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A unified approach of blow-up phenomena for two-dimensional singular Liouville systems

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Abstract. We consider generic 2×2 singular Liouville systems

$$(\star)\qquad \begin{cases} -\Delta u_1 = 2\lambda_1\, e^{u_1} - a\lambda_2\, e^{u_2} - 2\pi(\alpha_1-2)\delta_0 & \text{in } \Omega,\\ -\Delta u_2 = 2\lambda_2\, e^{u_2} - b\lambda_1\, e^{u_1} - 2\pi(\alpha_2-2)\delta_0 & \text{in } \Omega,\\ u_1=u_2=0 & \text{on } \partial\Omega, \end{cases}$$

where $\Omega \ni 0$ is a smooth bounded domain in $\mathbb{R}^2$ possibly having some symmetry with respect to the origin, δ_0 is the Dirac mass at 0, λ_1,λ_2 are small positive parameters and $a,b,\alpha_1,\alpha_2>0$.

We construct a family of solutions to $(\star)$ which blow up at the origin as $\lambda_1\to 0$ and $\lambda_2\to 0$ and whose local mass at the origin is a given quantity depending on a,b,α_1,α_2.

In particular, if $ab<4$ we get finitely many possible blow-up values of the local mass, whereas if $ab\geq 4$ we get infinitely many. The blow-up values are produced using an explicit formula which involves Chebyshev polynomials.

1. Introduction

In this paper we consider the system of singular Liouville equations

$$\begin{cases} -\Delta u_1 = 2\lambda_1\, e^{u_1} - a\lambda_2\, e^{u_2} - 2\pi(\alpha_1-2)\delta_0 & \text{in } \Omega,\\ -\Delta u_2 = 2\lambda_2\, e^{u_2} - b\lambda_1\, e^{u_1} - 2\pi(\alpha_2-2)\delta_0 & \text{in } \Omega,\\ u_1=u_2=0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^2$ is a smooth bounded domain which contains the origin, δ_0 is the Dirac mass at 0, λ_i are small positive parameters and the matrix

$$\mathcal{A} = \begin{pmatrix} 2 & -a\\ -b & 2\end{pmatrix} \quad \text{with } a,b>0. \tag{1.2}$$

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Liouville systems find applications in many fields of physics and mathematics, like theory of chemotaxis [20], theory of charged particle beams [36], theory of semi-conductors [48], Chern–Simons theory [38], [29], [47], [27], [39], [53], [54], and holomorphic projective curves [14], [19], [11], [39], [12], [26], [31].

It is not hard to see that any more general Liouville systems

$$\begin{cases} -\Delta u_1 = a_{11}\lambda_1\, e^{u_1} + a_{12}\lambda_2\, e^{u_2} - 2\pi(\alpha_1 - 2)\delta_0 & \text{in } \Omega, \\ -\Delta u_2 = a_{21}\lambda_2\, e^{u_2} + a_{21}\lambda_1\, e^{u_1} - 2\pi(\alpha_2 - 2)\delta_0 & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

with $a_{12}, a_{21} < 0 < a_{11}, a_{22}$, can be brought back to (1.1) just by a rescaling of the parameters λ_1 and λ_2.

Using Green's function

$$\begin{cases} -\Delta G(\cdot, y) = \delta_y & \text{in } \Omega, \\ G(\cdot, y) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and its decomposition

$$G(x,y) = \frac{1}{2\pi}\log|x-y| + H(x,y), \quad x, y \in \Omega, \tag{1.4}$$

with $H(x,y)$ smooth, we can eliminate the singularity on the right-hand side of (1.1) and rewrite the system as

$$\begin{cases} -\Delta u_1 = 2\lambda_1\, h_1\, e^{u_1} - a\,\lambda_2\, h_2\, e^{u_2} & \text{in } \Omega, \\ -\Delta u_2 = 2\lambda_2\, h_2\, e^{u_2} - b\lambda_1\, h_1\, e^{u_1} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where

$$h_i(x) = |x|^{\alpha_i - 2}\, e^{-2\pi(\alpha_i - 2)H(x,0)} \quad \text{for } i = 1, 2, \tag{1.6}$$

and $e^{-2\pi(\alpha_i-2)H(x,0)}$ is smooth and positive.

One of the most important and challenging issues concerning Liouville systems (1.1) or (1.5) is the blow-up phenomena. A point $x_0 \in \overline{\Omega}$ is called a blow-up point if a sequence of solutions $u_n = (u_{1,n}, u_{2,n})$ satisfies

$$\max_{i=1,2}\ \max_{B_r(x_0)\cap\Omega} u_{i,n} = \max_{i=1,2} u_{i,n}(x_n) \underset{n\to+\infty}{\longrightarrow} +\infty \quad \text{and} \quad x_n \underset{n\to+\infty}{\longrightarrow} x_0.$$

Knowing the asymptotic behavior of blowing-up solutions near the blow-up points is the first step in applying topological or variational methods to get solutions to the Liouville systems. In particular, the first main issue is to determine the set of critical masses of solutions with bounded energy, i.e., $\max_{i=1,2} \lambda_{i,n} \int_\Omega h_i e^{u_{i,n}} \le C$ for some C.

We define the local masses at the blow-up point x_0 as

$$m_i(x_0) := \lim_{r\to 0}\lim_{n\to+\infty} \lambda_{i,n} \int_{B_r(x_0)} h_i\, e^{u_{i,n}} \quad \text{for } i = 1, 2. \tag{1.7}$$

The local masses have been widely studied in the last years.

When the system (1.1) reduces to a single singular Liouville equation

$$\begin{cases} -\Delta u = \lambda e^u - 2\pi(\alpha-2)\delta_0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.8}$$

the local mass has been completely characterized. In particular, in the regular case, i.e., $\alpha = 2$, all the blow-up points are internal to Ω, they are simple and the local mass equals 8π (see Brezis and Merle [13], Nagasaki and Suzuki [50], Li and Shafrir [40]). In fact, in this case there is only one bubbling profile: after some rescaling, the bubble approaches a solution of the Liouville equation

$$\begin{cases} -\Delta U = e^U & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^U < +\infty. \end{cases} \tag{1.9}$$

In the singular case, i.e., $\alpha \neq 2$, the local mass around the origin is $4\pi\alpha$ and the corresponding bubbling profile, after some rescaling, is given by solution to the singular Liouville equation

$$\begin{cases} -\Delta U = |\cdot|^{\alpha-2}\, e^U & \text{in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |\cdot|^{\alpha-2}\, e^U < +\infty. \end{cases} \tag{1.10}$$

(see Bartolucci and Tarantello [6] and Bartolucci, Chen, Lin and Tarantello [3]).

The knowledge of the bubbling profile is the main step in finding existence and multiplicity results concerning the equation (1.8). Indeed, bubbling solutions with multiple concentration points have been built by Baraket and Pacard [2], del Pino, Kowalczyk and Musso [24] and Esposito, Grossi and Pistoia [28] in the regular case and by del Pino, Esposito and Musso [22] and D'Aprile [21] in the singular case. Moreover, a degree formula has been obtained by Chen and Lin [17] and [18], and Malchiodi [43], whereas solutions have also been found through variational methods by Bartolucci and Malchiodi [5], Bartolucci, De Marchis and Malchiodi [4], Carlotto and Malchiodi [15], Djadli [25] and Malchiodi and Ruiz [45].

The natural generalization to (1.8) is the 2×2 system (1.1) when the matrix $\mathcal{A} = (a_{ij})_{2\times 2}$ is as in (1.2). In particular, when $\mathcal{A}$ is the Cartan matrix of a simple Lie algebra we get the well-known Toda system. In this case, since the rank of the simple Lie Algebra is 2, there are three types of corresponding Cartan matrices of rank 2:

$$\mathcal{A}_2 = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathcal{B}_2 = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}, \quad \mathcal{G}_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}. \tag{1.11}$$

In the regular $\mathcal{A}_2$-Toda system, i.e., $\alpha_1 = \alpha_2 = 2$, Jost, Lin and Wang in [34] found that the local masses can only take 5 values. Moreover, all these blow-up values can occur as shown by Musso, Pistoia and Wei in [49] (see also Ao and Wang, [1]).

In the singular case Lin, Wei and Zhang in [42] found that only 5 possible values are allowed for the local masses, provided the singularities α_1 and α_2 satisfy a suitable condition (which also include the regular case) (see Example 1.7).

Recently, Lin and Zhang in [41] found that only 7 possible values are allowed for the local masses in the regular $\mathcal{B}_2$-Toda system (see Example 1.8) and 11 possible values are allowed for the local masses in the regular $\mathcal{G}_2$-Toda system (see Example 1.9) under some extra assumptions.

Solutions to the regular A_2 Toda system have been found both through the computation of the degree by Lin, Wei and Yang in [37] and variationally by Battaglia, Jevnikar, Malchiodi and Ruiz in [9], Jevnikar, Kallel and Malchiodi [32], Malchiodi and Ndiaye [44] and Malchiodi and Ruiz [46]. Variational solutions have also been found for the A_2 Toda system in Battaglia, Jevnikar, Malchiodi and Ruiz in [9], Battaglia in [7] and Battaglia and Malchiodi in [10] and for the B_2 and G_2 systems by Battaglia in [8].

At this stage, two questions naturally arise:

(Q1) *Which are the values of the local masses at the origin for the system* (1.1) *for a general matrix $\mathcal{A}$ with or without singular sources?*

(Q2) *Are these values attained?*

In this paper we focus on the second question and we give a partial answer. More precisely, we build solutions to the system (1.1), whose components blows-up at the origin and whose local masses are quantized in terms of a, b, α_1 and α_2. In particular, if $\det(\mathcal{A}) \leq 0$ we find infinitely many possible values for the local masses. We also provide an explicit formula involving Chebyshev polynomials which produces blow-up values of the local masses (see Remark 1.5). We also conjecture that these are the only admissible values when the blowing-up profile of each component resembles one or more bubbles solving the scalar Liouville equations (1.8). Indeed, they coincide with the known ones when the matrix $\mathcal{A}$ is as in (1.11) (see Examples 1.7, 1.8 and 1.9).

Let us state our main result.

For any integer $\ell \in \mathbb{N}$, we introduce the polynomials

$$(1.12)\qquad \begin{cases} P_0(t) = 0, \\ P_1(t) = 1, \\ P_2(t) = 1, \\ P_\ell(t) = \prod_{i=1}^{[(\ell-1)/2]} \left(t - 2 - 2\cos\frac{2\pi i}{\ell}\right) & \text{if } \ell \geq 3, \end{cases}$$

and the real numbers $\beta_\ell = \beta_\ell(a, b, \alpha_1, \alpha_2)$ defined as follows:

$$(1.13)\qquad \beta_\ell = \begin{cases} \alpha_1 P_\ell(ab) + a\,\alpha_2\, P_{\ell-1}(ab) & \text{if } \ell \text{ is odd}, \\ b\,\alpha_1 P_\ell(ab) + \alpha_2\, P_{\ell-1}(ab) & \text{if } \ell \text{ is even}. \end{cases}$$

Then we define the (possibly infinite) integer

$$(1.14)\qquad k_{\max} = k_{\max}(a, b, \alpha_1, \alpha_2) := \sup\{k : \beta_\ell > 0, \forall\, \ell = 1, \dots, k\}.$$

By (1.12) and (1.13) it immediately follows that $k_{\max} \geq 2$, and we also deduce that $k_{\max} = +\infty$ if $ab \geq 4$. In Remark 2.5, we find the following expression of $k_{\max}$ in terms of a and b when $ab < 4$:

$$(1.15)\qquad k_{\max} = \begin{cases} \frac{2\pi}{\arccos(ab/2-1)} & \text{if } \frac{2\pi}{\arccos(ab/2-1)} \in \mathbb{N}, \\ \left[\frac{2\pi}{\arccos(ab/2-1)}\right] & \text{if } \beta_{\left[\frac{2\pi}{\arccos(ab/2-1)}\right]+1} < 0, \\ \left[\frac{2\pi}{\arccos(ab/2-1)}\right] + 1 & \text{if } \beta_{\left[\frac{2\pi}{\arccos(ab/2-1)}\right]+1} > 0. \end{cases}$$

Definition 1.1. Set $\mathcal{I} := \{\ell \in \{1,\dots,k\} \ : \ \beta_\ell \in 2\mathbb{N}\}$. We say that Ω is *compatible* if

$$e^{\frac{2\pi}{\mathfrak{m}}\iota}\,\Omega := \left\{\left(x_1\cos\frac{2\pi}{\mathfrak{m}} - x_2\sin\frac{2\pi}{\mathfrak{m}}, x_1\sin\frac{2\pi}{\mathfrak{m}} + x_2\cos\frac{2\pi}{\mathfrak{m}}\right) \ : \ (x_1,x_2)\in\Omega\right\} = \Omega,$$

where $\mathfrak{m} := \operatorname{lcm}\{\mathfrak{m}_\ell \in \mathbb{N} : \beta_\ell/\mathfrak{m}_\ell \notin 2\mathbb{N},\ \ell \in \mathcal{I}\}$. In particular, if $\mathcal{I} = \emptyset$ any smooth bounded domain Ω containing the origin is *compatible*.

Remark 1.2. The integer $\mathfrak{m}$ introduce in the previous definition is not uniquely defined, since it depends on the choice of $\mathfrak{m}_1,\dots,\mathfrak{m}_k$, which are not unique. In the definition of compatibility we want the equality to hold true for *at least one* of such possible $\mathfrak{m}$'s.

Theorem 1.3. *Let $k\in\mathbb{N}$, $k \le k_{\max}$ be fixed, let $\beta_1,\dots,\beta_k$ be defined by (1.13), and let $\Omega \ni 0$ be a smooth bounded domain which is* compatible *in the sense of Definition 1.1.*

Then, there exists $\overline{\lambda} = \overline{\lambda}(k) > 0$ such that, for λ satisfying

$$(1.16)\quad \begin{cases} \lambda_1,\lambda_2 \in \left(0,\overline{\lambda}\right) & \text{if } k < k_{\max}, \\ \lambda_1,\lambda_2 \in \left(0,\overline{\lambda}\right),\ \lambda_2 \le \lambda_1^{\frac{\gamma-\beta_{k_{\max}+1}}{\beta_{k_{\max}}}} & \text{if } k = k_{\max} \text{ is odd}, \\ \lambda_1,\lambda_2 \in \left(0,\overline{\lambda}\right),\ \lambda_1 \le \lambda_2^{\frac{\gamma-\beta_{k_{\max}+1}}{\beta_{k_{\max}}}} & \text{if } k = k_{\max} \text{ is even}, \end{cases} \quad \text{for some } \gamma > 0,$$

the problem (1.1) has a solution $u = u_\lambda = (u_{1,\lambda}, u_{2,\lambda})$.

Moreover, there holds

$$(1.17)\qquad m_1(0) = 2\pi \sum_{j=0}^{[(k-1)/2]} \beta_{2j+1} \quad \text{and} \quad m_2(0) = 2\pi \sum_{j=0}^{[(k-2)/2]} \beta_{2j+2},$$

where we agree that $m_2(0) = 0$ if $k = 1$. Moreover, if G is the Green's function defined in (1.3), we have, as $\lambda \to 0$,

$$(1.18)\quad u_1 \to [2m_1(0) - am_2(0)]\,G(\cdot,0) \quad \text{and} \quad u_2 \to [2m_2(0) - bm_1(0)]\,G(\cdot,0)$$

weakly in $W^{1,q}(\Omega)$ for any $q < 2$ and strongly in $C^\infty_{\mathrm{loc}}(\Omega\setminus\{0\})$.

Remark 1.4. We point out that if Ω is a symmetric domain according to in Definition 1.1, then the functions h_j defined in (1.6) and the solutions found in Theorem 1.3 inherit the symmetry of the domain Ω, namely they satisfy the symmetry condition $u(e^{\frac{2\pi}{\mathfrak{m}}\iota}x) = u(x)$ for any $x \in \Omega$, where $\mathfrak{m}$ is as in Definition 1.1.

Remark 1.5. As far as we know, this is the first result which gives a clear relation between the local masses at the origin and their possible number and the values of the entries of the matrix $\mathcal{A}$ in (1.2) and the values of the singularities α_1 and α_2. Indeed, we can express the masses in (1.17) in terms of the value of the polynomials $P_\ell(t)$ at $t = ab$ as (see Remark 2.7):

$$m_1(0) = \begin{cases} 2\pi a\, P_{[(k-1)/2]}(ab)\left(b\alpha_1 P_{[(k-1)/2]}(ab) + \alpha_2\, P_{[(k-3)/2]}(ab)\right) \\ \qquad\qquad\qquad\qquad \text{if } k \in (4\mathbb{N}+1)\cup(4\mathbb{N}+2), \\ 2\pi\, P_{[(k-1)/2]}(ab)\left(\alpha_1 Pv(ab) + a\,\alpha_2\, P_{[(k-3)/2]}(ab)\right) \\ \qquad\qquad\qquad\qquad \text{if } k \in (4\mathbb{N}+3)\cup 4\mathbb{N}, \end{cases} \tag{1.19}$$

and

$$m_2(0) = \begin{cases} 2\pi\, P_{[(k-2)/2]}(ab)\left(b\alpha_1 P_{[k/2]}(ab) + \alpha_2\, P_{[(k-2)/2]}(ab)\right) \\ \qquad\qquad\qquad\qquad \text{if } k \in 4\mathbb{N}\cup(4\mathbb{N}+1), \\ 2\pi b\, P_{[(k-2)/2]}(ab)\left(\alpha_1 P_{[k/2]}(ab) + a\,\alpha_2\, P_{[(k-2)/2]}(ab)\right) \\ \qquad\qquad\qquad\qquad \text{if } k \in (4\mathbb{N}+2)\cup(4\mathbb{N}+3), \end{cases} \tag{1.20}$$

where the range of k is between 1 and the number $k_{\max}$ defined in (1.14).

Remark 1.6. The bubbling profile of each component resembles a sum (with alternating sign) of bubbles solutions to different singular Liouville problems (1.10): all the bubbles are centered at the origin and the rate of concentration of each bubble at the origin is slower than the previous one, namely

$$u_1 \sim w_1 - \frac{a}{2}w_2 + w_3 - \frac{a}{2}w_4 + \cdots \quad \text{and} \quad u_2 \sim -\frac{b}{2}w_1 + w_2 - \frac{b}{2}w_3 + w_4 + \cdots \tag{1.21}$$

where

$$\begin{aligned} w_i(x) &:= \log 2\beta_i^2 \frac{\delta_i^{\beta_i}}{(\delta_i^{\beta_i} + |x|^{\beta_i})^2} \quad x \in \mathbb{R}^2,\ \delta_i > 0 \\ &\text{solves} \quad -\Delta w_i = |\cdot|^{\beta_i - 2} e^{w_i} \text{ in } \mathbb{R}^2 \end{aligned} \tag{1.22}$$

and δ_i/δ_{i+1} approaches zero. The construction of a solution with such a profile is possible as long as the exponents β_i's are positive and that is why we need to introduce the maximal number of bubbles $k_{\max}$ in (1.14). Moreover, each bubble w_ℓ scaled with δ_i turns out to be a singular source for the equation solved by the bubble w_i whenever $\ell < i$.

Therefore, the choice of each β_i takes into account the singular sources present in the equation and all the singular sources generated by the interactions between the bubbles w_i and all the previous ones. This fact leads to choose β_i as in (2.3) to ensure that the prescribed profile is *almost* a solution to system (1.1) (as carefully proved in Lemma 2.2 and Lemma 4.1).

This kind of construction is strongly inspired by the bubble-tower construction in Musso, Pistoia and Wei [49] (see also Grossi and Pistoia [30]), where the regular $\mathcal{A}_2$-Toda system was studied. Nevertheless, the general case turns out to be rather delicate.

In particular, the interaction between the two components is much more involved because the concentration of each bubble is affected by all the other previous bubbles, not only the ones for which the same component concentrates. Even and odd bubbles affect the concentration in opposite ways.

Moreover, we will need some rather involved symmetry condition, which are needed to invert a linearized operator and strongly depend on the values of β_i. Finally, the presence of singularities gives weaker regularity properties and makes some estimates more subtle.

In the following examples we describe how our result can be applied to classical problems.

Example 1.7. If $a = b = 1$, the system (1.5) becomes the well-known *A_2-Toda system*:

$$\begin{cases} -\Delta u_1 = 2\lambda_1 h_1 e^{u_1} - \lambda_2 h_2 e^{u_2} & \text{in } \Omega, \\ -\Delta u_2 = 2\lambda_2 h_2 e^{u_2} - \lambda_1 h_1 e^{u_1} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

We have $ab = 1$ and by (1.15) we compute $k_{\max} = 3$. Moreover,

$$\begin{cases} P_1(1) = 1, \\ P_2(1) = 1, \\ P_3(1) = -1 - 2\cos\frac{2\pi}{3} = 0. \end{cases}$$

Then, by (1.13) and (1.17) (possibly exchanging the role of the components) we deduce the following configurations for $(m_1(0), m_2(0))$:

- if $k = 1$ we get $2\pi(\alpha_1, 0)$ and $2\pi(0, \alpha_2)$,
- if $k = 2$ we get $2\pi(\alpha_1, \alpha_1 + \alpha_2)$ and $2\pi(\alpha_1 + \alpha_2, \alpha_2)$,
- if $k = 3$ we get $2\pi(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2)$.

In [42] Lin, Wei and Zhang show that, for suitable values of α_1, α_2 (including the regular case $\alpha_1 = \alpha_2 = 2$), the only possible values are the five above. Therefore, Theorem 1.3 shows in particular the sharpness of their classification.

For the regular Toda system, Theorem 1.3 was already proved by Musso, Pistoia and Wei [49].

Example 1.8. The case $a = 1$, $\alpha_1 = \alpha_2 = b = 2$ is the *B_2-Toda system*

$$\begin{cases} -\Delta u_1 = 2\lambda_1 e^{u_1} - \lambda_2 e^{u_2} & \text{in } \Omega, \\ -\Delta u_2 = 2\lambda_2 e^{u_2} - 2\lambda_1 e^{u_1} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

We have $ab = 2$ and by (1.15) we compute $k_{\max} = 4$. Moreover,

$$\begin{cases} P_1(2) = 1, \\ P_2(2) = 1, \\ P_3(2) = -2\cos\frac{2\pi}{3} = 1, \\ P_4(2) = -2\cos\frac{\pi}{2} = 0. \end{cases}$$

Then, by (1.13), (1.17) we deduce the following configurations for $(m_1(0), m_2(0))$:

- if $k = 1$ we get $2\pi(\beta_1, 0) = 2\pi(2, 0)$,
- if $k = 2$ we get $2\pi(\beta_1, \beta_2) = 2\pi(2, 6)$,

- if $k = 3$ we get $2\pi(\beta_1 + \beta_3, \beta_2) = 2\pi(6, 6)$,
- if $k = 4$ we get $2\pi(\beta_1 + \beta_3, \beta_2 + \beta_4) = 2\pi(6, 8)$,

and exchanging the role of the components (i.e., $b = 1$ and $a = 2$),

- if $k = 1$ we get $2\pi(0, \beta_1) = 2\pi(0, 2)$,
- if $k = 2$ we get $2\pi(\beta_2, \beta_1) = 2\pi(4, 2)$,
- if $k = 3$ we get $2\pi(\beta_2, \beta_1 + \beta_3) = 2\pi(4, 8)$,
- if $k = 4$ we get $2\pi(\beta_2 + \beta_4, \beta_1 + \beta_3) = 2\pi(6, 8)$.

In [41] Lin and Zhang show that no other values are admissible in case of blow up. Theorem 1.3 shows the sharpness of their classification.

Example 1.9. The case $a = 1$, $b = 3$, $\alpha_1 = \alpha_2 = 2$ is the G_2-*Toda system*:

$$\begin{cases} -\Delta u_1 = 2\lambda_1 e^{u_1} - \lambda_2 e^{u_2} & \text{in } \Omega, \\ -\Delta u_2 = 2\lambda_2 e^{u_2} - 3\lambda_1 e^{u_1} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

We have $ab = 3$ and by (1.15) we compute $k_{\max} = 6$. Moreover,

$$\begin{cases} P_1(3) = 1, \\ P_2(3) = 1, \\ P_3(3) = 1 - 2\cos\frac{2\pi}{3} = 2, \\ P_4(3) = 1 - 2\cos\frac{\pi}{2} = 1, \\ P_5(3) = \left(1 - 2\cos\frac{2\pi}{5}\right)\left(1 - 2\cos\frac{4\pi}{5}\right) = 1 \\ P_6(3) = \left(1 - 2\cos\frac{\pi}{3}\right)\left(1 - 2\cos\frac{2\pi}{3}\right) = 0. \end{cases}$$

Then, by (1.13), (1.17) we deduce the following configurations for $(m_1(0), m_2(0))$:

- if $k = 1$ we get $2\pi(\beta_1, 0) = 2\pi(2, 0)$,
- if $k = 2$ we get $2\pi(\beta_1, \beta_2) = 2\pi(2, 8)$,
- if $k = 3$ we get $2\pi(\beta_1 + \beta_3, \beta_2) = 2\pi(8, 8)$,
- if $k = 4$ we get $2\pi(\beta_1 + \beta_3, \beta_2 + \beta_4) = 2\pi(8, 18)$,
- if $k = 5$ we get $2\pi(\beta_1 + \beta_3 + \beta_5, \beta_2 + \beta_4) = 2\pi(12, 18)$,
- if $k = 6$ we get $2\pi(\beta_1 + \beta_3 + \beta_5, \beta_2 + \beta_4 + \beta_6) = 2\pi(12, 20)$,

and exchanging the role of the components (i.e., $b = 1$ and $a = 3$),

- if $k = 1$ we get $2\pi(0, \beta_1) = 2\pi(0, 2)$,
- if $k = 2$ we get $2\pi(\beta_2, \beta_1) = 2\pi(4, 2)$,
- if $k = 3$ we get $2\pi(\beta_2, \beta_1 + \beta_3) = 2\pi(4, 12)$,
- if $k = 4$ we get $2\pi(\beta_2 + \beta_4, \beta_1 + \beta_3) = 2\pi(10, 12)$,
- if $k = 5$ we get $2\pi(\beta_2 + \beta_4, \beta_1 + \beta_3 + \beta_5) = 2\pi(10, 20)$,
- if $k = 6$ we get $2\pi(\beta_2 + \beta_4 + \beta_6, \beta_1 + \beta_3 + \beta_5) = 2\pi(12, 20)$.

In [41] Lin and Zhang found the previous blow-up values under some extra assumptions. Theorem 1.3 shows that these blow-up values are attained.

Example 1.10. The case $ab = 4$ is particularly interesting: in Remark 2.4 this is the borderline scenario to have an infinite number of blow-up values. This fact is related to the matrix of the coefficients in (1.1) being singular.

In fact, if we consider the system

$$\begin{cases} -\Delta u_1 = 2\lambda_1\, h_1\, e^{u_1} - a\,\lambda_2\, h_2\, e^{u_2} & \text{in } \Omega, \\ -\Delta u_2 = 2\lambda_2\, h_2\, e^{u_2} - \frac{4}{a}\lambda_1\, h_1\, e^{u_1} & \text{in } \Omega, \\ u_1 = u_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

then a suitable linear combination of the two equation gives

$$\begin{cases} -\Delta\left(u_1 + \frac{a}{2}\, u_2\right) = 0 & \text{in } \Omega, \\ u_1 + \frac{a}{2}\, u_2 = 0 & \text{on } \partial\Omega, \end{cases}$$

which means $u_2 = -\frac{2}{a}\, u_1$; therefore, in this case (1.5) is equivalent to the scalar equation

$$\begin{cases} -\Delta u = 2\lambda_1\, h_1\, e^{u} - a\,\lambda_2\, h_2\, e^{-\frac{2}{a}u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case (see Remark 2.6),

$$P_\ell(4) = \prod_{i=1}^{[(\ell-1)/2]} 2\left(1 - \cos\frac{2\pi i}{\ell}\right) = \begin{cases} \ell & \text{if } \ell \text{ is odd}, \\ \dfrac{\ell}{2} & \text{if } \ell \text{ is even}. \end{cases}$$

Therefore, using Remark 2.7, the infinitely many blow-up masses are

$$\begin{aligned}
&2\pi(\alpha_1, 0) \\
&2\pi\left(\alpha_1, \frac{4}{a}\,\alpha_1 + \alpha_2\right) \\
&2\pi\left(4\alpha_1 + a\,\alpha_2, \frac{4}{a}\,\alpha_1 + \alpha_2\right) \\
&2\pi\left(4\alpha_1 + a\,\alpha_2, \frac{12}{a}\,\alpha_1 + 4\alpha_2\right) \\
&\dots \\
&2\pi\left((\ell+1)^2\alpha_1 + \frac{a}{2}\ell(\ell+1)\alpha_2, \frac{2}{a}\ell(\ell+1)\alpha_1 + \ell^2\alpha_2\right) \\
&2\pi\left((\ell+1)^2\alpha_1 + \frac{a}{2}\ell(\ell+1)\alpha_2, \frac{2}{a}(\ell+1)(\ell+2)\alpha_1 + (\ell+1)^2\alpha_2\right) \\
&\dots \\
&2\pi(0, \alpha_2) \\
&2\pi(\alpha_1 + a\,\alpha_2, \alpha_2) \\
&2\pi\left(\alpha_1 + a\,\alpha_2, \frac{4}{a}\,\alpha_1 + 4\alpha_2\right)
\end{aligned}$$

$$\begin{aligned}
&2\pi\Big(4\alpha_1+3a\,\alpha_2,\frac{4}{a}\,\alpha_1+4\alpha_2\Big)\\
&\dots\\
&2\pi\Big(\ell^2\alpha_1+\frac{a}{2}\ell(\ell+1)\alpha_2,\frac{2}{a}\ell(\ell+1)\alpha_1+(\ell+1)^2\alpha_2\Big)\\
&2\pi\Big((\ell+1)^2\alpha_1+\frac{a}{2}(\ell+1)(\ell+2)\alpha_2,\frac{2}{a}\ell(\ell+1)\alpha_1+(\ell+1)^2\alpha_2\Big)\\
&\dots
\end{aligned}$$

The case $\alpha_1 = \alpha_2 = a = 2$ is known as *sinh-Gordon equation*. The above-mentioned values are shown to be the only admissible ones for any blow-up, as showed by Jost, Wang, Ye and Zhou in [35]. Moreover, all such values had already proved to be attained by Grossi and Pistoia in [30], where Theorem 1.3 is proved in this particular case.

The case $\alpha_1 = \alpha_2$, $a = 1$ is known as *Tzitzeica equation*. Jevnikar and Yang [33] proved that no other value, besides those above, can occur for blow-up masses.

Moreover, for $\alpha_1 = \alpha_2 = 2$, the above-mentioned blow-up values are attained for any $a > 0$, as Pistoia and Ricciardi have recently showed in [51].

The proof of our result (Section 3) relies on a contraction mapping argument. In Section 2 we give a more precise description of the leading term (1.21), in Section 4 we estimate the error terms and in Section 5 we study the linear theory.

The symmetry introduced in Definition 1.1 is a technical condition used in the linear theory which ensures the non-degeneracy in a one-codimensional space of the bubble w_i defined by (1.22) even when the parameter β_i is even (see Appendix A).

2. The ansatz

For any $\beta > 0$, let

$$w_\delta^\beta(x) := \log 2\beta^2\,\frac{\delta^\beta}{(\delta^\beta+|x|^\beta)^2}\qquad x\in\mathbb{R}^2,\ \delta>0$$

be the solutions to the singular Liouville problem in the whole plane, namely

$$\begin{cases} -\Delta w_\delta^\beta = |\cdot|^{\beta-2}e^{w_\delta^\beta} & \text{in } \mathbb{R}^2,\\ \int_{\mathbb{R}^2}|\cdot|^{\beta-2}e^{w_\delta^\beta}<+\infty. \end{cases}$$

For any integer $k\in[1,k_{\max}]$ we will look for a solution to problem (1.1) as

$$u_\lambda = W_\lambda+\phi_\lambda=(W_{1,\lambda}+\phi_{1,\lambda},W_{2,\lambda}+\phi_{2,\lambda}). \tag{2.1}$$

The components of the main term W_λ are defined as

$$\begin{cases} W_{1,\lambda}=\mathrm{P}w_1-\dfrac{a}{2}\mathrm{P}w_2+\cdots=\displaystyle\sum_{j=0}^{[(k-1)/2]}\mathrm{P}w_{2j+1}-\frac{a}{2}\sum_{j=0}^{[(k-2)/2]}\mathrm{P}w_{2j+2}\,,\\[2ex] W_{2,\lambda}=-\dfrac{b}{2}\mathrm{P}w_1+\mathrm{P}w_2-\cdots=\displaystyle\sum_{j=0}^{[(k-2)/2]}\mathrm{P}w_{2j+2}-\frac{b}{2}\sum_{j=0}^{[(k-1)/2]}\mathrm{P}w_{2j+1}\,, \end{cases}$$

where we agree that if $k=1$ the second sum in $W_{1,\lambda}$ and the first sum in $W_{2,\lambda}$ are zero.

Moreover, $w_\ell := w_{\delta_\ell}^{\beta_\ell}$, and the projection $\mathrm{P}\colon H^1(\Omega)\to H^1_0(\Omega)$ is defined by

$$\begin{cases} -\Delta(\mathrm{P}u) = -\Delta u & \text{in } \Omega,\\ \mathrm{P}u = 0 & \text{on } \partial\Omega.\end{cases} \tag{2.2}$$

The β_ℓ's are defined by recurrence as

$$\begin{cases} \beta_1 = \alpha_1,\\ \beta_2 = b\,\alpha_1 + \alpha_2,\\ \beta_{2j+1} = a\sum_{i=0}^{j-1}\beta_{2i+2} - 2\sum_{i=0}^{j-1}\beta_{2i+1} + \alpha_1 = a\,\beta_{2j} - \beta_{2j-1},\\ \beta_{2j+2} = b\sum_{i=0}^{j}\beta_{2i+1} - 2\sum_{i=0}^{j-1}\beta_{2i+2} + \alpha_2 = b\,\beta_{2j+1} - \beta_{2j}.\end{cases} \tag{2.3}$$

Actually, the two definitions of β_ℓ's given in (1.13) and in (2.3) match perfectly. That will be proved in Section 2.2.

The concentration parameters δ_ℓ's satisfy

$$\begin{aligned} &-\beta_{2j+1}\log\delta_{2j+1} - 2\sum_{i=j+1}^{[(k-1)/2]}\beta_{2i+1}\log\delta_{2i+1} + a\sum_{i=j}^{[(k-2)/2]}\beta_{2i+2}\log\delta_{2i+2}\\ &-\log\left(2\beta_{2j+1}^2\right) + 2\pi\Big(2\sum_{i=0}^{[(k-1)/2]}\beta_{2i+1} - a\sum_{i=0}^{[(k-2)/2]}\beta_{2i+2} - \alpha_1 + 2\Big)H(0,0)\\ &+\log(2\lambda_1) = 0,\\ &-2\sum_{i=j+1}^{[(k-2)/2]}\beta_{2i+2}\log\delta_{2i+2} + b\sum_{i=j+1}^{[(k-1)/2]}\beta_{2i+1}\log\delta_{2i+1} - \log\left(2\beta_{2j+2}^2\right)\\ &+2\pi\Big(2\sum_{i=0}^{[(k-2)/2]}\beta_{2i+2} - b\sum_{i=0}^{[(k-1)/2]}\beta_{2i+1} - \alpha_2 + 2\Big)H(0,0) + \log(2\lambda_2) = 0.\end{aligned} \tag{2.4}$$

The choice of β_ℓ's and δ_ℓ's is motivated by Lemma 2.2.

It is useful to point out that by (2.4) we easily deduce that

$$\delta_{2j+1} = d_{2j+1}\begin{cases} \lambda_1^{\frac{P_{k-2j}(ab)}{\beta_{2j+1}}}\lambda_2^{\frac{aP_{k-2j-1}(ab)}{\beta_{2j+1}}} & \text{if } k \text{ is odd},\\ \lambda_1^{\frac{P_{k-2j-1}(ab)}{\beta_{2j+1}}}\lambda_2^{\frac{aP_{k-2j}(ab)}{\beta_{2j+1}}} & \text{if } k \text{ is even},\end{cases}$$

and

$$\delta_{2j+2} = d_{2j+2}\begin{cases} \lambda_1^{\frac{bP_{k-2j-1}(ab)}{\beta_{2j+2}}}\lambda_2^{\frac{P_{k-2j-2}(ab)}{\beta_{2j+2}}} & \text{if } k \text{ is odd},\\ \lambda_1^{\frac{bP_{k-2j-2}(ab)}{\beta_{2j+2}}}\lambda_2^{\frac{P_{k-2j-1}(ab)}{\beta_{2j+2}}} & \text{if } k \text{ is even},\end{cases}$$

which implies

$$\frac{\delta_\ell}{\delta_{\ell+1}} = \frac{d_\ell}{d_{\ell+1}} \begin{cases} \lambda_1^{\frac{\beta_{k+1}}{\beta_\ell\beta_{\ell+1}}} \lambda_2^{\frac{\beta_k}{\beta_\ell\beta_{\ell+1}}} & \text{if } k \text{ is odd,} \\ \lambda_1^{\frac{\beta_k}{\beta_\ell\beta_{\ell+1}}} \lambda_2^{\frac{\beta_{k+1}}{\beta_\ell\beta_{\ell+1}}} & \text{if } k \text{ is even.} \end{cases}$$

We want to have $\delta_\ell/\delta_{\ell+1} \to 0$ as $\lambda \to 0$ for any ℓ, i.e., each bubble is slower than the previous one; this is always satisfied if $\beta_k + 1 > 0$, namely $k < k_{\max}$, otherwise we need the additional condition in (1.16). The condition (1.16) also ensures that $\delta_\ell/\delta_{\ell+1} = O(|\lambda|^\gamma)$ for some $\gamma > 0$, which will be useful in some estimates throughout the paper.

Finally, the remainder term ϕ_λ in (2.1) belongs to the following space:

$$\mathbf{H} := \left\{\phi = (\phi_1, \phi_2) \in H_0^1(\Omega) \times H_0^1(\Omega) : \phi_i(e^{\frac{2\pi}{\mathfrak{m}}\iota} x) = \phi_i(x)\ \forall x \in \Omega\, i = 1, 2\right\},$$

where $\mathfrak{m}$ is as in Definition 1.1. We agree that if $\mathfrak{m} = 1$ than $\mathbf{H}$ is nothing but the space $H_0^1(\Omega) \times H_0^1(\Omega)$.

The space $H_0^1(\Omega) \times H_0^1(\Omega)$ is equipped with the norm

$$\|(u_1, u_2)\| := \|u_1\| + \|u_2\|, \quad \text{where} \quad \|u\| := \Big(\int_\Omega |\nabla u|^2\Big)^{1/2}.$$

Moreover, we also consider the space $L^p(\Omega) \times L^p(\Omega)$, with $p > 1$, equipped with the norm

$$\|(u_1, u_2)\|_p := \|u_1\|_p + \|u_2\|_p, \quad \text{where} \quad \|u\|_p := \Big(\int_\Omega |u|^p\Big)^{1/p}.$$

2.1. The choice of concentration parameters

For any integer $\ell = 1, \dots, k$, we introduce the function Θ_ℓ which reads if ℓ is odd, i.e., $\ell = 2j + 1$, as

$$\begin{aligned} \Theta_{2j+1}(y) = \Big(&\sum_{i=0}^{[(k-1)/2]} \mathrm{P}w_{2i+1} - w_{2j+1} - \frac{a}{2} \sum_{i=0}^{[(k-2)/2]} \mathrm{P}w_{2i+2}\Big)(\delta_{2j+1} y) \\ &- (\beta_{2j+1} - \alpha_1) \log|\delta_{2j+1} y| - 2\pi(\alpha_1 - 2) H(\delta_{2j+1} y, 0) + \log(2\lambda_1), \end{aligned} \tag{2.5}$$

and if ℓ is even, i.e., $\ell = 2j + 2$, as

$$\begin{aligned} \Theta_{2j+2}(y) = \Big(&\sum_{i=0}^{[(k-2)/2]} \mathrm{P}w_{2i+2} - w_{2j+2} - \frac{b}{2} \sum_{i=0}^{[(k-1)/2]} \mathrm{P}w_{2i+1}\Big)(\delta_{2j+2} y) \\ &- (\beta_{2j+2} - \alpha_2) \log|\delta_{2j+2} y| - 2\pi(\alpha_2 - 2) H(\delta_{2j+2} y, 0) + \log(2\lambda_2). \end{aligned} \tag{2.6}$$

We agree that if $k = 1$ the second sum in (2.5) and the first sum in (2.6) are zero. We shall estimate each functions Θ_ℓ on the corresponding scaled annulus

$$\frac{\mathcal{A}_\ell}{\delta_\ell} = \left\{ y \in \frac{\Omega}{\delta_\ell} : \frac{\sqrt{\delta_{\ell-1}\delta_\ell}}{\delta_\ell} \le |y| \le \frac{\sqrt{\delta_\ell \delta_{\ell+1}}}{\delta_\ell} \right\}$$

where

$$\mathcal{A}_\ell := \big\{x \in \Omega :\ \sqrt{\delta_{\ell-1}\delta_\ell} \le |x| \le \sqrt{\delta_\ell\,\delta_{\ell+1}}\,\big\},$$

where we agree that $\delta_0 = 0$ and $\delta_{k+1} = +\infty$.

We recall the following estimate, which has been proved in [30].

Lemma 2.1.

$$\begin{aligned} \mathrm{P}w_\ell &= w_\ell - \log(2\beta_\ell^2\,\delta_\ell^{\beta_\ell}) + 4\pi\beta_\ell\,H(\cdot,0) + O(\delta_\ell^{\beta_\ell}) \\ &= -2\log(\delta_\ell^{\beta_\ell} + |\cdot|^{\beta_\ell}) + 4\pi\beta_\ell\,H(\cdot,0) + O(\delta_\ell^{\beta_\ell}), \end{aligned} \tag{2.7}$$

and, for any $i,\ell = 1,\dots,k$, $\mathrm{P}w_i(\delta_\ell y)$ *equals*

$$\begin{cases} -2\beta_i\log(\delta_\ell|y|) + 4\pi\beta_i H(0,0) + O\big(\frac{1}{|y|^{\beta_i}}\big(\frac{\delta_i}{\delta_\ell}\big)^{\beta_i}\big) + O(\delta_\ell|y|) + O(\delta_i^{\beta_i}) & \text{if } i<\ell, \\ -2\beta_i\log\delta_i - 2\log\big(1+|y|^{\beta_i}\big) + 4\pi\beta_i H(0,0) + O(\delta_i|y|) + O(\delta_i^{\beta_i}) & \text{if } i=\ell, \\ -2\beta_i\log\delta_i + 4\pi\beta_i H(0,0) + O\big(|y|^{\beta_i}\big(\frac{\delta_\ell}{\delta_i}\big)^{\beta_i}\big) + O(\delta_\ell|y|) + O(\delta_i^{\beta_i}) & \text{if } i>\ell. \end{cases} \tag{2.8}$$

Lemma 2.2. *Assume* β_ℓ *and* δ_ℓ *are defined respectively by* (2.3) *and* (2.4).

Then there exists $\gamma_0>0$ *such that, for any* $\ell=1,\dots,k$,

$$|\Theta_\ell(y)| = O(\delta_\ell|y| + |\lambda|^{\gamma_0}) \quad \text{for any } y\in\frac{\mathcal{A}_\ell}{\delta_\ell}, \tag{2.9}$$

and in particular

$$\sup_{\mathcal{A}_\ell/\delta_\ell} |\Theta_\ell| = O(1). \tag{2.10}$$

Proof. We will prove the lemma only for odd ℓ, i.e., $\ell = 2j+1$, since the same argument works in the general case. We can also restrict ourselves to consider the case of an odd k.

We can estimate $\mathrm{P}w_\ell$ by using Lemma 2.1 and then H by the mean value theorem, which gives $H(\delta_\ell y,0) = H(0,0) + O(\delta_\ell|y|)$:

$$\begin{aligned} &\Theta_{2j+1}(y) \\ &= \mathrm{P}w_{2j+1}(\delta_{2j+1}y) - w_{2j+1}(\delta_{2j+1}y) + \sum_{i=0}^{j-1}\mathrm{P}w_{2i+1}(\delta_{2j+1}y) + \sum_{i=j+1}^{m}\mathrm{P}w_{2i+1}(\delta_{2j+1}y) \\ &\quad - \frac{a}{2}\sum_{i=0}^{j-1}\mathrm{P}w_{2i+2}(\delta_{2j+1}y) - \frac{a}{2}\sum_{i=j}^{m-1}\mathrm{P}w_{2i+1}(\delta_{2j+1}y)(\beta_{2j+1}-\alpha_1)\log|\delta_{2j+1}y| \\ &\quad - 2\pi(\alpha_1-2)H(\delta_{2j+1}y,0) + \log(2\lambda_1) \\ &= -\log\big(2\beta_{2j+1}^2\big) - \beta_{2j+1}\log\delta_{2j+1} + 4\pi\beta_{2j+1}H(0,0) + O(\delta_{2j+1}|y|) \\ &\quad + O\big(\delta_{2j+1}^{\beta_{2j+1}}\big) + \sum_{i=0}^{j-1}\Big(-2\beta_{2i+1}\log(\delta_{2j+1}|y|) + 4\pi\beta_{2i+1}H(0,0) \end{aligned}$$

$$
\begin{aligned}
&+ O\Big(\frac{1}{|y|^{\beta_{2i+1}}}\Big(\frac{\delta_{2i+1}}{\delta_{2j+1}}\Big)^{\beta_{2i+1}}\Big) + O(\delta_{2j+1}|y|) + O\big(\delta_{2i+1}^{\beta_{2i+1}}\big)\Big) \\
&+ \sum_{i=j+1}^{l}\Big(- 2\beta_{2i+1}\log\delta_{2i+1} + 4\pi\beta_{2i+1}H(0,0) + O\Big(|y|^{\beta_{2i+1}}\Big(\frac{\delta_{2j+1}}{\delta_{2i+1}}\Big)^{\beta_{2i+1}}\Big) \\
&+ O(\delta_{2j+1}|y|) + O\big(\delta_{2i+1}^{\beta_{2i+1}}\big)\Big) - \frac{a}{2}\sum_{i=0}^{j-1}\Big(- 2\beta_{2i+2}\log(\delta_{2j+1}|y|) + 4\pi\beta_{2i+2}H(0,0) \\
&+ O\Big(\frac{1}{|y|^{\beta_{2i+2}}}\Big(\frac{\delta_{2i+2}}{\delta_{2j+1}}\Big)^{\beta_{2i+2}}\Big) + O(\delta_{2j+1}|y|) + O\big(\delta_{2i+2}^{\beta_{2i+2}}\big)\Big) \\
&- \frac{a}{2}\sum_{i=j}^{l-1}\Big(- 2\beta_{2i+2}\log\delta_{2i+2} + 4\pi\beta_{2i+2}H(0,0) + O\Big(|y|^{\beta_{2i+2}}\Big(\frac{\delta_{2j+1}}{\delta_{2i+2}}\Big)^{\beta_{2i+2}}\Big) \\
&+ O(\delta_{2j+1}|y|) + O\big(\delta_{2i+2}^{\beta_{2i+2}}\big)\Big) - 2\pi(\alpha_1 - 2)H(\delta_{2j+1}y, 0) \\
&- (\beta_{2j+1} - \alpha_1)\log|\delta_{2j+1}y| + \log(2\lambda_1) \\
= &- \log(2\beta_{2j+1}) - \beta_{2j+1}\log\delta_{2j+1} - 2\sum_{i=j+1}^{m}\beta_{2i+1}\log\delta_{2i+1} + a\sum_{i=j+1}^{m-1}\beta_{2i+2}\log\delta_{2i+2} \\
&+ 2\pi\Big(2\sum_{i=0}^{m}\beta_{2i+1} - a\sum_{i=0}^{m-1}\beta_{2i+2} - \alpha_1 + 2\Big)H(0,0) + \log(2\lambda_1) \\
&+ \Big(a\sum_{i=0}^{j-1}\beta_{2i+1} - 2\sum_{i=0}^{j-1}\beta_{2i+2} + \alpha_1 - \beta_{2j+1}\Big)\log(\delta_{2j+1}|y|) + O(\delta_{2j+1}|y|) \\
&+ \sum_{i=1}^{2m+1} O(\delta_i^{\beta_i}) + \sum_{i=1}^{2j} O\Big(\frac{1}{|y|^{\beta_i}}\Big(\frac{\delta_i}{\delta_{2j+1}}\Big)^{\beta_i}\Big) + \sum_{i=2j+2}^{2m+1} O\Big(|y|^{\beta_i}\Big(\frac{\delta_{2j+1}}{\delta_i}\Big)^{\beta_i}\Big) \\
= &\, O(\delta_{2j+1}|y|) + \sum_{i=1}^{2m+1} O(\delta_i^{\beta_i}) + \sum_{i=1}^{2j} O\Big(\frac{1}{|y|^{\beta_i}}\Big(\frac{\delta_i}{\delta_{2j+1}}\Big)^{\beta_i}\Big) + \sum_{i=2j+2}^{2m+1} O\Big(|y|^{\beta_i}\Big(\frac{\delta_{2j+1}}{\delta_i}\Big)^{\beta_i}\Big) \\
= &\, O(\delta_{2j+1}|y|) + \sum_{i=1}^{2m+1} O(\delta_i^{\beta_i}) + \sum_{i=1}^{2j} O\Big(\Big(\frac{\delta_i^2}{\delta_{2j}\delta_{2j+1}}\Big)^{\beta_i/2}\Big) \\
&+ \sum_{i=2j+2}^{2m+1} O\Big(\Big(\frac{\delta_{2j+1}\delta_{2j+2}}{\delta_i^2}\Big)^{\beta_i/2}\Big) \\
= &\, O(\delta_{2j+1}|y|) + \sum_{i=1}^{2m+1} O(\delta_i^{\beta_i}) + \sum_{i=1}^{2j} O\Big(\Big(\frac{\delta_{2j}}{\delta_{2j+1}}\Big)^{\beta_i/2}\Big) + \sum_{i=2j+2}^{2m+1} O\Big(\Big(\frac{\delta_{2j+1}}{\delta_{2j+2}}\Big)^{\beta_i/2}\Big) \\
= &\, O(\delta_{2j+1}|y|) + O\big(\min_i \delta_i^{\beta_i}\big) + O\Big(\min_{i,\ell}\Big(\frac{\delta_\ell}{\delta_{\ell+1}}\Big)^{\beta_i/2}\Big) \\
= &\, O(\delta_{2j+1}|y|) + O(|\lambda|^{\gamma_0}) \\
= &\, O(\delta_{2j+1}|y| + |\lambda|^{\gamma_0}),
\end{aligned}
$$

where we used that $\sqrt{\delta_{2j}/\delta_{2j+1}} \le |y| \le \sqrt{\delta_{2j+1}/\delta_{2j+2}}$ and that $\delta_i \le \delta_{2j}$ and $\delta_{2j+2} \le \delta_{i'}$ for any $i < 2j+1 < i'$.

Formula (2.10) follows straightforwardly from (2.9), since $\delta_\ell |y| = O(1)$ for any $y \in \mathcal{A}_\ell/\delta_\ell$. □

2.2. Chebyshev polynomials and the β_ℓ's

In this sub-section we shall prove that the β_ℓ's defined in (1.13) and in (2.3) coincide.

Let us introduce the polynomials

$$\begin{cases} P_1(t) = 1, \\ P_2(t) = 1, \\ \quad\vdots \\ P_{2j+1}(t) = \sum_{i=0}^{j} (-1)^{j+i} \binom{j+i}{2i} t^i, \\ P_{2j+2}(t) = \sum_{i=0}^{j} (-1)^{j+i} \binom{j+i+1}{2i+1} t^i, \\ \quad\vdots \end{cases} \tag{2.11}$$

By induction, is not difficult to check that the real numbers defined in (2.3) satisfy (1.13), since

$$\begin{cases} \beta_1 = \alpha_1, \\ \beta_2 = b\alpha_1 + \alpha_2, \\ \quad\vdots \\ \beta_{2j+1} = \alpha_1 \sum_{i=0}^{j} (-1)^{j+i} \binom{j+i}{2i} a^i b^i + \alpha_2 \sum_{i=0}^{j-1} (-1)^{j+i-1} \binom{j+i}{2i+1} a^{i+1} b^i \\ \qquad = \alpha_1 P_{2j+1}(ab) + a\,\alpha_2\, P_{2j}(ab), \\ \beta_{2j+2} = \alpha_1 \sum_{i=0}^{j} (-1)^{j+i} \binom{j+i+1}{2i+1} a^i b^{i+1} + \alpha_2 \sum_{i=0}^{j} (-1)^{j+i} \binom{j+i}{2i} a^i b^i \\ \qquad = b\,\alpha_1 P_{2j+2}(ab) + \alpha_2\, P_{2j+1}(ab), \\ \quad\vdots \end{cases} \tag{2.12}$$

Therefore, the problem reduces to prove that the polynomials defined in (2.11) coincide with the polynomial defined in (1.12).

Now, the polynomials defined in (2.11) can be expressed in terms of Chebyshev's polynomials

$$\begin{cases} T_0(x) = 1, \\ T_1(x) = x, \\ T_{\ell+1}(x) = 2x\, T_\ell(x) - T_{\ell-1}(x) \quad \text{if } \ell \geq 2. \end{cases} \tag{2.13}$$

Lemma 2.3. *Let P_ℓ be defined by (1.12) and T_ℓ be defined by (2.13).*

Then, for any $j \in \mathbb{N}$ and $x \in \mathbb{R}$ it holds:

$$\begin{aligned} T_{2j+1}(x) &= 1 + (x-1)\left(P_{2j+1}(2x+2)\right)^2, \\ T_{2j+2}(x) &= 1 + \left(2x^2 - 2\right)\left(P_{2j+2}(2x+2)\right)^2. \end{aligned} \tag{2.14}$$

Proof. We proceed by induction. We can easily see that the proposition is true for $\ell = 1, 2$.

Let us now assume the proposition to hold for any positive integer up to $2j$ and let us show it still holds true for $2j+1$ and $2j+2$.

First of all, by induction we can easily show that P_ℓ verifies the following properties:

$$P_{2j+1}(t) = tP_{2j}(t) - P_{2j-1}(t), \quad P_{2j+2}(t) = P_{2j+1}(t) - P_{2j}(t), \tag{2.15}$$

and also

$$\begin{aligned} &\left(P_{2j+1}(t)\right)^2 + t\left(P_{2j}(t)\right)^2 - t\, P_{2j+1}(t)\, P_{2j}(t) - 1 = 0, \\ &\left(P_{2j+2}(t)\right)^2 + \left(P_{2j+1}(t)\right)^2 - t\, P_{2j+2}(t)\, P_{2j+1}(t) - 1 = 0. \end{aligned} \tag{2.16}$$

Using (2.15) and (2.16) we get, for odd indexes:

$$\begin{aligned} T_{2j+1}(x) &= 2x\, T_{2j}(x) - T_{2j-1}(x) \\ &= 2x + \left(4x^3 - 4x\right) P_{2j}(2x+2)^2 - 1 - (x-1)P_{2j-1}(2x+2)^2 \\ &= 1 + (x-1)\left(\left(4x^2 + 4x\right) P_{2j}(2x+2)^2 - P_{2j-1}(2x+2)^2 + 2\right) \\ &= 1 + (x-1)\left(P_{2j+1}(2x+2)^2 - 2\left((2x+2)P_{2j}(2x+2)^2 - P_{2j-1}(2x+2)^2\right.\right. \\ &\quad \left.\left. -\ (2x+2)P_{2j}(2x+2)P_{2j-1}(2x+2) - 1\right)\right) \\ &= 1 + (x-1)P_{2j+1}(2x+2)^2; \end{aligned}$$

similarly, for even indexes:

$$\begin{aligned} T_{2j+2}(x) &= 2x\, T_{2j+1}(x) - T_{2j}(x) \\ &= 2x + \left(2x^2 - 2x\right) P_{2j+1}(2x+2)^2 - 1 - \left(2x^2 - 2\right) P_{2j}(2x+2)^2 \\ &= 1 + (2x-2)\left(xP_{2j+1}(2x+2)^2 - (x+1)P_{2j}(2x+2)^2 + 1\right) \\ &= 1 + (2x-2)\left((x+1)P_{2j+2}(2x+2)^2 - \left(P_{2j+1}(2x+2)^2\right.\right. \\ &\quad \left.\left. +\ (2x+2)P_{2j}(2x+2)^2 - (2x+2)P_{2j+2}(2x+2)P_{2j}(2x+2) - 1\right)\right) \\ &= 1 + \left(2x^2 - 2\right) P_{2j+2}(2x+2)^2. \end{aligned}$$

□

Remark 2.4. By using the properties of Chebyshev's polynomials (see for instance [52]), we easily find the explicit expression of T_ℓ as

$$T_\ell(x) = 1 + 2^{\ell-1} \prod_{i=1}^{\ell} \Big(x - \cos\frac{2\pi i}{\ell}\Big),$$

which can be rewritten, if $\ell = 2j+1$ is odd or $\ell = 2j+2$ is even, respectively as

$$\begin{aligned} T_{2j+1}(x) &= 1 + 2^{2j}(x-1) \prod_{i=1}^{j} \Big(x - \cos\frac{2\pi i}{2j+1}\Big)^2, \\ T_{2j+2}(x) &= 1 + 2^{2j+1}\left(x^2-1\right) \prod_{i=1}^{j} \Big(x - \cos\frac{\pi i}{j+1}\Big)^2. \end{aligned} \tag{2.17}$$

Now, if we compare (2.14) and (2.17) we get the explicit expression for P_ℓ given in (1.12).

Remark 2.5. If $ab = 2\cos\left(\frac{2\pi}{k}\right) + 2$ for some $k \in \mathbb{N}$, then we have $P_\ell(ab) > 0$ for any $\ell = 1, \dots, k-1$ and $P_k(ab) = 0 > P_{k+1}(ab)$; hence, by the definitions (1.13) of β_ℓ and (1.14) of $k_{\max}$ we get $k_{\max} = k = 2\pi/\arccos(ab/2-1)$.

On the other hand, if $2\cos\left(\frac{2\pi}{k}\right) + 2 < ab < 2\cos\left(\frac{2\pi}{k+1}\right) + 2$, then $P_\ell(ab) > 0$ for $\ell = 1, \dots, k-1$ and $P_k(ab), P_{k+1}(ab) < 0$; hence, $\beta_\ell > 0$ for $\ell \le k-1$ and $\beta_{k+1} < 0$, so $k_{\max}$ could be either $k-1$ or k.

Finally, if $ab \ge 4$, then clearly $P_\ell(ab) > 0$ for all ℓ, hence $b_\ell > 0$ and $k_{\max} = +\infty$.

Remark 2.6. By (2.14) we immediately deduce that

$$\begin{aligned} (P_{2j+1}(4))^2 &= T'_{2j+1}(1) = (2j+1)^2, \\ (P_{2j+2}(4))^2 &= \frac{1}{4}\, T'_{2j+2}(1) = \frac{(2j+2)^2}{4} = (j+1)^2, \end{aligned}$$

because the Chebyshev's polynomials satisfy (by induction, for instance) $T'_\ell(1) = \ell^2$ for any $\ell \ge 1$.

Remark 2.7. The validity of (1.19) and (1.20) follows by the fact that the coefficients β_ℓ verify the following properties (by induction, for instance):

$$\begin{aligned} \sum_{i=0}^{2j+1} \beta_{2i+1} &= P_{2j+1}(ab)\,(\alpha_1 P_{2j+1}(ab) + a\,\alpha_2\, P_{2j}(ab)), \\ \sum_{i=0}^{2j+2} \beta_{2i+1} &= a\, P_{2j+2}(ab)\,(b\,\alpha_1 P_{2j+2}(ab) + \alpha_2\, P_{2j+1}(ab)), \\ \sum_{i=0}^{2j+1} \beta_{2i+2} &= P_{2j+1}(ab)\,(b\,\alpha_1 P_{2j+2}(ab) + \alpha_2\, P_{2j+1}(ab)), \\ \sum_{i=0}^{2j+2} \beta_{2i+2} &= b\, P_{2j+2}(ab)\,(\alpha_1 P_{2j+3}(ab) + a\,\alpha_2\, P_{2j+2}(ab)). \end{aligned}$$

3. Proof of the main theorem

In this section, we prove the existence of a solution to system (1.1) using a contraction mapping argument and we study its properties.

Proposition 3.1. *There exist* $\gamma, R, \overline{\lambda} > 0$ *such that for any* $\lambda \in \left(0, \overline{\lambda}\right) \times \left(0, \overline{\lambda}\right)$ *there exists a unique* $\phi_\lambda = (\phi_{1,\lambda}, \phi_{2,\lambda}) \in \mathbf{H}$ *such that:*

- $W_\lambda + \phi_\lambda$ *solves* (1.1), *namely*

$$\begin{cases} -\Delta(W_{1,\lambda} + \phi_{1,\lambda}) = 2\lambda_1 h_1 e^{W_{1,\lambda}+\phi_{1,\lambda}} - a\,\lambda_2 h_2 e^{W_{2,\lambda}+\phi_{2,\lambda}} & \text{in } \Omega, \\ -\Delta(W_{2,\lambda} + \phi_{2,\lambda}) = 2\lambda_2 h_2 e^{W_{2,\lambda}+\phi_{2,\lambda}} - b\,\lambda_1 h_1 e^{W_{1,\lambda}+\phi_{1,\lambda}} & \text{in } \Omega; \end{cases}$$

- $\|\phi_\lambda\| \le R\,|\lambda|^\gamma \log \frac{1}{|\lambda|}$.

Proof. We point out that $W_\lambda + \phi_\lambda$ solves (1.1) if and only if

$$\mathcal{L}_\lambda \phi = \mathcal{N}_\lambda(\phi) - \mathcal{S}_\lambda \phi - \mathcal{R}_\lambda.$$

where the linear operator $\mathcal{L}_\lambda \colon H_0^1(\Omega) \times H_0^1(\Omega) \to L^p(\Omega) \times L^p(\Omega)$ is defined by

$$\mathcal{L}_\lambda(\phi) := \begin{pmatrix} -\Delta\phi_1 - \left(\mathcal{L}_1\phi_1 - \frac{a}{2}\mathcal{L}_2\phi_2\right) \\ -\Delta\phi_2 - \left(\mathcal{L}_2\phi_2 - \frac{b}{2}\mathcal{L}_1\phi_1\right) \end{pmatrix}, \tag{3.1}$$

with

$$\mathcal{L}_1 = \sum_{j=0}^{[(k-1)/2]} 2\beta_{2j+1}^2 \frac{\delta_{2j+1}^{\beta_{2j+1}} |\cdot|^{\beta_{2j+1}-2}}{\left(\delta_{2j+1}^{\beta_{2j+1}} + |\cdot|^{\beta_{2j+1}}\right)^2} \quad \mathcal{L}_2 = \sum_{j=0}^{[(k-2)/2]} 2\beta_{2j+2}^2 \frac{\delta_{2j+2}^{\beta_{2j+2}} |\cdot|^{\beta_{2j+2}-2}}{\left(\delta_{2j+2}^{\beta_{2j+2}} + |\cdot|^{\beta_{2j+2}}\right)^2},$$

the error function $\mathcal{R}_\lambda \in L^p(\Omega) \times L^p(\Omega)$ is defined by

$$\mathcal{R}_\lambda := \begin{pmatrix} \mathcal{R}_{1,\lambda} \\ \mathcal{R}_{2,\lambda} \end{pmatrix} = \begin{pmatrix} -\Delta W_{1,\lambda} - 2\lambda_1 h_1 e^{W_{1,\lambda}} + a\,\lambda_2 h_2 e^{W_{2,\lambda}} \\ -\Delta W_{2,\lambda} - 2\lambda_2 h_2 e^{W_{2,\lambda}} + b\lambda_1 h_1 e^{W_{1,\lambda}} \end{pmatrix}, \tag{3.2}$$

the error linear operator $\mathcal{S}_\lambda : H_0^1(\Omega) \times H_0^1(\Omega) \to L^p(\Omega) \times L^p(\Omega)$ is defined by

$$\mathcal{S}_\lambda \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} := \begin{pmatrix} \mathcal{S}_1\phi_1 - \frac{a}{2}\mathcal{S}_2\phi_2 \\ \mathcal{S}_2\phi_2 - \frac{b}{2}\mathcal{S}_1\phi_1 \end{pmatrix}, \tag{3.3}$$

with

$$\mathcal{S}_1 = \sum_{j=0}^{[(k-1)/2]} |\cdot|^{\beta_{2j+1}} e^{w_{2j+1}} - 2\lambda_1 h_1 e^{W_{1,\lambda}} \quad \mathcal{S}_2 = \sum_{j=0}^{[(k-2)/2]} |\cdot|^{\beta_{2j+2}} e^{w_{2j+2}} - 2\lambda_2 h_2 e^{W_{2,\lambda}},$$

and the quadratic term $\mathcal{N}_\lambda \colon H_0^1(\Omega) \times H_0^1(\Omega) \to L^p(\Omega) \times L^p(\Omega)$ is defined by

$$\mathcal{N}_\lambda(\phi) := \begin{pmatrix} 2\lambda_1 h_1 e^{W_{1,\lambda}}(e^{\phi_1} - 1 - \phi_1) - a\,\lambda_2 h_2 e^{W_{2,\lambda}}(e^{\phi_2} - 1 - \phi_2) \\ 2\lambda_2 h_2 e^{W_{2,\lambda}}(e^{\phi_2} - 1 - \phi_2) - b\lambda_1 h_1 e^{W_{1,\lambda}}(e^{\phi_1} - 1 - \phi_1) \end{pmatrix}. \tag{3.4}$$

Since Proposition 5.1 ensures that $\mathcal{L}_\lambda\colon \mathbf{H}\to\mathbf{H}$ is invertible, this is equivalent to requiring ϕ_λ to be a fixed point of the map

$$\mathcal{T}_\lambda : \phi \mapsto (\mathcal{L}_\lambda)^{-1}(\mathcal{N}_\lambda(\phi) - \mathcal{S}_\lambda\phi - \mathcal{R}_\lambda);$$

therefore, the existence of such a ϕ_λ will follow by showing that $\mathcal{T}_\lambda$ is a contraction on the ball

$$B_{\gamma,\lambda,R} := \Big\{\phi \in \mathbf{H}:\ \|\phi\| \le R|\lambda|^\gamma \log\frac{1}{|\lambda|}\Big\},$$

for γ, λ small enough and R large enough.

We first show that $\mathcal{T}_\lambda$ maps $B_{\gamma,\lambda,R}$ into itself.

We will use Proposition 5.1 to get estimates on $\mathcal{L}_\lambda$ and Lemmas 4.1, 4.2, and 4.3 to estimate $\mathcal{N}_\lambda(\phi)$, $\mathcal{S}_\lambda\phi$ and $\mathcal{R}_\lambda$, respectively.

With the notation of these lemmas, we take $\gamma \le \max\{\gamma_1,\gamma_2\}$ and p so close to 1 that all such lemmas apply and $\gamma_3(1-p)+\gamma > 0$. We then take C as in Lemma 4.3 and $\overline{\lambda}$ so small that $e^C R^2 \overline{\lambda}^{\gamma_3(1-p)+\gamma}(\log 1/\overline{\lambda})^2 \le 1$. Finally, we take $R > 0$ greater than the three constants which define the O in Lemmas 4.1, 4.2, and 4.3, times the C appearing in Proposition 5.1.

Notice that these choices imply that $R|\lambda|^\gamma \log\frac{1}{|\lambda|} \le 1$; therefore

$$\begin{aligned}
\|\mathcal{T}_\lambda(\phi)\| &\le C\log\frac{1}{|\lambda|}\big(\|\mathcal{N}_\lambda(\phi)\|_p + \|\mathcal{S}_\lambda\phi\|_p + \|\mathcal{R}_\lambda\|_p\big)\\
&\le C\log\frac{1}{|\lambda|}\big(|\lambda|^{\gamma_3(1-p)}\|\phi\|^2 e^{C\|\phi\|^2} + |\lambda|^{\gamma_2}\|\phi\| + |\lambda|^{\gamma_1}\big)\\
&\le C\log\frac{1}{|\lambda|}\Big(R^2 e^C\Big(\log\frac{1}{|\lambda|}\Big)^2|\lambda|^{\gamma_3(1-p)+2\gamma} + |\lambda|^\gamma\Big)\\
&\le C\,|\lambda|^\gamma\log\frac{1}{|\lambda|}\,x \le R\,|\lambda|^\gamma\log\frac{1}{|\lambda|}.
\end{aligned}$$

Moreover, we also get

$$\begin{aligned}
\|\mathcal{T}_\lambda(\phi)\ - \mathcal{T}_\lambda(\phi')\| &\le C\log\frac{1}{|\lambda|}\big(\|\mathcal{N}_\lambda(\phi) - \mathcal{N}_\lambda(\phi')\|_p + \|\mathcal{S}_\lambda(\phi-\phi')\|_p\big)\\
&\le C\log\frac{1}{|\lambda|}\Big(|\lambda|^{\gamma_3(1-p)}\|\phi-\phi'\|(\|\phi\|+\|\phi'\|)e^{C\left(\|\phi\|^2+\|\phi\|^2\right)} + |\lambda|^{\gamma_2}\|\phi-\phi'\|\Big)\\
&\le C\Big(2R\,e^{2C}\,|\lambda|^{\gamma_3(1-p)+\gamma}\Big(\log\frac{1}{|\lambda|}\Big)^2 + |\lambda|^{\gamma_2}\log\frac{1}{|\lambda|}\Big)\|\phi-\phi'\|\\
&\le C\Big(2\,\frac{e^C}{R} + \overline{\lambda}^{\gamma_2}\log\frac{1}{\overline{\lambda}}\Big)\|\phi-\phi'\|;
\end{aligned}$$

with the constant multiplying $\|\phi-\phi'\|$ being smaller than 1, after taking larger R and/or smaller $\overline{\lambda}$, if needed. This concludes the proof. □

Proof of Theorem 1.3. By Proposition 3.1 we get $u_\lambda = W_\lambda + \phi_\lambda$ which solves (1.1).

Let us prove (1.17).

We basically show that ϕ_λ is negligible in this computations, thanks to the estimates from Proposition 4.3 and in particular (4.5). Then, we compare $W_{i,\lambda}$ with w_{2j+i} using the estimate (4.2) from Lemma 4.2:

$$\begin{aligned}
&\left|\lambda_i\int_{B_r(0)} h_i e^{u_{i,\lambda}} - 2\pi\sum_{j=0}^{[(k-i)/2]}\beta_{2j+i}\right| \\
&= \left|\lambda_i\int_{B_r(0)} h_i e^{W_{i,\lambda}+\phi_{i,\lambda}} - \frac{1}{2}\sum_{j=0}^{[(k-i)/2]}\int_{\mathbb{R}^2} 2\beta_{2j+i}^2\frac{|\cdot|^{\beta_{2j+i}-2}}{(1+|\cdot|^{\beta_{2j+i}})^2}\right| \\
&\le \int_\Omega \lambda_i h_i e^{W_{i,\lambda}}\left|e^{\phi_{i,\lambda}}-1\right| + \left|\lambda_i\int_{B_r(0)} h_i e^{W_{i,\lambda}} - \frac{1}{2}\sum_{j=0}^{[(k-i)/2]}\int_{\mathbb{R}^2} 2\beta_{2j+i}^2\frac{|\cdot|^{\beta_{2j+i}-2}}{(1+|\cdot|^{\beta_{2j+i}})^2}\right| \\
&\le \int_\Omega \lambda_i h_i\, e^{W_{i,\lambda}}|\phi_{i,\lambda}|e^{\phi_{i,\lambda}} \\
&\quad + \left|\lambda_i\int_{B_r(0)} h_i e^{W_{i,\lambda}} - \frac{1}{2}\sum_{j=0}^{[(k-i)/2]}\int_{B_{\frac{r}{\delta_{2j+i}}}(0)} 2\beta_{2j+i}^2\frac{|\cdot|^{\beta_{2j+i}-2}}{(1+|\cdot|^{\beta_{2j+i}})^2}\right| + o(1) \\
&\le \left\|\lambda_i h_i e^{W_{i,\lambda}}\right\|_p \|\phi_{i,\lambda}\|_q \left\|e^{\phi_{i,\lambda}}\right\|_{\frac{pq}{pq-p-q}} \\
&\quad + \int_{B_r(0)}\left|\lambda_i h_i e^{W_{i,\lambda}} - \frac{1}{2}\sum_{j=0}^{[(k-i)/2]}|\cdot|^{\beta_{2j+i}-2}e^{w_{2j+i}}\right| + o(1) \\
&\le CR\,|\lambda|^{\gamma_3(1-p)+\gamma}\log\frac{1}{|\lambda|} + C\,|\lambda|^{\gamma_1} + o(1) \underset{\lambda\to 0}{\longrightarrow} 0.
\end{aligned}$$

Since this holds true for any r, then letting r tend to 0 we find the value of $m_i(0)$.

Finally, we prove (1.18).

First of all, u_λ is bounded in $W^{1,q}(\Omega)\times W^{1,q}(\Omega)$ for any $q<2$, because $W^{1,q/(q-1)}(\Omega)\hookrightarrow C\left(\overline{\Omega}\right)$, hence for any $\varphi\in W^{1,q/(q-1)}(\Omega)$ with $\|\varphi\|_{W^{1,q/(q-1)}(\Omega)}\le 1$, we have

$$\left|\int_\Omega \nabla u_{i,\lambda}\cdot\nabla\varphi\right| = \left|\int_\Omega(-\Delta u_{i,\lambda})\varphi\right| \le C\Big(\lambda_1\int_\Omega h_1\, e^{u_{1,\lambda}} + \lambda_2\int_\Omega h_2\, e^{u_{2,\lambda}}\Big)\|\varphi\|_\infty \le C.$$

From (2.7) we get $\mathrm{P}w_\ell(x)\to 4\pi\beta_\ell G(\cdot,0)$ as $\lambda\to 0$ pointwise in $\Omega\setminus\{0\}$. Since $\|\phi_\lambda\|\to 0$ as $\lambda\to 0$, from the definition of u_λ and m_1,m_2 we deduce that the weak limit of u_λ in $W^{1,q}(\Omega)$ must be the one in (1.18).

Moreover, from (2.8) and the definition of $W_{i,\lambda}$ we deduce that the latter are both bounded in $L^\infty_{\mathrm{loc}}(\Omega\setminus\{0\})$. Therefore, for any $\mathcal{K}\Subset\Omega\setminus\{0\}$,

$$\begin{aligned}
\int_\mathcal{K}|-\Delta u_{i,\lambda}|^q &\le C\left(\int_\mathcal{K}|\cdot|^{q(\alpha_1-2)}e^{q(W_{1,\lambda}+\phi_{1,\lambda})} + \int_\mathcal{K}|\cdot|^{q(\alpha_2-2)}e^{q(W_{2,\lambda}+\phi_{2,\lambda})}\right) \\
&\le C\,e^{q\|W_\lambda\|_\infty}\left(\int_\mathcal{K} e^{q\phi_{1,\lambda}} + \int_\mathcal{K} e^{q\phi_{2,\lambda}}\right) \le C.
\end{aligned}$$

Therefore, a standard bootstrap method will imply convergence in $C^\infty(\mathcal{K})$ hence, being $\mathcal{K}$ arbitrary, in $C^\infty_{\mathrm{loc}}(\Omega\setminus\{0\})$. □

4. The error terms

In this section we estimate in the L^p norm the function $\mathcal{R}_\lambda$ defined in (3.2), the linear operator $\mathcal{S}_\lambda$ defined in (3.3) and the quadratic term $\mathcal{N}_\lambda$ defined in (3.4).

Roughly speaking, both $\mathcal{R}_\lambda$ and $\mathcal{S}_\lambda$ will decay as a power of λ if p is close enough to 1. On the other hand, the norm of $\mathcal{N}_\lambda$ will diverge as λ goes to 0, but its growth will be slow for small p.

The estimates for $\mathcal{S}_\lambda$ and $\mathcal{N}_\lambda$ will require mostly the same calculations as the ones needed for $\mathcal{R}_\lambda$.

4.1. The function $\mathcal{R}_\lambda$

Lemma 4.1. *Let $\mathcal{R}_\lambda$ be defined by (3.2). There exists $p_0 > 1$ and $\gamma_1 > 0$ such that for any $p \in [1, p_0)$,*

$$\|\mathcal{R}_\lambda\|_p = O(|\lambda|^{\gamma_1}).$$

Proof. We will only provide estimates for $\mathcal{R}_{1,\lambda}$; the estimates for $\mathcal{R}_{2,\lambda}$ are similar.

First of all, by the very definition of $W_{i,\lambda}$ and triangular inequalities, we can split the L^p norm of $\mathcal{R}_{1,\lambda}$ in the following way:

$$\begin{aligned}
&\int_\Omega |\mathcal{R}_{1,\lambda}|^p \\
&= \int_\Omega \Big| \sum_{j=0}^{[(k-1)/2]} |\cdot|^{\beta_{2j+1}-2} e^{w_{2j+1}} - \frac{a}{2} \sum_{j=0}^{[(k-2)/2]} |\cdot|^{\beta_{2j+2}-2} e^{w_{2j+2}} \\
&\quad - 2\lambda_1 h_1 e^{\sum_{m=0}^{[(k-1)/2]} \mathrm{P}w_{2m+1} - \frac{a}{2}\sum_{m=0}^{[(k-2)/2]} \mathrm{P}w_{2m+2}} \\
&\quad + a\,\lambda_2 h_2 e^{\sum_{m=0}^{[(k-2)/2]} \mathrm{P}w_{2m+2} - \frac{b}{2}\sum_{m=0}^{[(k-1)/2]} \mathrm{P}w_{2m+1}} \Big|^p \\
&\le C \int_\Omega \Big| \sum_{j=0}^{[(k-1)/2]} |\cdot|^{\beta_{2j+1}-2} e^{w_{2j+1}} - 2\lambda_1 h_1 e^{\sum_{m=0}^{[(k-1)/2]} \mathrm{P}w_{2m+1} - \frac{a}{2}\sum_{m=0}^{[(k-2)/2]} \mathrm{P}w_{2j+2}} \Big|^p \\
&\quad + C \int_\Omega \Big| \sum_{j=0}^{[(k-2)/2]} |\cdot|^{\beta_{2j+2}-2} e^{w_{2j+2}} - 2\lambda_2 h_2 e^{\sum_{m=0}^{[(k-2)/2]} \mathrm{P}w_{2m+2} - \frac{b}{2}\sum_{m=0}^{[(k-1)/2]} \mathrm{P}w_{2m+1}} \Big|^p \\
&\le C \sum_{j=0}^{[(k-1)/2]} \underbrace{\int_{A_{2j+1}} \Big| |\cdot|^{\beta_{2j+1}-2} e^{w_{2j+1}} - 2\lambda_1 h_1 e^{\sum_{m=0}^{[(k-1)/2]} \mathrm{P}w_{2m+1} - \frac{a}{2}\sum_{m=0}^{[(k-2)/2]} \mathrm{P}w_{2m+2}} \Big|^p}_{=: I'_{2j+1}} \\
&\quad + C \sum_{j=0}^{[(k-1)/2]} \sum_{i=1, i\neq 2j+1}^{k} \underbrace{\int_{\mathcal{A}_i} \Big| |\cdot|^{\beta_{2j+1}-2} e^{w_{2j+1}} \Big|^p}_{=: I''_{i,2j+1}} \\
&\quad + C \sum_{i=0}^{[(k-2)/2]} \underbrace{\int_{\mathcal{A}_{2i+2}} \Big| \lambda_1 h_1 e^{\sum_{m=0}^{[(k-1)/2]} \mathrm{P}w_{2m+1} - \frac{a}{2}\sum_{m=0}^{[(k-2)/2]} \mathrm{P}w_{2m+2}} \Big|^p}_{=: I'''_{2i+2}}
\end{aligned}$$

$$+C\sum_{j=0}^{[(k-2)/2]}\underbrace{\int_{\mathcal{A}_{2j+2}}\left||\cdot|^{\beta_{2j+2}-2}e^{w_{2j+2}}-2\lambda_2\,h_2\,e^{\sum_{m=0}^{[(k-2)/2]}\mathrm{P}w_{2m+2}-\frac{b}{2}\sum_{m=0}^{[(k-1)/2]}\mathrm{P}w_{2m+1}}\right|^p}_{=:I'_{2j+2}}$$

$$+C\sum_{j=0}^{[(k-2)/2]}\sum_{i=1,i\neq 2j+2}^{k}\underbrace{\int_{\mathcal{A}_i}\left||\cdot|^{\beta_{2j+2}-2}e^{w_{2j+2}}\right|^p}_{=:I''_{i,2j+2}}$$

$$+C\sum_{i=0}^{[(k-1)/2]}\underbrace{\int_{\mathcal{A}_{2i+1}}\left|\lambda_2\,h_2\,e^{\sum_{m=0}^{[(k-2)/2]}\mathrm{P}w_{2m+2}-\frac{b}{2}\sum_{m=0}^{[(k-1)/2]}\mathrm{P}w_{2m+1}}\right|^p}_{=:I'''_{2i+1}}.$$

Now we suffice to estimate separately each of $I'_\ell, I''_{i,\ell}, I'''_i$. To handle with I'_ℓ we use the definition (2.5) of Θ_ℓ and Lemma 2.2:

$$\begin{aligned}
I'_\ell &= \int_{\mathcal{A}_\ell}\left||x|^{\beta_\ell-2}e^{w_\ell(x)}\left(1-e^{\Theta_\ell(x/\delta_\ell)}\right)\right|^p\mathrm{d}x\\
&= \left(2\beta_\ell^2\right)^p\delta_\ell^{2-2p}\int_{\mathcal{A}_\ell/\delta_\ell}\frac{|y|^{(\beta_\ell-2)p}}{(1+|y|^{\beta_\ell})^{2p}}\left|1-e^{\Theta_\ell(y)}\right|^p\mathrm{d}y\\
&\le \left(2\beta_\ell^2\right)^p\delta_\ell^{2-2p}\int_{\mathcal{A}_\ell/\delta_\ell}\frac{|y|^{(\beta_\ell-2)p}}{(1+|y|^{\beta_\ell})^{2p}}\left|\Theta_\ell(y)\right|^p e^{p|\Theta_\ell(y)|}\mathrm{d}y\\
&\le C\,\delta_\ell^{2-2p}\int_{\frac{\mathcal{A}_\ell}{\delta_\ell}}\frac{|y|^{(\beta_\ell-2)p}}{(1+|y|^{\beta_\ell})^{2p}}|\Theta_\ell(y)|^p\mathrm{d}y\le C\,\delta_\ell^{2-2p}\int_{\frac{\mathcal{A}_\ell}{\delta_\ell}}\frac{|y|^{(\beta_\ell-2)p}}{(1+|y|^{\beta_\ell})^{2p}}\left|\delta_\ell y+|\lambda|^{\gamma_0}\right|^p\mathrm{d}y\\
&\le C\Big(\delta_\ell^{2-\min\{1,2-\frac{\beta_\ell}{2}\}p}\int_{\frac{\mathcal{A}_\ell}{\delta_\ell}}\frac{|y|^{\max\left\{\frac32\beta_\ell-2,\beta_\ell-1\right\}p}}{(1+|y|^{\beta_\ell})^{2p}}\mathrm{d}y+\delta_\ell^{2-2p}|\lambda|^{p\gamma_0}\int_{\frac{\mathcal{A}_\ell}{\delta_\ell}}\frac{|y|^{(\beta_\ell-2)p}}{(1+|y|^{\beta_\ell})^{2p}}\mathrm{d}y\Big)\\
&\le C\left(\delta_\ell^{2-\min\{1,2-\beta_\ell/2\}p}+\delta_\ell^{2-2p}|\lambda|^{p\gamma_0}\right),
\end{aligned}$$

which can be estimated by a power of $|\lambda|$ if p is close enough to 1.

Concerning $I''_{i,\ell}$, we have:

$$\begin{aligned}
I''_{i,\ell} &= \int_{\mathcal{A}_i}\left|\frac{2\beta_\ell^2\delta_\ell^{\beta_\ell}|x|^{\beta_\ell-2}}{(\delta_\ell^{\beta_\ell}+|x|^{\beta_\ell})^2}\right|^p\mathrm{d}x=\left(2\beta_\ell^2\right)^p\delta_\ell^{2-2p}\int_{B_{\sqrt{\frac{\delta_i\delta_{i+1}}{\delta_\ell}}}\setminus B_{\sqrt{\frac{\delta_{i-1}\delta_i}{\delta_\ell}}}}\frac{|y|^{(\beta_\ell-2)p}}{(1+|y|^{\beta_\ell})^{2p}}\mathrm{d}y\\
&\le C\,\delta_\ell^{2-2p}\begin{cases}\left(\sqrt{\delta_i\delta_{i+1}}/\delta_\ell\right)^{(\beta_\ell-2)p+2} & \text{if } i<\ell\\ \left(\delta_\ell/\sqrt{\delta_{i-1}\delta_i}\right)^{-(\beta_\ell+2)p+2} & \text{if } \ell>i\end{cases}\\
&\le C\,\delta_\ell^{2-2p}\begin{cases}\left(\delta_{\ell-1}/\delta_\ell\right)^{\frac{(\beta_\ell-2)p+2}{2}} & \text{if } i<\ell\\ \left(\delta_\ell/\delta_{\ell+1}\right)^{\frac{-(\beta_\ell+2)p+2}{2}} & \text{if } \ell>i\end{cases}\\
&\le C\,\delta_\ell^{2-2p}|\lambda|^{\gamma'},
\end{aligned}\tag{4.1}$$

which is still bounded by a power of λ for small p.

Finally, for I'''_{2i+2}, we use (2.7), the fact that $\delta_\ell \le |y| \le \delta_{\ell'}$ for any $y \in \mathcal{A}_{2i+2}$ and $\ell < 2i+2 < \ell'$, and then the properties (2.4) of δ_ℓ's and (2.12) of β_ℓ:

$$\begin{aligned}
&I'''_{2i+2}\\
&\le C\int_{\mathcal{A}_{2i+2}}\Big|\lambda_1|x|^{\alpha_1-2}\prod_{m=0}^{[(k-1)/2]}\frac{1}{\big(\delta_{2m+1}^{\beta_{2m+1}}+|x|^{\beta_{2m+1}}\big)^2}\prod_{m=0}^{[(k-2)/2]}\big(\delta_{2m+2}^{\beta_{2m+2}}+|x|^{\beta_{2m+2}}\big)^a\Big|^p \mathrm{d}x\\
&\le C\,\lambda_1^p\prod_{m=i+1}^{[(k-1)/2]}\delta_{2m+1}^{-2\beta_{2m+1}p}\prod_{m=i+1}^{[(k-2)/2]}\delta_{2m+2}^{a\beta_{2m+2}p}.\\
&\quad\cdot\int_{\mathcal{A}_{2i+2}}|x|^{(\alpha_1-2)p-2\sum_{m=0}^{i}\beta_{2m+1}p+a\sum_{m=0}^{i-1}\beta_{2m+2}p}\left(\delta_{2i+2}^{\beta_{2i+2}}+|x|^{\beta_{2i+2}}\right)^{ap}\mathrm{d}x\\
&=C\,\delta_{2i+3}^{-\beta_{2i+3}p}\int_{\mathcal{A}_{2i+2}}|x|^{-(\beta_{2i+1}+2)p}\left(\delta_{2i+2}^{\beta_{2i+2}}+|x|^{\beta_{2i+2}}\right)^{ap}\mathrm{d}x\\
&=C\,\delta_{2i+3}^{-\beta_{2i+3}p}\delta_{2i+2}^{2+(-2-\beta_{2i+1}+a\beta_{2i+2})p}\int_{\frac{\mathcal{A}_{2i+2}}{\delta_{2i+2}}}|y|^{-(\beta_{2i+1}+2)p}\left(1+|y|^{\beta_{2i+2}}\right)^{ap}\mathrm{d}y\\
&=C\left(\frac{\delta_{2i+2}}{\delta_{2i+3}}\right)^{\beta_{2i+3}p}\delta_{2i+2}^{2-2p}\int_{B_{\sqrt{\frac{\delta_{2i+3}}{\delta_{2i+2}}}}\setminus B_{\sqrt{\frac{\delta_{2i+1}}{\delta_{2i+2}}}}}|y|^{-(\beta_{2i+1}+2)p}\left(1+|y|^{\beta_{2i+2}}\right)^{ap}\mathrm{d}y\\
&\le C\left(\frac{\delta_{2i+2}}{\delta_{2i+3}}\right)^{\beta_{2i+3}p}\delta_{2i+2}^{2-2p}\left(\left(\frac{\delta_{2i+1}}{\delta_{2i+2}}\right)^{\frac{2-(\beta_{2i+1}+2)p}{2}}+\left(\frac{\delta_{2i+3}}{\delta_{2i+2}}\right)^{\frac{2+(\beta_{2i+3}-2)p}{2}}\right)\\
&=C\left(\frac{\delta_{2i+2}}{\delta_{2i+3}}\right)^{\beta_{2i+3}p}\delta_{2i+2}^{2-2p}\left(\left(\frac{\delta_{2i+2}}{\delta_{2i+3}}\right)^{\frac{(2-(\beta_{2i+1}+2)p)\beta_{2i+3}}{2\beta_{2i+1}}}+\left(\frac{\delta_{2i+2}}{\delta_{2i+3}}\right)^{\frac{-2-(\beta_{2i+3}-2)p}{2}}\right)\\
&\le C\,\delta_{2i+2}^{2-2p}\left(\frac{\delta_{2i+2}}{\delta_{2i+3}}\right)^{\frac{\beta_{2i+3}(\beta_{2i+1}p+2-2p)}{2\beta_{2i+1}}}\le C\,\delta_{2i+2}^{2-2p}|\lambda|^{\gamma'}\le C\,|\lambda|^{\gamma_1}.
\end{aligned}$$

This argument has to be slightly modified when $k=2l+2$; in this case, none of the two products in the second line appear and therefore we have λ_1^p in place of $\delta_{2i+3}^{-\beta_{2i+3}p}$:

$$\begin{aligned}
I'''_k &\le C\,\lambda_1^p\,\delta_k^{2+(\beta_{k+1}-2)p}\int_{B_{\mathrm{diam}\Omega/\delta_k}\setminus B_{\sqrt{\delta_{k-1}/\delta_k}}}|y|^{-(\beta_{k-1}+2)p}\left(1+|y|^{\beta_k}\right)^{ap}\mathrm{d}y\\
&\le C\,\lambda_1^p\,\delta_k^{2+(\beta_{k+1}-2)p}\left(\left(\frac{\delta_{k-1}}{\delta_k}\right)^{\frac{2-(\beta_{k-1}+2)p}{2}}+\delta_k^{-2-(\beta_{k+1}-2)p}\right)\\
&\le C\,\lambda_1^p\left(\delta_k^{2+(\beta_{k+1}-2)p}\left(\frac{\delta_{k-1}}{\delta_k}\right)^{\frac{2-(\beta_{k-1}+2)p}{2}}+1\right)\le C\,|\lambda|^{\gamma_1}.
\end{aligned}$$

The same argument works for I'''_{2i+1}, with a slight modification needed now for I'''_1: this time in the second line we do not have any of the sums in the power of $|x|$ and we get:

$$I'''_1 \le C\,\delta_2^{-\beta_2 p}\int_{A_1}|x|^{(\alpha_2-2)p}(\delta_1^{\beta_1}+|x|^{\beta_1})^{bp}\mathrm{d}x$$

$$\leq C\Big(\frac{\delta_1}{\delta_2}\Big)^{\beta_2 p}\delta_2^{2-2p}\int_{B_{\sqrt{\delta_2/\delta_1}}}|y|^{(\alpha_2-2)p}\left(1+|y|^{\beta_1}\right)^{bp}\mathrm{d}y$$
$$\leq C\Big(\frac{\delta_1}{\delta_2}\Big)^{\frac{(\beta_2+2)p-2}{2}}\delta_2^{2-2p}\leq C|\lambda|^{\gamma_1};$$

this concludes the proof. □

4.2. The linear operator $\mathcal{S}_\lambda$

Lemma 4.2. *Let $\mathcal{S}_\lambda$ be defined by (3.3). There exists $p_0>1$ and $\gamma_2>0$ such that for any $p\in[1,p_0)$,*

$$\|\mathcal{S}_\lambda\phi\|_p=O(|\lambda|^{\gamma_2}\|\phi\|).$$

Proof. We can estimate $\|\mathcal{S}_\lambda\|_p$ by arguing as in Lemma 4.1:

$$\int_\Omega\Big|\sum_{j=0}^{[(k-1)/2]}|\cdot|^{\beta_{2j+1}}e^{w_{2j+1}}-2\lambda_1 h_1 e^{W_{1,\lambda}}\Big|^p$$
$$=\int_\Omega\Big|\sum_{j=0}^{[(k-1)/2]}|\cdot|^{\beta_{2j+1}-2}e^{w_{2j+1}}-2\lambda_1 h_1 e^{\sum_{m=0}^{[(k-1)/2]}\mathrm{P}w_{2m+1}-\frac{a}{2}\sum_{m=0}^{[(k-2)/2]}\mathrm{P}w_{2j+2}}\Big|^p$$
$$\leq C\sum_{j=0}^{[(k-1)/2]}\int_{\mathcal{A}_{2j+1}}\Big||\cdot|^{\beta_{2j+1}-2}e^{w_{2j+1}}$$
$$-2\lambda_1 h_1 e^{\sum_{m=0}^{[(k-1)/2]}\mathrm{P}w_{2m+1}-\frac{a}{2}\sum_{m=0}^{[(k-2)/2]}\mathrm{P}w_{2m+2}}\Big|^p$$
$$+C\sum_{j=0}^{[(k-1)/2]}\sum_{i=1,i\neq 2j+1}^{k}\int_{\mathcal{A}_i}\Big||\cdot|^{\beta_{2j+1}-2}e^{w_{2j+1}}\Big|^p$$
$$+C\sum_{i=0}^{[(k-2)/2]}\int_{\mathcal{A}_{2i+2}}\Big|\lambda_1 h_1 e^{\sum_{m=0}^{[(k-1)/2]}\mathrm{P}w_{2m+1}-\frac{a}{2}\sum_{m=0}^{[(k-2)/2]}\mathrm{P}w_{2m+2}}\Big|^p$$
$$\leq C\Big(\sum_{j=0}^{[(k-1)/2]}I'_{2j+1}+\sum_{j=0}^{[(k-1)/2]}\sum_{i=1,i\neq 2j+1}^{k}I''_{i,2j+1}+\sum_{i=0}^{[(k-2)/2]}I'''_{2i+2}\Big)\leq C\,|\lambda|^{\gamma_1};\tag{4.2}$$

and the same estimates also work the other components of $\mathcal{S}_\lambda$.

Then we suffice to apply the Hölder and Sobolev inequalities, with q so close to 1 that the previous estimates hold for $\|S_\lambda\|_{pq}$:

$$\|\mathcal{S}_\lambda\phi\|_p\leq\|\mathcal{S}_\lambda\|_{pq}\|\phi\|_{\frac{pq}{q-1}}\leq C|\lambda|^{\gamma_2}\|\phi\|.$$ □

4.3. The quadratic term $\mathcal{N}_\lambda$

Lemma 4.3. *Let $\mathcal{N}_\lambda$ be defined by (3.4).*

There exists $p_0>1, C>0$ and $\gamma_3>0$ such that, for any $p\in[1,p_0)$,

$$\|\mathcal{N}_\lambda(\phi)-\mathcal{N}_\lambda(\phi')\|_p=O\big(|\lambda|^{\gamma_3(1-p)}\|\phi-\phi'\|(\|\phi\|+\|\phi'\|)e^{C(\|\phi\|^2+\|\phi'\|^2)}\big),\tag{4.3}$$

and in particular

$$\|\mathcal{N}_\lambda(\phi)\|_p = O\big(|\lambda|^{\gamma_3(1-p)}\|\phi\|^2 e^{C\|\phi\|^2}\big). \tag{4.4}$$

Proof. By writing

$$\mathcal{N}_\lambda(\phi) - \mathcal{N}_\lambda(\phi') = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 h_1 e^{W_{1,\lambda}}(e^{\phi_1} - e^{\phi_1'} - \phi_1 + \phi_1') \\ \lambda_2 h_2 e^{W_{2,\lambda}}(e^{\phi_2} - e^{\phi_2'} - \phi_2 + \phi_2') \end{pmatrix},$$

we suffice to provide L^p estimates for $\lambda_i h_i e^{W_{i,\lambda}}(e^{\phi_i} - e^{\phi_i'} - \phi_i + \phi_i')$ for $i = 1, 2$.

By the elementary inequality

$$|e^t - e^s - t + s| \le |t-s|\,(|t| + |s|)\, e^{|t|+|s|}, \quad \forall\, t, s \in \mathbb{R}$$

and the Hölder, Sobolev and Moser–Trudinger inequalities, we get

$$\begin{aligned}
&\int_\Omega \big|\lambda_i h_i e^{W_{i,\lambda}}\big(e^{\phi_i} - e^{\phi_i'} - \phi_i + \phi_i'\big)\big|^p \\
&\le \int_\Omega \big|\lambda_i h_i e^{W_{i,\lambda}}\big|^p\, |\phi_i - \phi_i'|^p\, \big(|\phi_i|^p + |\phi_i'|^p\big)\, e^{p\left(|\phi_i| + |\phi_i'|\right)} \\
&\le \Big(\int_\Omega \big|\lambda_i h_i e^{W_{i,\lambda}}\big|^{pq}\Big)^{1/q} \Big(\int_\Omega |\phi_i - \phi_i'|^{ps}\Big)^{1/s} \Big(\Big(\int_\Omega |\phi_i|^{ps}\Big)^{1/s} + \Big(\int_\Omega |\phi_i'|^{ps}\Big)^{1/s}\Big) \\
&\quad \cdot \Big(\int_\Omega e^{\frac{pqrs}{qrs-qr-qs-rs}\left(|\phi_i| + |\phi_i'|\right)}\Big)^{1-1/q-1/r-1/s} \\
&\le C\Big(\int_\Omega \big|\lambda_i h_i e^{W_{i,\lambda}}\big|^{pq}\Big)^{1/q} \|\phi_i - \phi_i'\|\, (\|\phi_i\| + \|\phi_i'\|)\, e^{\frac{p^2 qrs}{qrs-qr-qs-rs}\left(\|\phi_i\| + \|\phi_i'\|\right)^2};
\end{aligned}$$

therefore, we just have to estimate $\lambda_i h_i e^{W_{i,\lambda}}$ in $L^p(\Omega)$.

The computations from Lemma 4.1 and (2.10) yield:

$$\begin{aligned}
&\int_\Omega \big|\lambda_i h_i e^{W_{i,\lambda}}\big|^p \\
&\le C \sum_{j=0}^{[(k-i)/2]} \int_{\mathcal{A}_{2j+i}} \big||x|^{\beta_{2j+i}-2} e^{w_{2j+i}(x) + \Theta_{2j+i}(x/\delta_{2j+i})}\big|^p \mathrm{d}x + C \sum_{j=0}^{[(k-3+i)/2]} I'''_{2j-3+i} \\
&\le \beta_{2j+i}^p \sum_{j=0}^{[(k-i)/2]} \delta_{2j+i}^{2-2p} \int_{\frac{\mathcal{A}_{2j+i}}{\delta_{2j+i}}} \frac{|y|^{(\beta_{2j+i}-2)p}}{(1+|y|^{\beta_{2j+i}})^{2p}} e^{p|\Theta_{2j+i}(y)|} \mathrm{d}y + o(1) \\
&\le C \sum_{j=0}^{[(k-i)/2]} \delta_{2j+i}^{2-2p} \int_{\frac{\mathcal{A}_{2j+i}}{\delta_{2j+i}}} \frac{|y|^{(\beta_{2j+i}-2)p}}{(1+|y|^{\beta_{2j+i}})^{2p}} \mathrm{d}y + o(1) \\
&\le C \sum_{j=0}^{[(k-i)/2]} \delta_{2j+i}^{2-2p} \le C\,|\lambda|^{\gamma_3(1-p)} \qquad (4.5)
\end{aligned}$$

hence (4.3) is proved.

Formula (4.4) just follows from (4.3) after setting $\phi' = 0$. □

5. Linear theory

In this section we develop a linear theory for the linear operator $\mathcal{L}_\lambda$ defined in (3.1).

The following proposition, whose proof will take up the whole section, is inspired by [30] (Proposition 4.1) and [49] (Proposition 4.1).

Proposition 5.1. *For any $p>1$ there exists $\overline{\lambda}>0$ and $C>0$ such that for any $\lambda\in\big(0,\overline{\lambda}\big)\times\big(0,\overline{\lambda}\big)$ and any $\psi\in\mathbf{H}$ there exists a unique $\phi\in\mathbf{H}$ solution of*

$$\mathcal{L}_\lambda\phi=\psi\quad \text{on } \Omega,$$

satisfying

$$\|\phi\|\le C\log\frac{1}{|\lambda|}\|\psi\|_p$$

Proof. Suppose the statement is not true. This means that there exist $p>1$ and sequences $\{\lambda_n\}_{n\in\mathbb{N}}\subset\mathbb{R}^2_{>0}$, $\{\psi_n\}_{n\in\mathbb{N}}\subset\mathbf{H}$, $\{\phi_n\}_{n\in\mathbb{N}}\subset\mathbf{H}$ such that

$$(5.1)\qquad \begin{cases} -\Delta\phi_{n,1}-\mathcal{L}_{n,1}\,\phi_{n,1}+\frac{a}{2}\mathcal{L}_{n,2}\,\phi_{n,2}=\psi_{n,1},\\ -\Delta\phi_{n,2}-\mathcal{L}_{n,2}\,\phi_{n,2}+\mathcal{L}_{n,1}\,\phi_{n,1}=\psi_{n,2},\\ \lambda_n\to 0 \text{ as } n\to+\infty,\\ \|\phi_n\|=1,\\ \log\frac{1}{|\lambda_n|}\|\psi_n\|_p\to 0 \text{ as } n\to+\infty,, \end{cases}$$

where $\delta_{n,\ell}$ is defined as in (2.4) with $\lambda_{n,1},\lambda_{n,2}$ in place of λ_1,λ_2 and

$$\mathcal{L}_{n,1}=\sum_{j=0}^{[(k-1)/2]}2\,\beta^2_{n,2j+1}\frac{\delta_{n,2j+1}^{\beta_{n,2j+1}}|\cdot|^{\beta_{n,2j+1}-2}}{\left(\delta_{n,2j+1}^{\beta_{n,2j+1}}+|\cdot|^{\beta_{n,2j+1}}\right)^2}$$

$$\mathcal{L}_{n,2}=\sum_{j=0}^{[(k-2)/2]}2\,\beta^2_{n,2j+2}\frac{\delta_{n,2j+2}^{\beta_{n,2j+2}}|\cdot|^{\beta_{n,2j+2}-2}}{\left(\delta_{n,2j+2}^{\beta_{n,2j+2}}+|\cdot|^{\beta_{n,2j+2}}\right)^2}.$$

We will divide the proof in six steps.

Step 1. *For any $i=1,2,\ j=0,\dots,[(k-i)/2]$,*

$$\int_\Omega\frac{\delta_{n,2j+i}^{\beta_{2j+i}}|\cdot|^{\beta_{2j+i}-2}}{\left(\delta_{n,2j+i}^{\beta_{2j+i}}+|\cdot|^{\beta_{2j+i}}\right)^2}|\phi_{n,i}|^2=O(1).$$

If we multiply both sides of the first equation in (5.1) by $\phi_{n,1}$ and both sides of the second equation by $\frac{a}{2}\phi_{n,1}$ and then we sum the two equalities we get

$$\begin{aligned}\left(1-\frac{ab}{4}\right)&\sum_{j=0}^{[(k-1)/2]}2\beta^2_{2j+1}\int_\Omega\frac{\delta_{n,2j+1}^{\beta_{2j+1}}|\cdot|^{\beta_{2j+1}-2}}{\left(\delta_{n,2j+1}^{\beta_{2j+1}}+|\cdot|^{\beta_{2j+1}}\right)^2}|\phi_{n,1}|^2\\ &=\int_\Omega|\nabla\phi_{n,1}|^2-\int_\Omega\psi_{n,1}\,\phi_{n,1}+\frac{a}{2}\int_\Omega\nabla\phi_{n,1}\cdot\nabla\phi_{n,2}-\frac{a}{2}\int_\Omega\psi_{n,2}\,\phi_{n,1}\\ &\le C\left(\|\phi_n\|^2+\|\psi_n\|_p\|\phi_n\|\right)\le C;\end{aligned}$$

similarly, by multiplying the first equation in (5.1) by $\frac{b}{2}\phi_{n,1}$, the second equation by $\phi_{n,2}$ and then summing, we get

$$\left(1-\frac{ab}{4}\right)\sum_{j=0}^{[(k-2)/2]} 2\beta_{2j+2}^2 \int_\Omega \frac{\delta_{n,2j+2}^{\beta_{2j+2}}|\cdot|^{\beta_{2j+2}-2}}{\left(\delta_{n,2j+2}^{\beta_{2j+2}}+|\cdot|^{\beta_{2j+2}}\right)^2}|\phi_{n,2}|^2 = O(1).$$

Therefore, the claim is proved if $ab\neq 4$.

On the other hand, if $ab=4$, then summing the first equation in (5.1) and the second equation multiplied by $a/2=2/b$ gives

$$\begin{cases} -\Delta\left(\phi_{n,1}+\frac{a}{2}\phi_{n,2}\right)=\psi_{n,1}+\frac{a}{2}\psi_{n,2} & \text{in } \Omega,\\ \phi_{n,1}+\frac{a}{2}\phi_{n,2}=0 & \text{on } \partial\Omega, \end{cases}$$

hence standard regularity theory yields

$$\left\|\phi_{n,1}+\frac{a}{2}\phi_{n,2}\right\|_\infty \le C\left\|\psi_{n,1}+\frac{a}{2}\psi_{n,2}\right\|_p = o(1).$$

Therefore, multiplying the first equation in (5.1) by $\phi_{n,1}$, the second equation by $\frac{a^2}{4}\phi_{n,2}=\frac{4}{b^2}\phi_{n,2}$ and then summing we get

$$\begin{aligned}
&2\sum_{j=0}^{[(k-1)/2]} 2\beta_{2j+1}^2 \int_\Omega \frac{\delta_{n,2j+1}^{\beta_{2j+1}}|\cdot|^{\beta_{2j+1}-2}}{\left(\delta_{n,2j+1}^{\beta_{2j+1}}+|\cdot|^{\beta_{2j+1}}\right)^2}|\phi_{n,1}|^2\\
&\quad+\frac{a^2}{2}\sum_{j=0}^{[(k-2)/2]} 2\beta_{2j+2}^2 \int_\Omega \frac{\delta_{n,2j+2}^{\beta_{2j+2}}|\cdot|^{\beta_{2j+2}-2}}{\left(\delta_{n,2j+2}^{\beta_{2j+2}}+|\cdot|^{\beta_{2j+2}}\right)^2}|\phi_{n,2}|^2\\
&=\int_\Omega\Bigg(\sum_{j=0}^{[(k-1)/2]} 2\beta_{2j+1}^2 \frac{\delta_{n,2j+1}^{\beta_{2j+1}}|\cdot|^{\beta_{2j+1}-2}}{\left(\delta_{n,2j+1}^{\beta_{2j+1}}+|\cdot|^{\beta_{2j+1}}\right)^2}\phi_{n,1}\\
&\quad+\frac{a}{2}\sum_{j=0}^{[(k-2)/2]} 2\beta_{2j+2}^2 \frac{\delta_{n,2j+2}^{\beta_{2j+2}}|\cdot|^{\beta_{2j+2}-2}}{\left(\delta_{n,2j+2}^{\beta_{2j+2}}+|\cdot|^{\beta_{2j+2}}\right)^2}\phi_{n,2}\Bigg)\left(\phi_{n,1}+\frac{a}{2}\phi_{n,2}\right)\\
&\quad+\int_\Omega|\nabla\phi_{n,1}|^2-\int_\Omega\psi_{n,1}\,\phi_{n,1}+\frac{a^2}{4}\int_\Omega|\nabla\phi_{n,2}|^2-\frac{a^2}{4}\int_\Omega\psi_{n,2}\phi_{n,2}\\
&\le\Bigg(\sum_{j=0}^{[(k-1)/2]} 2\beta_{2j+1}^2 \int_\Omega \frac{\delta_{n,2j+1}^{\beta_{2j+1}}|\cdot|^{\beta_{2j+1}-2}}{\left(\delta_{n,2j+1}^{\beta_{2j+1}}+|\cdot|^{\beta_{2j+1}}\right)^2}|\phi_{n,1}|\\
&\quad+\frac{a}{2}\sum_{j=0}^{[(k-2)/2]} 2\beta_{2j+2}^2 \int_\Omega \frac{\delta_{n,2j+2}^{\beta_{2j+2}}|\cdot|^{\beta_{2j+2}-2}}{\left(\delta_{n,2j+2}^{\beta_{2j+2}}+|\cdot|^{\beta_{2j+2}}\right)^2}|\phi_{n,2}|\Bigg)^{1/2}\left\|\phi_{n,1}+\frac{a}{2}\phi_{n,2}\right\|_\infty+C
\end{aligned}$$

(5.2)

$$\le o(1)\Bigg(\sum_{j=0}^{[(k-1)/2]} 2\beta_{2j+1}^2 \int_\Omega \frac{\delta_{n,2j+1}^{\beta_{2j+1}}|\cdot|^{\beta_{2j+1}-2}}{\left(\delta_{n,2j+1}^{\beta_{2j+1}}+|\cdot|^{\beta_{2j+1}}\right)^2}|\phi_{n,1}|^2$$

$$+\frac{a}{2}\sum_{j=0}^{[(k-2)/2]}2\beta_{2j+2}^2\int_\Omega\frac{\delta_{n,2j+2}^{\beta_{2j+2}}|\cdot|^{\beta_{2j+2}-2}}{\left(\delta_{n,2j+2}^{\beta_{2j+2}}+|\cdot|^{\beta_{2j+2}}\right)^2}|\phi_{n,2}|^2\Bigg)^{1/2}+C;$$

therefore,

$$\int_\Omega\frac{\delta_{n,2j+i}^{\beta_{2j+i}}|\cdot|^{\beta_{2j+i}-2}}{(\delta_{n,2j+i}^{\beta_{2j+i}}+|\cdot|^{\beta_{2j+i}})^2}|\phi_{n,i}|^2\leq C$$

for all i,ℓ.

Step 2: The sequence $\widetilde{\phi}_{n,\ell}$, defined by $\widetilde{\phi}_{n,2j+i}(y):=\phi_{n,i}(\delta_{n,2j+i}y)$, converges to $\mu_\ell\frac{1-|\cdot|^{\beta_\ell}}{1+|\cdot|^{\beta_\ell}}$ for some μ_ℓ, weakly in $H_{\beta_\ell}(\mathbb{R}^2)$ and strongly in $L_{\beta_\ell}(\mathbb{R}^2)$, where

$$L_\beta(\mathbb{R}^2):=\left\{u\in L^2_{\mathrm{loc}}(\mathbb{R}^2):\ \frac{|\cdot|^{\frac{\beta-2}{2}}}{1+|\cdot|^\beta}u\in L^2(\mathbb{R}^2)\right\},$$

$$\|u\|_{L_\beta}:=\left\|\frac{|\cdot|^{\frac{\beta-2}{2}}}{1+|\cdot|^\beta}u\right\|_{L^2(\mathbb{R}^2)};$$

$$H_\beta(\mathbb{R}^2):=\left\{u\in H^1_{\mathrm{loc}}(\mathbb{R}^2):\ |\nabla u|+\frac{|\cdot|^{\frac{\beta-2}{2}}}{1+|\cdot|^\beta}u\in L^2(\mathbb{R}^2)\right\},$$

$$\|u\|_{H_\beta}:=\left(\|\nabla u\|^2_{L^2(\mathbb{R}^2)}+\left\|\frac{|\cdot|^{\frac{\beta-2}{2}}}{1+|\cdot|^\beta}u\right\|^2_{L^2(\mathbb{R}^2)}\right)^{1/2}.$$

First of all, because of Step 1, $\widetilde{\phi}_{n,\ell}$ is bounded in $H_{\beta_\ell}(\mathbb{R}^2)$:

$$\begin{aligned}\int_{\Omega/\delta_{n,2j+i}}\left|\nabla\widetilde{\phi}_{n,2j+i}(y)\right|^2\mathrm{d}y&=\delta_{n,2j+i}^2\int_{\Omega/\delta_{n,2j+i}}|\nabla\phi_{n,i}(\delta_{n,2j+i}y)|^2\,\mathrm{d}y\\&=\int_\Omega|\nabla\phi_{n,i}(x)|^2\,\mathrm{d}x=1,\end{aligned}$$

$$\begin{aligned}\int_{\Omega/\delta_{n,2j+i}}\frac{|y|^{\beta_{2j+i}-2}}{(1+|y|^{\beta_{2j+i}})^2}\left|\widetilde{\phi}_{n,2j+i}(y)\right|^2\mathrm{d}y&=\int_\Omega\frac{\delta_{n,2j+i}^{\beta_{2j+i}}|x|^{\beta_{2j+i}-2}}{\left(\delta_{n,2j+i}^{\beta_{2j+i}}+|x|^{\beta_{2j+i}}\right)^2}|\phi_{n,i}(x)|^2\,\mathrm{d}x\\&=O(1).\end{aligned}$$

Therefore, $\widetilde{\phi}_{n,\ell}\to\widetilde{\phi}_\ell$ as $n\to+\infty$ in $H_{\beta_\ell}(\mathbb{R}^2)$ for some $\widetilde{\phi}_\ell\in H_{\beta_\ell}(\mathbb{R}^2)$; moreover, the embedding $L_{\beta_\ell}(\mathbb{R}^2)\hookrightarrow H_{\beta_\ell}(\mathbb{R}^2)$ is compact (see [30], Proposition 6.1; the result is stated only for $\alpha\geq 2$ but the same argument works for any $\alpha>0$). From this we get $\widetilde{\phi}_{n,\ell}\to\widetilde{\phi}_\ell$ as $n\to+\infty$ in $L_{\beta_\ell}(\mathbb{R}^2)$.

The function $\widetilde{\phi}_{n,\ell}$ solves

$$\begin{cases}-\Delta\widetilde{\phi}_{n,\ell}=2\beta_\ell^2\dfrac{|\cdot|^{\beta_\ell-2}}{(1+|\cdot|^{\beta_\ell})^2}\widetilde{\phi}_{n,\ell}+\rho_{n,\ell} & \text{in }\Omega/\delta_{n,\ell},\\ \widetilde{\phi}_{n,\ell}=0 & \text{on }\partial\left(\Omega/\delta_{n,\ell}\right),\end{cases}$$

where

$$\rho_{n,2j+1}(y) := \sum_{i=0,i\neq j}^{[(k-1)/2]} 2\beta_{2i+1}^2 \frac{\delta_{n,2i+1}^{\beta_{2i+1}}\,\delta_{n,2j+1}^{\beta_{2i+1}}\,|y|^{\beta_{2i+1}-2}}{\left(\delta_{n,2i+1}^{\beta_{2i+1}}+\delta_{n,2j+1}^{\beta_{2i+1}}\,|y|^{\beta_{2i+1}}\right)^2}\,\phi_{n,1}(\delta_{n,2j+1}y)$$
$$-\frac{a}{2}\sum_{i=0}^{[(k-2)/2]} 2\beta_{2i+2}^2 \frac{\delta_{n,2i+2}^{\beta_{2i+2}}\,\delta_{n,2j+1}^{\beta_{2i+2}}\,|y|^{\beta_{2i+2}-2}}{\left(\delta_{n,2i+2}^{\beta_{2i+2}}+\delta_{n,2j+1}^{\beta_{2i+2}}|y|^{\beta_{2i+2}}\right)^2}\,\phi_{n,2}(\delta_{n,2j+1}y)$$
$$+\,\delta_{n,2j+1}^2\,\psi_{n,1}(\delta_{n,2j+1}y)$$

and

$$\rho_{n,2j+2}(y) := \sum_{i=0,i\neq j}^{[(k-2)/2]} 2\beta_{2i+2}^2 \frac{\delta_{n,2i+2}^{\beta_{2i+2}}\,\delta_{n,2j+2}^{\beta_{2i+2}}\,|y|^{\beta_{2i+2}-2}}{\left(\delta_{n,2i+2}^{\beta_{2i+2}}+\delta_{n,2j+2}^{\beta_{2i+2}}|y|^{\beta_{2i+2}}\right)^2}\,\phi_{n,2}(\delta_{n,2j+2}y)$$
$$-\frac{b}{2}\sum_{i=0}^{[(k-1)/2]} 2\beta_{2i+1}^2 \frac{\delta_{n,2i+1}^{\beta_{2i+1}}\,\delta_{n,2j+2}^{\beta_{2i+1}}\,|y|^{\beta_{2i+1}-2}}{\left(\delta_{n,2i+1}^{\beta_{2i+1}}+\delta_{n,2j+2}^{\beta_{2i+1}}|y|^{\beta_{2i+1}}\right)^2}\,\phi_{n,1}(\delta_{n,2j+2}y)$$
$$+\,\delta_{n,2j+2}^2\,\psi_{n,2}(\delta_{n,2j+2}y).$$

Let us show that $\rho_{n,\ell}\to 0$ as $n\to+\infty$ in $L^1_{\mathrm{loc}}\left(\mathbb{R}^2\setminus\{0\}\right)$.

Any compact set $\mathcal{K}\Subset\mathbb{R}^2\setminus\{0\}$ will be contained, for large n, in

$$A_{n,\ell}/\delta_{n,\ell} := \left\{y\in\Omega/\delta_{n,\ell} :\ \sqrt{\delta_{n,\ell-1}/\delta_{n,\ell}}\le |y|\le\sqrt{\delta_{n,\ell}/\delta_{n,\ell+1}}\right\};$$

therefore, by the estimate (4.1),

$$\int_{\mathcal{K}}|\rho_{n,\ell}| \le \int_{A_{n,\ell}/\delta_{n,\ell}} |\rho_{n,\ell}(y)|\,\mathrm{d}y$$
$$\le C\sum_{i=0,i\neq j}^{k}\int_{A_{n,\ell}/\delta_{n,\ell}} \frac{\delta_{n,i}^{\beta_i}\delta_{n,\ell}^{\beta_i}|y|^{\beta_i-2}}{\left(\delta_{n,i}^{\beta_i}+\delta_{n,\ell}^{\beta_i}|y|^{\beta_i}\right)^2}\left(|\phi_{n,1}(\delta_{n,\ell}y)|+|\phi_{n,2}(\delta_{n,\ell}y)|\right)\mathrm{d}y$$
$$+\,\delta_{n,\ell}^2\int_{\Omega/\delta_{n,\ell}}\left(|\psi_{n,1}(\delta_{n,\ell}y)|+|\psi_{n,2}(\delta_{n,\ell}y)|\right)\mathrm{d}y$$
$$= C\sum_{i=0,i\neq j}^{k}\int_{A_{n,\ell}} \frac{\delta_{n,i}^{\beta_i}|x|^{\beta_i-2}}{\left(\delta_{n,i}^{\beta_i}+|x|^{\beta_i}\right)^2}\left(|\phi_{n,1}(x)|+|\phi_{n,2}(x)|\right)\mathrm{d}x$$
$$+\,C\int_{\Omega}\left(|\psi_{n,1}(x)|+|\psi_{n,2}(x)|\right)\mathrm{d}x$$
$$\le C\left(\sum_{i=0,i\neq j}^{k}\int_{A_{n,\ell}}\left|\frac{\delta_{n,i}^{\beta_i}|x|^{\beta_i-2}}{\left(\delta_{n,i}^{\beta_i}+|x|^{\beta_i}\right)^2}\right|^q\mathrm{d}x\right)^{1/q}\|\phi_n\|_{q/(q-1)}+C\,\|\psi_n\|_p$$
$$\le C\,|\lambda_n|^{\gamma_1}\|\phi_n\|+C\,\|\psi_n\|_p \underset{n\to+\infty}{\longrightarrow} 0.$$

Therefore, the weak limit $\widetilde{\phi}_\ell$ must be a solution of

$$-\Delta\widetilde{\phi}_\ell = 2\beta_\ell^2 \frac{|\cdot|^{\beta_\ell-2}}{(1+|\cdot|^{\beta_\ell})^2}\,\widetilde{\phi}_\ell \quad \text{in } \mathbb{R}^2\setminus\{0\}.$$

Finally, by the properties of weak convergence we get $\int_{\mathbb{R}^2}|\nabla\widetilde{\phi}_\ell|^2 \le 1$, so $\widetilde{\phi}_\ell$ must be a solution on the whole plane; by Proposition A.1 we get

$$\widetilde{\phi}_\ell = \mu_\ell \frac{1-|\cdot|^{\beta_\ell}}{1+|\cdot|^{\beta_\ell}}.$$

Step 3. *As $n\to+\infty$, and for all ℓ's,*

$$\sigma_{n,\ell} := \log\frac{1}{|\lambda_n|}\int_{\Omega/\delta_{n,\ell}} 2\beta_\ell^2 \frac{|\cdot|^{\beta_\ell-2}}{(1+|\cdot|^{\beta_\ell})^2}\,\widetilde{\phi}_{n,\ell}\to 0.$$

Define

$$Z_{n,\ell} := \frac{\delta_{n,\ell}^{\beta_\ell}-|\cdot|^{\beta_\ell}}{\delta_{n,\ell}^{\beta_\ell}+|\cdot|^{\beta_\ell}},$$

which solves (see Theorem A.1)

$$-\Delta Z_{n,\ell} = 2\beta_\ell^2 \frac{\delta_{n,\ell}^{\beta_\ell}|\cdot|^{\beta_\ell-2}}{(\delta_{n,\ell}^{\beta_\ell}+|\cdot|^{\beta_\ell})^2}\,Z_{n,\ell} \quad \text{in } \mathbb{R}^2;$$

consider now its projection $\mathrm{P}Z_{n,\ell}$ on $H_0^1(\Omega)$, namely (see (2.2)) the solution of

$$\text{(5.3)}\qquad \begin{cases} -\Delta(\mathrm{P}Z_{n,\ell}) = 2\beta_\ell^2 \dfrac{\delta_{n,\ell}^{\beta_\ell}|\cdot|^{\beta_\ell-2}}{(\delta_{n,\ell}^{\beta_\ell}+|\cdot|^{\beta_\ell})^2}\,Z_{n,\ell} & \text{in } \Omega,\\ \mathrm{P}Z_{n,\ell}=0 & \text{on } \partial\Omega. \end{cases}$$

As in Lemma 2.1, the maximum principle gives

$$\text{(5.4)}\qquad \mathrm{P}Z_{n,\ell} = Z_{n,\ell}+1+O\big(\delta_{n,\ell}^{\beta_\ell}\big) = \frac{2\delta_{n,\ell}^{\beta_\ell}}{\delta_{n,\ell}^{\beta_\ell}+|\cdot|^{\beta_\ell}}+O\big(\delta_{n,\ell}^{\beta_\ell}\big),$$

hence

$$\text{(5.5)}\qquad \mathrm{P}Z_{n,i}(\delta_{n,\ell}y) = \begin{cases} 2\,\dfrac{(\delta_{n,i}/\delta_{n,\ell})^{\beta_i}}{\big((\delta_{n,i}/\delta_{n,\ell})^{\beta_i}+|y|^{\beta_i}\big)}+O\big(\delta_{n,i}^{\beta_i}\big) & \text{if } i<\ell,\\ \dfrac{2}{1+|y|^{\beta_i}}+O\big(\delta_{n,i}^{\beta_i}\big) & \text{if } i=\ell,\\ 2-2\dfrac{(\delta_{n,\ell}/\delta_{n,i})^{\beta_i}\,|y|^{\beta_i}}{\big(1+(\delta_{n,\ell}/\delta_{n,i})^{\beta_i}|y|^{\beta_i}\big)}+O\big(\delta_{n,i}^{\beta_i}\big) & \text{if } i>\ell. \end{cases}$$

Recall now the first equation of (5.1) and multiply it by $\log\frac{1}{|\lambda_n|}\mathrm{P}Z_{n,2i+1}$; then, multiply by $\log\frac{1}{|\lambda_n|}\,\phi_{n,1}$ the equation (5.3) satisfied by $\mathrm{P}Z_{n,2i+1}$ and subtract the two quantities: we get

$$\begin{aligned}
0 &= \underbrace{\log\frac{1}{|\lambda_n|}\int_\Omega 2\beta_{2i+1}^2\frac{\delta_{n,2i+1}^{\beta_{2i+1}}|\cdot|^{\beta_{2i+1}-2}}{\big(\delta_{n,2i+1}^{\beta_{2i+1}}+|\cdot|^{\beta_{2i+1}}\big)^2}\,\phi_{n,1}(\mathrm{P}Z_{n,2i+1}-Z_{n,2i+1})}_{=:I'_{n,2i+1}}\\
&\quad+\sum_{j=0,j\neq i}^{[(k-1)/2]}\underbrace{\log\frac{1}{|\lambda_n|}\int_\Omega 2\beta_{2j+1}^2\frac{\delta_{n,2j+1}^{\beta_{2j+1}}|\cdot|^{\beta_{2j+1}-2}}{\big(\delta_{n,2j+1}^{\beta_{2j+1}}+|\cdot|^{\beta_{2j+1}}\big)^2}\,\phi_{n,1}\mathrm{P}Z_{n,2i+1}}_{=:I''_{n,2i+1,2j+1}}\\
&\quad-\frac{a}{2}\sum_{j=0}^{[(k-2)/2]}\underbrace{\log\frac{1}{|\lambda_n|}\int_\Omega 2\beta_{2j+2}^2\frac{\delta_{n,2j+2}^{\beta_{2j+2}}|\cdot|^{\beta_{2j+2}-2}}{\big(\delta_{n,2j+2}^{\beta_{2j+2}}+|\cdot|^{\beta_{2j+2}}\big)^2}\phi_{n,2}\mathrm{P}Z_{n,2i+1}}_{=:I''_{n,2i+1,2j+2}}\\
&\quad+\underbrace{\log\frac{1}{|\lambda_n|}\int_\Omega\psi_{n,1}\mathrm{P}Z_{n,2i+1}}_{=:I'''_{n,2i+1}}.
\end{aligned}$$

To estimate $I'_{n,2i+1}$ we use (5.4), then the boundedness in $L_{\beta_{2i+1}}(\mathbb{R}^2)$ and the definitions of $\delta_{n,i}$:

$$\begin{aligned}
I'_{n,2i+1} &= \log\frac{1}{|\lambda_n|}\int_{\Omega/\delta_{n,2i+1}}2\beta_{2i+1}^2\frac{|y|^{\beta_{2i+1}-2}}{(1+|y|^{\beta_{2i+1}})^2}\,\widetilde{\phi}_{n,2i+1}(y)(\mathrm{P}Z_{n,2i+1}(\delta_{n,2i+1}y)\\
&\qquad-Z_{n,2i+1}(\delta_{n,2i+1}y))\mathrm{d}y\\
&= \log\frac{1}{|\lambda_n|}\int_{\Omega/\delta_{n,2i+1}}2\beta_{2i+1}^2\frac{|y|^{\beta_{2i+1}-2}}{(1+|y|^{\beta_{2i+1}})^2}\,\widetilde{\phi}_{n,2i+1}(y)\mathrm{d}y\\
&\qquad+O\Big(\delta_{n,2i+1}^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}\int_{\Omega/\delta_{n,2i+1}}2\beta_{2i+1}^2\frac{|y|^{\beta_{2i+1}-2}}{(1+|y|^{\beta_{2i+1}})^2}\big|\widetilde{\phi}_{n,2i+1}(y)\big|\mathrm{d}y\Big)\\
&= \sigma_{n,2i+1}+O\Big(\delta_{n,2i+1}^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}\big\|\widetilde{\phi}_{n,2i+1}\big\|_{L_{\beta_{2i+1}}}\Big)=\sigma_{n,2i+1}+o(1).
\end{aligned}$$

Concerning the terms in $I''_{n,2i+1,2j+1}$, we proceed differently depending whether $j<i$ or $j>i$: in the former case, using (5.5) and choosing q very close to 1, we get

$$\begin{aligned}
&I''_{n,2i+1,2j+1}\\
&= \log\frac{1}{|\lambda_n|}\int_{\Omega/\delta_{n,2j+1}}2\beta_{2j+1}^2\frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\,\widetilde{\phi}_{n,2j+1}(y)\,\mathrm{P}Z_{n,2i+1}(\delta_{n,2j+1}y)\,\mathrm{d}y\\
&= 2\log\frac{1}{|\lambda_n|}\int_{\Omega/\delta_{n,2j+1}}2\beta_{2j+1}^2\frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\,\widetilde{\phi}_{n,2j+1}(y)\,\mathrm{d}y-2\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}+\beta_{2i+1}-2}}{(1+|y|^{\beta_{2j+1}})^2\big(1+\big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\big)^{\beta_{2i+1}}|y|^{\beta_{2i+1}}\big)}\,\widetilde{\phi}_{n,2j+1}(y)\,\mathrm{d}y\\
&\quad + O\Big(\delta_{n,2i+1}^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}\int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2\frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\big|\widetilde{\phi}_{n,2j+1}(y)\big|\,\mathrm{d}y\Big)\\
&= 2\sigma_{n,2j+1} + o(1) + O\bigg(\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}\big\|\widetilde{\phi}_{n,2j+1}\big\|_{q/(q-1)}\\
&\quad\cdot\Big(\int_{\mathbb{R}^2}\Big|\frac{|y|^{\beta_{2j+1}+\beta_{2i+1}-2}}{(1+|y|^{\beta_{2j+1}})^2\big(1+(\delta_{n,2j+1}/\delta_{n,2i+1})^{\beta_{2i+1}}|y|^{\beta_{2i+1}}\big)}\Big|^q\mathrm{d}y\Big)^{1/q}\bigg) + o(1)\\
&= 2\sigma_{n,2j+1} + O\bigg(\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}\,\delta_{n,2j+1}^{-2(1-1/q)}\|\phi_{n,1}\|_{q/(q-1)}\\
&\quad\cdot\Big(\int_{\mathbb{R}^2\setminus B_1(0)}\frac{|y|^{(\beta_{2i+1}-\beta_{2j+1}-2)q}}{\big(1+(\delta_{n,2j+1}/\delta_{n,2i+1})^{\beta_{2i+1}}|y|^{\beta_{2i+1}}\big)^q}\Big)^{1/q}\bigg) + o(1)\\
&= 2\sigma_{n,2j+1} + O\bigg(\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}\|\phi_n\|\\
&\quad\cdot\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\min\{0,\beta_{2j+1}-\beta_{2i+1}+2(1-1/q)\}}\delta_{n,2j+1}^{-2(1-1/q)}\bigg) + o(1)\\
&= 2\sigma_{n,2j+1} + o(1);
\end{aligned}
$$

in the latter case,

$$
\begin{aligned}
&I''_{n,2i+1,2j+1}\\
&\quad= \log\frac{1}{|\lambda_n|}\int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2\frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\,\widetilde{\phi}_{n,2j+1}(y)\mathrm{P}Z_{n,2i+1}(\delta_{n,2j+1}y)\mathrm{d}y\\
&\quad= \Big(\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}\Big)^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}\cdot\\
&\qquad\cdot\int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2\frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2\big(\big(\delta_{n,2i+1}/\delta_{n,2j+1}\big)^{\beta_{2i+1}}+|y|^{\beta_{2i+1}}\big)}\,\widetilde{\phi}_{n,2j+1}(y)\,\mathrm{d}y\\
&\qquad+ O\Big(\delta_{n,2i+1}^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}\int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2\frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\,\big|\widetilde{\phi}_{n,2j+1}(y)\big|\,\mathrm{d}y\Big)\\
&\quad= O\bigg(\Big(\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}\Big)^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}\,\delta_{n,2j+1}^{-2(1-1/q)}\,\|\phi_{n,1}\|_{q/(q-1)}\\
&\qquad\cdot\Big(\int_{B_1(0)}\frac{|y|^{(\beta_{2j+1}-2)q}}{\big((\delta_{n,2i+1}/\delta_{n,2j+1})^{\beta_{2i+1}}+|y|^{\beta_{2i+1}}\big)^q}\,\mathrm{d}y\Big)^{1/q}\bigg) + o(1)\\
&\quad= O\bigg(\Big(\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}\Big)^{\beta_{2i+1}}\log\frac{1}{|\lambda_n|}\,\delta_{n,2j+1}^{-2(1-1/q)}\,\|\phi_n\|\\
&\qquad\cdot\Big(\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}\Big)^{\min\{0,\beta_{2j+1}-\beta_{2i+1}+2(1-1/q)\}}\bigg) + o(1) \underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

The same argument shows that

$$I'''_{n,2i+1,2j+2} = \begin{cases} 2\sigma_{n,2j+2} + o(1) & \text{if } j < i \\ o(1) & \text{if } j \geq i \end{cases} .$$

Finally, since $\|\mathrm{P}Z_{n,\ell}\|_\infty \leq C$,

$$|I'''_{n,2i+1}| \leq \log \frac{1}{|\lambda_n|} \|\psi_{n,1}\|_1 \|\mathrm{P}Z_{n,2i+1}\|_\infty \leq C \log \frac{1}{|\lambda_n|} \|\psi_n\|_p \underset{n\to+\infty}{\longrightarrow} 0.$$

Therefore, we get

$$\sigma_{n,2i+1} + 2\sum_{j=0}^{i-1} \sigma_{n,2j+1} - a \sum_{j=0}^{i-1} \sigma_{n,2j+2} = o(1); \tag{5.6}$$

a similar argument yields

$$\sigma_{n,2i+2} + 2\sum_{j=0}^{i-1} \sigma_{n,2j+2} - b \sum_{j=0}^{i} \sigma_{n,2j+1} = o(1). \tag{5.7}$$

Putting (5.6) and (5.7) together we get $\sigma_{n,i} = o(1)$ for all i's.

Step 4: $\mu_\ell = 0$ *for all* j*'s.*

We recall the solution $\mathrm{P}w_{n,\ell} = \mathrm{P}w^{\beta_\ell}_{\delta_{n,\ell}}$ of

$$\begin{cases} -\Delta(\mathrm{P}w_{n,\ell}) = 2\beta_\ell^2 \dfrac{\delta_{n,\ell}^{\beta_\ell} |\cdot|^{\beta_\ell - 2}}{\left(\delta_{n,\ell}^{\beta_\ell} + |\cdot|^{\beta_\ell}\right)^2} & \text{in } \Omega, \\ \mathrm{P}w_{n,\ell} = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.8}$$

We multiply by $\mathrm{P}w_{n,2i+1}$ the first equation of (5.1), then we multiply by $\phi_{n,1}$ the equation (5.8) satisfied by $\mathrm{P}w_{n,2i+1}$; their difference gives

$$\begin{aligned} 0 = &\sum_{j=0}^{[(k-1)/2]} \underbrace{\int_\Omega 2\beta_{2j+1}^2 \frac{\delta_{n,2j+1}^{\beta_{2j+1}} |\cdot|^{\beta_{2j+1}-2}}{\left(\delta_{n,2j+1}^{\beta_{2j+1}} + |\cdot|^{\beta_{2j+1}}\right)^2} \phi_{n,1} \mathrm{P}w_{n,2i+1}}_{=:J'_{n,2i+1,2j+1}} \\ &- \frac{a}{2} \sum_{j=0}^{[(k-2)/2]} \underbrace{\int_\Omega 2\beta_{2j+2}^2 \frac{\delta_{n,2j+2}^{\beta_{2j+2}} |\cdot|^{\beta_{2j+2}-2}}{\left(\delta_{n,2j+2}^{\beta_{2j+2}} + |\cdot|^{\beta_{2j+2}}\right)^2} \phi_{n,2} \mathrm{P}w_{n,2i+1}}_{=:J'_{n,2i+1,2j+2}} + \underbrace{\int_\Omega \psi_{n,1} \mathrm{P}w_{n,2i+1}}_{=:J''_{n,2i+1}} \\ &- \underbrace{\int_\Omega 2\beta_{2i+1}^2 \frac{\delta_{n,2i+1}^{\beta_{2i+1}} |\cdot|^{\beta_{2i+1}-2}}{\left(\delta_{n,2i+1}^{\beta_{2i+1}} + |\cdot|^{\beta_{2i+1}}\right)^2} \phi_{n,1}}_{=:J'''_{n,2i+1}} . \end{aligned}$$

We start by estimating $J'_{n,i,\ell}$, considering as before only the case of odd indexes.

For $\ell < i$ we use (2.7), the definition of $\delta_{n,\ell}$ and the vanishing of $\sigma_{n,\ell}$. Notice that, to handle with $\mathrm{P}w_i$, (2.8) would not suffice hence we need sharper estimates for the logarithmic term.

$$\begin{aligned}
&J'_{n,2i+1,2j+1}\\
&= \int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\, \widetilde{\phi}_{n,2j+1}(y)\mathrm{P}w_{n,2i+1}(\delta_{n,2j+1}y)\mathrm{d}y\\
&= (-2\beta_{2i+1}\log\delta_{n,2i+1} + 4\pi\beta_{2i+1}H(0,0))\\
&\quad \cdot \int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\, \widetilde{\phi}_{n,2j+1}(y)\mathrm{d}y\\
&\quad - 2\int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\, \widetilde{\phi}_{n,2j+1}(y)\\
&\quad \cdot \log\Big(1+\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\beta_{2i+1}}|y|^{\beta_{2i+1}}\Big)\mathrm{d}y\\
&\quad + O\Big(\delta_{n,2j+1}\int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-1}}{(1+|y|^{\beta_{2j+1}})^2}\big|\widetilde{\phi}_{n,2j+1}(y)\big|\mathrm{d}y\Big)\\
&\quad + O\Big(\delta_{n,2i+1}^{\beta_{2i+1}}\int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\big|\widetilde{\phi}_{n,2j+1}(y)\big|\mathrm{d}y\Big)\\
&= O\Big(\log\frac{1}{|\lambda_n|}\Big)\Big|\int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\, \widetilde{\phi}_{n,2j+1}(y)\mathrm{d}y\Big|\\
&\quad + O\Big(\Big(\int_{\Omega/\delta_{n,2i+1}} \frac{|y|^{(\beta_{2j+1}-2)q}}{(1+|y|^{\beta_{2j+1}})^{2q}}\log\Big(1+\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\beta_{2i+1}}|y|^{\beta_{2i+1}}\Big)^q\Big)^{1/q}\\
&\qquad \cdot \big\|\widetilde{\phi}_{n,2j+1}\big\|_{q/(q-1)}\Big)\\
&\quad + O\Big(\delta_{n,2i+1}\Big(\int_{\Omega/\delta_{n,2i+1}} \frac{|y|^{(\beta_{2j+1}-1)q}}{(1+|y|^{\beta_{2j+1}})^{2q}}\Big)^{1/q}\big\|\widetilde{\phi}_{n,2j+1}\big\|_{q/(q-1)}\Big) + O\big(\delta_{n,2i+1}^{\beta_{2i+1}}\big)\\
&= O(|\sigma_{n,2j+1}|)\\
&\quad + O\Big(\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\beta_{2i+1}}\Big(\int_{B_{\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}}(0)} \frac{|y|^{(\beta_{2j+1}+\beta_{2i+1}-2)q}}{(1+|y|^{\beta_{2j+1}})^{2q}}\mathrm{d}y\Big)^{1/q}\delta_{n,2j+1}^{-2(1-1/q)}\|\phi_n\|\Big)\\
&\quad + O\Big(\Big(\int_{\mathbb{R}^2\setminus B_{\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}}(0)} \frac{|y|^{(\beta_{2j+1}-2)q}}{(1+|y|^{\beta_{2j+1}})^{2q}}\log\big(1+|y|^{\beta_{2i+1}}\big)^q\mathrm{d}y\Big)^{1/q}\\
&\qquad \cdot \delta_{n,2j+1}^{-2(1-1/q)}\|\phi_n\|\Big)\\
&\quad + O\Big(\delta_{n,2i+1}\delta_{n,2i+1}^{\min\{0,\beta_{2i+1}+1-\frac{2}{q}\}}\delta_{n,2j+1}^{-2(1-1/q)}\|\phi_n\|\Big) + o(1)
\end{aligned}$$

$$= O\Big(\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\beta_{2i+1}}\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\min\{0,\beta_{2i+1}-\beta_{2j+1}+2(1-1/q)\}}\delta_{n,2j+1}^{-2(1-1/q)}\Big)$$
$$+ O\Big(\Big(\frac{\delta_{n,2j+1}}{\delta_{n,2i+1}}\Big)^{\frac{\beta_{2j+1}}{2}+2(1-1/q)}\delta_{n,2j+1}^{-2(1-1/q)}\Big) + o(1) = o(1).$$

In the other cases, some terms will vanish by the same arguments as before, but some others will not. To estimate the latter terms, we will use the convergence of $\widetilde{\phi}_{n,\ell}$ in $L_{\beta_{2i+1}}$ and the following equalities, which can be proved by direct computation:

$$\int_{\mathbb{R}^2} 2\beta_\ell^2 \frac{|y|^{\beta_\ell-2}}{(1+|y|^{\beta_\ell})^2}\frac{1-|y|^{\beta_\ell}}{1+|y|^{\beta_\ell}}\log\big(1+|y|^{\beta_\ell}\big)\,\mathrm{d}y = -2\pi\beta_\ell;$$
$$\int_{\mathbb{R}^2} 2\beta_\ell^2 \frac{|y|^{\beta_\ell-2}}{(1+|y|^{\beta_\ell})^2}\frac{1-|y|^{\beta_\ell}}{1+|y|^{\beta_\ell}}\log|y|\mathrm{d}y = -4\pi.$$

When $j = i$ we have

$$\begin{aligned}
&J'_{n,2i+1,2i+1}\\
&= \int_{\Omega/\delta_{n,2i+1}} 2\beta_{2i+1}^2\frac{|y|^{\beta_{2i+1}-2}}{(1+|y|^{\beta_{2i+1}})^2}\,\widetilde{\phi}_{n,2i+1}(y)\mathrm{P}w_{n,2i+1}(\delta_{n,2i+1}y)\,\mathrm{d}y\\
&= (-2\beta_{2i+1}\log\delta_{n,2i+1} + 4\pi\beta_{2i+1}H(0,0))\\
&\quad\cdot\int_{\Omega/\delta_{n,2i+1}} 2\beta_{2i+1}^2\,\frac{|y|^{\beta_{2i+1}-2}}{(1+|y|^{\beta_{2i+1}})^2}\,\widetilde{\phi}_{n,2i+1}(y)\,\mathrm{d}y\\
&\quad+\int_{\Omega/\delta_{n,2i+1}} 2\beta_{2i+1}^2\,\frac{|y|^{\beta_{2i+1}-2}}{(1+|y|^{\beta_{2i+1}})^2}\,\widetilde{\phi}_{n,2i+1}(y)\log\big(1+|y|^{\beta_{2i+1}}\big)\,\mathrm{d}y\\
&\quad+O\Big(\delta_{n,2i+1}\int_{\Omega/\delta_{n,2i+1}} 2\beta_{2i+1}^2\,\frac{|y|^{\beta_{2i+1}-1}}{(1+|y|^{\beta_{2i+1}})^2}\,\big|\widetilde{\phi}_{n,2i+1}(y)\big|\,\mathrm{d}y\Big)\\
&\quad+O\Big(\delta_{n,2i+1}^{\beta_{2i+1}}\int_{\Omega/\delta_{n,2i+1}} 2\beta_{2i+1}^2\frac{|y|^{\beta_{2i+1}-2}}{(1+|y|^{\beta_{2i+1}})^2}\,\big|\widetilde{\phi}_{n,2i+1}(y)\big|\,\mathrm{d}y\Big)\\
&= -2\int_{\Omega/\delta_{n,2i+1}} 2\beta_{2i+1}^2\frac{|y|^{\beta_{2i+1}-2}}{(1+|y|^{\beta_{2i+1}})^2}\,\widetilde{\phi}_{n,2i+1}(y)\log\big(1+|y|^{\beta_{2i+1}}\big)\,\mathrm{d}y + o(1)\\
&= -2\mu_{2i+1}\int_{\mathbb{R}^2} 2\beta_{2i+1}^2\frac{|y|^{\beta_{2i+1}-2}}{(1+|y|^{\beta_{2i+1}})^2}\frac{1-|y|^{\beta_{2i+1}}}{1+|y|^{\beta_{2i+1}}}\log\big(1+|y|^{\beta_{2i+1}}\big) + o(1)\\
&= 4\beta_{2i+1}\mu_{2i+1} + o(1).
\end{aligned}$$

Similarly, if $j > i$,

$$\begin{aligned}
&J'_{n,2i+1,2j+1}\\
&= \int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2\frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\,\widetilde{\phi}_{n,2j+1}(y)\mathrm{P}w_{n,2i+1}(\delta_{n,2j+1}y)\mathrm{d}y\\
&= (-2\beta_{2i+1}\log\delta_{n,2j+1} + 4\pi\beta_{2i+1}H(0,0))
\end{aligned}$$

$$
\begin{aligned}
&\cdot \int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\, \widetilde{\phi}_{n,2j+1}(y)\,\mathrm{d}y \\
&- 2\beta_{2i+1} \int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\, \widetilde{\phi}_{n,2j+1}(y) \log|y|\mathrm{d}y \\
&- 2 \int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2}\, \widetilde{\phi}_{n,2j+1}(y) \cdot \\
&\cdot \log\Big(\Big(\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}\Big)^{\beta_{2i+1}} \frac{1}{|y|^{\beta_{2i+1}}} + 1\Big)\,\mathrm{d}y \\
&+ O\Big(\delta_{n,2j+1} \int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-1}}{(1+|y|^{\beta_{2j+1}})^2} \big|\widetilde{\phi}_{n,2j+1}(y)\big|\mathrm{d}y\Big) \\
&+ O\Big(\delta_{n,2i+1}^{\beta_{2i+1}} \int_{\Omega/\delta_{n,2j+1}} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2} \big|\widetilde{\phi}_{n,2j+1}(y)\big|\,\mathrm{d}y\Big) \\
= &-2\beta_{2j+1}\,\mu_{2j+1} \int_{\mathbb{R}^2} 2\beta_{2j+1}^2 \frac{|y|^{\beta_{2j+1}-2}}{(1+|y|^{\beta_{2j+1}})^2} \frac{1-|y|^{\beta_{2j+1}}}{1+|y|^{\beta_{2j+1}}} \log|y| + o(1) \\
&+ O\bigg(\bigg(\int_{\Omega/\delta_{n,2i+1}} \frac{|y|^{(\beta_{2j+1}-2)q}}{(1+|y|^{\beta_{2j+1}})^{2q}} \log\Big(\Big(\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}\Big)^{\beta_{2i+1}} \frac{1}{|y|^{\beta_{2i+1}}} + 1\Big)^q\bigg)^{1/q} \\
&\cdot \big\|\widetilde{\phi}_{n,2j+1}\big\|_{q/(q-1)}\bigg) \\
= &\; 8\pi\beta_{2j+1}\,\mu_{2j+1} + o(1) \\
&+ O\bigg(\bigg(\int_{B_{\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}}(0)} \frac{|y|^{(\beta_{2j+1}-2)q}}{(1+|y|^{\beta_{2j+1}})^{2q}} \log\Big(1+\frac{1}{|y|^{\beta_{2i+1}}}\Big)^q \mathrm{d}y\bigg)^{1/q} \delta_{n,2j+1}^{-2(1-1/q)} \|\phi_n\|\bigg) \\
&+ O\bigg(\Big(\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}\Big)^{\beta_{2i+1}} \bigg(\int_{\mathbb{R}^2\setminus B_{\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}}(0)} \frac{|y|^{(\beta_{2j+1}-\beta_{2i+1}-2)q}}{(1+|y|^{\beta_{2j+1}})^{2q}}\,\mathrm{d}y\bigg)^{1/q} \delta_{n,2j+1}^{-2(1-1/q)} \|\phi_n\|\bigg) \\
= &\; 8\pi\beta_{2j+1}\,\mu_{2j+1} + O\bigg(\Big(\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}\Big)^{\frac{\beta_{2j+1}}{2}-2(1-1/q)} \delta_{n,2j+1}^{-2(1-1/q)}\bigg) \\
&+ O\bigg(\Big(\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}\Big)^{\beta_{2i+1}} \Big(\frac{\delta_{n,2i+1}}{\delta_{n,2j+1}}\Big)^{\min\{0,\beta_{2j+1}-\beta_{2i+1}-2(1-1/q)\}} \delta_{n,2j+1}^{-2(1-1/q)}\bigg) + o(1) \\
= &\; 8\pi\beta_{2j+1}\,\mu_{2j+1} + o(1)
\end{aligned}
$$

The term $J''_{n,2i+1}$ vanishes because, by Lemma 2.1, $\|\mathrm{P}w_{n,\ell}\|_\infty = O(\log 1/|\lambda_n|)$, therefore

$$
\big|J''_{n,2i+1}\big| \le \|\psi_{n,1}\|_1 \|\mathrm{P}w_{n,2i+1}\|_\infty \le C\log\frac{1}{|\lambda_n|}\|\psi_n\|_p \underset{n\to+\infty}{\longrightarrow} 0.
$$

Finally, Step 3 gives

$$
J'''_{n,2i+1} = \int_{\Omega/\delta_{n,2i+1}} 2\beta_{2i+1}^2 \frac{|y|^{\beta_{2i+1}-2}}{(1+|y|^{\beta_{2i+1}})^2}\, \widetilde{\phi}_{n,2i+1}(y)\,\mathrm{d}y = \frac{\sigma_{n,2i+1}}{\log\frac{1}{|\lambda_n|}} \underset{n\to+\infty}{\longrightarrow} 0.
$$

Putting all these estimates together, repeating the computations for even indexes and passing to the limit gives

$$4\pi\beta_{2i+1}\mu_{2i+1} + 8\pi \sum_{j=i+1}^{[(k-1)/2]} \beta_{2j+1}\mu_{2j+1} - 4\pi a \sum_{j=i}^{[(k-2)/2]} \beta_{2j+2}\mu_{2j+2} = 0,$$

$$4\pi\beta_{2i+2}\mu_{2i+2} + 8\pi \sum_{j=i+1}^{[(k-2)/2]} \beta_{2j+2}\mu_{2j+2} - 4\pi b \sum_{j=i+1}^{[(k-1)/2]} \beta_{2j+1}\mu_{2j+1} = 0,$$

from which we get $\mu_\ell = 0$ for all j's.

Step 5. $\phi_n \to 0$ *as* $n \to +\infty$ *in* $L^\infty(\Omega)^2$.

We fix $x \in \Omega$ and we estimate $\phi_{n,i}(x)$, using Green's representation formula. We provide the estimate only for $i = 1$:

$$\begin{aligned}
|\phi_{n,1}(x)| &= \Big| \sum_{j=0}^{[(k-1)/2]} \int_\Omega G(x,y) 2\beta_{2j+1}^2 \frac{\delta_{n,2j+1}^{\beta_{2j+1}} |y|^{\beta_{2j+1}-2}}{\big(\delta_{n,2j+1}^{\beta_{2j+1}} + |y|^{\beta_{2j+1}}\big)^2} \phi_{n,1}(y)\,\mathrm{d}y \\
&\quad - \frac{a}{2} \sum_{j=0}^{[(k-2)/2]} \int_\Omega G(x,y)\, 2\beta_{2j+2}^2 \frac{\delta_{n,2j+2}^{\beta_{2j+2}} |y|^{\beta_{2j+2}-2}}{\big(\delta_{n,2j+2}^{\beta_{2j+2}} + |y|^{\beta_{2j+2}}\big)^2} \phi_{n,2}(y)\,\mathrm{d}y \\
&\quad + \int_\Omega G(x,y)\,\psi_{n,1}(y)\,\mathrm{d}y \Big| \\
&\le \sum_{i=1}^{2} \sum_{j=0}^{[(k-i)/2]} \Big| \int_\Omega G(x,y) \frac{\delta_{n,2j+i}^{\beta_{2j+i}} |y|^{\beta_{2j+i}-2}}{\big(\delta_{n,2j+i}^{\beta_{2j+i}} + |y|^{\beta_{2j+i}}\big)^2} \phi_{n,i}(y)\,\mathrm{d}y \Big| \\
&\quad + \Big| \int_\Omega G(x,y)\,\psi_{n,1}(y)\,\mathrm{d}y \Big| \\
&\le \sum_{j=0}^{k} \Big| \int_{\Omega/\delta_{n,\ell}} G(x, \delta_{n,\ell} z) \frac{|z|^{\beta_\ell - 2}}{(1+|z|^{\beta_\ell})^2} \widetilde{\phi}_{n,\ell}(z)\,\mathrm{d}z \Big| + \sup_{x\in\Omega} \|G(x,\cdot)\|_{\frac{p}{p-1}} \|\psi_n\|_p \\
&\le \sum_{j=0}^{k} \underbrace{\Big| \int_{\Omega/\delta_{n,\ell}} \log|x - \delta_{n,\ell} z| \frac{|z|^{\beta_\ell - 2}}{(1+|z|^{\beta_\ell})^2} \widetilde{\phi}_{n,\ell}(z)\,\mathrm{d}z \Big|}_{:=K'_{n,\ell}} \\
&\quad + \sum_{j=0}^{k} \underbrace{\int_{\Omega/\delta_{n,\ell}} |H(x, \delta_{n,\ell} z)| \frac{|z|^{\beta_\ell - 2}}{(1+|z|^{\beta_\ell})^2} \big|\widetilde{\phi}_{n,\ell}(z)\big|\,\mathrm{d}z}_{:=K''_{n,\ell}} + o(1).
\end{aligned}$$

To estimate $K''_{n,\ell}$ we apply some weighted Sobolev inequalities to $\widetilde{\phi}_{n,\ell}$: since it is bounded in $H_{\beta_\ell}(\mathbb{R}^2)$ and tends to 0 in $L_{\beta_\ell}(\mathbb{R}^2)$, then for any $q \ge 2$,

$$\int_{\Omega/\delta_{n,\ell}} \frac{|z|^{\beta_\ell - 2}}{(1+|z|^{\beta_\ell})^2} \big|\widetilde{\phi}_{n,\ell}(z)\big|^q \,\mathrm{d}z \underset{n\to+\infty}{\longrightarrow} 0.$$

Therefore, for a suitable q,

$$
\begin{aligned}
K''_{n,\ell} &\le (H(0,0)+|x|)\int_{\Omega/\delta_{n,\ell}} \frac{|z|^{\beta_\ell-2}}{(1+|z|^{\beta_\ell})^2}\left|\widetilde{\phi}_{n,\ell}(z)\right|\mathrm{d}z \\
&\quad + \delta_{n,\ell}\int_{\Omega/\delta_{n,\ell}} \frac{|z|^{\beta_\ell-1}}{(1+|z|^{\beta_\ell})^2}\left|\widetilde{\phi}_{n,\ell}(z)\right|\mathrm{d}z \\
&\le C\left(\int_{\Omega/\delta_{n,\ell}} \frac{|z|^{\beta_\ell-2}}{(1+|z|^{\beta_\ell})^2}\,\mathrm{d}z\right)^{1/2}\left(\int_{\Omega/\delta_{n,\ell}} \frac{|z|^{\beta_\ell-2}}{(1+|z|^{\beta_\ell})^2}\left|\widetilde{\phi}_{n,\ell}(z)\right|^2\mathrm{d}z\right)^{1/2} \\
&\quad + \delta_{n,\ell}\left(\int_{\Omega/\delta_{n,\ell}} \frac{|z|^{\beta_\ell-2+\frac{q}{q-1}}}{(1+|z|^{\beta_\ell})^2}\,\mathrm{d}z\right)^{1-1/q}\left(\int_{\Omega/\delta_{n,\ell}} \frac{|z|^{\beta_\ell-2}}{(1+|z|^{\beta_\ell})^2}\left|\widetilde{\phi}_{n,\ell}(z)\right|^q\mathrm{d}z\right)^{1/q} \\
&\le C\left\|\widetilde{\phi}_{n,\ell}\right\|_{L_{\beta_\ell}(\mathbb{R}^2)} + \delta_{n,\ell}^{\min\{1,\beta_\ell(1-1/q)\}}o(1) \underset{n\to+\infty}{\longrightarrow} 0.
\end{aligned}
$$

To deal with $K'_{n,\ell}$ we use that $\sigma_{n,\ell}\to 0$ as $n\to+\infty$ (see Step 3) and that $\delta_{n,\ell}$ is given by powers of $\lambda_{n,i}$; in particular, we will distinguish whether $|x|$ is smaller or larger than $\delta_{n,\ell}$:

$$
\begin{aligned}
|K'_{n,\ell}| &\le \left|\int_{\Omega/\delta_{n,\ell}} \log\left|\frac{x-\delta_{n,\ell}z}{\max\{\delta_{n,\ell},|x|\}}\right|\frac{|z|^{\beta_\ell-2}}{(1+|z|^{\beta_\ell})^2}\,\widetilde{\phi}_{n,\ell}(z)\,\mathrm{d}z\right| \\
&\quad + |\log\max\{\delta_{n,\ell},|x|\}|\left|\int_{\Omega/\delta_{n,\ell}} \frac{|z|^{\beta_\ell-2}}{(1+|z|^{\beta_\ell})^2}\,\widetilde{\phi}_{n,\ell}(z)\,\mathrm{d}z\right| \\
&\le \left(\int_{\Omega/\delta_{n,\ell}} \left|\log\frac{|x/\delta_{n,\ell}-z|}{\max\{1,|x/\delta_{n,\ell}|\}}\right|^2\frac{|z|^{\beta_\ell-2}}{(1+|z|^{\beta_\ell})^2}\,\mathrm{d}z\right)^{1/2} \\
&\quad \cdot\left(\int_{\Omega/\delta_{n,\ell}} \frac{|z|^{\beta_\ell-2}}{(1+|z|^{\beta_\ell})^2}\left|\widetilde{\phi}_{n,\ell}(z)\right|^2\mathrm{d}z\right)^{1/2} + \frac{\max\left\{\log\frac{1}{\delta_{n,\ell}},\log\operatorname{diam}\Omega\right\}}{\log 1/|\lambda_n|}|\sigma_{n,\ell}| \\
&\le \underbrace{\left(\int_{\mathbb{R}^2}\left|\log\frac{|z'|}{\max\{1,|x/\delta_{n,\ell}|\}}\right|^2\frac{|z'-x/\delta_{n,\ell}|^{\beta_\ell-2}}{\left(1+|z'-x/\delta_{n,\ell}|^{\beta_\ell}\right)^2}\,dz'\right)^{1/2}}_{K'''_\ell(x/\delta_{n,\ell})} o(1)+o(1).
\end{aligned}
$$

The claim will follow by showing that $K'''_\ell(x')$ is uniformly bounded for $x'\in\mathbb{R}^2$.

Taking a cue from [16] (Lemma 1.1), we split the integral in the ball of radius $2\max\{1,|x'|\}$ and its complementary: in the ball, we just apply a Hölder inequality with suitable exponents and then a dilatation; in its exterior, we use the monotonicity of the logarithm and the fact that x' it is somehow negligible with respect to z':

$$
\begin{aligned}
|z'-x'| &\ge |z'|-|x'| \ge |z'|-\max\{1,|x'|\} \ge \tfrac{1}{2}|z'| \\
|z'-x'| &\le |z'|+|x'| \le |z'|+\max\{1,|x'|\} \le \tfrac{3}{2}|z'|.
\end{aligned}
$$

We get, for a suitable $q>1$:

$$\begin{aligned}
K_\ell'''(x) &= \int_{B_{2\max\{1,|x'|\}}(0)} \Big|\log\frac{|z'|}{\max\{1,|x'|\}}\Big|^2 \frac{|z'-x'|^{\beta_\ell-2}}{(1+|z'-x'|^{\beta_\ell})^2}\,\mathrm{d}z' \\
&\quad + \int_{\mathbb{R}^2\setminus B_{2\max\{1,|x'|\}}(0)} \Big|\log\frac{|z'|}{\max\{1,|x'|\}}\Big|^2 \frac{|z'-x'|^{\beta_\ell-2}}{(1+|z'-x'|^{\beta_\ell})^2}\,\mathrm{d}z' \\
&\le \Big(\int_{B_{2\max\{1,|x'|\}}(0)} \Big|\log\frac{|z'|}{\max\{1,|x'|\}}\Big|^{2q/(q-1)}\mathrm{d}z'\Big)^{1-1/q} \\
&\qquad \cdot \Big(\int_{\mathbb{R}^2} \frac{|z'-x'|^{(\beta_\ell-2)q}}{(1+|z'-x'|^{\beta_\ell})^{2q}}\,\mathrm{d}z'\Big)^{1/q} \\
&\quad + C\int_{\mathbb{R}^2\setminus B_{2\max\{1,|x'|\}}(0)} \Big|\log\frac{|z'|}{\max\{1,|x'|\}}\Big|^2 \frac{|z'|^{\beta_\ell-2}}{\big(1+|z'/2|^{\beta_\ell}\big)^2}\,\mathrm{d}z' \\
&\le C\Big(\int_{B_2(0)} |\log|y'||^{\frac{2q}{q-1}}\mathrm{d}y'\Big)^{1-1/q} + C\int_{\mathbb{R}^2\setminus B_2(0)} |\log|z'||^2 \frac{|z'|^{\beta_\ell-2}}{\big(1+|z'/2|^{\beta_\ell}\big)^2}\,\mathrm{d}z' \\
&\le C.
\end{aligned}$$

Step 6: A contradiction arises.

We multiply each equation of (5.1) by the respective $\phi_{n,i}$ and we sum the two of them. We get:

$$\begin{aligned}
1 &= \int_\Omega |\nabla\phi_{n,1}|^2 + \int_\Omega |\nabla\phi_{n,2}|^2 \\
&= \sum_{j=0}^{[(k-1)/2]} \int_\Omega 2\beta_{2j+1}^2 \frac{\delta_{n,2j+1}^{\beta_{2j+1}}|\cdot|^{\beta_{2j+1}-2}}{\big(\delta_{n,2j+1}^{\beta_{2j+1}}+|\cdot|^{\beta_{2j+1}}\big)^2}\,\phi_{n,1}^2 \\
&\quad - \frac{a}{2}\sum_{j=0}^{[(k-2)/2]} \int_\Omega 2\beta_{2j+2}^2 \frac{\delta_{n,2j+2}^{\beta_{2j+2}}|\cdot|^{\beta_{2j+2}-2}}{\big(\delta_{n,2j+2}^{\beta_{2j+2}}+|\cdot|^{\beta_{2j+2}}\big)^2}\,\phi_{n,1}\phi_{n,2} + \int_\Omega \psi_{n,1}\,\phi_{n,1} \\
&\quad + \sum_{j=0}^{[(k-2)/2]} \int_\Omega 2\beta_{2j+2}^2 \frac{\delta_{n,2j+2}^{\beta_{2j+2}}|\cdot|^{\beta_{2j+2}-2}}{\big(\delta_{n,2j+2}^{\beta_{2j+2}}+|\cdot|^{\beta_{2j+2}}\big)^2}\phi_{n,2}^2 \\
&\quad - \frac{b}{2}\sum_{j=0}^{[(k-1)/2]} \int_\Omega 2\beta_{2j+1}^2 \frac{\delta_{n,2j+1}^{\beta_{2j+1}}|\cdot|^{\beta_{2j+1}-2}}{\big(\delta_{n,2j+1}^{\beta_{2j+1}}+|\cdot|^{\beta_{2j+1}}\big)^2}\,\phi_{n,1}\phi_{n,2} + \int_\Omega \psi_{n,2}\phi_{n,2} \\
&\le C\sum_{j=0}^{k}\int_\Omega \frac{\delta_{n,\ell}^{\beta_\ell}|\cdot|^{\beta_\ell-2}}{\big(\delta_{n,\ell}^{\beta_\ell}+|\cdot|^{\beta_\ell}\big)^2}\|\phi_n\|_{L^\infty(\Omega)}^2 + \|\psi_n\|_p\,\|\phi_n\|_{\frac{p}{p-1}} \\
&\le C\big(\|\phi_n\|_{L^\infty(\Omega)}^2 + \|\psi_n\|_p\,\|\phi_n\|_{\frac{p}{p-1}}\big) \underset{n\to+\infty}{\longrightarrow} 0;
\end{aligned}$$

which is a contradiction. □

A. Appendix

We prove here a classification result for entire solutions of a scalar linearized problem.

Proposition A.1. *Assume $\alpha > 0$, $m \in \mathbb{N}$ and $\alpha/m \notin 2\mathbb{N}$. Then, any solution ϕ of*

$$\text{(A.1)} \qquad \begin{cases} -\Delta\phi = 2\alpha^2 \dfrac{|\cdot|^{\alpha-2}}{(1+|\cdot|^\alpha)^2}\,\phi & \text{in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |\nabla\phi|^2 < +\infty, \\ \phi\big(e^{\frac{2\pi}{m}\iota}\,\cdot\big) = \phi, \end{cases}$$

satisfies, for some $\mu \in \mathbb{R}$,

$$\phi = \mu\,\frac{1-|\cdot|^\alpha}{1+|\cdot|^\alpha}.$$

Proof. We argue as Baraket and Pacard do in [2], Proposition 1, where the case $\alpha = 2$ is covered (see also Del Pino, Esposito and Musso [23]).

By writing any solution ϕ of (A.1) as a Fourier decomposition,

$$\phi(x) = \sum_{n\in\mathbb{Z}} \phi_n(|x|)\,e^{in\theta},$$

we see that each of the ϕ_n solves the following ODE:

$$\text{(A.2)} \qquad \partial_\rho^2\phi_n(\rho) + \frac{1}{\rho}\,\partial_\rho\phi_n(\rho) - \frac{n^2}{\rho^2}\,\phi_n(\rho) + \frac{2\alpha^2\rho^{\alpha-2}}{(1+\rho^\alpha)^2}\,\phi_n(\rho).$$

Integrating by parts, ϕ_n must satisfy

$$\int_0^{+\infty}\Big(|\partial_\rho\phi_n(\rho)|^2 + \Big(\frac{n^2}{\rho^2} - \frac{2\alpha^2\rho^{\alpha-2}}{(1+\rho^\alpha)^2}\Big)\,\phi_n(\rho)^2\Big)\rho\,\mathrm{d}\rho = 0;$$

since

$$\frac{n^2}{\rho^2} - \frac{2\alpha^2\rho^{\alpha-2}}{(1+\rho^\alpha)^2} \geq \frac{1}{\rho^2}\Big(n^2 - \frac{\alpha^2}{2}\Big),$$

we must have $\phi_n \equiv 0$ for $|n| \geq \alpha/\sqrt{2}$. In particular, ϕ is a finite combination of the ϕ_n's.

It is easy to see that each solution of (A.2) is a linear combination of the fundamental solutions

$$\phi_{n,+}(\rho) = \rho^n\,\frac{\alpha+2n-(\alpha-2n)\rho^\alpha}{1+\rho^\alpha}, \qquad \phi_{n,-}(\rho) = \rho^{-n}\,\frac{\alpha-2n-(\alpha+2n)\rho^\alpha}{1+\rho^\alpha}.$$

Since we are looking for bounded solutions of (A.1), here we are allowed to take only bounded solution of (A.2).

If α is not an even integer, the condition is satisfied only by

$$\phi_{0,+}(\rho) = \phi_{0,-}(\rho) = \frac{1-\rho^\alpha}{1+\rho^\alpha},$$

hence $\phi(x) = \phi_0(|x|)$ is an integer multiple of its and the Proposition is proved.

On the other hand, if $\alpha \in 2\mathbb{N}$, then $\phi_{\alpha/2,+}(\rho) = 2\alpha\frac{\rho^{\alpha/2}}{1+\rho^\alpha}$ is also allowed, therefore in this case $\phi(x)$ is a combination of the following functions:

$$\begin{aligned} \phi_0(|x|) &= \frac{1-|x|^\alpha}{1+|x|^\alpha} \\ \frac{1}{2\alpha}\,\phi_{\alpha/2}(|x|)\,\cos\Big(\frac{\alpha}{2}\,\theta\Big) &= \frac{|x|^{\alpha/2}}{1+|x|^\alpha}\,\cos\Big(\frac{\alpha}{2}\,\theta\Big) \\ \frac{1}{2\alpha}\,\phi_{\alpha/2}(|x|)\,\sin\Big(\frac{\alpha}{2}\,\theta\Big) &= \frac{|x|^{\alpha/2}}{1+|x|^\alpha}\,\sin\Big(\frac{\alpha}{2}\,\theta\Big). \end{aligned}$$

Anyway, the latter two functions do not satisfy the symmetry requirement if m is as in the assumptions, therefore ϕ must again be a multiple of $\phi_0(|x|)$. The proof is completed. □

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