrability properties of solutions of $D_\| u = f$ on Ω when $\Omega \subset \mathbb{R}^n$ is a bounded strongly Lipschitz domain. We recall that,

according to Theorem 11.2 in [22], there exists $0 < \varepsilon' \leq 1$ depending on the geometry of Ω such that for all $r \in (2 - \varepsilon', 2 + \varepsilon')$, there is a constant $C > 0$ with

$$(7.1) \qquad \|u\|_{B^{r,r^{\sharp}}_{1/r}(\Omega,\Lambda)} \leq C \left(\|u\|_r + \|du\|_r + \|\delta u\|_r + \|\nu \lrcorner u\|_{L^r(\partial\Omega,\Lambda)} \right),$$

where $r^{\sharp} := \max\{2, r\}$. This estimate is also true if we replace $\|\nu \lrcorner u\|_{L^r(\partial\Omega,\Lambda)}$ in (7.1) by $\|\nu \wedge u\|_{L^r(\partial\Omega,\Lambda)}$. Applying Corollary 2 in [25], p. 36, we can show that the embedding

$$(7.2) \qquad\qquad\qquad B^{r,r^{\sharp}}_{1/r} \hookrightarrow L^{r^*}$$

holds as long as $r^{\sharp} \leq r^*$. In particular (7.2) is true for all $r \geq 2(n-1)/n = 2 - 2/n$. Combining (7.1) and (7.2), we obtain

$$(7.3) \qquad \|u\|_{r^*} \leq C \left(\|u\|_r + \|du\|_r + \|\delta u\|_r + \|\nu \lrcorner u\|_{L^r(\partial\Omega,\Lambda)} \right)$$

and

$$(7.4) \qquad \|u\|_{r^*} \leq C \left(\|u\|_r + \|du\|_r + \|\delta u\|_r + \|\nu \wedge u\|_{L^r(\partial\Omega,\Lambda)} \right)$$

for all $r \in \left(2 - \min\{\varepsilon', 2/n\}, 2 + \varepsilon' \right)$. By Theorem 4.3, we know that there exits $\varepsilon > 0$ such that the Hodge decompositions (4.8) and (4.9) hold for all $p \in \left((2+\varepsilon)', 2+\varepsilon \right)$. Let $\alpha := \min\{\varepsilon, \varepsilon', 2/(n-2)\} > 0$. Remark that this particular choice of α ensures that (7.3), (7.4), (4.8) and (4.9) hold in the interval $\left((2+\alpha)', 2+\alpha \right)$.

We have the following result.

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded strongly Lipschitz domain. Then we can estimate the Hodge exponents associated to the Hodge decompositions (4.8) and (4.9) as follows:*

$$p_H \leq \left((2+\alpha)^* \right)' = \frac{(2+\alpha)n}{n(1+\alpha)+1} < \frac{2n}{n+1} = (2^*)'$$
$$< \frac{2n}{n-1} = 2^* < \frac{(2+\alpha)n}{n-1} = (2+\alpha)^* \leq p^H,$$

where $\alpha > 0$ was defined just above: (7.3), (7.4), (4.8) and (4.9) hold in the interval $\left((2+\alpha)', 2+\alpha \right)$. In particular, in dimension $n = 2$ we have that $p_H < 4/3 < 4 < p^H$, and in dimension $n = 3$ we have that $p_H < 3/2 < 3 < p^H$.

Before proving this theorem, we first give some properties of the null space of the operator $D_{\|}$ or $D_{\perp}$.

Lemma 7.2. *Let $r \in \left((2+\alpha)', 2+\alpha \right)$. Let $\mathsf{N}^r(D)$ be the null space of $D = D_{\|}$ or $D_{\perp}$ endowed with the L^r-norm. Then the projection $P \colon L^r(\Omega,\Lambda) \to \mathsf{N}^r(D)$ maps $L^r(\Omega,\Lambda)$ to $L^{r^*}(\Omega,\Lambda)$. Moreover, $\mathsf{N}^r(D) = \mathsf{N}^{r^*}(D)$ with equivalent norms and the projection P extends to a bounded operator from $L^p(\Omega,\Lambda)$ to $L^{p^S}(\Omega,\Lambda) \cap \mathsf{N}^p(D)$ for all $p \in J_\alpha$ where J_α denotes the open interval around $(2^*)' = \frac{2n}{n+1}$ given by $\left(((2+\alpha)^*)', (((2+\alpha)')^*)' \right)$.*

Proof. Let $r \in \big((2+\alpha)', 2+\alpha\big)$. The projection $P \colon L^r(\Omega, \Lambda) \to \mathsf{N}^r(D)$ coming from the Hodge decomposition (4.8) (or (4.9)) satisfies, thanks to (7.3) (or (7.4)),

$$\|Pu\|_{r^*} \le C \|Pu\|_r \le C' \|u\|_r$$

since for $v \in \mathsf{N}^r(D)$, we have that $dv = 0$, $\delta v = 0$ in Ω and $\nu \lrcorner\, v = 0$ (or $\nu \wedge v = 0$) on $\partial\Omega$. This proves that P maps $L^r(\Omega, \Lambda)$ to $L^{r^*}(\Omega, \Lambda)$.

It is clear that $\mathsf{N}^{r^*}(D) \hookrightarrow \mathsf{N}^r(D)$ since we assumed that Ω was bounded. Conversely, let $v \in \mathsf{N}^r(D)$. Then we have that $dv = 0$, $\delta v = 0$ in Ω and $\nu \lrcorner\, v = 0$ (or $\nu \wedge v = 0$) on $\partial\Omega$, and thanks to (7.3) (or (7.4)), $v \in L^{r^*}(\Omega, \Lambda)$ and $\|v\|_{r^*} \lesssim \|v\|_r$, which proves that $\mathsf{N}^r(D) \hookrightarrow \mathsf{N}^{r^*}(D)$ and therefore $\mathsf{N}^{r^*}(D) = \mathsf{N}^r(D)$ with equivalent norms.

Let now $p \in J_\alpha$. We want to prove that P maps $L^p(\Omega, \Lambda)$ to $L^{p^S}(\Omega, \Lambda)$. Since P maps $L^r(\Omega, \Lambda)$ to $L^{r^*}(\Omega, \Lambda)$ for all $r \in \big((2+\alpha)', 2+\alpha\big)$, its adjoint maps $L^{\frac{nq}{n+q-1}}(\Omega, \Lambda)$ to $L^q(\Omega, \Lambda)$ for all $q \in \big((2+\alpha)', 2+\alpha\big)$. We know moreover that P is a projection, so that $P = P' = P^2$. Therefore, we obtain by composition that P maps $L^{\frac{nq}{n+q-1}}(\Omega, \Lambda)$ to $L^{\frac{nq}{n-1}}(\Omega, \Lambda)$ for all $q \in \big((2+\alpha)', 2+\alpha\big)$. If we let $p = \frac{nq}{n+q-1}$, it is easy to check that $p^S = \frac{nq}{n-1}$ and the result is proved. $\qquad\square$

To prove Theorem 7.1, we need the following lemma which gives a partial right inverse of $D_\parallel$ (or $D_\perp$) in $L^p(\Omega, \Lambda)$.

Lemma 7.3. *Let $p \in J_\alpha$ (J_α was defined in Lemma 7.2). Then any $u \in L^p(\Omega, \Lambda)$ can be decomposed as*

$$u = D_\parallel Tu + Ku = D_\perp Su + Lu, \tag{7.5}$$

where

$$T, S : L^p(\Omega, \Lambda) \to L^{p^S}(\Omega, \Lambda) \cap \begin{cases} \mathsf{D}^p(D_\parallel) \\ \mathsf{D}^p(D_\perp) \end{cases} \tag{7.6}$$

and

$$K, L : L^p(\Omega, \Lambda) \to L^{p^S}(\Omega, \Lambda) \cap \begin{cases} \mathsf{N}^p(D_\parallel) \\ \mathsf{N}^p(D_\perp) \end{cases} \tag{7.7}$$

are bounded linear operators.

Proof. Let $D := D_\parallel$ or $D_\perp$. Let $r \in \big((2+\alpha)', 2+\alpha\big)$. We denote by $\mathsf{D}^r(D)$ the domain and $\mathsf{R}^r(D)$ the range of D, both endowed with the L^r-norm. We have that

$$\mathsf{D}^r(D) = \begin{cases} \mathsf{D}^r(d) \cap \mathsf{D}^r(\underline\delta) & \text{if } D = D_\parallel, \\ \mathsf{D}^r(\delta) \cap \mathsf{D}^r(\underline d) & \text{if } D = D_\perp, \end{cases}$$

and since the Hodge decompositions (4.8)–(4.9) hold in $L^r(\Omega, \Lambda)$, the projection onto the null space of D, $P \colon L^r(\Omega, \Lambda) \to \mathsf{N}^r(D)$, is bounded and the operator $D(\mathrm{I} - P) \colon \mathsf{D}^r(D) \to \mathsf{R}^r(D)$ is invertible (one-to-one and onto); we denote by

$\widetilde{T} \colon \mathsf{R}^r(D) \to \mathsf{D}^r(D)$ its inverse. Let $p := (r^*)'$: p belongs to J_α, and $p^S = r^*$ by (2.1). From now on, we assume that $D = D_\|$ (the case $D = D_\perp$ can be treated similarly). We define $T := (\mathrm{I} - P)\widetilde{T}(\mathrm{I} - P)$ and $K := P$. It is clear that T maps $L^r(\Omega, \Lambda)$ to itself and that, thanks to (7.3),

$$\|Tu\|_{r^*} \leq C \left(\|Tu\|_r + \|dTu\|_r + \|\delta Tu\|_r \right) \leq C \|u\|_r, \quad \forall\, u \in L^r(\Omega, \Lambda),$$

which proves also, by duality (T is self-adjoint in $L^2(\Omega, \Lambda)$),

(7.8) $$\|Tu\|_r \leq C \|u\|_{\frac{nr}{n+r-1}}, \quad \forall\, r \in \big((2+\alpha)', 2+\alpha \big).$$

It remains to prove that these operators T and K satisfy (7.5) and the mapping properties (7.6) and (7.7). The fact that $K = P$ satisfies (7.7) is a direct consequence of Lemma 7.2. Next, let $u \in L^r(\Omega, \Lambda)$, $(2+\alpha)' < r < 2+\alpha$. Since $D_\| P = 0$ and $D_\| \widetilde{T} v = v$ for all $v \in \mathsf{R}^r(D_\|)$, we have that

$$D_\| Tu = D_\|(\mathrm{I} - P)\widetilde{T}(\mathrm{I} - P)u = (\mathrm{I} - P)u = u - Ku,$$

which proves (7.5) for $u \in L^r(\Omega, \Lambda)$. The last step in this proof is to show that T maps $L^p(\Omega, \Lambda)$ to $L^{p^S}(\Omega, \Lambda) \cap \mathsf{D}^p(D_\|)$ for all $p \in J_\alpha$. Let $u \in L^2(\Omega, \Lambda) \cap L^p(\Omega, \Lambda)$ and denote by $w \in \dot{W}^{1,p}(\mathbb{R}^n, \Lambda)$ the solution of

$$(d + \delta)w = \begin{cases} (\mathrm{I} - P)u & \text{in } \Omega \\ 0 & \text{outside } \Omega \end{cases} \quad \in L^p(\mathbb{R}^n, \Lambda).$$

We have that $\|w_{|\Omega}\|_{p^S} + \|w_{|\partial\Omega}\|_{L^{\frac{(n-1)p}{n-p}}(\partial\Omega, \Lambda)} \leq C \|u\|_p$. Let now $v := Tu - w_{|\Omega}$. The function v satisfies

$$\begin{cases} (d + \delta)v = 0 & \text{in } \Omega, \\ \nu \lrcorner\, v = -\nu \lrcorner\, w \in B^{p,p}_{1-1/p}(\partial\Omega, \Lambda) \hookrightarrow L^{\frac{(n-1)p}{n-p}}(\partial\Omega, \Lambda). \end{cases}$$

Let $q = (n-1)p/(n-p)$, so that $q^* = np/(n-p) = p^S$; in particular, $q \in ((2+\alpha)', 2+\alpha)$. By (7.3), since $dv + \delta v = 0$, we have that

$$\|v\|_{q^*} \leq C \left(\|v\|_q + \|\nu \lrcorner\, v\|_{L^q(\partial\Omega, \Lambda)} \right)$$

and therefore, using (7.8) and the fact that $nq/(n+q-1) = p$,

$$\|Tu\|_{p^S} \lesssim \left(\|Tu\|_q + \|u\|_p \right) \lesssim \|u\|_p,$$

which ends the proof.   □

Corollary 7.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded strongly Lipschitz domain. Then the operators $D_\|$ and $D_\perp$ admit a bounded holomorphic functional calculus on $L^p(\Omega, \Lambda)$ for all p in the interval $\big(((2+\alpha)^*)', (2+\alpha)^* \big)$.*

Proof. The proof follows the lines of the proof of Theorem 5.1. Conditions (A) of Theorem 9.1 and (B) of Theorem 9.2 hold for $X_p = L^p(\Omega, \Lambda)$ and $A = B = D_\|$ (or $D_\perp$), using the potentials (T, K) (or (S, L)) defined in Lemma 7.3.   □

Proof of Theorem 7.1. It is an immediate consequence of Corollary 7.4 and (ii) of Theorem 5.1.   □

8. Hodge-Laplacian and Hodge–Stokes operators

Direct applications of the results in Section 5 are the following properties of the Hodge-Laplacian $-\Delta_{\parallel} = D_{\parallel}{}^2$ and the Hodge–Stokes operator $S_{\parallel}$ defined as the part of $-\Delta_{\parallel}$ in $\mathsf{N}^2(\underline{\delta})$ extended as sectorial operators in $L^p(\Omega, \Lambda)$ and in $\mathsf{N}^p(\underline{\delta})$.

Corollary 8.1. *Suppose Ω is a very weakly Lipschitz domain in $\mathbb{R}^n$. Define $-\Delta_{\parallel} = D_{\parallel}{}^2$ in $L^2(\Omega, \Lambda)$. If $p_H < p < p^H$, then $-\Delta_{\parallel}$ is sectorial of angle 0 in $L^p(\Omega, \Lambda)$ and for all $\mu \in (0, \pi/2)$, $-\Delta_{\parallel}$ admits a bounded $S^{\circ}_{\mu+}$ holomorphic functional calculus in $L^p(\Omega, \Lambda)$.*

Let us mention that the first part of this corollary (sectoriality of $-\Delta_{\parallel}$) has been proved in [24] in the case of a bounded strongly Lipschitz domain.

Corollary 8.2. *Suppose Ω is a very weakly Lipschitz domain in $\mathbb{R}^n$. Define $S_{\parallel} := D_{\parallel}{}^2$ in $\mathsf{R}^2(\underline{\delta}, \Omega)$. If $\max\{1, (p_H)_S\} < p < p^H$, then $S_{\parallel}$ is sectorial of angle 0 in $\mathsf{R}^p(\underline{\delta})$ and for all $\mu \in (0, \pi/2)$, $S_{\parallel}$ admits a bounded $S^{\circ}_{\mu+}$ holomorphic functional calculus in $\mathsf{R}^p(\underline{\delta})$.*

9. General L^p extrapolation results

In Section 5, we used the following extrapolation results, but are presenting them separately, as they are general results which could be useful in other contexts. In them, $L^p(\Omega) := L^p(\Omega, \mathbb{C}^N)$, where Ω is an open subset of $\mathbb{R}^n$, and N is a positive integer.

Theorem 9.1. *Let $q \in [1, \infty)$, $\max\{1, q_S\} \le p < q$ and $0 \le \omega < \mu < \pi/2$. Let A be a bisectorial operator of angle ω in L^q such that the family of the resolvents $\{(\mathrm{I} + zA)^{-1}, z \in \mathbb{C} \setminus S_{\mu}\}$ has L^q-L^q off-diagonal bounds. Assume that B is an unbounded operator in L^q such that $\{(\mathrm{I} + zA)^{-1}zB, z \in \mathbb{C} \setminus S_{\mu}\}$ has L^q-L^q off-diagonal bounds.*

(A) *Assume that X_p is a closed subspace of $L^p(\Omega)$ such that for all $u \in X_p$, there exist $w, v \in L^q(\Omega)$ with $w \in \mathsf{D}^p(B)$, $\|w\|_q, \|v\|_q \lesssim \|u\|_p$ and $u = Bw + v$.*

Moreover, assume that for each $t \in (0, \operatorname{diam}\Omega]$ there exists a family $\{Q_k^t, k \in \mathbb{Z}^n\}$ of open subsets of Ω with the property that

$$|Q_k^t| \lesssim t^n, \quad \mathbb{1}_{\Omega} \le \sum_k \mathbb{1}_{Q_k^t} \le N\,\mathbb{1}_{\Omega},$$

$$\sup_j \sum_k e^{-\varepsilon \operatorname{dist}(Q_k^t, Q_j^t)/t} = \sup_k \sum_j e^{-\varepsilon \operatorname{dist}(Q_k^t, Q_j^t)/t} \le C_{\varepsilon}$$

for all $\varepsilon > 0$, where C_{ε} does not depend on t, and for all $u \in X_p$, there exist $w_k, v_k \in L^q(\Omega)$ such that $w_k \in \mathsf{D}^p(B)$ for all k, and w_k, v_k satisfy

$$\operatorname{sppt}_{\Omega} w_k, \operatorname{sppt}_{\Omega} v_k \subset Q_k^t, \quad \|w_k\|_q, \|v_k\|_q \lesssim t^{1-n(1/p-1/q)} \|\mathbb{1}_{Q_k^t} u\|_p,$$

$$u = \sum_k \left(Bw_k + \tfrac{1}{t}v_k\right).$$

Then there exists a constant $M_{p,\mu}$ such that

$$\left\|(I + zA)^{-1}u\right\|_p \le M_{p,\mu} \|u\|_p, \quad \forall\, z \in \mathbb{C} \setminus S_\mu, \ \forall\, u \in X_p \cap L^q(\Omega) \ .$$

Proof. For $z \in \mathbb{C} \setminus S_\mu$, let $t = \min\{|z|, \operatorname{diam}\Omega\} \in (0, \operatorname{diam}\Omega]$, $u \in X_p$.

If $t = \operatorname{diam}\Omega$, then let w and v be as in the first part of Assumption (A): $u = Bw + v$, and therefore

$$(I + zA)^{-1}u = \frac{1}{z}(I + zA)^{-1}zBw + (I + zA)^{-1}v,$$

so that, thanks to the boundedness of $(I + zA)^{-1}$ and $(I + zA)^{-1}zB$ in $L^q(\Omega)$,

$$\|(I + zA)^{-1}u\|_p \lesssim (\operatorname{diam}\Omega)^{n(1/p-1/q)} \left(\frac{1}{\operatorname{diam}\Omega} \|w\|_q + \|v\|_q\right) \lesssim \|u\|_p.$$

If $t < \operatorname{diam}\Omega$, then let Q_k^t, w_k, v_k as in the statement of the theorem. Then, using the L^q-L^q off diagonal bounds for $(I + zA)^{-1}$ and $(I + zA)^{-1}zB$ we have that for all $u \in X_p \cap L^q$,

$$\|(I + zA)^{-1}u\|_p \le \left(\sum_j \int_{Q_j^t} |(I + zA)^{-1}u|^p\right)^{1/p}$$

$$\lesssim \left[\sum_j \left(\|(I + zA)^{-1}u\|_{L^q(Q_j^t)} |Q_j^t|^{1/p-1/q}\right)^p\right]^{1/p}$$

(by Hölder's inequality)

$$\lesssim \left[\sum_j \left(\sum_k \|(I + zA)^{-1}(tBw_k + v_k)\|_{L^q(Q_j^t)} \, t^{-1+n(\frac{1}{p}-\frac{1}{q})}\right)^p\right]^{1/p}$$

(since $u = \sum_k (Bw_k + \frac{1}{t}v_k)$ and $|Q_j^t| \lesssim t^n$)

$$\lesssim \left[\sum_j \left(\sum_k e^{-c\operatorname{dist}(Q_j^t, Q_k^t)/|z|}\left(\frac{t}{|z|} \|w_k\|_q + \|v_k\|_q\right) t^{-1+n(1/p-1/q)}\right)^p\right]^{1/p}$$

(by off-diagonals bounds)

$$\lesssim C_c \left[\sum_k \left(\left(\frac{t}{|z|} \|w_k\|_q + \|v_k\|_q\right) t^{-1+n(1/p-1/q)}\right)^p\right]^{1/p}$$

(by Schur's lemma and the fact that $t/|z| \le 1$)

$$\lesssim C_c \left[\sum_k \|\mathbb{1}_{Q_k^t} u\|_p^p\right]^{1/p}$$

(by the L^q bounds for w_k and v_k and since $t \le |z|$)

$$\lesssim \|u\|_p,$$

where we have used, in the last estimate, the finite overlapping property of the cubes Q_k^t.   □

Theorem 9.2. *Suppose that all the hypotheses of Theorem* 9.1 *hold, but with* (A) *replaced by*:

(B) *Assume that X_p is a closed subspace of $L^p(\Omega)$ such that there is a Calderón–Zygmund type decomposition: for all $\alpha > 0$ and all $u \in X_p$ there exist functions $g, w_k, v_k \in L^q(\Omega)$, $t_k > 0$ and cubes $Q_k = Q(x_k, t_k) \subset \mathbb{R}^n$ of center x_k and sidelength t_k such that*

$$\|g\|_p \lesssim \|u\|_p, \quad \|g\|_\infty \leq \alpha,$$

$$\mathbb{1}_\Omega \leq \sum_k \mathbb{1}_{Q_k} \leq N\, \mathbb{1}_\Omega, \ \|\mathbb{1}_{Q_k \cap \Omega} u\|_p \lesssim \alpha |Q_k|^{1/p}, \ \sum_k |Q_k| \lesssim \frac{1}{\alpha^p} \|u\|_p^p,$$

$$\text{sppt}\, w_k, \text{sppt}\, v_k \subset Q_k \cap \Omega, \ w_k \in \mathsf{D}_{L^p}(B),$$

$$\|w_k\|_q, \|v_k\|_q \lesssim t_k^{1-n(1/p-1/q)} \|\mathbb{1}_{Q_k \cap \Omega} u\|_p, \quad and \quad u = g + \sum_k \left(Bw_k + \tfrac{1}{t_k} v_k\right).$$

If A admits a bounded S_μ° holomorphic functional calculus in $L^q(\Omega)$, then $f(A)$ is bounded from $X_p \cap L^q(\Omega)$ to the weak L^p space $L_w^p(\Omega)$ defined as follows:

$$L_w^p(\Omega) := \Big\{ u : \Omega \to \Lambda \text{ measurable } ;$$

$$\|u\|_{p,w} := \left(\sup_{\alpha > 0} \alpha^p \big| \big\{ x \in \Omega; |u(x)| > \alpha \big\} \big| \right)^{1/p} < \infty \Big\}$$

i.e., for each $\theta \in (\omega, \mu)$ there exists $K_{p,\theta}$ such that

$$\|f(A)u\|_{p,w} \leq K_{p,\theta} \|f\|_\infty \|u\|_p \quad \forall u \in X_p \cap L^q(\Omega), \ \forall f \in \Psi(S_\mu^\circ).$$

Proof. The idea of the proof presented below is inspired by the techniques developed in [16]. The starting point is a Calderón–Zygmund like decomposition as (B) in the statement.

It suffices to prove the result when $\|f\|_\infty = 1$. So assume henceforth that $\|f\|_\infty = 1$.

We proceed in several steps. Let $f \in \Psi(S_\mu^\circ)$. Let $\alpha > 0$, $u \in X_p$, and write

$$u = g + \sum_k \left(Bw_k + \tfrac{1}{t_k} v_k\right)$$

as in the statement of the theorem.

Step 1. *The part involving g.*

We have that $g \in L^q(\Omega)$ with the estimate

$$\|g\|_q \leq \|g\|_\infty^{1-p/q} \|g\|_p^{p/q} \lesssim \alpha^{1-p/q} \|u\|_p^{p/q}.$$

Using the boundedness of $f(A)$ on $L^q(\Omega)$, we have

$$\alpha^p \big| \big\{ x \in \Omega : |f(A)g(x)| > \alpha \big\} \big| \lesssim \alpha^p \frac{1}{\alpha^q} \|f(A)g\|_q^q \lesssim \alpha^{p-q} \|g\|_q^q,$$

which shows, using the bound just proven for $\|g\|_q$,

$$(9.1) \qquad \alpha^p \big| \{ x \in \Omega : |f(A)g(x)| > \alpha \} \big| \lesssim \|u\|_p{}^p$$

Step 2. *On the subsets* $2Q_k \cap \Omega = Q(x_k, 2t_k) \cap \Omega$.

We denote by E the set $\cup_k (2Q_k \cap \Omega)$. We have the estimate

$$|E| \leq \sum_k |2Q_k| \lesssim \frac{1}{\alpha^p} \|u\|_p^p,$$

so that

$$(9.2) \qquad \alpha^p \, |E| \lesssim \|u\|_p^p.$$

Step 3. *We claim that for all* $m \geq 1$,

$$\Big\| \sum_k R_k^m \big(Bw_k + \tfrac{1}{t_k} v_k \big) \Big\|_q \lesssim \alpha \Big| \bigcup_k Q_k \Big|^{1/q},$$

where

$$R_k := (\mathrm{I} + it_k A)^{-1}$$

and for $M \geq 1$ *to be chosen later,*

$$(9.3) \quad \alpha^p \big| \big\{ x \in \Omega \setminus E : \big| f(A) \sum_k \big(\mathrm{I} - (\mathrm{I} - R_k)^M \big) \big(Bw_k + \tfrac{1}{t_k} v_k \big) \big| > \alpha \big\} \big| \lesssim \|u\|_p^p.$$

Indeed, let $h \in L^{q'}(\Omega)$ with $\|h\|_{q'} = 1$. We have that

$$\Big| \int_\Omega \big\langle \sum_k R_k^m \big(Bw_k + \tfrac{1}{t_k} v_k \big), h \big\rangle \Big|$$

$$\leq \Big| \int_\Omega \big\langle \sum_k \tfrac{1}{t_k} w_k, t_k B^* R_k^* (R_k^*)^{m-1} h \big\rangle \Big| + \Big| \int_\Omega \big\langle \sum_k \tfrac{1}{t_k} v_k (R_k^*)^m h \big\rangle \Big|,$$

(taking the adjoints)

$$\leq \sum_k \Big(\frac{1}{t_k} \|w_k\|_q \|t_k B^* R_k^* (R_k^*)^{m-1} h\|_{L^{q'}(Q_k \cap \Omega)} + \frac{1}{t_k} \|v_k\|_q \|(R_k^*)^m h\|_{L^{q'}(Q_k \cap \Omega)} \Big).$$

For each k, we denote by A_{kj}, $j \geq 1$, the annulus $2^j Q_k \setminus 2^{j-1} Q_k$ and by $A_{k0} = Q_k$, so that $\mathbb{R}^n = \bigcup_{j \geq 0} Q_{kj}$. For each k, we decompose h as

$$h = \sum_{j \geq 0} \mathbb{1}_{A_{kj} \cap \Omega} h$$

and we obtain

$$\Big| \int_\Omega \langle \sum_k R_k^m \big(B w_k + \frac{1}{t_k} v_k \big), h \rangle \Big|$$

$$\lesssim \sum_k \frac{1}{t_k} \big(\|w_k\|_q + \|v_k\|_q \big) \Big(\sum_j e^{-c2^j} \|h\|_{L^{q'}(\Omega \cap A_{kj})} \Big)$$

(thanks to the off-diagonal bounds satisfied by R_k^*, $t_k B^* R_k^*$
and compositions of them)

$$\lesssim \sum_k \frac{1}{t_k} \| \mathbb{1}_{Q_k} u \|_p t_k^{1-n(1/p-1/q)} \Big[\sum_j e^{-c2^j} 2^{jn/q'} t_k^{n/q'} \Big(\fint_{2^j Q_k} |\tilde{h}|^{q'} \Big)^{1/q'} \Big]$$

(using the bounds for w_k and v_k in L^q, denoting by $\tilde{h}$ the extension
by zero to $\mathbb{R}^n$ of h and using the fact that $|2^j Q_k| = 2^{jn} t_k^n$)

$$\lesssim \sum_k \alpha\, t_k^n \inf_{x \in Q_k} \big(\mathcal{M}(|\tilde{h}|^{q'})(x) \big)^{1/q'} \Big(\sum_j e^{-c2^j} 2^{jn/p'} \Big)$$

(since $\| \mathbb{1}_{Q_k} \tilde{u} \|_p \lesssim \alpha t_k^{n/p}$ and using the maximal function $\mathcal{M}$ in $\mathbb{R}^n$

$$\lesssim \alpha \sum_k \int_{Q_k} \big(\mathcal{M}(|\tilde{h}|^{q'}) \big)^{1/q'} \quad \text{(since } t_k^n \inf_{x \in Q_k} |f|(x) \lesssim \int_{Q_k} |f|\text{)}$$

$$\lesssim \alpha \int_{\bigcup_k Q_k} \big(\mathcal{M}(|\tilde{h}|^{q'}) \big)^{1/q'} \quad \text{(by the finite overlap property of the } Q_k\text{)}$$

$$\lesssim \alpha \Big| \bigcup_k Q_k \Big|^{1/q} \| |\tilde{h}|^{q'} \|_1^{1/q'} \quad \text{(thanks to the estimate: } \int_F \big(\mathcal{M}|\varphi| \big)^{1/q'} \lesssim |F|^{1/q} \|\varphi\|_1^{1/q'},$$

$$\text{see, e.g., Lemma 5.16 in [14]).}$$

$$\lesssim \alpha \Big| \bigcup_k Q_k \Big|^{1/q}.$$

To prove (9.3) we now use the fact that $f(A)$ is bounded in L^q and we obtain

$$\alpha^p \Big| \big\{ x \in \Omega \setminus E : |f(A) \sum_k \big(\mathrm{I} - (\mathrm{I} - R_k)^M \big) \big(B w_k + t \frac{1}{t_k} v_k \big)(x)| > \alpha \big\} \Big|$$

$$\lesssim \alpha^p \frac{1}{\alpha^q} \Big\| \sum_k \big(\mathrm{I} - (\mathrm{I} - R_k)^M \big) \big(B w_k + \frac{1}{t_k} v_k \big) \Big\|_q^q$$

$$\lesssim \alpha^{p-q} \Big(\sum_{m=1}^M \binom{M}{m} \Big) \Big\| \sum_k R_k^m \big(B w_k + \frac{1}{t_k} v_k \big) \Big\|_q^q \lesssim \alpha^p \Big| \bigcup_k Q_k \Big| \lesssim \|u\|_p^p.$$

Step 4. Estimate of $\big\| \sum_k f(A)(\mathrm{I} - R_k)^M \big(B w_k + \frac{1}{t_k} v_k \big) \big\|_{L_w^p(\Omega \setminus E)}$.

Let $\theta \in (\omega, \mu)$. Recall that for each $b \in L^p(\Omega)$, the definition of the functional
calculus gives

$$f(A)(\mathrm{I} - R_k)^M b = \frac{1}{2\pi i} \int_{\partial S_\theta^\circ} f(z) \Big(1 - \frac{1}{1 + it_k z} \Big)^M (z\mathrm{I} - A)^{-1} b \, \mathrm{d}z.$$

Using the change of variable $z = \frac{1}{t}e^{\pm i(\theta - \pi)}$ and $z = \frac{1}{t}e^{\pm i\theta}$ we obtain for $b_k := t_k B w_k + v_k$

$$
\begin{aligned}
&f(A)(I - R_k)^M \left(\tfrac{1}{t_k} b_k\right) \\
&= \frac{1}{2\pi i} \sum_{\varphi = \pm\theta, \pm(\pi-\theta)} \pm \int_0^\infty \frac{1}{t_k} f(t^{-1}e^{i\varphi}) \left(\frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}}\right)^M (I - te^{-i\varphi}A)^{-1} b_k \, \frac{dt}{t} \\
&= \sum_{\varphi = \pm\theta, \pm(\pi-\theta)} \pm \frac{1}{2\pi i} \int_0^{2t_k} \frac{1}{t_k} f(t^{-1}e^{i\varphi}) \left(\frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}}\right)^M (I - te^{-i\varphi}A)^{-1} b_k \, \frac{dt}{t} \\
&\quad + \sum_{\varphi = \pm\theta, \pm(\pi-\theta)} \pm \frac{1}{2\pi i} \int_{2t_k}^\infty \frac{1}{t_k} f(t^{-1}e^{i\varphi}) \left(\frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}}\right)^M (I - te^{-i\varphi}A)^{-1} b_k \, \frac{dt}{t} \\
&\text{(9.4)} \quad = \sum_{\varphi = \pm\theta, \pm(\pi-\vartheta)} \left(F_{1,\varphi}^k(b_k) + F_{2,\varphi}^k(b_k)\right).
\end{aligned}
$$

Step 4.1. For $\varphi = \pm\theta$ or $\varphi = \pm(\pi - \theta)$, we claim that

$$
\text{(9.5)} \qquad \alpha^p \left|\left\{x \in \Omega \setminus E : \left|\sum_k F_{1,\varphi}^k(b_k)(x)\right| > \alpha\right\}\right| \lesssim \|u\|_p^p.
$$

Let $h \in L^{p'}(\Omega)$ with $\|h\|_{p'} = 1$. As before, for $j \geq 1$, we denote by A_{kj} the annulus $2^j Q_k \setminus 2^{j-1} Q_k$. Using the representation of $F_{1,\varphi}^k(b_k)$, we have that

$$
\begin{aligned}
&\left|\int_{\Omega \setminus E} \left\langle \sum_k F_{1,\varphi}^k(b_k), h\right\rangle\right| \\
&\leq \frac{1}{2\pi} \left|\sum_k \int_\Omega \int_0^{2t_k} f(t^{-1}e^{i\varphi}) \left(\frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}}\right)^M \left\langle w_k, tB^*(I - te^{-i\varphi}A^*)^{-1}(\mathbb{1}_{\Omega \setminus E} h)\right\rangle \frac{dt}{t^2}\right| \\
&\quad + \frac{1}{2\pi} \left|\sum_k \int_\Omega \int_0^{2t_k} \frac{t}{t_k} f(t^{-1}e^{i\varphi}) \left(\frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}}\right)^M \left\langle v_k, (I - te^{-i\varphi}A^*)^{-1}(\mathbb{1}_{\Omega \setminus E} h)\right\rangle \frac{dt}{t^2}\right|
\end{aligned}
$$

(using the definition of b_k and duality)

$$
\begin{aligned}
&= \frac{1}{2\pi} \left|\sum_k \int_0^{2t_k} \sum_{j \geq 2} \int_\Omega f(t^{-1}e^{i\varphi}) \left(\frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}}\right)^M \right. \\
&\qquad\qquad\qquad \left. \cdot \left\langle w_k, tB^*(I - te^{-i\varphi}A^*)^{-1}(\mathbb{1}_{(\Omega \cap A_{kj}) \setminus E} h)\right\rangle \frac{dt}{t^2}\right| \\
&\quad + \frac{1}{2\pi} \left|\sum_k \int_0^{2t_k} \sum_{j \geq 2} \int_\Omega \frac{t}{t_k} f(t^{-1}e^{i\varphi}) \left(\frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}}\right)^M \right. \\
&\qquad\qquad\qquad \left. \cdot \left\langle v_k, (I - te^{-i\varphi}A^*)^{-1}(\mathbb{1}_{(\Omega \cap A_{kj}) \setminus E} h)\right\rangle \frac{dt}{t^2}\right|
\end{aligned}
$$

(where we have decomposed $\mathbb{1}_{\Omega \setminus E} h$ as $\sum_{j \geq 2} \mathbb{1}_{(\Omega \cap A_{kj}) \setminus E} h$) .

We then obtain, denoting by $\tilde{h}$ the extension by 0 to $\mathbb{R}^n$ of h,

$$\left| \int_{\Omega \setminus E} \left\langle \sum_k F_{1,\varphi}^k(b_k), h \right\rangle \right|$$

$$\lesssim \|f\|_\infty \frac{1}{(\cos\theta)^M} \sum_k \left(\|w_k\|_q + \|v_k\|_q \right) \int_0^{2t_k} \sum_{j \geq 2} e^{-c2^{j-1}t_k/t} \|\mathbb{1}_{A_{kj}}\tilde{h}\|_{q'} \frac{dt}{t^2}$$

(using the off-diagonal bounds for $(I + zA^*)^{-1}$ and for $zB^*(I + zA^*)^{-1}$,

the estimate $\left| \frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}} \right| \leq 1/|\cos\varphi| = 1/\cos\vartheta$, the fact that $t/t_k \leq 2$ for $t \in [0, 2t_k]$

and the estimate $2^{j-1}t_k/t \geq 2^j/8 + 2^j t_k/(4t)$ if $0 < t < 2t_k$)

$$\lesssim \sum_k \sum_{j \geq 2} \alpha\, t_k^{n/p} t_k^{1 - n(\frac{1}{p} - \frac{1}{q})} e^{-c2^j/8} \int_0^{2t_k} e^{-\frac{c}{4}2^j t_k/t}\, 2^{nj/q'} |Q_k|^{1/q'} \left(\fint_{2^j Q_k} |\tilde{h}|^{q'} \right)^{1/q'} \frac{dt}{t^2}$$

(where we have used the bounds for w_k and v_k in L^q

and the fact that $\|\mathbb{1}_{Q_k} u\|_p \lesssim \alpha\, t_k^{n/p}$)

$$\lesssim \alpha \sum_k |Q_k| \inf_{x \in Q_k} \left(\mathcal{M}(|\tilde{h}|^{p'})(x) \right)^{1/p'} \left(\sum_{j \geq 2} 2^{nj/q'} e^{-c2^j/8} \int_0^{2t_k} \frac{2^j t_k}{t} e^{-\frac{c}{4}(2^j t_k/t)} \frac{dt}{t} \right)$$

(since $p' > q'$ and $|Q_k|^{1/n} \sim t_k$)

and so

$$\left| \int_{\Omega \setminus E} \left\langle \sum_k F_{1,\varphi}^k(b_k), h \right\rangle \right|$$

$$\lesssim \alpha \sum_k \int_{Q_k} \left(\mathcal{M}(|\tilde{h}|^{p'}) \right)^{1/p'} \left(\sum_{j \geq 2} 2^{nj/q'} e^{-c2^j/8} \int_{2^{j-1}}^\infty e^{-cs/4}\, ds \right)$$

$$(9.6) \qquad \lesssim \alpha \left| \bigcup_k Q_k \right|^{1/p} \qquad \text{(where we conclude as in Step 4, using the change of variable } s = 2^j t_k/t \text{ in the integral with respect to } t \text{ and the fact that the sum over } j \text{ converges).}$$

The estimate (9.6) shows that $\left\| \sum_k F_{1,\varphi}^k(b_k) \right\|_{L^p(\Omega \setminus E, \Lambda)} \lesssim \alpha \left| \bigcup_k Q_k \right|^{1/p}$. We can now prove (9.5). We have that

$$\alpha^p \left| \left\{ x \in \Omega \setminus E : \left| \sum_k F_{1,\varphi}^k(b_k)(x) \right| > \alpha \right\} \right| \lesssim \left\| \sum_k F_{1,\varphi}^k(b_k) \right\|_{L^p(\Omega \setminus E)}^p$$

$$\lesssim \alpha^p \left| \bigcup_k Q_k \right| \lesssim \|u\|_p^p.$$

Step 4.2. For $\varphi = \pm\theta$ or $\varphi = \pm(\pi - \theta)$ and $M > n/q'$, we claim that

$$(9.7) \qquad \alpha^p \left| \left\{ x \in \Omega \setminus E : \left| \sum_k F_{2,\varphi}^k(b_k)(x) \right| > \alpha \right\} \right| \lesssim \|u\|_p^p.$$

Let $h \in L^{p'}(\Omega)$ with $\|h\|_{p'} = 1$. We proceed as in the previous step and we obtain

$$\left| \int_{\Omega \setminus E} \left\langle \sum_k F_{2,\varphi}^k(b_k), h \right\rangle \right|$$

$$\leq \frac{1}{2\pi} \left| \sum_k \int_{2t_k}^{\infty} f(t^{-1}e^{i\varphi}) \left(\frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}} \right)^M \right.$$

$$\cdot \sum_{j \geq 2} \int_{\Omega} \left\langle w_k, tB^*(I - te^{-i\varphi}A^*)^{-1} (\mathbb{1}_{(\Omega \cap A_{kj}) \setminus E} h) \right\rangle \frac{dt}{t^2} \Big|$$

$$+ \frac{1}{2\pi} \left| \sum_k \int_{2t_k}^{\infty} \frac{t}{t_k} f(t^{-1}e^{i\varphi}) \left(\frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}} \right)^M \right.$$

$$\cdot \sum_{j \geq 2} \int_{\Omega} \left\langle v_k, (I - te^{-i\varphi}A^*)^{-1} (\mathbb{1}_{(\Omega \cap A_{kj}) \setminus E} h) \right\rangle \frac{dt}{t^2} \Big|$$

(where we have used duality and the decomposition $\mathbb{1}_{\Omega \setminus E} h = \sum_{j \geq 2} \mathbb{1}_{(\Omega \cap A_{kj}) \setminus E} h$).

And so

$$\left| \int_{\Omega \setminus E} \left\langle \sum_k F_{2,\varphi}^k(b_k), h \right\rangle \right|$$

$$\lesssim \|f\|_{\infty} \sum_k \left(\|w_k\|_p + \|v_k\|_p \right) \int_{2t_k}^{\infty} \left(\frac{2t_k}{t} \right)^{M-1} \sum_{j \geq 2} e^{-c 2^{j-1} t_k/t} \|\mathbb{1}_{A_{kj}} \tilde{h}\|_{q'} \frac{dt}{t^2}$$

(thanks to the L^q-L^q off-diagonal bounds satisfied by $(I + zA^*)^{-1}$

and $zB^*(I + zA^*)^{-1}$ and the fact that $\left| \frac{it_k e^{i\varphi}}{t + it_k e^{i\varphi}} \right| \leq 2t_k/t$ if $t \geq 2t_k$)

$$\lesssim \alpha \sum_k |Q_k| \inf_{x \in Q_k} \left(\mathcal{M}(|\tilde{h}|^{p'}) \right)^{1/p'} \left(\sum_{j \geq 2} 2^{nj/q'} t_k \int_{2t_k}^{\infty} \left(\frac{t_k}{t} \right)^M e^{-\frac{c}{2}(2^j t_k/t)} \frac{dt}{t^2} \right)$$

where we have used the same arguments as for the proof of (9.5).

To estimate the sum over $j \geq 2$, we change the variable $s := 2^j t_k/t$ in the integral and we obtain

$$\sum_{j \geq 2} 2^{nj/q'} \int_{2t_k}^{\infty} \left(\frac{t_k}{t} \right)^M e^{-\frac{c}{2}(2^j tk/t)} \frac{dt}{t} = \sum_{j \geq 2} 2^{nj/q'} \int_0^{2^{j-1}} 2^{-jM} s^M e^{-\frac{c}{2}s} \frac{ds}{s}$$

$$\leq \left(\int_0^{\infty} s^{M-1} e^{-\frac{c}{2}s} \, ds \right) \left(\sum_{j \geq 2} 2^{nj/q'} 2^{-jM} \right) < \infty.$$

The sum over j is finite since we have chosen $M > n/q'$. Therefore, we obtain as in the proof of (9.6)

$$\left| \int_{\Omega \setminus E} \left\langle \sum_k F_{2,\varphi}^k(b_k), h \right\rangle \right| \lesssim \alpha \sum_k \int_{Q_k} \left(\mathcal{M}(|\tilde{h}|^{p'}) \right)^{1/p'} \lesssim \alpha \left| \bigcup_k Q_k \right|^{1/p}.$$

This proves (9.7) the same way we proved (9.5).

Step 5. Conclusion: $f(A)$ *maps* $X_p \cap L^q$ *to* $L_w^p(\Omega)$.

Indeed, for all $\beta > 0$ we have for $\alpha = \beta/11$,

$$
\{x \in \Omega : |f(A)u(x)| > \beta\}
$$
$$
\subset \{x \in \Omega : |f(A)g(x)| > \alpha\} \cup E
$$
$$
\cup \left\{x \in \Omega \setminus E : \left|f(A)\sum_k (I - (I - R_k)^M)(Bw_k + \tfrac{1}{t_k}v_k)(x)\right| > \alpha\right\}
$$
$$
\cup \left(\bigcup_{\varphi=\pm\theta,\pm(\pi-\theta)} \left\{x \in \Omega \setminus E : \left|\sum_k F_{1,\varphi}^k(b_k)(x)\right| > \alpha\right\}\right)
$$
$$
\cup \left(\bigcup_{\varphi=\pm\theta,\pm(\pi-\theta)} \left\{x \in \Omega \setminus E : \left|\sum_k F_{2,\varphi}^k(b_k)(x)\right| > \alpha\right\}\right).
$$

We can estimate the size of each of the sets on the left hand side of the previous decomposition thanks to (9.1), (9.2), (9.3), (9.5) and (9.7). We prove that for all $u \in X_p$ and all $\beta > 0$, we have that

$$
\beta^p |\{x \in \Omega : |f(A)u(x)| > \beta\}| \lesssim \|u\|_p^p,
$$

which is exactly the claim. $\qquad\square$

A. Deferred proofs

Recall the statement of Proposition 2.14:

Let $\Omega \subset \mathbb{R}^n$ be a bounded strongly Lipschitz domain. Then there exists a biLipschitz map $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ where $\phi^{-1}(\Omega) = \Omega'$ is a smooth domain in $\mathbb{R}^n$ satisfying $\phi(\mathbb{R}^n \setminus \overline{\Omega'}) = \mathbb{R}^n \setminus \overline{\Omega}$ and $\phi(\partial\Omega') = \phi(\partial\Omega)$.

Proof. Let $\eta \in \mathscr{C}_c^\infty(\mathbb{R}^{n-1})$ such that $\eta \geq 0$, sppt $\eta \subset B_{n-1}(0,1)$ and $\int_{\mathbb{R}^{n-1}} \eta = 1$. For $\varepsilon > 0$, define $\eta_\varepsilon(x') = \varepsilon^{-(n-1)}\eta(x'/\varepsilon)$ for all $x' \in \mathbb{R}^{n-1}$. By definition of a strongly Lipschitz domain, there is a covering of $\partial\Omega$ by N open sets $V_j \subset \mathbb{R}^n$ $(j = 1, \dots, N)$ with the following properties:

$$
\chi_j \in \mathscr{C}_c^\infty(\mathbb{R}^n, [0,1]), \quad V_j = \{x \in \mathbb{R}^n; \chi_j(x) = 1\}, \quad \text{sppt}\, \chi_j \subset U_j,
$$
$$
U_j = E_j\Big(\prod_{k=1}^n [a_k, b_k]\Big), \quad \text{where } a_k, b_k \in \mathbb{R} \text{ and } E_j \text{ is a Euclidian transformation,}
$$
$$
\rho_j(x) = E_j(x', x_n - g_j(x')), \ \forall\, x = (x', x_n) \in \mathbb{R}^n,
$$
$$
g_j \colon \mathbb{R}^{n-1} \to \mathbb{R} \text{ Lipschitz continuous}, \quad \Omega \cap U_j = \rho_j(\mathbb{R}^{n-1} \times (0, +\infty)) \cap U_j.
$$

We fix now $j \in \{1, \dots, N\}$ and omit to write the subscript j. For the sake of simplicity, we assume that E_j is the identity on $\mathbb{R}^n$; if this is not the case, the modifications in the following proof are easy. We define

$$
\alpha \colon \mathbb{R}^n \to \mathbb{R}^n, \quad \alpha(x) = \big(x', x_n - \chi(x)(g(x') - g_\varepsilon(x'))\big), \ x = (x', x_n),
$$

where $g_\varepsilon = \eta_\varepsilon * g - \varepsilon M$ for $\varepsilon < 1/M$, $M := \|\nabla g\|_\infty$. The map α is Lipschitz continuous by construction and we have, in particular, for all $x' \in \mathbb{R}^{n-1}$,

$$|\eta_\varepsilon * g(x') - g(x')| = \left| \int_{\mathbb{R}^{n-1}} \eta(y')\big(g(x' - \varepsilon y') - g(x')\big)\,\mathrm{d}y' \right|$$

$$\leq \int_{\mathbb{R}^{n-1}} \eta(y')\, M\, \varepsilon\, |y'|\,\mathrm{d}y' \leq \varepsilon M,$$

so that $\varepsilon M - \big(\eta_\varepsilon * g(x') - g(x')\big) \geq 0$ for all $x' \in \mathbb{R}^{n-1}$. Moreover we have the following properties:

(i) It is straightforward to see that if $x = (x', x_n) \in V \cap \partial\Omega$, then $\chi(x) = 1$, $x_n = g(x')$ and therefore $\alpha(x) = (x', g_\varepsilon(x'))$, which defines a piece of a smooth hypersurface. We have moreover that if $x \in \mathbb{R}^n \setminus U$, then $\chi(x) = 0$ and then $\alpha(x) = x$.

(ii) The map $\alpha \colon \mathbb{R}^n \to \mathbb{R}^n$ is invertible. Indeed, let $x' \in \mathbb{R}^{n-1}$. The function $h_{x'} \colon t \mapsto t - \chi(x', t)(g(x') - g_\varepsilon(x'))$ is smooth on $\mathbb{R}$ and its derivative is given by $h'_{x'}(t) = 1 - \partial_n \chi(x', t)(g(x') - g_\varepsilon(x'))$. Choosing $\varepsilon > 0$ small enough such that

$$\sup_{x' \in \mathbb{R}^{n-1}, t \in \mathbb{R}} |\partial_n \chi(x', t)| \leq \frac{1}{4\varepsilon M},$$

we have that $1/2 \leq h'_{x'}(t) \leq 3/2$, for all $t \in \mathbb{R}$, $x' \in \mathbb{R}^{n-1}$, so that $h_{x'} \colon \mathbb{R} \to h_{x'}(\mathbb{R})$ is strictly increasing, invertible and its inverse is smooth (in the variable t). For $|t|$ large, $\chi(x', t) = 0$. This implies that $h_{x'}(t) \to -\infty$ as $t \to -\infty$, and $h_{x'}(t) \to +\infty$ as $t \to +\infty$, and then $h_{x'}(\mathbb{R}) = \mathbb{R}$. Therefore, the map α is invertible, its inverse given by

$$\alpha^{-1} \colon \mathbb{R}^n \to \mathbb{R}^n, \quad \alpha^{-1}(y', y_n) = \big(y', h_{y'}^{-1}(y_n)\big).$$

Moreover, since $h_{x'}$ is strictly increasing, we have that $\alpha(\mathbb{R}^n \setminus \overline{\Omega}) = \mathbb{R}^n \setminus \overline{\alpha(\Omega)}$ and $\alpha(\partial\Omega) = \partial\big(\alpha(\Omega)\big)$.

(iii) The map α^{-1} is Lipschitz continuous. The Jacobian $n \times n$ matrix of α at a point $x = (x', t)$ is given by

$$J_\alpha(x', t) = \left(\begin{array}{c|c} \mathrm{I}_{n-1} & \nabla_{x'}\big(x' \mapsto h_{x'}(t)\big) \\ \hline 0 & 1 - \partial_n \chi(x', t)\big(g(x') - g_\varepsilon(x')\big) \end{array} \right)$$

This matrix is invertible, its inverse at a point $(x', t) = \alpha^{-1}(y', y_n)$ is given by

$$J_\alpha(x', t)^{-1} = \left(\begin{array}{c|c} \mathrm{I}_{n-1} & -\frac{\nabla_{x'}\big(x' \mapsto h_{x'}(t)\big)}{1 - \partial_n \chi(x', t)\big(g(x') - g_\varepsilon(x')\big)} \\ \hline 0 & \frac{1}{1 - \partial_n \chi(x', t)\big(g(x') - g_\varepsilon(x')\big)} \end{array} \right) = J_{\alpha^{-1}}(y', y_n)$$

which is bounded on $\mathbb{R}^n$. Therefore α^{-1} is Lipschitz continuous.

Following this construction for all $j = 1, \ldots, N$, we finally obtain

$$\alpha := \alpha_N \circ \cdots \circ \alpha_1 : \mathbb{R}^n \to \mathbb{R}^n \quad \text{is a biLipschitz map}$$

for which $\alpha(\Omega) = \Omega'$ is a smooth domain. Letting $\phi = \alpha^{-1}$ proves the claim made in Proposition 2.14. $\qquad \square$

The following result shows a property of smooth domains. We did not use it in this paper, but it seems to us to be of independent interest and can justify, a posteriori, together with Proposition 2.14, the classical assumption that for Ω a bounded strongly Lipschitz domain, $x \in \partial\Omega$, $r > 0$, the domain $B(x, r) \cap \Omega$ has the same Lipschitz constant as Ω (see, e.g., [24], §5).

Lemma A.1. *Let Ω' be a smooth domain in $\mathbb{R}^n$. For $x_0 \in \partial\Omega'$ and $r > 0$, we consider $B(x_0, r) \cap \Omega'$. Then there exists a smooth domain (of class $\mathscr{C}^3$) $Q_r \subset \mathbb{R}^n$ such that*

$$B(x_0, r) \cap \Omega' \subset Q_r \subset B(x_0, 2r) \cap \Omega'.$$

Proof. We define $G : \mathbb{R}^n \to \mathbb{R}$ by

$$G(x) = 2r^2 \operatorname{dist}(x, \mathbb{R}^n \setminus \Omega')^2 - \max\{0, (|x - x_0|^2 - r^2)^2\}, \quad x \in \mathbb{R}^n.$$

The function G is of class $\mathscr{C}^3$. We define $Q_r := G^{-1}(0, +\infty)$. Then Q_r is of class $\mathscr{C}^3$. It remains to verify that $B(x_0, r) \cap \Omega' \subset Q_r \subset B(x_0, 2r) \cap \Omega'$.

(i) If $x \in B(x_0, r) \cap \Omega'$, then $\operatorname{dist}(x, \mathbb{R}^n \setminus \Omega')^2 > 0$ and $\max\{0, (|x - x_0|^2 - r^2)^2\} = 0$. Therefore, $G(x) > 0$, and $x \in Q_r$.

(ii) If $x \in \mathbb{R}^n \setminus \Omega'$, then $\operatorname{dist}(x, \mathbb{R}^n \setminus \Omega')^2 = 0$ and therefore $G(x) \geq 0$ which implies that $x \notin Q_r$.

(iii) If $x \in \Omega'$ with $|x - x_0| \geq 2r$, then $\operatorname{dist}(x, \mathbb{R}^n \setminus \Omega')^2 \leq |x - x_0|^2$ and

$$\max\{0, (|x - x_0|^2 - r^2)^2\} = (|x - x_0|^2 - r^2)^2.$$

Therefore,

$$G(x) \leq 2r^2 |x - x_0|^2 - (|x - x_0|^2 - r^2)^2 \leq 4r^2 |x - x_0|^2 - |x - x_0|^4 \leq 0$$

so that $x \notin Q_r$.

This proves the properties of Q_r. $\qquad \square$

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