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Lifting weighted blow-ups

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Abstract. Let $f\colon X \to Z$ be a local, projective, divisorial contraction between normal varieties of dimension n with $\mathbb{Q}$-factorial singularities.

Let $Y \subset X$ be a f-ample Cartier divisor and assume that $f_{|Y}\colon Y \to W$ has a structure of a weighted blow-up. We prove that $f\colon X \to Z$, as well, has a structure of weighted blow-up.

As an application we consider a local projective contraction $f\colon X \to Z$ from a variety X with terminal $\mathbb{Q}$-factorial singularities, which contracts a prime divisor E to an isolated $\mathbb{Q}$-factorial singularity $P \in Z$, such that $-(K_X + (n-3)L)$ is f-ample, for a f-ample Cartier divisor L on X. We prove that (Z, P) is a hyperquotient singularity and f is a weighted blow-up.

1. Introduction

Let X be a normal variety over $\mathbb{C}$ and let $n = \dim X$. A *contraction* is a surjective morphism $\varphi\colon X \to Z$ with connected fibres onto a normal variety Z. If Z is affine then $f\colon X \to Z$ will be called a *local contraction*.

We always assume that f is *projective*, that is, we assume the existence of f-ample Cartier divisors L.

If f is birational and its exceptional set is an irreducible divisor, then it is called *divisorial*. We say that the contraction is $\mathbb{Q}$-*factorial* if X and Z have $\mathbb{Q}$-factorial singularities. Note that if X is $\mathbb{Q}$-factorial and f is a divisorial contraction of an extremal ray (in the sense of Mori theory), then Z is also $\mathbb{Q}$-factorial (see Corollary 3.18 in [22]).

A fundamental example of local contraction in algebraic geometry is the blow-up of $\mathbb{C}^n = \operatorname{Spec} \mathbb{C}[x_1, \dots, x_n]$ at 0. More generally, given $\sigma = (a_1, \dots, a_n) \in \mathbb{N}^n$ such that $a_i > 0$ and $m \in \mathbb{N}$, one can define the σ-*blow-up* (or the weighted blow-up with weight σ) of a hyperquotient singularity $Z\colon ((g = 0) \subset \mathbb{C}^n)/\mathbb{Z}_m(a_1, \dots, a_n)$. The definition is given in Section 2, in accordance with Section 10 in [21].

The main goal of the paper is to prove the following theorem.

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Theorem 1.1. *Let $f\colon X \to Z$ be a local, projective, divisorial and $\mathbb{Q}$-factorial contraction, which contracts an irreducible divisor E to an isolated $\mathbb{Q}$-factorial singularity $P \in Z$. Assume that* $\dim X \geq 4$.

Let $Y \subset X$ be a f-ample Cartier divisor such that $f' = f_{|Y}\colon Y \to f(Y) = W$ is a $\sigma' = (a_1, \dots, a_{n-1})$-blow-up, $\pi_{\sigma'}\colon Y \to W$.

Then $f\colon X \to Z$ is a $\sigma = (a_1, \dots, a_{n-1}, a_n)$-blow-up, $\pi_\sigma\colon X \to Z$, where a_n is such that $Y \sim_f -a_n E$ ($\sim_f$ means linearly equivalent over f).

We apply the above theorem to the study of birational contractions which appear in a minimal model program (MMP) with scaling on polarized pairs.

More precisely, if X is a variety with terminal $\mathbb{Q}$-factorial singularities and L is an ample Cartier divisor on X, the pair (X, L) is called a *polarized pair*. Given a non negative rational number r, there exists an effective $\mathbb{Q}$-divisor Δ^r on X such that $\Delta^r \sim_{\mathbb{Q}} rL$ and (X, Δ^r) is Kawamata log terminal. Consider the pair (X, Δ^r) and the $\mathbb{Q}$-Cartier divisor $K_X + \Delta^r \sim_{\mathbb{Q}} K_X + rL$.

By Theorem 1.2 and Corollary 1.3.3 of [4], we can run a $K_X + \Delta^r$*-minimal model program (MMP) with scaling*. This type of MMP was studied in deeper details in the case $r \geq (n-2)$ in [1].

To perform such a program one needs to understand local birational maps (divisorial or small contractions), $f\colon X \to Z$, which are contractions of an extremal rays $R := \mathbb{R}^+[C] \subset N_1(X/Z)$, where C is a rational curve such that $(K_X + rL)\cdot C < 0$ for a f-ample Cartier divisor L. We will call these maps Fano–Mori contractions or *contractions for a MMP*.

In [2] we classified local birational contractions for a MMP if $r \geq (n-2)$: they are σ-blow-up of a smooth point with $\sigma = (1, 1, b, \dots, b)$, where b is a positive integer.

In [3], Theorem 1.1, we proved that if $r > (n-3) > 0$ then one can find a general divisor $X' \in |L|$ which is a variety with at most $\mathbb{Q}$-factorial terminal singularities and such that $f_{|X'}\colon X' \to f(X') =: Z'$ is a contraction of an extremal ray $R' := \mathbb{R}^+[C']$ such that $(K_{X'} + (r-1)L')\cdot C' < 0$, where $L' := L_{|X'}$.

On the other hand, a very hard program, aimed to classify local divisorial contractions to a point for a MMP in dimension 3, was started long ago by Y. Kawamata ([19]); it was further carried on by M. Kawakita, T. Hayakawa and J. A. Chen (see, among other papers, [17], [18], [14], [15], [16], [10], [11], [12], [5]). They are all weighted blow-ups of (particular) cyclic quotient or hyperquotient singularities, and this should be the case for the few remaining ones. It is reasonable to make the following:

Assumption 1.2. *The divisorial contractions to a point for a MMP in dimension* 3 *are weighted blow-ups.*

The next result is a consequence, via a standard induction procedure, called the *Apollonius method*, of Theorem 1.1, the above quoted Theorem 1.1 in [3] and Assumption 1.2 in dimension 3.

Theorem 1.3. *Let X be a variety with $\mathbb{Q}$-factorial terminal singularities of dimension $n \geq 3$ and let $f\colon X \to Z$ be a local, projective, divisorial contraction which contracts a prime divisor E to an isolated $\mathbb{Q}$-factorial singularity $P \in Z$ such that $-(K_X + (n-3)L)$ is f-ample, for a f-ample Cartier divisor L on X.*

Then $P \in Z$ is a hyperquotient singularity.

Moreover, if we assume that 1.2 holds, f is a weighted blow-up.

2. Weighted blow-ups

We recall the definition of weighted blow-up; our notation is compatible with that of Section 10 in [21] and of Section 3 in [10].

Let $\sigma = (a_1, \ldots, a_n) \in \mathbb{N}^n$ such that $a_i > 0$ and $\gcd(a_1, \ldots, a_n) = 1$; let $M = \operatorname{lcm}(a_1, \ldots, a_n)$.

The *weighted projective space* with weight $(a_1, \ldots, a_n)$, denoted by $\mathbb{P}(a_1, \ldots, a_n)$, can be defined either as

$$\mathbb{P}(a_1, \ldots, a_n) := (\mathbb{C}^n - \{0\})/\mathbb{C}^*,$$

where $\xi \in \mathbb{C}^*$ acts by $\xi(x_1, \ldots, x_n) = (\xi^{a_1}x_1, \ldots, \xi^{a_n}x_n)$, or as

$$\mathbb{P}(a_1, \ldots, a_n) := \operatorname{Proj}_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n],$$

where $\mathbb{C}[x_1, \ldots, x_n]$ is the polynomial algebra over $\mathbb{C}$ graded by the condition $deg(x_i) = a_i$, for $i = 1, \ldots, n$.

A *cyclic quotient singularity*, denoted by $\mathbb{C}^n/\mathbb{Z}_m(a_1, \ldots, a_n) := X$, is an affine variety defined as the quotient of $\mathbb{C}^n$ by the action $(x_1, ..., x_n) \to (\epsilon^{a_1}x_1, ..., \epsilon^{a_n}x_n)$, where ϵ is a primitive m-th root of unity. Equivalently X is isomorphic to the spectrum $\operatorname{Spec} \mathbb{C}[x_1, \ldots, x_n]^{\mathbb{Z}_m}$ of the ring of invariant monomials under the group action.

Let $Q \in Y\colon (g = 0) \subset \mathbb{C}^{n+1}$ be a hypersurface singularity with a $\mathbb{Z}^m$ action. The point $P \in Y/\mathbb{Z}^m := X$ is called a *hyperquotient singularity*. In suitable local analytic coordinates, the action on Y extends to an action on $\mathbb{C}^{n+1}$ (in fact it acts on the tangent space $T_{Y,Q}$) and we can assume that $\mathbb{Z}_m$ acts diagonally by $\epsilon\colon (x_0, \ldots, x_n) \to (\epsilon^{a_0}x_0, \ldots, \epsilon^{a_n}x_n)$, where ϵ is a primitive m-th root of unity. Since Y is fixed by the action of $\mathbb{Z}_m$, it follows that g is an eigenfunction, so that $\epsilon\colon g \to \epsilon^e g$. We define the *type* of the hyperquotient singularity $P \in X$ with the symbol $\frac{1}{m}(a_0, \ldots, a_n; e)$. Note that if $m = 1$ this is simply a hypersurface singularity, while if $g = x_0$ this is a cyclic quotient singularity.

Let $X = \mathbb{C}^n/\mathbb{Z}_m(a_1, \ldots, a_n)$ be a cyclic quotient singularity and consider the rational map

$$\varphi\colon X \to \mathbb{P}(a_1, \ldots, a_n)$$

given by $(x_1, \ldots, x_n) \mapsto (x_1 : \cdots : x_n)$.

Definition 2.1. The *weighted blow-up* of $X = \mathbb{C}^n/\mathbb{Z}_m(a_1, \ldots, a_n)$ with weight $\sigma = (a_1, \ldots, a_n)$ (or simply the σ-blow-up), $\overline{X}$, is defined as the closure in $X \times \mathbb{P}(a_1, \ldots, a_k)$ of the graph of φ, together with the morphism $\pi_\sigma\colon \overline{X} \to X$ given by the projection on the first factor.

The weighted blow-up can be described by the theory of torus embeddings, as in section 10 of [21]. Namely, let $e_i = (0, \dots, 1, \dots, 0)$ for $i = 1, \dots, n$ and let $e = 1/m(a_1, \dots, a_n)$. Then X is the toric variety which corresponds to the lattice $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n + \mathbb{Z}e$ and the cone $C(X) = \mathbb{Q}_+e_1 + \cdots + \mathbb{Q}_+e_n$ in $\mathbb{Q}^n$, where $\mathbb{Q}_+ = \{z \in \mathbb{Q} : z \geq 0\}$.

We denote with $\pi_\sigma : \overline{X} \to X$ the proper birational morphism from the normal toric variety $\overline{X}$ corresponding to the cone decomposition of $C(X)$ consisting of $C_i = \Sigma_{j \neq i}\mathbb{Q}_+e_j + \mathbb{Q}_+e$, for $i = 1, \dots, n$, and their intersections.

The following facts can be easily checked in many ways, for instance via toric geometry (see also section 10 in [21] or section 3 in [10]).

- The map π_σ is birational and contracts an exceptional irreducible divisor $E \cong \mathbb{P}(a_1, \dots, a_k)$ to $0 \in X$.
- Let $(y_1 : \dots : y_n)$ be homogeneous coordinates on $\mathbb{P}(a_1, \dots, a_n)$. For any $1 \leq i \leq k$ consider the open affine subset $U_i = \overline{X} \cap \{y_i \neq 0\}$; these affine open subset are described as follows:
$$U_i \cong \mathrm{Spec}\mathbb{C}[\bar{x}_1, \dots, \bar{x}_n]/\mathbb{Z}_{a_i}(-a_1, \dots, m, \dots, -a_n).$$
The morphism $\varphi_{\sigma|U_i} : U_i \to X$ is given by
$$(\bar{x}_1, \dots, \bar{x}_n) \mapsto (\bar{x}_1\bar{x}_i^{a_1/m}, \dots, \bar{x}_i^{a_i/m}, \dots, \bar{x}_k\,\bar{x}_i^{a_k/m}).$$
- In the affine set U_i the divisor E is defined by $\{\bar{x}_i = 0\}$; it is a $\mathbb{Q}$-Cartier divisor and $\mathcal{O}_{\overline{X}}(-aE) \otimes \mathcal{O}_E = \mathcal{O}_{\mathbb{P}}(ma)$, for a divisible by Πa_i.
The divisor $H := -ME$ is actually Cartier, it is generated over π_σ by global sections and it is the generator of $\mathrm{Pic}(\overline{X}/X) = \mathbb{Z} =< H >$.
- Let $L = aH$, for a a positive integer; clearly L is σ-ample. We have
$$R^1\pi_{\sigma *}\mathcal{O}_Y(iL) = H^1(\overline{X}, iL) = 0$$
for every $i \in \mathbb{Z}$.

We now use Grothendieck's language to give a different characterization of the σ-weighted blow-up.

For a a positive integer, let $L = aH = -aME$. The divisor L is a π_σ-ample Cartier divisor.

Consider the graduated $\mathbb{C}[x_1, \dots, x_n]^{\mathbb{Z}_m}$-algebra $\bigoplus_{d \geq 0} \pi_*\mathcal{O}_X(dL)$. The construction in section (8.8) of [7] gives
$$\overline{X} = \mathrm{Proj}_X\left(\mathcal{O}_X \oplus \bigoplus_{d>0} \pi_*\mathcal{O}_X(dL)\right) \to X.$$

Consider now the function
$$\sigma\text{-wt} : \mathbb{C}[x_1, \dots, x_n] \to \mathbb{Q}$$
defined as follows: on a monomial $M = x_1^{s_1} \dots x_n^{s_n}$, we put
$$\sigma\text{-wt}(M) := \sum_{i=1}^{n} s_i a_i/m;$$

for a general $f = \sum_I \alpha_I M_I$, where $\alpha_I \in \mathbb{C}$ and M_I are monomials, we set

$$\sigma\text{-wt}(f) := \min\{\sigma\text{-wt}(M_I) : \alpha_I \neq 0\}.$$

Definition 2.2. For a rational number k, the σ-weighted ideal $I^\sigma(k)$ is defined as

$$I^\sigma(k) = \{g \in \mathbb{C}[x_1, \dots, x_n] : \sigma\text{-wt}(g) \geq k\} = \Big(x_1^{s_1} \cdots x_n^{s_n} : \sum_{j=1}^{n} s_j a_j/m \geq k\Big).$$

The set $I^\sigma(k)$ is a an ideal in $\mathbb{C}[x_1, \dots, x_n]$ and therefore also in $\mathbb{C}[x_1, \dots, x_n]^{\mathbb{Z}_m}$; in particular $\mathbb{C}[x_1, \dots, x_n]^{\mathbb{Z}_m} \oplus \bigoplus_{k \in \mathbb{N}, d>0} I^\sigma(k)$ is a $\mathbb{C}[x_1, \dots, x_n]^{\mathbb{Z}_m}$-graded module.

The next lemma follows straightforward from the above discussion; see also Lemma 3.5 in [10].

Lemma 2.3. *Let $\pi_\sigma \colon \overline{X} \to X$ be a σ-blow-up, E the exceptional divisor; let D be the $\mathbb{Q}$-Cartier Weil divisor defined by a $\mathbb{Z}_m$-semi invariant $f \in \mathbb{C}[x_1, \dots, x_n]$. Then we have*

$$\pi_\sigma^*(D) = \overline{D} + (\sigma\text{-}wt(f))E,$$

where $\overline{D}$ is the proper transform of D.

In particular, for every integer a, we have $\pi_* \mathcal{O}_{\overline{X}}(-aE) = I^\sigma(a)$.

The Grothendieck set-up and Lemma 2.3 imply immediately the following characterization of weighted blow-up.

Proposition 2.4. *Let $X = \mathbb{C}^n/\mathbb{Z}_m(a_1, \dots, a_n)$ and b a positive integer multiple of $M = \operatorname{lcm}(a_1, \dots, a_n)$. The weighted blow-up of X with weight σ defined above, $\pi_\sigma \colon \overline{X} \to X$, is given by*

$$\overline{X} = \operatorname{Proj}_X \Big(\mathcal{O}_X \oplus \bigoplus_{d \in \mathbb{N}, d\ >0} I^\sigma(db)\Big).$$

Remark 2.5. The above characterization of $\overline{X}$ does not depend on the the choice of b as a positive multiple of M; in fact taking Proj of *truncated* graded algebras we obtain isomorphic objects (see for instance Exercise 5.13 or 7.11, Chapter II in [9]).

Note that it is not true that $I^\sigma(db) = I^\sigma(b)^d$: see for instance Example 3.5 in [2]. However this is true if b is chosen big enough; this can be proved, for instance, following the proof of Theorem 7.17 in [9].

If this is the case we have that $\overline{X} = \operatorname{Proj}_X \big(\mathcal{O}_X \oplus \bigoplus_{d \in \mathbb{N}, d\ >0} I^\sigma(b)^d\big)$; that is, $\overline{X}$ is the blowing-up of $X = \mathbb{C}^n/\mathbb{Z}_m(a_1, \dots, a_n)$ with respect to the coherent ideal $I^\sigma(b)$ (see the definition in Section 7, Chapter II, [9]).

Definition 2.6. Let $X \colon ((g = 0) \subset \mathbb{C}^{n+1})/\mathbb{Z}_m(a_0, \dots, a_n)$ be a hyperquotient singularity and let $\pi \colon \overline{\mathbb{C}^{n+1}/\mathbb{Z}_m(a_0, \dots, a_n)} \to \mathbb{C}^{n+1}/\mathbb{Z}_m(a_0, \dots, a_n)$ be the $\sigma = (a_0, \dots, a_n)$-blow-up. Let $\overline{X}$ be the proper transform of X via π and call again, by abuse, π its restriction to $\overline{X}$. Then $\pi \colon \overline{X} \to X$ is also called the *weighted blow-up of X with weight $\sigma = (a_1, \dots, a_n)$* (or simply the σ-blow-up).

The above Proposition 2.4, together with Corollary 7.15 in Chapter II of [9], implies the following.

Proposition 2.7. *Let* $X\colon ((g=0)\subset \mathbb{C}^{n+1})/\mathbb{Z}_m(a_0,\dots,a_n)$ *be a hyperquotient singularity and let* $i\colon X\to \mathbb{C}^{n+1}/\mathbb{Z}_m(a_0,\dots,a_n)$ *be the inclusion.*

Then

$$\overline{X}=\mathrm{Proj}_X\Big(\mathcal{O}_X\oplus\bigoplus_{d\in\mathbb{N},d>0}J^\sigma(db)\Big)\to X,$$

where $J^\sigma(db):=i^{-1}\big(I^\sigma(db)\big)^{\cdot}\mathcal{O}_X$.

If b *is big enough, then*

$$\overline{X}=\mathrm{Proj}_X\Big(\mathcal{O}_X\oplus\bigoplus_{d\in\mathbb{N},d>0}J^\sigma(b)^d\Big)\to X.$$

3. Lifting cyclic quotient singularities

In this section we consider affine varieties Z and W; we think at them as germs of complex spaces around a point P, (Z,P) and (W,P). We assume that $P\in Z$ is an isolated $\mathbb{Q}$-factorial singularities; $\mathbb{Q}$-factoriality in this case depends on the analytic type of the singularity.

Proposition 3.1. *Let* Z *be an affine variety of dimension* $n\geq 4$ *and assume that* Z *has an isolated* $\mathbb{Q}$*-factorial singularity at* $P\in Z$.

Assume that $(W,P)\subset(Z,P)$ *is a Weil divisor which is a cyclic quotient singularity, i.e.,* $W=\mathbb{C}^{n-1}/\mathbb{Z}_m(a_1,\dots,a_{n-1})$.

Then Z *is a cyclic quotient singularity, i.e.,* $Z=\mathbb{C}^n/\mathbb{Z}_m(a_1,\dots,a_{n-1},a_n)$, *where* $a_n\in\mathbb{Z}$ *is defined in the proof.*

Proof. Assume first that W is a Cartier divisor, i.e., W is given as a zero locus of a regular function f, $W\colon (f=0)\subset Z$. The map $f\colon Z\to\mathbb{C}$ is flat, since $\dim_{\mathbb{C}}\mathbb{C}=1$. Quotient singularities of dimension bigger or equal then three are rigid, by a fundamental theorem of M. Schlessinger ([26]). Since Z has an isolated singularity and $\dim W=n-1\geq 3$, it implies that W is smooth, i.e., $m=1$. A variety containing a smooth Cartier divisor is smooth along it, therefore, eventually shrinking around P, Z is also smooth.

In the general case, since Z is $\mathbb{Q}$-factorial, we can assume that there exists a minimal positive integer r such that rW is Cartier (r is the index of W). Following Proposition 3.6 in [25], we can take a Galois cover $\pi\colon Z'\to Z$, with group $\mathbb{Z}_r$, such that Z' is normal, π is etale over $Z\setminus P$, $\pi^{-1}(P)=:Q$ is a single point and the $\mathbb{Q}$-divisor $\pi^*W:=W'$ is Cartier, $W':(f'=0)\subset Z'$.

Our assumption on W implies that $r|m$, that is, $m=r\cdot s$, and that $W'=\mathbb{C}^{n-1}/\mathbb{Z}_s(a_1,\dots,a_{n-1})$. By the first part of the proof we have that $s=1$, i.e., W' and Z' are smooth.

Taking possibly a smaller neighborhood of Q, we can assume that, if $W'=\mathbb{C}^{n-1}$ with coordinates $(x_1,\dots,x_{n-1})$, then $Z'=\mathbb{C}^n$, with coordinates $(x_1,\dots,x_{n-1},x_n)$, where $x_n:=f'$.

The action of $\mathbb{Z}_m$ on $\mathbb{C}^n$, which extends the one on $\mathbb{C}^{n-1}$, fixes W', therefore f' is an eigenfunction; that is for a primitive m-root of unity ϵ there exists $a_n \in \mathbb{N}$ such that $\epsilon\colon f' \to \epsilon^{a_n} f'$.

Therefore the Galois cover $\pi\colon Z' = \mathbb{C}^n \to Z$ is exactly the cover of the cyclic quotient singularity $Z = \mathbb{C}^n/\mathbb{Z}_m(a_1, \dots, a_{n-1}, a_n)$. □

If $n = 3$, the above proposition is false, as the following example shows.

Example 3.2. Let $Z' = \mathbb{C}^4/\mathbb{Z}_r(a, -a, 1, 0)$; let (x, y, z, t) be coordinates in $\mathbb{C}^4$ and assume $(a, r) = 1$. Let $Z \subset Z'$ be the hypersurface given as the zero set of the function $f := xy + z^{rm} + t^n$, with $m \geq 1$ and $n \geq 2$. This is a terminal singularity which is not a cyclic quotient (it is a terminal hyperquotient singularity); in the classification of terminal singularities it is described in Theorem (12.1) of [24] (see also section 6 of [25]).

However the surface $W := Z \cap (t = 0)$, which is the surface in $\mathbb{C}^3/\mathbb{Z}_r(a, -a, 1)$ given as the zero set of $(xy + z^{rm})$, is a cyclic quotient singularity of the type $\mathbb{C}^2/\mathbb{Z}_{r^2m}(a, rm - a)$.

We give a proof of this last fact for the interested reader. Let $\overline{W}$ be the surface in $\mathbb{C}^3$, with coordinate (x, y, z), given as the zero set of the function $xy + z^{rm}$. $\overline{W}$ has a singularity of type A_{rm-1}, which is a cyclic quotient singularity of type $\overline{W} = \mathbb{C}^2/\mathbb{Z}_{rm}(1, -1)$.

Let (ξ, η) be the coordinate of $\mathbb{C}^2$ and let $\epsilon = e^{\frac{2\pi i}{r^2 m}}$ a r^2m root of unit; note that ϵ^r is a rm root of unit. The action of $\mathbb{Z}_{rm}$ on $\mathbb{C}^2$ can be described as $\epsilon^r(\xi, \eta) = (\epsilon^r \xi, \epsilon^{-r} \eta)$. A base for $\mathbb{C}[\xi, \eta]^{\mathbb{Z}_{rm}}$, the spectrum of the ring of invariant monomials under the group action, is given by $(\xi^{rm}, \eta^{rm}, \xi{\cdot}\eta)$ and therefore $\overline{W} = \operatorname{Spec}(\xi^{rm}, \eta^{rm}, \xi{\cdot}\eta)$. Let $(x, y, z) = (\xi^{rm}, \eta^{rm}, \xi{\cdot}\eta)$, then W is obtained as the quotient of $\overline{W}$ by the action of $\mathbb{Z}_r$ with weights $(a, -a, 1)$ given by $\epsilon^{rm}(x, y, z) = (\epsilon^{rma} x, \epsilon^{-rma} y, \epsilon^{rm} z)$. It is easy to check that this action can be lifted directly to $\mathbb{C}^2$ as the action: $\epsilon(\xi, \eta) = (\epsilon^a \xi, \epsilon^{rm-a} \eta)$. This extends the previously defined $\mathbb{Z}_{rm}$-action on $\mathbb{C}^2$ and has W as quotient.

Proposition 3.3. *Let Z be an affine variety of dimension $n \geq 4$ with an isolated $\mathbb{Q}$-factorial singularity at $P \in Z$. Assume also that $(W, P) \subset (Z, P)$ is a Weil divisor which has a hyperquotient singularity at P.*

Then (Z, P) is a hyperquotient singularity.

Proof. Let $W : (g = 0) \subset \mathbb{C}^n/\mathbb{Z}_m(a_1, \dots, a_n)$.

As in the previous proof we assume first that W is a Cartier divisor, i.e., W is given as the zero locus of a regular function f. The map $f\colon Z \to \mathbb{C}$ is flat and it gives a deformation of W. Since W is a hypersurface singularity, its infinitesimal deformations are all embedded deformations, i.e., they extend to a deformation of the ambient space. That is, there exists a flat map $\tilde{f}\colon \tilde{Z} \to \mathbb{C}$ such that $\tilde{f}^{-1}(0) = \mathbb{C}^n/\mathbb{Z}_m(a_1, \dots, a_n)$, Z is a hypersurface in $\tilde{Z}$, i.e., $Z\colon (\tilde{g} = 0) \subset \tilde{Z}$, and $\tilde{f}_{|Z} = f$.

By Schlessinger's theorem ([26]), this deformation $\tilde{f}$ is rigid, therefore $\tilde{Z} = \mathbb{C}^n/\mathbb{Z}_m(a_1, \dots, a_n) \times \mathbb{C} = \mathbb{C}^{n+1}/\mathbb{Z}_m(a_1, \dots, a_n, 0)$.

Thus $Z\colon (\tilde{g} = 0) \subset \mathbb{C}^{n+1}/\mathbb{Z}_m(a_1, \dots, a_n, 0)$.

In the general case, as in [25], Proposition 3.6, we take the $\mathbb{Z}_r$-Galois cover $\pi\colon Z' \to Z$, such that Z' is normal, π is etale over $Z \setminus P$, $\pi^{-1}(P) =: Q$ is a single point and the $\mathbb{Q}$-divisor $\pi^* W := W'$ is a Cartier divisor: $W' : (f' = 0) \subset Z'$.

The map $W' \to W$ is an etale cover of W ramified at P and it depends on (a subgroup of) the local fundamental group $\pi_1(W \setminus \{0\})$. By our assumption on the dimensions and the Lefschetz theorem, this is equal to $\pi_1(\mathbb{C}^n/\mathbb{Z}_m(a_1,\dots,a_n)\setminus \{0\}) = \mathbb{Z}_m$. Therefore the etale cover extends to $\mathbb{C}^n/\mathbb{Z}_m(a_1,\dots,a_n)$ and we have that $W'\colon (g' = 0) \subset \mathbb{C}^n/\mathbb{Z}_s(a_1,\dots,a_n)$, with $m = r{\cdot}s$. By the first part of the proof, $Z'\colon (\tilde{g}' = 0) \subset \mathbb{C}^{n+1}/\mathbb{Z}_s(a_1,\dots,a_n,0)$. Therefore $Z\colon (\tilde{g} := \tilde{g}' \circ \pi^{-1} = 0) \subset \mathbb{C}^{n+1}/\mathbb{Z}_m(a_1,\dots,a_n,a_{n+1})$. □

4. Lifting weighted blow-ups

This section is dedicated to the proof of Theorem 1.1; therefore $f\colon X \to Z$ will be a local, projective, divisorial contraction which contracts an irreducible divisor E to $P \in Z$. We assume that X (as a projective variety over Z) and Z (as affine variety) are $\mathbb{Q}$-factorial; factoriality on Z depends only on the analytic type of the singularities, on X also on their relative position.

By assumption, $Y \subset X$ is a f- ample Cartier divisor such that $f' = f_{|Y}\colon Y \to f(Y) = W$ is a $\sigma' = (a_1,\dots,a_{n-1})$-blow-up, $\pi_{\sigma'}\colon Y \to W$.

In particular $W = (g = 0) \subset \mathbb{C}^{n-1}/\mathbb{Z}_m(a_1,\dots,a_{n-1})$, possibly with $g \equiv 0$. Proposition 3.3 implies that $Z = (\tilde{g} = 0) \subset \mathbb{C}^n/\mathbb{Z}_m(a_1,\dots,a_{n-1},a_n)$. Note that $W = f(Y)$ is given as $(x_n = 0) \subset Z$.

We have also $\mathrm{Pic}(Y/W) = < L_{|E} >$, where $L = -ME$, $M = \mathrm{lcm}(a_1,\dots,a_{n-1})$. By the relative Lefschetz theorem, $\mathrm{Pic}(X/Z) = \mathrm{Pic}(Y/W) = < L >$; note that we simply use the injectivity of the restriction map $\mathrm{Pic}(X/Z) \longrightarrow \mathrm{Pic}(Y/W)$, true even in the singular case (see for instance p. 305 of [20], or [8]).

Since Y is Cartier and ample, there exists a positive integer a such that $\mathcal{O}_X(Y) \sim_f aL$. We claim that $a_n = aM$. To show this consider the $\sigma := (a_1,\dots,a_n)$-blow up of Z, $\tilde{f}\colon \tilde{X} \to Z$. Let $\tilde{E}$ be the exceptional divisor. Note that Y sits in $\tilde{X}$ as an ample divisor, therefore by the Lefschetz theorem there exists a Cartier divisor $\tilde{L}$ on $\tilde{X}$ which extends $L_{|E'}$, $\tilde{L} = -M\tilde{E}$ and $Y = -aM\tilde{E}$. Since $\tilde{f}(\tilde{Y}) : (x_n = 0)$, by Lemma 2.3 we compute that $a_n = \sigma\text{-wt}(x_n) = aM$.

The map f is proper, so, as in Section 2, we can apply Grothendieck's language, section 8 of [7], to say that

$$X = \mathrm{Proj}_Z\Big(\mathcal{O}_Z \oplus \bigoplus_{d>0} I_d\Big),$$

where $I_d := f_*\mathcal{O}_X(-d(ME)) = f_*\mathcal{O}_X(dL)$.

Note that, since E is effective, $I_d = f_*\mathcal{O}_X(dL) \subset \mathcal{O}_Z \subset \mathbb{C}^n[x_1,\dots,x_n]$ is an ideal for positive d and $I_d = f_*\mathcal{O}_X(dL) = \mathcal{O}_Z \subset \mathbb{C}^n[x_1,\dots,x_n]$ for non positive d.

By Propositions 2.4 and 2.7, X will be the weighted blow-up if for positive d

$$f_*\mathcal{O}_X(dL) = i^{-1}\Big(x_1^{s_1}\cdots x_n^{s_n} : \sum_{j=1}^{n} s_j a_j \geq db\Big)\cdot\mathcal{O}_Z$$

where $b = M$, s_i are non negative integers and $i\colon Z \to \mathbb{C}^n/\mathbb{Z}_m(a_1,\ldots,a_n)$ is the inclusion.

We now mimic the proof of Theorem 3.6 in [23].

Consider the exact sequence

$$0 \to \mathcal{O}_X(iL - aL) \to \mathcal{O}_X(iL) \to \mathcal{O}_Y(iL) \to 0, \tag{4.1}$$

for every integer i.

We have noticed in Section 2 that $R^1 f'_*\mathcal{O}_Y(iL) = 0$ for $i \in \mathbb{Z}$. Therefore, by (4.1), we obtain surjections $R^1 f_*\mathcal{O}_X((i-aj)L) \to R^1 f_*\mathcal{O}_X(iL)$, $i,j \in \mathbb{Z}, j \geq 0$. On the other hand $R^1 f_*\mathcal{O}_X(-jL) = 0$ for sufficiently large j. Hence we obtain

$$R^1 f_*\mathcal{O}_X(iL) = 0 \quad \text{for every integer } i.$$

Let $\mathcal{O}_Z = \big(\mathbb{C}[x_1,\ldots,x_n]/(\tilde{g})\big)^{\mathbb{Z}_m}$. All above implies the following exact sequences of $\mathcal{O}_Z$-algebras:

$$0 \to f_*\mathcal{O}_X((i-a)L) \to f_*\mathcal{O}_X(iL) \to f_*\mathcal{O}_Y(iL) \to 0. \tag{4.2}$$

In particular, for $i = a$, we have

$$0 \to \mathcal{O}_Z \to f_*\mathcal{O}_X(aL) \to f_*\mathcal{O}_Y(aL) \to 0.$$

Let θ be the image of 1 by the map $\mathcal{O}_Z \to f_*\mathcal{O}_X(aL)$; then (4.2) becomes

$$0 \to f_*\mathcal{O}_X((i-a)L) \stackrel{\times\theta}{\to} f_*\mathcal{O}_X(iL) \to f_*\mathcal{O}_Y(iL) \to 0; \tag{4.3}$$

here, $\times\theta$ is exactly $\times(x_n)$.

We will prove, by induction on d, that

$$f_*\mathcal{O}_X(dL) = \Big(x_1^{s_1}\cdots x_n^{s_n} : \sum_{j=1}^{n} s_j a_j \geq db\Big)\cdot\mathcal{O}_Z.$$

By assumption we have that

$$f_*\mathcal{O}_Y(dL) = \Big(x_1^{s_1}\cdots x_n^{s_n} : \sum_{j=1}^{n-1} s_j a_j \geq db\Big)\cdot\mathcal{O}_W$$

where $s_j \in \mathbb{N}$.

By induction on d, we can assume that

$$f_*\mathcal{O}_X((d-a)L) = \Big(x_1^{s_1}\cdots x_n^{s_n} : \sum_{j=1}^{n} s_j a_j \geq (d-a)b\Big)\cdot\mathcal{O}_Z,$$

the case $d - a \leq 0$ being trivial.

Let $g = x_1^{s_1} \cdots x_n^{s_n} \in f_*\mathcal{O}_X(dL)$ be a monomial.

If $s_n \geq 1$ then, looking at the sequence (4.3), g comes from $f_*\mathcal{O}_X((d-a)L)$ by the multiplication by (x_n); therefore

$$\sum_{j=1}^{n} s_j a_j = \sum_{j=1}^{n-1} s_j a_j + s_n a_n \geq (d-a)b + s_n a_n \geq db - ab + ab = db.$$

If $s_n = 0$, then $g \in f_*\mathcal{O}_Y(dL)$ and so

$$\sum_{j=1}^{n} s_j a_j = \sum_{j=1}^{n-1} s_j a_j \geq db.$$

The non-monomial case follows immediately.

5. Application to MMP with scaling

The proof of Theorem 1.3, as explained in the introduction, follows via a standard induction procedure using Theorem 1.1, Theorem 1.1 in [3] and, for dimension 3, assuming 1.2. It is actually very similar to the proof of Theorem 1.2.A in [3], we rewrite it for the reader's convenience.

Proof of Theorem 1.3. Let $f\colon X \to Z$ be a local projective, divisorial contraction which contracts a prime divisor E to $P \in Z$ as in the theorem.

The *nef-value* of the pair $(f: X \to Z, L)$ is defined as $\tau_f(X, L) := \inf\{t \in \mathbb{R} : K_X + tL \text{ is } f\text{-nef}\}$. By the rationality theorem of Kawamata (Theorem 3.5 in [22]), $\tau_f(X, L) := \tau$ is a rational non-negative number. Moreover f is an adjoint contraction supported by $K_X + \tau L$, that is $K_X + \tau L \sim_f \mathcal{O}_X$ ($\sim_f$ stays for numerical equivalence over f).

By our assumption, $\tau > (n-3)$. Therefore $\tau + 3 > n > n - 1 = \dim E$ and, by Proposition 3.3.2 in [3], there exists a section of L not vanishing along E; in particular $|L|$ is not empty.

Let $H_i \in |L|$ be general divisors for $i = 1, \ldots, n-3$. By Theorem 1.1 in [3], quoted in the introduction, for any i, H_i is a variety with terminal singularities and the morphism $f_i = f_{|H_i} : H_i \to f(H_i) =: Z_i$ is a local contraction supported by $K_{H_i} + (\tau - 1)L_{|H_i}$. Since Z is terminal and $\mathbb{Q}$-factorial (see Corollaries 3.36 and 3.43 in [22]), then the Z_i's are $\mathbb{Q}$-Cartier divisors on Z.

For any $t = n-3, \ldots, 0$ define $Y_t = \cap_{i=1}^{n-3-t} H_i$ and $g_t = f_{|Y_t} : Y_t \to f(Y_t) = W_t$; in particular $Y_{n-3} = X$, $g_{n-3} = f$ and $W_{n-3} = Z$.

By induction on t, applying Theorem 1.1 in [3], one sees that, for any $t = n-4, \ldots, 0$, Y_t is terminal and $g_t\colon Y_t \to W_t$ is a local Fano–Mori contraction supported by $K_{Y_t} + (\tau - (n-3-t)L_{|Y_t}$. Therefore W_t is a terminal variety (by Corollary 3.43 in [22]) and it is a $\mathbb{Q}$-Cartier divisor in W_{t+1}, because intersection of $\mathbb{Q}$-Cartier divisors (by construction $W_t = \cap_{i=1}^{n-3-t} Z_i$).

Set $L_t := L_{|W_t}$. By Proposition 3.3.4 of [3], $Bs|L_t|$ has dimension at most 1; by Bertini's theorem (see Theorem 6.3 in [13]), $E_t := Y_t \cap E$ is a prime divisor. E_t is the intersection of $\mathbb{Q}$-Cartier divisors and hence it is $\mathbb{Q}$-Cartier.

Let $X'' = Y_0$ and $f'' = g_0$; by what said above, $f'' \colon X'' \to Z''$ is a divisorial contraction from a 3-fold X'' with terminal singularities, which contracts a prime $\mathbb{Q}$-Cartier divisor E'' to a point $P \in Z''$. Using the classification in dimension 3 of terminal $\mathbb{Q}$-factorial singularities ([24]) and of divisorial contractions (for a summary see [5]), one can see that Z'' has a hyperquotient singularity at P, which is actually contained in a special list.

By Proposition 3.3 and by induction on t, also Z has a hyperquotient singularity at P.

Assume now (1.2), that is that f'' is a weighted blow-up of P; applying Theorem 1.1 inductively on t, we have that f is a weighted blow-up of a hyperquotient singularities. □

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