



Multiplicative dependence of the translations of algebraic numbers

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Abstract. In this paper, we first prove that given pairwise distinct algebraic numbers $\alpha_1, \dots, \alpha_n$, the numbers $\alpha_1 + t, \dots, \alpha_n + t$ are multiplicatively independent for all sufficiently large integers t . Then, for a pair (a, b) of distinct integers, we study how many pairs $(a + t, b + t)$ are multiplicatively dependent when t runs through the set integers \mathbb{Z} . Assuming the *ABC* conjecture we show that there exists a constant C_1 such that for any pair $(a, b) \in \mathbb{Z}^2$, $a \neq b$, there are at most C_1 values of $t \in \mathbb{Z}$ such that $(a + t, b + t)$ are multiplicatively dependent. For a pair $(a, b) \in \mathbb{Z}^2$ with difference $b - a = 30$ we show that there are 13 values of $t \in \mathbb{Z}$ for which the pair $(a + t, b + t)$ is multiplicatively dependent. We further conjecture that 13 is the largest number of such translations for any such pair (a, b) and prove this for all pairs (a, b) with difference at most 10^{10} .

1. Introduction

Given $n \geq 1$ non-zero complex numbers $z_1, \dots, z_n \in \mathbb{C}^*$, we say that they are *multiplicatively dependent* if there exists a non-zero integer vector $(k_1, \dots, k_n) \in \mathbb{Z}^n$ for which

$$(1.1) \quad z_1^{k_1} \cdots z_n^{k_n} = 1.$$

Otherwise (if there is no such non-zero integer vector (k_1, \dots, k_n)), we say that the numbers z_1, \dots, z_n are *multiplicatively independent*. Consequently, a vector in \mathbb{C}^n is called *multiplicatively dependent* (resp. *independent*) if its coordinates are all non-zero and are multiplicatively dependent (resp. independent). To avoid confusion, the vectors with zero coordinates, like $(0, 1)$, are not considered to be multiplicatively dependent (although, by convention, $0^0 1^1 = 1$) or independent.

In [8], several asymptotic formulas for the number of multiplicatively dependent vectors of algebraic numbers of fixed degree (or lying in a fixed number field)

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and bounded height have been obtained. In an ongoing project [14], the authors continue to study multiplicatively dependent vectors from the viewpoint of their density and sparsity. By contrast, in this paper, aside from the multiplicative dependence and independence of a given set of algebraic numbers, we also want to investigate the multiplicative dependence and independence of their translations. More generally, the authors in [7] study multiplicative dependence of values of rational functions in some special cases. We remark that a method on deciding the multiplicative independence of complex numbers in a finitely generated field has been proposed by Richardson [12].

In Section 3, we prove a result (Theorem 3.1) which implies that given pairwise distinct algebraic numbers $\alpha_1, \dots, \alpha_n, n \geq 2$, for each sufficiently large integer t , the algebraic numbers $\alpha_1 + t, \dots, \alpha_n + t$ are multiplicatively independent. This is in fact a special case of Theorem 1' in [2]. A weaker version of this statement given in Lemma 2.1 of [4] was used in [4], and so it is an additional motivation for Theorem 3.1. In particular, by Theorem 3.1, for an integer vector (a_1, \dots, a_n) whose coordinates are pairwise distinct, there are only finitely many integers t for which the numbers $a_1 + t, \dots, a_n + t$ are multiplicatively dependent. So, a natural question is to estimate the number of such integers t corresponding to a given integer vector. In this paper, we investigate in detail the case of dimension $n = 2$ by presenting some explicit formulas, upper bounds and several conjectures. See Theorems 4.2, 4.3, 4.4 and 4.8. For example, we conjecture that for any pair of distinct integers $(a, b) \in \mathbb{Z}^2$, the number of such integer translations t is at most 13, which is in fact related to two special forms of Pillai's equation. The pair $(a, b) = (1, 31)$ is an example which has exactly 13 integer translations leading to multiplicatively dependent vectors (see Section 4).

2. Preliminaries

For the convenience of the reader, we recall some basic concepts and results in this section, which are used later on.

For any algebraic number α of degree $\deg \alpha = m \geq 1$, let

$$f(x) = a_m x^m + \dots + a_1 x + a_0$$

be the minimal polynomial of α over the integers \mathbb{Z} , where $a_m > 0$. Suppose that f is factored as

$$f(x) = a_m (x - \alpha_1) \cdots (x - \alpha_m)$$

over the complex numbers \mathbb{C} . The *height* of α , also known as the *absolute Weil height* of α and denoted by $H(\alpha)$, is defined by

$$H(\alpha) = \left(a_m \prod_{i=1}^m \max\{1, |\alpha_i|\} \right)^{1/m}.$$

Besides, we define the *house* of α to be the maximum of the modulus of its conjugates:

$$\overline{|\alpha|} = \max\{|\alpha_1|, \dots, |\alpha_m|\};$$

see Section 3.4 of [16]. Clearly, if $|a_0/a_m| \geq 1$ we have

$$H(\alpha) \leq a_m^{1/m} \overline{|\alpha|}.$$

In particular, for any algebraic integer $\alpha \neq 0$ we have $H(\alpha) \leq \overline{|\alpha|}$.

The next result shows that if algebraic numbers $\alpha_1, \dots, \alpha_n$ are multiplicatively dependent, then one can find a relation as in (1.1), where the exponents $k_i, i = 1, \dots, n$, are not too large; see for example Theorem 3 of [5] or Theorem 1 of [11].

Lemma 2.1. *Let $n \geq 2$, and let $\alpha_1, \dots, \alpha_n$ be multiplicatively dependent non-zero algebraic numbers of height at most $H \geq 2$ and contained in a number field K of degree D over the rational numbers \mathbb{Q} . Then, there are $k_1, \dots, k_n \in \mathbb{Z}$, not all zero, and a positive number c_1 which depends only on n , such that*

$$(2.1) \quad \alpha_1^{k_1} \cdots \alpha_n^{k_n} = 1$$

and

$$(2.2) \quad \max_{1 \leq i \leq n} |k_i| \leq c_1 D^n (\log(D + 1))^{3(n-1)} (\log H)^{n-1}.$$

Furthermore, if K is totally real, then there are integers k_1, \dots, k_n , not all zero, as in (2.1) and a positive number c_2 which depends only on n such that

$$(2.3) \quad \max_{1 \leq i \leq n} |k_i| \leq c_2 (\log H)^{n-1}.$$

Proof. Let $w(K)$ be the number of roots of unity in K . Note that for Euler’s totient function φ we have $\varphi(m) \gg m/\log \log m$ for any $m \geq 3$. Since $\varphi(w(K)) \leq D$, we obtain $w(K) \ll D \log \log(3D)$. Then, using Theorem 3 (A) in [5], we can get (2.2). In the same fashion, (2.3) follows directly from Theorem 3 (B) in [5]. \square

The following statement is Mihăilescu’s theorem (previously known as Catalan’s conjecture) [6], which roughly says that $(2^3, 3^2)$ is the only case of two consecutive powers of natural integers.

Lemma 2.2 ([6]). *The equation*

$$b^y - a^x = 1$$

with unknowns $b \geq 1, y \geq 2, a \geq 1$ and $x \geq 2$, has only one integer solution $(a, b, x, y) = (2, 3, 3, 2)$.

We also need the following classical result due to Siegel [15].

Lemma 2.3 ([15]). *Let $f(x)$ be a polynomial in $\mathbb{Z}[x]$. If f has at least three simple roots, then the equation $y^2 = f(x)$ has only finitely many integer solutions (x, y) .*

3. Multiplicative independence

In the following theorem, we confirm the multiplicative independence among the translations of algebraic numbers. Actually, we can do more than it was claimed at the beginning.

Theorem 3.1. *Let $\alpha_1, \dots, \alpha_n$ be pairwise distinct algebraic numbers, and let $d = [\mathbb{Q}(\alpha_1, \dots, \alpha_n) : \mathbb{Q}]$. Then, there is a positive constant $C = C(n, \alpha_1, \dots, \alpha_n)$ such that for any algebraic integer t of degree at most $\overline{|t|}^{1/(nd+1)}$ and with $\overline{|t|} \geq C$, the following n algebraic numbers: $\alpha_1 + t, \dots, \alpha_n + t$, are multiplicatively independent.*

We remark that the exponent $1/(nd+1)$ for $\overline{|t|}$ here is not optimal and is chosen for the sake of simplicity.

Proof. The result is trivial for $n = 1$. Assume that $n \geq 2$. Without loss of generality, we can further assume that

$$(3.1) \quad |t| = \overline{|t|}.$$

Indeed, if $|t| \neq \overline{|t|}$, then there is a Galois isomorphism σ of the Galois closure of $\mathbb{Q}(\alpha_1, \dots, \alpha_n, t)$ over \mathbb{Q} such that $|\sigma(t)| = \overline{|t|}$. Then, it suffices to verify the multiplicative independence of the algebraic numbers $\sigma(\alpha_1) + \sigma(t), \dots, \sigma(\alpha_n) + \sigma(t)$.

Take $|t|$ large enough. Then, we can assume that $\alpha_i + t \neq 0$ and, moreover,

$$||1 + \alpha_i/t| - 1| < \varepsilon, \quad i = 1, 2, \dots, n,$$

for a sufficiently small $\varepsilon > 0$. For a complex number z , let $\arg(z) \in (-\pi, \pi]$ be the principal argument of z . Note that for $\varepsilon \leq 1/2$ and each $i = 1, 2, \dots, n$, we have

$$|\sin(\arg(1 + \alpha_i/t))| = \frac{|\sin(\arg(\alpha_i/t))| \cdot |\alpha_i/t|}{|1 + \alpha_i/t|} \leq 2|\alpha_i|/|t|.$$

Thus, using the fact that $|x| \leq 2|\sin x|$ for any $x \in [-\pi/2, \pi/2]$, we can further assume that the principal arguments satisfy

$$(3.2) \quad |\arg(1 + \alpha_i/t)| \leq 4|\alpha_i|/|t|, \quad i = 1, 2, \dots, n.$$

Besides, by the basic properties of the Weil height (see, e. g., [16]) and (3.1), we have

$$(3.3) \quad H(\alpha_i + t) \leq 2H(t)H(\alpha_i) \leq 2|t|H(\alpha_i), \quad i = 1, 2, \dots, n.$$

Here, $H(t) \leq |t|$, since t is an algebraic integer and $|t| = \overline{|t|}$, by (3.1).

For a contradiction, assume that $\alpha_1 + t, \dots, \alpha_n + t$ are multiplicatively dependent, that is, there is a non-zero vector $(k_1, \dots, k_n) \in \mathbb{Z}^n$ such that

$$(3.4) \quad (\alpha_1 + t)^{k_1} \cdots (\alpha_n + t)^{k_n} = 1.$$

Set

$$D = [\mathbb{Q}(\alpha_1, \dots, \alpha_n, t) : \mathbb{Q}].$$

Then, by the degree assumption on t , we find that

$$D \leq [\mathbb{Q}(t) : \mathbb{Q}] d \leq d |t|^{1/(nd+1)}.$$

By Lemma 2.1 (see (2.2)) and (3.3), we can further assume that the nonzero integers in (3.4) can be chosen such that

$$(3.5) \quad \max_{1 \leq i \leq n} |k_i| \leq c_3 |t|^{n/(nd+1)} (\log |t|)^{4(n-1)},$$

where c_3 depends only on $n, \alpha_1, \dots, \alpha_n$. (Note that d also depends on $\alpha_1, \dots, \alpha_n$.)

Observe first that if in (3.4) we have $S = \sum_{i=1}^n k_i \neq 0$, then, since each $|\alpha_i + t|$ is close to $|t|$, the absolute value of the left-hand side of (3.4) is either very large (if $S > 0$) or very small (if $S < 0$) provided that $|t|$ is large enough, which contradicts with (3.4). Indeed, by (3.4), we obtain

$$|t|^S = \prod_{i=1}^n |1 + \alpha_i/t|^{-k_i}.$$

Suppose that $S \neq 0$. Replacing (k_1, \dots, k_n) by $(-k_1, \dots, -k_n)$ if necessary, we can assume that $S > 0$, and hence $S \geq 1$. Then, using $|1 + \alpha_i/t|^{-k_i} \leq |1 + |\alpha_i|/|t||^{2|k_i|}$ for $|t|$ large enough and $|t| \leq |t|^S$, we deduce that

$$|t| \leq \prod_{i=1}^n (1 + |\alpha_i|/|t|)^{2|k_i|} \leq \exp\left(\frac{2}{|t|} \sum_{i=1}^n |k_i| |\alpha_i|\right).$$

By taking logarithms of both sides and using (3.5), we get the inequality

$$|t| \log |t| \leq c_4 |t|^{n/(nd+1)} (\log |t|)^{4(n-1)}$$

for some constant c_4 depending only on $n, \alpha_1, \dots, \alpha_n$. However, this inequality cannot hold for $|t|$ large enough, because $n/(nd+1) < 1$. Thus, we must have $S = 0$.

Now, by (3.4) combined with $\sum_{i=1}^n k_i = 0$, it follows that

$$(3.6) \quad (1 + \alpha_1/t)^{k_1} \cdots (1 + \alpha_n/t)^{k_n} = 1.$$

With our assumptions, by (3.2), we further deduce that

$$\sum_{i=1}^n |k_i \arg(1 + \alpha_i/t)| \leq \sum_{i=1}^n \frac{4|k_i \alpha_i|}{|t|},$$

which, by (3.5), is clearly less than π when $|t|$ is large enough. So, by taking logarithms of both sides of (3.6), we obtain

$$\sum_{i=1}^n k_i \log(1 + \alpha_i/t) = 0,$$

where “log” means the principal branch of the complex logarithm. Then, using the Taylor expansion we deduce that

$$(3.7) \quad \frac{1}{t} \sum_{i=1}^n k_i \alpha_i - \frac{1}{2t^2} \sum_{i=1}^n k_i \alpha_i^2 + \frac{1}{3t^3} \sum_{i=1}^n k_i \alpha_i^3 - \cdots = 0.$$

Multiplying both sides of (3.7) by t and using the bound (3.5), we get

$$(3.8) \quad \left| \sum_{i=1}^n k_i \alpha_i \right| \leq c_5 |t|^{(n-nd-1)/(nd+1)} (\log |t|)^{4(n-1)},$$

where c_5 is a constant depending only on n and $\alpha_1, \dots, \alpha_n$.

Assume that $\sum_{i=1}^n k_i \alpha_i \neq 0$. Then, by Liouville’s inequality (see Proposition 3.14 of [16]) and the upper bound (3.5), one can easily get that

$$(3.9) \quad \left| \sum_{i=1}^n k_i \alpha_i \right| \geq c_6 (|t|^{n/(nd+1)} (\log |t|)^{4(n-1)})^{1-d},$$

where c_6 is a constant depending only on n and $\alpha_1, \dots, \alpha_n$. Clearly, in view of $nd - n + 1 > n(d - 1)$, the two estimates (3.8) and (3.9) lead to a contradiction provided that $|t|$ is large enough. Hence, we must have

$$\sum_{i=1}^n k_i \alpha_i = 0.$$

Applying the same argument to (3.7), step by step, we obtain

$$\sum_{i=1}^n k_i \alpha_i^2 = 0, \quad \sum_{i=1}^n k_i \alpha_i^3 = 0, \quad \dots, \quad \sum_{i=1}^n k_i \alpha_i^n = 0.$$

This is a system of n linear equations with unknowns k_1, \dots, k_n . Notice that its coefficient matrix is the Vandermonde matrix with non-zero determinant, since $\alpha_i \neq \alpha_j$ for $1 \leq i \neq j \leq n$. So, we must have

$$k_1 = \dots = k_n = 0,$$

which contradicts to the assumption that (k_1, \dots, k_n) is a non-zero vector. This completes the proof of the theorem. □

Following the same arguments as in the proof of Theorem 3.1 and using the inequality (2.3) of Lemma 2.1 (instead of (2.2)), which yields

$$\max_{1 \leq i \leq n} |k_i| \leq c_7 (\log |t|)^{n-1}$$

instead of (3.5), we obtain the following.

Theorem 3.2. *Given $n \geq 2$ pairwise distinct totally real algebraic numbers $\alpha_1, \dots, \alpha_n$, there is a positive constant $C = C(n, \alpha_1, \dots, \alpha_n)$ such that for any totally real algebraic integer t with $|t| \geq C$, the following n algebraic numbers: $\alpha_1 + t, \dots, \alpha_n + t$, are multiplicatively independent.*

Theorem 3.1 implies the following corollary. (It also follows from Theorem 1’ of [2], by considering the line parameterized by $x - \alpha_1, \dots, x - \alpha_n$ as x varies.)

Corollary 3.3. *Given a positive integer m and $n \geq 2$ pairwise distinct algebraic numbers $\alpha_1, \dots, \alpha_n$, there is a positive constant $C = C(m, n, \alpha_1, \dots, \alpha_n)$ such that for any algebraic integer t of degree at most m and with $\overline{|t|} \geq C$, the following n algebraic numbers: $\alpha_1 + t, \dots, \alpha_n + t$, are multiplicatively independent.*

In particular, we have:

Corollary 3.4. *Given n pairwise distinct algebraic numbers $\alpha_1, \dots, \alpha_n$, there are only finitely many integers $t \in \mathbb{Z}$ for which the translated numbers $\alpha_1 + t, \dots, \alpha_n + t$ are multiplicatively dependent.*

On the other hand, for a fixed integer $t \in \mathbb{Z}$, there are infinitely many vectors $(\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ such that $(\alpha_1 + t, \dots, \alpha_n + t)$ is multiplicatively independent. For example, we can choose $\alpha_i = p_i - t$ for each i , where p_1, \dots, p_n are pairwise distinct rational primes.

4. Sets of multiplicatively dependent vectors

4.1. General setting

In this section, we focus our attention on vectors in \mathbb{Z}^2 which are multiplicatively dependent. This turns out to be related to Pillai’s equation, which is a quite typical kind of Diophantine equation and has been extensively studied; see, for example, [1], [3], and [13].

Starting from an integer vector $(a_1, \dots, a_n) \in \mathbb{Z}^n$, we can get a set of multiplicatively dependent vectors in \mathbb{Z}^n by adding $t \in \mathbb{Z}$ to each coordinate of the given vector. Corollary 3.4 implies that the set of such $t \in \mathbb{Z}$ is finite when the coordinates of the given vector are pairwise distinct, namely, $a_i \neq a_j$ for $i \neq j$. Now, a natural question is to estimate the size of the set of possible $t \in \mathbb{Z}$ for which the vector $(a_1 + t, \dots, a_n + t)$ is multiplicatively dependent (and thus contains no zero coordinates by definition). In this paper, we only consider the simplest case $n = 2$.

Given a vector $(a, b) \in \mathbb{Z}^2$ with $a \neq b$, note that either $(1, b - a + 1)$ or $(-1, b - a - 1)$ is multiplicatively dependent obtained from (a, b) by translation as above, because $b - a + 1$ and $b - a - 1$ cannot be zero at the same time. So, the set of all possible $t \in \mathbb{Z}$ only depends on the difference $b - a$, which is also called the *difference* of the set. For an integer $d \in \mathbb{Z}$, we denote by $\mathcal{M}(d)$ the set of multiplicatively dependent vectors in $(a, b) \in \mathbb{Z}^2$, $ab \neq 0$, with difference $d = b - a$. Corollary 3.4 implies that each set $\mathcal{M}(d)$, $d \neq 0$, is a finite set. Let us put

$$M(d) = |\mathcal{M}(d)|, \quad d \in \mathbb{Z},$$

where $|\mathcal{M}(d)|$ is the cardinality of the set $\mathcal{M}(d)$. One interesting direction is to study the size of $M(d)$, and especially whether the following maximum:

$$\max_{d \neq 0} M(d),$$

is finite. (Clearly, the set $\mathcal{M}(0)$ is infinite, because it consists of all pairs $(a, a) \in \mathbb{Z}^2$, $a \neq 0$.)

Note that for any multiplicatively dependent vector $(a, b) \in \mathbb{Z}^2$, we certainly have $(a, b) \in \mathcal{M}(b - a)$. So, the sets $\mathcal{M}(d), d \in \mathbb{Z}$, form a disjoint union of all the multiplicatively dependent vectors in \mathbb{Z}^2 . Since there is a one-to-one correspondence between the vectors in $\mathcal{M}(d)$ and those in $\mathcal{M}(-d)$ by the permutation of coordinates, we have

$$M(d) = M(-d)$$

for any $d \neq 0$. So, in the sequel we will always assume that $d \in \mathbb{N}$.

Before going further, let us emphasize the following useful fact about multiplicatively dependent vectors in \mathbb{Z}^2 . That is, if $(a, b) \in \mathbb{Z}^2, a \neq b$, is multiplicatively dependent, then there exists a positive integer g and two non-negative integers x, y such that $(a, b) = (\pm g^x, \pm g^y)$.

4.2. Some explicit formulas

We essentially relate $M(d)$ to counting integer solutions of two simple Pillai's equations in the lemma below.

Throughout, for any given integer $d \geq 1$ we say that an integer solution (g, x, y) of the equation

$$(4.1) \quad g^y + g^x = d, \quad g \geq 2 \quad \text{and} \quad y > x \geq 1$$

is *primitive* if g is not a perfect power. Let $N^+(d)$ be the number of primitive integer solutions of (4.1). Similarly, for any given integer $d \geq 1$ we say that an integer solution (g, x, y) of the equation

$$(4.2) \quad g^y - g^x = d, \quad g \geq 2 \quad \text{and} \quad y > x \geq 1$$

is *primitive* if g is not a perfect power. Let $N^-(d)$ be the number of primitive integer solutions of (4.2).

Lemma 4.1. *For any integer $d \geq 3$, we have*

$$(4.3) \quad M(d) = 2N^+(d) + 2N^-(d) + 4 + \delta(d),$$

where $\delta(d) = 1$ if d is even, and $\delta(d) = 0$ if d is odd.

Proof. Let

$$S_0 = \{(-d - 1, -1), (-d + 1, 1), (-1, d - 1), (1, d + 1)\},$$

$$S_1 = \{(-g^x, g^y), (-g^y, g^x) : (g, x, y) \text{ is a primitive solution of (4.1)}\}$$

and

$$S_2 = \{(g^x, g^y), (-g^y, -g^x) : (g, x, y) \text{ is a primitive solution of (4.2)}\}.$$

We claim that, if d is odd,

$$(4.4) \quad \mathcal{M}(d) = S_0 \cup S_1 \cup S_2,$$

and if d is even,

$$(4.5) \quad \mathcal{M}(d) = \{(-d/2, d/2)\} \cup S_0 \cup S_1 \cup S_2.$$

Evidently, $S_0 \subseteq \mathcal{M}(d)$. Also, $(-d/2, d/2) \in \mathcal{M}(d)$ if d is even. Let us count the vectors $(a, b) \in \mathcal{M}(d) \setminus (S_0 \cup \{(-d/2, d/2)\})$ with $ab < 0$. Then, $a < 0 < b$, so that such vectors (a, b) have a form of $(-g^x, g^y)$ or $(-g^y, g^x)$ for some positive integer $g \geq 2$ and two non-negative integers $x \leq y$. If d is even, then $(-d/2, d/2) \in \mathcal{M}(d)$, which corresponds to the case $g^x = g^y = d/2$, so this solution is not in $S_1 \cup S_2$, and since $d \geq 3$, we have $(-d/2, d/2) \notin S_0$. In case $x = 0$, that is, $g^x = 1$, we obtain two vectors $(-d + 1, 1), (-1, d - 1) \in \mathcal{M}(d)$, which are already in S_0 . Besides, if an integer vector (g, x, y) with $g = a^r \geq 2$ and $y > x \geq 1$ satisfies $g^y + g^x = d$, where a and r are positive integers, then (g, x, y) and (a, rx, ry) are different integer solutions of (4.1), but they produce the same vectors in $\mathcal{M}(d)$: $(-g^x, g^y)$ and $(-g^y, g^x)$. Thus, the sets $\{(-d/2, d/2)\} \cup S_0$ and S_1 are disjoint and, by the definition of S_1 , we have $|S_1| = 2N^+(d)$.

It remains to count the vectors $(a, b) \in \mathcal{M}(d)$ with $ab > 0$. Clearly, they have the form (g^x, g^y) or $(-g^y, -g^x)$ for some positive integer $g \geq 2$ and two non-negative integers x, y with $y > x \geq 0$. If $x = 0$, i.e., $g^x = 1$, we get two vectors $(-d - 1, -1), (1, d + 1) \in \mathcal{M}(d)$ which belong to S_0 . Now, by the same argument as the above, we see that the sets $\{(-d/2, d/2)\} \cup S_0$ and S_2 are disjoint and $|S_2| = 2N^-(d)$.

Finally, since the sets S_1 and S_2 are disjoint by their definitions (and each of them is disjoint from the set $\{(-d/2, d/2)\} \cup S_0$), we deduce (4.3), in view of (4.4), (4.5), $|S_0| = 4$, $|S_1| = 2N^+(d)$ and $|S_2| = 2N^-(d)$. \square

Lemma 4.1 transfers our problem to estimates for the quantities $N^+(d)$ and $N^-(d)$. Next, using the formulas (4.4) and (4.5) we give the explicit constructions for $\mathcal{M}(d)$ as well as the explicit values for their sizes $M(d)$ in some special cases.

Theorem 4.2. *We have*

- (i) $M(1) = 2, M(2) = 5$, and $M(2^r) = 7$ for any positive integer $r \geq 2$;
- (ii) $M(d) = 4$ for any odd integer $d \geq 3$.

Proof. It is straightforward to check that

$$\mathcal{M}(1) = \{(-2, -1), (1, 2)\},$$

and

$$\mathcal{M}(2) = \{(-4, -2), (-3, -1), (-1, 1), (1, 3), (2, 4)\}.$$

Now, we consider the set $\mathcal{M}(2^r)$, where $r \geq 2$. We first look at the equation (4.1) with $d = 2^r$. Notice that $g^x(g^{y-x} + 1) = 2^r$. Since $x \geq 1$ and $\gcd(g^x, g^{y-x} + 1) = 1$, the left-hand side $g^x(g^{y-x} + 1)$ has at least two distinct prime factors. So, there is no integer solution of the equation (4.1). Consequently, $N^+(2^r) = 0$.

Next, let us consider the equation (4.2) with $d = 2^r$. This time, in view of $g^x(g^{y-x} - 1) = 2^r$ and $x \geq 1$, we must have $g^x = 2^r$ and $g^{y-x} = 2$. Hence,

$(g, x, y) = (2, r, r + 1)$ is the only primitive integer solution of (4.2). It follows that $N^-(2^r) = 1$, which gives two vectors

$$(-2^{r+1}, -2^r), (2^r, 2^{r+1}) \in \mathcal{M}(2^r).$$

So, by Lemma 4.1, it follows that $M(2^r) = 2 \cdot 0 + 2 \cdot 1 + 4 + 1 = 7$ for $r \geq 2$, as claimed. This completes the proof of (i).

Now, let $d \geq 3$ be an odd integer. Considering the equation (4.1), we first note that, since $x \geq 1$, it is impossible to have $g^y + g^x = d$ for d odd, because $g^y + g^x$ is even. Similarly, there is also no integer solution of the equation (4.2) for d odd. Using $N^+(d) = N^-(d) = 0$, by Lemma 4.1, we obtain $M(d) = 4$, as claimed in (ii), and in fact

$$\mathcal{M}(d) = \{(-d - 1, -1), (-d + 1, 1), (-1, d - 1), (1, d + 1)\}$$

for each odd $d \geq 3$. □

To handle the case when d is the product of a power of 2 and a power of an odd prime, i.e., $d = 2^r p^s$, where $p \geq 3$ is a prime and $r, s \geq 1$, we shall use Mihăilescu’s theorem, that is, Lemma 2.2. Recall that a prime number p is said to be a *Fermat prime* if $p = 2^m + 1$ for some positive integer m , and consequently m must be a power of 2. So far, the only known Fermat primes are 3, 5, 17, 257 and 65537. Also, recall that a prime number p is called a *Mersenne prime* if $p = 2^m - 1$ for some positive integer m , and in fact m must be also a prime.

Theorem 4.3. *Let r and s be two positive integers. For $1 \leq r \leq 3$ we have*

$$M(2^r 3^s) = \begin{cases} 11 & \text{if } s = 1, \\ 9 & \text{if } s = 2, \\ 7 & \text{if } s \geq 3; \end{cases}$$

and for $r \geq 4$, we have

$$M(2^r 3^s) = \begin{cases} 9 & \text{if } s = 1, \\ 7 & \text{if } s = 2, \\ 5 & \text{if } s \geq 3. \end{cases}$$

Let $p \geq 5$ be a prime, and let r, s be two positive integers. Then,

$$M(2^r p^s) = \begin{cases} 9 & \text{if } s = 1, \text{ and either } p = 2^r + 1 \text{ or } p = 2^r - 1, \\ 7 & \text{if } s \geq 2, \text{ and either } p = 2^r + 1 \text{ or } p = 2^r - 1, \\ 7 & \text{if } s = 1, \text{ and either } p \text{ is a Fermat prime satisfying} \\ & p \neq 2^r + 1, \text{ or } p \text{ is a Mersenne prime satisfying} \\ & p \neq 2^r - 1, \\ 5 & \text{otherwise.} \end{cases}$$

Proof. By Lemma 4.1, it suffices to count primitive integer solutions of the equations (4.1) and (4.2).

Consider the equation (4.1) with $d = 2^r p^s$, where $p \geq 3$ is a prime. Since $y > x \geq 1$, from $g^y + g^x = g^x(g^{y-x} + 1) = 2^r p^s$, we must have

$$(4.6) \quad \text{either } \begin{cases} g^x = 2^r, \\ g^{y-x} + 1 = p^s, \end{cases} \quad \text{or} \quad \begin{cases} g^x = p^s, \\ g^{y-x} + 1 = 2^r. \end{cases}$$

In the first case, since g is not a perfect power, we must have $g = 2$ and $x = r$. The second equation $g^{y-x} + 1 = p^s$ becomes

$$(4.7) \quad 2^{y-r} + 1 = p^s.$$

By Lemma 2.2, in (4.7) we cannot have $s \geq 3$. Suppose that in (4.7) we have $s = 2$. Then, by Lemma 2.2, $p = 3$ and $y = r + 3$. This gives the unique primitive solution $(g, x, y) = (2, r, r + 3)$ of (4.1). If in (4.7) we have $s = 1$, then there is a unique primitive solution of (4.1) if and only if p is a Fermat prime. (Otherwise, (4.1) has no primitive solutions.) Consequently, the contribution of the “first case” into the quantity $N^+(2^r p^s)$ is one if $(p, s) = (3, 2)$ or if p is a Fermat prime and $s = 1$, and zero otherwise.

In the second case of (4.6), we must have $g = p$ and $x = s$. The second equation $g^{y-x} + 1 = 2^r$ becomes

$$(4.8) \quad p^{y-s} + 1 = 2^r.$$

Clearly, $r \geq 2$. Note that we cannot have $y - s \geq 2$ in (4.8), by Lemma 2.2. Hence, $y = s + 1$. This yields $p = 2^r - 1$. Hence, the contribution of the “second case” of (4.8) into the quantity $N^+(2^r p^s)$ is one if and only if $p = 2^r - 1$, where $r \geq 2$, and zero otherwise. Combining both these contributions we deduce that

$$(4.9) \quad N^+(2^r p^s) = \begin{cases} 2 & \text{if } p = 3, r = 2, s \in \{1, 2\}, \\ 1 & \text{if } p = 3, r \neq 2, s \in \{1, 2\}, \\ 1 & \text{if } p = 3, r = 2, s \geq 3, \\ 1 & \text{if } p \geq 5 \text{ is a Fermat prime and } s = 1, \\ 1 & \text{if } p = 2^r - 1 \text{ and } r \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Now, let us investigate the equation (4.2) with $d = 2^r p^s$. Since $y > x \geq 1$, by $g^y - g^x = g^x(g^{y-x} - 1) = 2^r p^s$, we must have

$$(4.10) \quad \text{either } \begin{cases} g^x = 2^r \\ g^{y-x} - 1 = p^s \end{cases} \quad \text{or} \quad \begin{cases} g^x = p^s \\ g^{y-x} - 1 = 2^r. \end{cases}$$

In the first case of (4.10), we obtain $(g, x) = (2, r)$, and the second equation $g^{y-x} - 1 = p^s$ becomes

$$(4.11) \quad 2^{y-r} - 1 = p^s.$$

Clearly, we must have $y - r \geq 2$. By Lemma 2.2, the equality in (4.11) can not hold for $s \geq 2$. For $s = 1$ there is a unique integer solution of (4.11) if and only if p is a Mersenne prime.

In the second case of (4.10), we obtain $(g, x) = (p, s)$. The second equation $g^{y-x} - 1 = 2^r$ becomes

$$p^{y-s} - 1 = 2^r.$$

For $r = 1$ we obtain $p = 3$ and $y = s + 1$. For $r = 3$, we must have $p = 3$ and $y = s + 2$. Then, for $r \in \mathbb{N} \setminus \{1, 3\}$, by Lemma 2.2, we must have $y = s + 1$ and so p is a Fermat prime of the form $p = 2^r + 1$. Therefore, as above, combining both contributions into $N^-(2^r p^s)$ we derive that

$$(4.12) \quad N^-(2^r p^s) = \begin{cases} 2 & \text{if } p = 3, s = 1, r \in \{1, 3\}, \\ 1 & \text{if } p = 3, s = 1, r \notin \{1, 3\}, \\ 1 & \text{if } p = 3, s \geq 2, r \in \{1, 3\}, \\ 1 & \text{if } p \geq 7 \text{ is a Mersenne prime and } s = 1, \\ 1 & \text{if } p = 2^r + 1 \text{ and } r \geq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, applying Lemma 4.1 and combining (4.9) with (4.12) first for $p = 3$ and then for $p \geq 5$, we conclude the proof. \square

Obviously, given an explicit value of d , following the arguments in the proof of Theorem 4.3 we can compute the exact value of $M(d)$. However, the argument can be quite complicated when d has many distinct prime factors. At the end of the paper we will present an algorithm which allows to calculate $M(d)$ for any given even integer $d \in \mathbb{N}$.

4.3. Unconditional upper bound

Note that in the above we have obtained the exact value of $M(d)$ when d is either odd or has at most two distinct prime factors. Now, we present an unconditional upper bound for $M(d)$ when d is even and has at least three distinct prime factors.

Theorem 4.4. *Suppose that an even integer $d \in \mathbb{N}$ has $m \geq 3$ distinct prime factors. Then*

$$(4.13) \quad M(d) \leq 2^{m+1} + 1.$$

Furthermore, if d is square-free, then

$$(4.14) \quad M(d) \leq \begin{cases} 13 & \text{if } m = 3, \\ 2^{m+1} + 7 - 4m & \text{if } m \geq 4. \end{cases}$$

Proof. We first define the subset of factors of d :

$$\mathcal{D}(d) = \{j : j \mid d, \gcd(j, d/j) = 1, 1 < j < d\}.$$

Since d has m distinct prime factors, where $m \geq 3$, we have

$$|\mathcal{D}(d)| = \binom{m}{1} + \binom{m}{2} + \dots + \binom{m}{m-1} = 2^m - 2.$$

From (4.1), since $1 \leq x < y$ and $d = g^x(g^{y-x} + 1)$, in view of $\gcd(g^x, g^{y-x} + 1) = 1$, we obtain $g^x \in \mathcal{D}(d)$. By the same argument, from (4.2) it follows that $g^x \in \mathcal{D}(d)$. However, since d is not of the form $2^r \cdot 3$, there are no positive integer $g \geq 2$ and non-negative integers x, u, v for which

$$d = g^x(g^u + 1) = g^x(g^v - 1).$$

This means that g^x counted as a primitive solution (g, x, y) in $N^+(d)$ and g^x similarly counted in $N^-(d)$ are distinct. Thus, we obtain

$$N^+(d) + N^-(d) \leq |\mathcal{D}(d)| = 2^m - 2.$$

Therefore, applying Lemma 4.1, we deduce that

$$M(d) = 2N^+(d) + 2N^-(d) + 5 \leq 2^{m+1} + 1.$$

This completes the proof of (4.13).

From the above discussion, we see that there is an injective map, say σ , from the primitive integer solutions of (4.1) or (4.2) to the set $\mathcal{D}(d)$ that sends (g, x, y) to g^x . To prove the second part in (4.14), we need to show that there are m elements in $\mathcal{D}(d)$ which are not in the image of σ when $m \geq 4$. Now, we assume that d is square-free with the following prime factorization:

$$d = p_1 p_2 \cdots p_m, \quad p_1 = 2 < p_2 < \cdots < p_m.$$

We first claim that the cases $g^x = d/p_i, 1 \leq i \leq m - 1$, cannot happen neither in (4.1) nor in (4.2). Indeed, fix p_i , where $i < m$. If the equation (4.1) has an integer solution with $g^x = d/p_i$, then we must have $g = d/p_i$ and $x = 1$. Thus, by $d = g^y + g^x$ and by the choice of p_i , we obtain $y = 1$, which contradicts to $y > x$. Similarly, we can show that the equation (4.2) has no integer solution (g, x, y) for which $g^x = d/p_i$. This proves the claim, and this claim actually shows that these $m - 1$ elements $(d/p_i, i = 1, 2, \dots, m - 1)$ in $\mathcal{D}(d)$ are not in the image of σ . Hence, we have

$$M(d) \leq 2(|\mathcal{D}(d)| - (m - 1)) + 5 = 2^{m+1} + 3 - 2m.$$

In particular, this implies the first part of (4.14) when $m = 3$.

To complete the proof, we only need to exclude $m - 2$ more cases when $m \geq 4$. For any $2 \leq i < m$, as the above, both equations (4.1) and (4.2) have no integer solution with $g^x = d/(p_1 p_i)$, where we need to use $m \geq 4$. So, this shows that these $m - 2$ elements $(d/(p_1 p_i), i = 2, 3, \dots, m - 1)$ in $\mathcal{D}(d)$ are not in the image of σ . This in fact completes the proof. \square

We remark that the estimate (4.14) is optimal in general. For example, $M(30) = 13$, which achieves the first upper bound in (4.14). In fact, $\mathcal{M}(30)$ consists of the following 13 vectors:

$$\begin{aligned} &(-15, 15), (-1, 29), (-29, 1), (1, 31), (-31, -1), (-5, 25), (-25, 5), \\ &(-3, 27), (-27, 3), (2, 32), (-32, -2), (6, 36), (-36, -6). \end{aligned}$$

Here, except for the five vectors in the set $\{(-15, 15)\} \cup S_0$, we have eight more vectors in view of

$$30 = 5^2 + 5 = 3^3 + 3 = 6^2 - 6 = 2^5 - 2,$$

so that $N^+(30) = N^-(30) = 2$.

4.4. Conditional upper bound

Actually, under the *ABC conjecture*, there is a uniform upper bound for $M(d)$ where $d \in \mathbb{N}$. To show this, we need some preparations.

Recall that the *ABC conjecture* asserts that for a given real $\varepsilon > 0$ there exists a constant K_ε depending only on ε such that for any non-zero integers A, B, C satisfying

$$A + B = C$$

and $\gcd(A, B) = 1$ we have

$$\max\{|A|, |B|, |C|\} \leq K_\varepsilon \left(\prod_{p|ABC} p \right)^{1+\varepsilon},$$

where p runs through all the (distinct) prime factors of ABC .

We first show an unconditional result, which is an analogue of Theorem 6.2 in [3].

Lemma 4.5. *Assume that x_1, x_2, y_1, y_2 are fixed positive integers with $x_1 > x_2, y_1 > y_2, x_1 > y_1, \gcd(x_1, x_2) = 1$ and $\gcd(y_1, y_2) = 1$. Then, the equation*

$$(4.15) \quad a^{x_1} + a^{x_2} = b^{y_1} + b^{y_2}$$

has only finitely many positive integer solutions (a, b) .

Proof. Note that, since $x_1 > y_1$ and $y_1 > y_2 \geq 1$, we have $x_1 > y_1 \geq 2$. If $y_1 \geq 3$, then, by Theorem 1 of [10], the equation

$$a^{x_1} + a^{x_2} = b^{y_1} + b^{y_2}$$

has only finitely many positive integer solutions (a, b) .

Next, let $y_1 = 2$. Then, $y_2 = 1$, and thus the equation (4.15) becomes

$$(4.16) \quad a^{x_1} + a^{x_2} = b^2 + b$$

with unknowns a, b . If $x_1 = 2x_2$, then, since $\gcd(x_1, x_2) = 1$, we must have $x_1 = 2$, which contradicts with $x_1 > y_1 = 2$. So, we can assume that $x_1 \neq 2x_2$. Then, using Theorem 2 of [10] and noticing $x_1 \geq 3$, we only need to consider the following cases:

$$(4.17) \quad (x_1, x_2) = (3, 1), (3, 2), (4, 1), (4, 3), (6, 2), \text{ and } (6, 4).$$

In order to apply Lemma 2.3, we rewrite (4.16) as

$$(4.18) \quad 4a^{x_1} + 4a^{x_2} + 1 = (2b + 1)^2.$$

For any case of (x_1, x_2) listed in (4.17), the left-hand side of (4.18) is in fact a polynomial in a . By computing its discriminant, one can see that it is non-zero, so the polynomial $4a^{x_1} + 4a^{x_2} + 1$ has at least three simple roots. Thus, by Lemma 2.3, the equation (4.18) has only finitely many integer solutions (a, b) . This completes the proof of the lemma. \square

The following lemma is a direct analogue of Theorem 6.1 in [3], where the equation $a^{x_1} - a^{x_2} = b^{y_1} - b^{y_2}$ instead of (4.19) have been considered.

Lemma 4.6. *Under the ABC conjecture, the equation*

$$(4.19) \quad a^{x_1} + a^{x_2} = b^{y_1} + b^{y_2}$$

has only finitely many positive integer solutions $(a, b, x_1, x_2, y_1, y_2)$ with $a > 1, b > 1, x_1 > x_2, y_1 > y_2$ and $a^{x_1} \neq b^{y_1}$.

Proof. First, applying the same arguments as those in Step 1 and Step 2 of the proof of Theorem 6.1 in [3], we can prove that, under the ABC conjecture, both x_1 and y_1 are bounded from above.

Next, let us fix positive integers x_1, x_2, y_1, y_2 , where $x_1 > x_2, y_1 > y_2$. If $\gcd(x_1, x_2) > 1$, then in (4.19) we can replace a by $a^{\gcd(x_1, x_2)}$. So, without loss of generality, we can assume that $\gcd(x_1, x_2) = 1$ and $\gcd(y_1, y_2) = 1$. If $x_1 = y_1$, then by $a^{x_1} \neq b^{y_1}$ we have $a \neq b$, say $a > b$, and so

$$a^{x_1} + a^{x_2} > a^{x_1} \geq (b + 1)^{x_1} = (b + 1)^{y_1} > b^{y_1} + b^{y_2},$$

which implies that there is no such integer solution (a, b) . Thus, we can further assume that $x_1 \neq y_1$, say, $x_1 > y_1$. Then, by Lemma 4.5, the equation

$$a^{x_1} + a^{x_2} = b^{y_1} + b^{y_2}$$

has only finitely many positive integer solutions (a, b) . This concludes the proof. \square

The next corollary follows from Theorem 6.1 of [3] and Lemma 4.6.

Corollary 4.7. *Under the ABC conjecture, for each sufficiently large d we have $N^+(d) \leq 1$ and $N^-(d) \leq 1$.*

Proof. By Lemma 4.6, under the ABC conjecture, there are only finitely many positive integer solutions of (4.19). So, excluding these solutions, for large enough d there will be no solutions $(a, b, x_1, x_2, y_1, y_2)$ of the equation $a^{x_1} + a^{x_2} = b^{y_1} + b^{y_2} = d$ with restrictions as in Lemma 4.6. This yields $N^+(d) \leq 1$ for d large enough. Similar argument implies $N^-(d) \leq 1$, by Theorem 6.1 of [3]. \square

We are now ready to give a conditional uniform upper bound for $M(d)$.

Theorem 4.8. *Under the ABC conjecture, there is a positive integer C_1 such that for any integer $d \in \mathbb{N}$ we have $M(d) \leq C_1$. Moreover, under the ABC conjecture, we have $M(d) \leq 9$ for d large enough.*

Proof. Take any d_1 such that for $d \geq d_1$ the two inequalities in Corollary 4.7 hold. Set $C_2 = \max_{1 \leq d < d_1} N^+(d)$ and $C_3 = \max_{1 \leq d < d_1} N^-(d)$. (Evidently, we have $C_2 < \infty$ and $C_3 < \infty$ by Theorems 4.2, 4.3 and 4.4.) Therefore, Lemma 4.1 implies that

$$M(d) \leq 2C_2 + 2C_3 + 5.$$

This proves the first assertion of the theorem with $C_1 = 2C_2 + 2C_3 + 5$. For $d \geq d_1$ we have $M(d) \leq 2 + 2 + 5 = 9$, by Corollary 4.7 and Lemma 4.1, which proves the second assertion of the theorem. \square

In Conjecture 4.11 below we predict that the integer C_1 in Theorem 4.8 can be chosen to be 13 according to the numerical data. Note that for d large enough the constant 9 of Theorem 4.8 would be best possible. To see this, we can take $d = 3 \cdot 2^r$ with $r \geq 4$. With this choice, by Theorem 4.3 we have $M(d) = 9$ for each such d . Also, we can take d of the form $n^2 + n$, where $n \geq 2$. Then, for each such d we have $N^+(d) \geq 1$. Indeed, this is true if n is not a perfect power. If it is, say $n = g^m$, where $m \geq 2$ and $g \geq 2$ is not a perfect power, we still have $N^+(d) \geq 1$ in view of $d = g^{2m} + g^m$. By the same argument, the inequality $N^-(d) \geq 1$ holds, since

$$d = n^2 + n = (n + 1)^2 - (n + 1).$$

Consequently, $M(d) \geq 9$ for each d of the form $n^2 + n$, $n \geq 2$.

4.5. Numerical data and conjectures

In this section, we want to design an algorithm for computing $M(d)$, $d \in \mathbb{N}$, and perform the corresponding computations.

From Theorem 4.2 (ii), we only need to compute $M(d)$ for positive even integers d . Based on Lemma 4.1, we design Algorithm 1 for this purpose. As one can see, the algorithm is very simple, and essentially it is also an algorithm to solve the equations (4.1) and (4.2). Here, we use PARI/GP [9] to implement this algorithm and make the corresponding computations.

When using Algorithm 1 to compute $M(d)$ for a large range of d , to speed up the computation and save the memory we can set A, B to be two zero vectors of size 2 in Step 2 of Algorithm 1, and then let the algorithm return the value of d if the size 2 is not big enough. Besides, in Step 3 of Algorithm 1 we use the

Algorithm 1 Computing $M(d)$

Require: positive even integer $d \geq 4$ (input).

Ensure: $M(d)$ (output).

- 1: Compute the prime factorization of d , say, $d = p_1^{r_1} p_2^{r_2} \cdots p_m^{r_m}$.
 - 2: Set A, B to be two zero vectors of size 2^m .
 - 3: Execute the subsequent three steps by running through all the factors a of d with $\gcd(a, d/a) = 1$.
 - 4: Given such a factor a of d , say $a = p_1^{r_1} \cdots p_j^{r_j}$, compute $r = \gcd(r_1, \dots, r_j)$ and $g = p_1^{r_1/r} \cdots p_j^{r_j/r}$.
 - 5: Divide $d - a$ repeatedly by g until the quotient is not greater than 1. Then, if the quotient is equal to 1, store a in the vector A .
 - 6: Divide $d + a$ repeatedly by g until the quotient is not greater than 1. Then, if the quotient is equal to 1, store a in the vector B .
 - 7: Count the number of distinct non-zero entries in A , say N_1 , and count the number of distinct non-zero entries in B , say N_2 . Return $M(d) = 2(N_1 + N_2) + 5$.
-

TABLE 1. Statistics of $M(d)$ for positive even integers d

$M(d)$	5	7	9	11	13
$d \leq 10^3$	380	79	33	7	1
$d \leq 10^4$	4653	233	103	10	1
$d \leq 10^5$	49177	488	323	11	1
$d \leq 10^6$	498015	963	1010	11	1
$d \leq 10^7$	4994967	1846	3175	11	1
$d \leq 10^8$	49986562	3410	10015	12	1
$d \leq 10^9$	499961918	6427	31642	12	1
$d \leq 10^{10}$	4999887540	12425	100022	12	1

binary representations of integers between 0 and $2^m - 1$ to run over all such 2^m factors of d . For example, the factor corresponding to the binary number $0\dots 011$ is $p_1^{r_1} p_2^{r_2}$.

In Table 1, the first row shows all the possible values of $M(d)$ for positive even integer $d \leq 10^{10}$. The second row gives the number of such integers $d \leq 10^3$ whose $M(d)$ correspond to the values in the first row. Other rows have similar meaning.

In particular, we have $M(30) = 13$, and $M(d) = 11$ if d is one of the following twelve integers:

$$6, 12, 24, 132, 210, 240, 252, 6480, 8190, 9702, 78120, 24299970.$$

In fact, these thirteen integers are of the form $n^2 + n$ except for $d = 24$ and $d = 252$. For example, $24299970 = 4929^2 + 4929$. Moreover, we used Algorithm 1 to test all the integers $d = n^2 + n$, where $4930 \leq n \leq 10^8$, and found no examples with $M(d) > 9$.

TABLE 2. The values of $d \leq 10^{10}$ with $N^+(d) = 2$

d	Primitive integer solutions (g, x, y) of (4.1)
12	$(2,2,3), (3,1,2)$
30	$(3,1,3), (5,1,2)$
36	$(2,2,5), (3,2,3)$
130	$(2,1,7), (5,1,3)$
132	$(2,2,7), (11,1,2)$
252	$(3,2,5), (6,2,3)$
9702	$(21,2,3), (98,1,2)$
65600	$(2,6,16), (40,2,3)$

Furthermore, from Table 1 and Theorem 4.2 we see that for any positive integer $d \leq 10^{10}$ we have

$$M(d) \leq 13.$$

From Table 1, one can also observe the following interesting phenomenon. Corresponding to the values 5, 7, 9, the quotients of the numbers of such integers d in two nearby rows are very close to 10, 2, 3, respectively.

Based on our computations, we pose two conjectures on the equations (4.1) and (4.2) as follows, which are of independent interest.

Conjecture 4.9. *For any given integer $d \geq 1$, we have $N^+(d) \leq 2$.*

Conjecture 4.10. *For any given integer $d \geq 1$, we have $N^-(d) \leq 2$.*

From our computations, it follows that Conjectures 4.9 and 4.10 are true for all positive integers $d \leq 10^{10}$. Moreover, it is likely that either $N^+(d) = 2$ or $N^-(d) = 2$ are very rare events. We collect the values of positive integers $d \leq 10^{10}$ for which either $N^+(d) = 2$ or $N^-(d) = 2$, and the corresponding primitive integer solutions of the equations (4.1) and (4.2) in Tables 2 and 3, respectively. In particular, one can see that 30 is the unique positive integer in the range $[1, 10^{10}]$ with $N^+(30) = 2$ and $N^-(30) = 2$. We emphasize that, by Corollary 4.7, under the *ABC* conjecture the inequalities $N^+(d) \leq 1$ and $N^-(d) \leq 1$ hold for each sufficiently large d . The last example in Table 3 corresponds to the solution $(x, y) = (30, 9859)$ on the hyperelliptic curve

$$y^2 = 4x^5 - 4x + 1.$$

Inserting $y = 2 \cdot 4930 - 1$ and $x = 30$ we get $4930^2 - 4930 = 30^5 - 30$.

From the proof of Theorem 4.8 we know that, under the *ABC* conjecture, there exists a positive integer $C_4 = \max\{C_2, C_3\}$, which is independent of d , such that each of the equations in Conjectures 4.9 and 4.10 has at most C_4 primitive integer solutions.

Under Conjectures 4.9 and 4.10 and in view of (4.3), for any integer $d \in \mathbb{N}$ we have

$$M(d) \leq 13,$$

TABLE 3. The values of $d \leq 10^{10}$ with $N^-(d) = 2$

d	Primitive integer solutions (g, x, y) of (4.2)
6	(2,1,3), (3,1,2)
24	(2,3,5), (3,1,3)
30	(2,1,5), (6,1,2)
120	(2,3,7), (5,1,3)
210	(6,1,3), (15,1,2)
240	(2,4,8), (3,1,5)
2184	(3,1,7), (13,1,3)
6480	(3,4,8), (6,4,5)
8190	(2,1,13), (91,1,2)
78120	(5,1,7), (280,1,2)
24299970	(30,1,5), (4930,1,2)

which is also compatible with our numerical data. So, in conclusion we suggest the following conjecture.

Conjecture 4.11. *For any $d \in \mathbb{N}$ we have $M(d) \leq 13$. Moreover, $M(d) = 13$ if and only if $d = 30$.*

In fact, the second part of Conjecture 4.11 asserts that 30 is the unique positive integer d satisfying $N^+(d) = N^-(d) = 2$.

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