Rev. Mat. Iberoam. **34** (2018), no. 4, 1853–1866
DOI 10.4171/RMI/1046

On Galois group of factorized covers of curves

Angel Carocca and Martha Romero Rojas

Abstract. Let $\mathcal{Y} \xrightarrow{\psi} \mathcal{X} \xrightarrow{\varphi} \mathbb{P}^1$ be a sequence of covers of compact Riemann surfaces. In this work we study and completely characterize the Galois group $\mathfrak{G}(\varphi\circ\psi)$ of $\varphi\circ\psi$ under the following properties: φ is a simple cover of degree m and ψ is a Galois unramified cover of degree n with abelian Galois group of type $(n_1, n_2, \ldots, n_s)$.

We prove that $\mathfrak{G}(\varphi\circ\psi) \cong (\mathbb{Z}_{n_1}\times\mathbb{Z}_{n_2}\times\cdots\times\mathbb{Z}_{n_s})^{m-1}\rtimes\mathbf{S}_m$. Furthermore, we give a natural geometric generator system of $\mathfrak{G}(\varphi\circ\psi)$ obtained by studying the action on the compact Riemann surface $\mathcal{Z}$ associated to the Galois closure of $\varphi\circ\psi$.

1. Introduction

Let $\mathcal{X}$ be a compact Riemann surface and $\varphi\colon \mathcal{X}\to\mathbb{P}^1$ a cover of degree m. The Galois closure of φ is a Galois cover $\widehat{\varphi}\colon \mathcal{Z}\to\mathbb{P}^1$ of smallest possible degree such that there exists a sequence of compact Riemann surfaces $\mathcal{Z} \xrightarrow{\widehat{\psi}} \mathcal{X} \xrightarrow{\varphi} \mathbb{P}^1$ with $\varphi\circ\widehat{\psi} = \widehat{\varphi}$ (up to equivalence the Galois closure is unique). Let $\mathbb{C}(\mathcal{X})$ be the field of meromorphic functions on $\mathcal{X}$. The Galois group $\mathfrak{G}(\varphi)$ of the cover φ is the Galois group associated to the Galois closure of the field extension $\mathbb{C}(\mathcal{X})/\mathbb{C}(\mathbb{P}^1)$. An elementary property of $\mathfrak{G}(\varphi)$ is that it has a natural representation as a transitive subgroup of the symmetric group $\mathbf{S}_m$.

The problem of determining the structure of the group $\mathfrak{G}(\varphi)$ in general was originally considered by O. Zariski [18].

Since then, many authors have worked on it imposing conditions on the cover. For instance, the cover φ is called *simple* if the fiber $\varphi^{-1}(p)$ over every branch point $p\in\mathbb{P}^1$ consists of exactly $m-1$ different points. In this case it is well known that the Galois group $\mathfrak{G}(\varphi)$ is isomorphic to the symmetric group $\mathbf{S}_m$ and $\mathfrak{G}(\varphi)$ is generated geometrically by transpositions, see [6], [11] and [14]. Related problems have been considered by other authors, [1], [3], [10], [11] and [14].

Mathematics Subject Classification (2010): Primary 14E20, 14H37; Secondary 14H55.
Keywords: Covers, Riemann surfaces, monodromy, automorphisms.

Now, consider $\mathcal{Y} \xrightarrow{\psi} \mathcal{X} \xrightarrow{\varphi} \mathbb{P}^1$ a sequence of covers of compact Riemann surfaces, and denote by $\mathfrak{G}(\varphi \circ \psi)$ the Galois group of the *factorized cover* $\varphi \circ \psi \colon \mathcal{Y} \to \mathbb{P}^1$. Many authors have studied the problem of determining the geometric structure of $\mathfrak{G}(\varphi \circ \psi)$ based on special properties of the covers φ and ψ, see for instance [2], [8], [9], [11], [12], [15] and [17]. Probably the most studied case of a factorized cover $\varphi \circ \psi$ is when $\psi \colon \mathcal{Y} \to \mathcal{X}$ is an unramified cover of degree two; the results obtained in this situation involve a systematic study of the Weyl groups WB_m and WD_m, see for instance [12] and [17].

Another interesting case of factorized covers $\varphi \circ \psi$ was studied by Biggers–Fried in [4]. Using results on meromorphic functions, they proved that if φ is a simple cover of degree m, and $\psi \colon \mathcal{Y} \to \mathcal{X}$ is an unramified Galois cover of degree n with cyclic Galois group, then

$$\mathfrak{G}(\varphi \circ \psi) \cong (\mathbb{Z}_n)^{m-1} \rtimes \mathbf{S}_m.$$

In this paper we extend the Biggers–Fried result by considering factorized covers $\varphi \circ \psi$ with φ a simple cover of degree m, and $\psi \colon \mathcal{Y} \to \mathcal{X}$ an unramified Galois cover with abelian Galois group of type $(n_1, n_2, \ldots, n_s)$. In this case, using group theoretical arguments, we prove that

$$\mathfrak{G}(\varphi \circ \psi) \cong (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s})^{m-1} \rtimes \mathbf{S}_m.$$

Furthermore, we give a natural geometric system of generators of $\mathfrak{G}(\varphi \circ \psi)$ as a transitive subgroup of $\mathbf{S}_{mn}$.

2. Preliminaries

In order to fix the notation, we start by recalling some basic properties on group action on compact Riemann surfaces. Let $\mathcal{X}$ be a compact Riemann surface of genus $\mathsf{g}(\mathcal{X})$ and G a finite group acting on $\mathcal{X}$. The quotient space $\mathcal{X}/G := \mathcal{X}_G$ is a smooth surface and the quotient projection $\mathcal{X} \to \mathcal{X}_G$ is a branched cover. This cover may be partially characterized by a vector of numbers $(\gamma; m_1, \ldots, m_r)$, where γ is the genus of $\mathcal{X}_G$, the integer $0 \le r \le 2\mathsf{g}(\mathcal{X}) + 2$ is the number of branch points of the cover, and the integers m_j are the orders of the cyclic subgroups G_j of G which fix points on $\mathcal{X}$. We call $(\gamma; m_1, \ldots, m_r)$ the *branching data* of G on $\mathcal{X}$. These numbers satisfy the Riemann–Hurwitz equation

$$\frac{2(\mathsf{g}(\mathcal{X}) - 1)}{|G|} = 2(\gamma - 1) + \sum_{j=1}^{r} \Big(1 - \frac{1}{m_j}\Big). \tag{2.1}$$

A $(2\gamma + r)$-tuple $(a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, c_1, \ldots, c_r)$ of elements of G is called a *generating vector of type* $(\gamma; m_1, \ldots, m_r)$ if

$$G = \Big\langle a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, c_1, \ldots, c_r \;\Big/ \prod_{i=1}^{\gamma} [a_i, b_i] \prod_{j=1}^{r} c_j = 1,\ |c_j| = m_j \text{ for } j = 1, \ldots, r,\ \mathcal{R} \Big\rangle, \tag{2.2}$$

where $\mathcal{R}$ is a set of appropriate relations on $\{a_1, \ldots, a_\gamma, b_1, \ldots, b_\gamma, c_1, \ldots, c_r\}$ and $[a_i,\ b_i] = a_i b_i a_i^{-1} b_i^{-1}$.

Riemann's existence theorem then tells us that (see [5]):

Theorem 2.1. *The group G acts on a surface $\mathcal{X}$ of genus $\mathsf{g}(\mathcal{X})$ with branching data $(\gamma; m_1, \ldots, m_r)$ if and only if G has a generating vector of type $(\gamma; m_1, \ldots, m_r)$ satisfying the Riemann–Hurwitz formula* (2.1).

Given an action of G on $\mathcal{X}$, the elements of a generating vector for this action will be called *geometric generators* for G.

For a subgroup $H \leq G$, the structure of the intermediate cover $\mathcal{X} \to \mathcal{X}_H$ is given by the signature of the action of H, as follows.

Proposition 2.2. *Let G be a finite group acting on a compact Riemann surface $\mathcal{X}$ with branching data $(\gamma; m_1, \ldots, m_r)$. For each $j = 1, \ldots, r$, consider the stabilizer G_j of the corresponding fixed points on $\mathcal{X}$.*

Then for each subgroup $H \leq G$ we have

$$\mathsf{g}(\,\mathcal{X}_H\,) = |\,G : H|(\gamma - 1) + 1 + \frac{1}{2}\sum_{j=1}^{r}(|\,G : H\,| - |\,H\backslash G/G_j\,|), \tag{2.3}$$

where $|\,H\backslash G/G_j\,|$ is the number of double cosets $H\backslash G/G_j$.

Proof. See [16]. □

3. Some properties of factorized covers

In this section we collect some properties of the Galois group of certain factorized covers. Let

$$\varphi : \mathcal{X} \to \mathbb{P}^1$$

be a simple cover of degree m; that is, the fiber $\varphi^{-1}(p)$ over every branch point $p \in \mathbb{P}^1$ consists of exactly $m-1$ different points. We recall that in this case the Galois group of φ is isomorphic to the symmetric group $\mathbf{S}_m$ and is geometrically generated by transpositions, see for instance [6], [11] or [14].

Now consider a Galois cover of degree n

$$\psi : \mathcal{Y} \to \mathcal{X}.$$

We denote by $\widehat{\varphi \circ \psi}$ the Galois cover of the factorized cover $\varphi \circ \psi$ and by $\mathfrak{G} = \mathfrak{G}(\varphi \circ \psi)$ the corresponding Galois group of $\varphi \circ \psi$. We begin with some general properties.

Proposition 3.1. *Let $\mathcal{Z}$ be the Riemann surface associated to $\widehat{\varphi \circ \psi}$. Then there are subgroups N and H of $\mathfrak{G}$ satisfying the following properties:*

1. *$\mathcal{Z}_N \cong \mathcal{Y}$, $\mathcal{Z}_H \cong \mathcal{X}$ and $\mathcal{Z}_{\mathfrak{G}} \cong \mathbb{P}^1$.*
2. *$N_{\mathfrak{G}} = \{1\}$, where $N_{\mathfrak{G}} = \mathrm{Core}_{\mathfrak{G}}(N)$. In particular, if $N \trianglelefteq G$, then $N = \{1\}$ and $\mathcal{Z} \cong \mathcal{Y}$.*

(3) *H is a maximal subgroup of $\mathfrak{G}$.*

(4) *$\mathfrak{G}/K \cong \boldsymbol{S}_m$, where $\boldsymbol{S}_m$ is the symmetric group of degree m and $K = H_{\mathfrak{G}}$.*

(5) *$N \trianglelefteq H$.*

Proof. (1) and (2) follow from the definition of Galois cover.

(3) and (4) follow since $\varphi\colon \mathcal{X} \to \mathbb{P}^1$ is a simple cover.

(5) follows since $\psi\colon \mathcal{Y} \to \mathcal{X}$ is a Galois cover. □

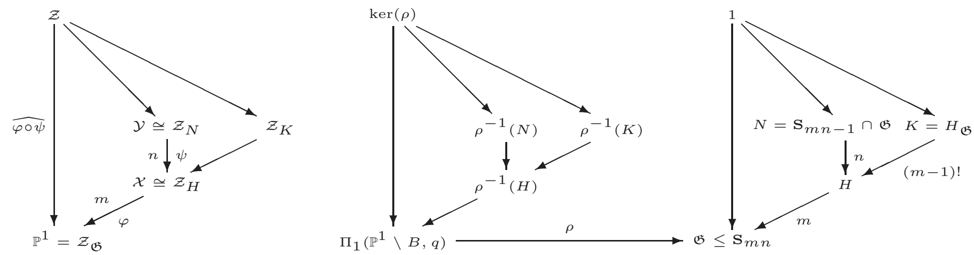
Remark 3.2. The monodromy representation of the cover $\varphi \circ \psi\colon \mathcal{Y} \to \mathbb{P}^1$ is the natural group homomorphism $\rho\colon \Pi_1(\mathbb{P}^1 \setminus B, q) \to \mathbf{S}_{mn}$ with transitive image in $\mathbf{S}_{nm}$. It is well known that this representation is equivalent to the permutational representation given by the action of $\mathfrak{G}$ on the right cosets of N in $\mathfrak{G}$. To obtain this representation, we consider $\{1 = x_1, x_2, \ldots, x_m\}$ a right transversal of H in $\mathfrak{G}$ and $\{1 = h_1, h_2, \ldots, h_n\}$ a right transversal of N in H. Then the set $\{h_j x_i \,/\, i = 1, \ldots, m,\ j = 1, \ldots, n\}$ is a right transversal of N in $\mathfrak{G}$.

Let $\Omega = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$, where $\Delta_i = \{Nh_1x_i, Nh_2x_i, \ldots, Nh_nx_i\}$ for $i = 1, \ldots, m$. Then $\mathfrak{G}$ acts transitively on Ω, $\mathfrak{G} \cong \mathfrak{G}/N_{\mathfrak{G}} \hookrightarrow \mathbf{S}_{nm}$ and $\mathfrak{G}$ acts transitively on the set $\{\Delta_1, \Delta_2, \ldots, \Delta_m\}$ with kernel $K = H_{\mathfrak{G}}$. We will use the same letters $\mathfrak{G}$, H and N to denote their corresponding images in $\mathbf{S}_{nm}$; we will also identify the set Δ_i with the set $\{i, m+i, 2m+i, \ldots, (n-1)m+i\} = \Delta_i$.

Thus for each $1 \le i \le m$ we have the following results:

- $N_i = x_i^{-1} N x_i$ stabilizes each point in Δ_i. Here $N_1 = N$.
- $H_i = x_i^{-1} H x_i$ stabilizes the set Δ_i. Here $H_1 = H$.
- $K = H_{\mathfrak{G}}$ stabilizes each set Δ_i.
- If $N \neq \{1\}$, then $H = \mathbf{N}_{\mathfrak{G}}(N)$, the normalizer of N in G. Since $N = \mathbf{S}_{mn-1} \cap \mathfrak{G}$ we have that $|\mathrm{Fix}(N)| = |\mathbf{N}_{\mathfrak{G}}(N) : N| = |H : N| = n$. Thus N stabilizes n points. In particular, $N = \bigcap_{j=1}^{n} \mathbf{Stb}(j) \cap \mathfrak{G}$, where $\mathbf{Stb}(j)$ is the stabilizer in $\mathbf{S}_{mn-1}$ of the point $j \in \Delta_1$.
- For $m \ge 3$, we have $K = H_{\mathfrak{G}} = \bigcap_{j=1}^{m-1} H^{g_j}$ and $K \neq \bigcap_{j=1}^{m-2} H^{g_j}$ for any subset $\{H^{g_1}, H^{g_2}, \ldots, H^{g_{m-1}}\}$ with $m-1$ distinct conjugates of H in $\mathfrak{G}$.

The following diagram illustrates the relationship between covers and subgroups:



where $\mathcal{Z}$ is the Riemann surface associated to $\widehat{\varphi \circ \psi}$.

Proposition 3.3. *Let* $\{g_1, g_2, \dots, g_r\}$ *be a set of geometric generators of* $\mathfrak{G}$ *given by the action on* $\mathcal{Z}$ *and* $G_i = \langle g_i \rangle$. *Then the following properties hold*:

(1) *The cover* $\psi\colon \mathcal{Y} \to \mathcal{X}$ *is unramified if and only*

$$|H : N|\,|H\backslash\mathfrak{G}/G_i| = |N\backslash\mathfrak{G}/G_i| \quad \textit{for all } 1 \le i \le r.$$

(2) *If the cover* $\psi\colon \mathcal{Y} \to \mathcal{X}$ *is unramified, then* $K \cap G_i = \{1\}$, *for all* $1 \le i \le r$.

Proof. (1) Suppose that ψ is unramified. By the Riemann–Hurwitz formula (2.1), we have

$$\mathsf{g}(\mathcal{Y}) = |H : N|(\mathsf{g}(\mathcal{X}) - 1) + 1.$$

On the other hand, using the formula (2.3) (Proposition 2.2), we have

$$\mathsf{g}(\mathcal{Y}) = -|\mathfrak{G} : N| + 1 + \frac{1}{2}\sum_{i=1}^{r}(|\mathfrak{G} : N| - |N\backslash\mathfrak{G}/G_i|), \quad \text{and}$$
$$\mathsf{g}(\mathcal{X}) = -|\mathfrak{G} : H| + 1 + \frac{1}{2}\sum_{i=1}^{r}(|\mathfrak{G} : H| - |H\backslash\mathfrak{G}/G_i|).$$

Hence

$$\sum_{i=1}^{r}|N\backslash\mathfrak{G}/G_i| = |H : N|\sum_{i=1}^{r}|H\backslash\mathfrak{G}/G_i|.$$

Since $|N\backslash\mathfrak{G}/G_i| \le |H : N||H\backslash\mathfrak{G}/G_i|$, we conclude that, for every $1 \le i \le r$, $|H : N|\,|H\backslash\mathfrak{G}/G_i| = |N\backslash\mathfrak{G}/G_i|$.

Now, suppose $|N\backslash\mathfrak{G}/G_i| = |H : N|\,|H\backslash\mathfrak{G}/G_i|$ for each $i = 1, \dots, r$. Since

$$\mathsf{g}(\mathcal{Y}) = -|\mathfrak{G} : N| + 1 + \frac{1}{2}\sum_{i=1}^{r}(|\mathfrak{G} : N| - |N\backslash\mathfrak{G}/G_i|),$$

we obtain $\mathsf{g}(\mathcal{Y}) = |H : N|(\mathsf{g}(\mathcal{X}) - 1) + 1$. Hence by the Riemann–Hurwitz formula we conclude that the cover $\psi\colon \mathcal{Y} \to \mathcal{X}$ is unramified.

(2) Assume that ψ is unramified. Then $|H : N|\,|H\backslash\mathfrak{G}/G_i| = |N\backslash\mathfrak{G}/G_i|$.

Let $C^{\mathfrak{G}}(g)$ be the conjugacy class of g in $\mathfrak{G}$ and C a complete set of representatives of the conjugacy classes of $\mathfrak{G}$. Applying a well-known formula for the cardinality of double cosets ([13], p. 55), we have

$$\frac{|H|}{|N|}\frac{|\mathfrak{G}|}{|H||G_i|}\sum_{g\in C}\frac{|(C^{\mathfrak{G}}(g)\cap H|\,|C^{\mathfrak{G}}(g)\cap G_i|}{|C^{\mathfrak{G}}(g)|} = \frac{|\mathfrak{G}|}{|N||G_i|}\sum_{g\in C}\frac{|(C^{\mathfrak{G}}(g)\cap N|\,|C^{\mathfrak{G}}(g)\cap G_i|}{|C^{\mathfrak{G}}(g)|}.$$

Hence

$$\sum_{g\in C}\frac{(|C^{\mathfrak{G}}(g)\cap G_i|)\,(|C^{\mathfrak{G}}(g)\cap H| - |C^{\mathfrak{G}}(g)\cap N|)}{|C^{\mathfrak{G}}(g)|} = 0.$$

Since $N \leq H$ for all $g \in \mathfrak{G}$, we have that $C^{\mathfrak{G}}(g) \cap N \subseteq C^{\mathfrak{G}}(g) \cap H$ and

$$\frac{(|C^{\mathfrak{G}}(g) \cap G_i|)\,(|C^{\mathfrak{G}}(g) \cap H| - |C^{\mathfrak{G}}(g) \cap N|)}{|C^{\mathfrak{G}}(g)|} = 0.$$

Therefore, for all $g \in C$ we have

$$|C^{\mathfrak{G}}(g) \cap G_i| = 0 \quad \text{or} \quad |C^{\mathfrak{G}}(g) \cap H| = |C^{\mathfrak{G}}(g) \cap N|.$$

Let $1 \neq k \in K = H_{\mathfrak{G}}$. If we assume $|C^{\mathfrak{G}}(k) \cap N| = |C^{\mathfrak{G}}(k) \cap H|$ then $C^{\mathfrak{G}}(k) = C^{\mathfrak{G}}(k) \cap H = C^{\mathfrak{G}}(k) \cap N$. Hence $C^{\mathfrak{G}}(k) \subseteq N_{\mathfrak{G}} = \{1\}$, a contradiction.

Therefore, for all $1 \neq k \in K$ we have that $|C^{\mathfrak{G}}(k) \cap G_i| = 0$ and $K \cap G_i = \{1\}$. □

Proposition 3.4. *Suppose $\psi\colon \mathcal{Y} \to \mathcal{X}$ is an unramified cover. Then the following properties hold.*

(1) *The action of $\mathfrak{G}$ on $\mathcal{Z}$ induces a geometric presentation of $\mathfrak{G}$ given by*

$$\mathfrak{G} = \left\langle g_1, g_2, \ldots, g_r \;\Big/\; \prod_{i=1}^{r} g_i = 1,\;\; g_i^2 = 1, \quad i = 1, 2, \ldots, r,\; \mathcal{R} \right\rangle,$$

where $\mathcal{R}$ is a set of appropriate relations on $\{g_1, g_2, \ldots, g_r\}$.

(2) *For $K = H_{\mathfrak{G}}$, the corresponding action of $\mathfrak{G}/K$ on $\mathcal{Z}_K$ induces a geometric presentation of $\mathfrak{G}/K$ given by*

$$\mathfrak{G}/K = \left\langle g_1K, g_2K, \ldots, g_rK \;\Big/\; \prod_{i=1}^{r} g_i = 1,\; g_i^2 = 1,\; i = 1, 2, \ldots, r,\; \mathcal{R}' \right\rangle \cong \boldsymbol{S}_m,$$

where $\mathcal{R}'$ is a set of appropriate relations on the set of cosets $\{Kg_1, \ldots, Kg_r\}$.

(3) *The corresponding image in $\boldsymbol{S}_{mn}$ for each g_i has a cycle structure given as a product of n disjoint transpositions.*

(4) *The corresponding image in $\boldsymbol{S}_m$ for each g_iK is a transposition.*

(5) *$H \trianglelefteq \mathfrak{G}$ if and only if $m = 2$.*

Proof. Since $\varphi\colon \mathcal{X} \to \mathbb{P}^1$ is a simple cover and $\psi\colon \mathcal{Y} \to \mathcal{X}$ is an unramified cover we have that the fiber under the factorized covering $\varphi \circ \psi$ of each branch point $b \in \mathbb{P}^1$ is given by $(\varphi \circ \psi)^{-1}(b) = \{p_1, p_2, \ldots, p_{n(m-1)}\}$. Then the cycle structure of $\varphi \circ \psi$ at b is an $n(m-1)$-tuple $(2^n, 1^{nm-2n})$ where the ramification index of $\varphi \circ \psi$ at each of the points $p_1, p_2, \ldots, p_n$ is 2 and at each of the points $p_{n+1}, p_{n+2}, \ldots, p_{mn-2n}$ is 1. This implies that the corresponding image in $\mathbf{S}_{mn}$ for each geometric generator of $\mathfrak{G}$, given by the action of $\mathfrak{G}$ on $\mathcal{Z}$, has cycle structure as a product of n disjoint transpositions and that the corresponding image in $\mathbf{S}_m$ of each g_iK is a transposition. In this way we conclude (1), (2), (3) and (4).

(5) Assume $H \trianglelefteq G$. By Proposition 3.3, we have $\langle g_1 \rangle \not\leq H$. Hence $\langle g_1 \rangle H = \mathfrak{G}$, since H is a maximal subgroup of $\mathfrak{G}$. Hence $m = 2$. □

Remark 3.5. According to Proposition 3.4, if $\psi\colon \mathcal{Y} \to \mathcal{X}$ is an unramified cover, then the corresponding image in $\mathbf{S}_{mn}$ for each geometric generator g_i of $\mathfrak{G}$, given by the action of $\mathfrak{G}$ on $\mathcal{Z}$, has cycle structure as a product of n disjoint transpositions. By Proposition 3.3, we have

- if $m \geq 3$, then each $G_i = \langle g_i \rangle$ is contained in the intersection of $m-2$ different conjugates of N in $\mathfrak{G}$ and G_i is not contained in $K = H_{\mathfrak{G}}$;
- if $m = 2$, then each $G_i = \langle g_i \rangle$ is not contained in H.

Proposition 3.6. *Suppose $\psi\colon \mathcal{Y} \to \mathcal{X}$ is an unramified cover. Then the following are equivalent*:

(1) $N = \{1\}$,

(2) $H \trianglelefteq \mathfrak{G}$ (*or, equivalently,* $m = 2$).

Proof. By Proposition 3.4, $\mathfrak{G}$ has a geometric presentation of the form

$$\mathfrak{G} = \langle g_1, g_2, \ldots, g_r \ \ / \ \ g_i^2 = 1, \quad \text{for all} \ \ i = 1, \ldots, r, \quad \mathcal{R} \rangle,$$

and we can assume that $g_1 \notin H$.

Assume $N = \{1\}$. Then g_1 does not fix points of the set $\Omega = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m$ and interchanges Δ_1 with Δ_k. Since g_1 has cycle structure as a product of n disjoint transpositions and $g_1 \notin H$, we obtain that $m = 2$ and $H \trianglelefteq \mathfrak{G}$.

Suppose $H \trianglelefteq \mathfrak{G}$ (or equivalently $m = 2$). By Proposition 3.3 we have $\langle g_1 \rangle \not\leq H$. Hence $\langle g_1 \rangle H = \mathfrak{G}$. For each $2 \leq i \leq r$ we can write $g_i = h_i g_1$ with $h_i \in H$ according to Remark 3.5. Since $g_i^2 = 1$ we have $g_1 h_i g_1 = {h_i}^{-1}$ and $\mathfrak{G} = \langle g_1, g_2, \ldots, g_r \rangle = \langle h_2, h_3, \ldots, h_r \rangle \langle g_1 \rangle$. Therefore $H = \langle h_2, h_3, \ldots, h_r \rangle$ and g_1 normalizes each subgroup of H. So, $N = \{1\}$, since $N \trianglelefteq H$ and $N_{\mathfrak{G}} = \{1\}$. □

4. On the structure of Galois Group of factorized covers

In this section we will determine the structure of the Galois group of certain factorized covers. We recall the notation: $\varphi\colon \mathcal{X} \to \mathbb{P}^1$ is a simple cover of degree m, the cover $\psi\colon \mathcal{Y} \to \mathcal{X}$ is Galois of degree n and $\widehat{\varphi \circ \psi}$ is the Galois cover of the factorized cover $\varphi \circ \psi$. The group $\mathfrak{G}$ is the corresponding Galois group of $\varphi \circ \psi$. Also $\mathcal{Z}$ is the compact Riemann surface associated to $\widehat{\varphi \circ \psi}$ and the subgroups N and H of $\mathfrak{G}$ correspond to $\mathcal{Z}_N \cong \mathcal{Y}$ and $\mathcal{Z}_H \cong \mathcal{X}$, respectively. We are using the same letters $\mathfrak{G}$, H and N to denote their images in $\mathbf{S}_{nm}$, given by the permutational representation of $\mathfrak{G}$ on the set of the right cosets of N in $\mathfrak{G}$. Here we identify the set $\Delta_i = \{Nh_1x_i, Nh_2x_i, \ldots, Nh_nx_i\}$ with the set $\{i, m+i, 2m+i, \ldots, (n-1)m+i\} = \Delta_i$, and we will use the same letters g_i to denote the corresponding image in $\mathbf{S}_{nm}$ for each geometric generator g_i.

Collecting some results of the previous section, we have the following.

Remark 4.1. As proved in the previous section, if $\psi\colon \mathcal{Y} \to \mathcal{X}$ is unramified and $m = 2$, then $N = \{1\}$ and $H \trianglelefteq \mathfrak{G}$. Also, for any geometric generator $g \in \mathfrak{G}$ we have $\mathfrak{G} = H\langle g \rangle$ and $ghg = h^{-1}$ for all $h \in H$, since $g^2 = 1$.

Therefore, for the remainder of this section we consider $m \geq 3$. Now, we are able to prove the following.

Proposition 4.2. *If $\psi\colon \mathcal{Y} \to \mathcal{X}$ is unramified, then $\mathfrak{G}$ contains a subgroup L which is isomorphic to $\boldsymbol{S}_m$ and L has at least m different conjugates in $\mathfrak{G}$.*

Proof. Let $N_1, N_2, \ldots, N_m$ be the set of the m different conjugates of N in $\mathfrak{G}$. As follows from Remark 3.5, we can choose elements $g_1 \in (N_3 \cap N_4 \cap \cdots \cap N_m) \setminus K$, $g_2 \in (N_1 \cap N_4 \cap \cdots \cap N_m) \setminus K, \ldots,$ $g_{m-1} \in (N_1 \cap N_2 \cap \cdots \cap N_{m-3} \cap N_{m-2}) \setminus K$. The corresponding image of g_i in $\mathbf{S}_{mn}$ may be written as follows:

$$g_i = (i \quad i+1)(m+i \quad m+i+1) \cdots ((n-1)m+i \quad (n-1)m+i+1).$$

In this way we have

- $g_i^2 = 1$, for all $i = 1, \ldots, m-1$.
- $(g_i g_{i+1})^3 = 1$, for all $i = 1, \ldots, m-2$.
- $g_i g_j = g_j g_i$, when $|i-j| \geq 2$.

Hence $L = \langle g_1, g_2, \ldots, g_{m-1} \;/\; g_i^2 = (g_i g_{i+1})^3 = [g_i, g_j] = 1\,,\, |i-j| \geq 2 \rangle \cong \mathbf{S}_m$, according to the Coxeter presentation for the symmetric group $\mathbf{S}_m$, see [7].

Since for each $1 \leq i < m$ we have $g_i \notin H_i \cup H_{i+1}$ and since $\mathfrak{G}$ acts transitively on the set $\{\Delta_1, \Delta_2, \ldots, \Delta_m\}$, we obtain that L has at least m different conjugates in $\mathfrak{G}$. □

In [4] Biggers and Fried studied factorized covers $\varphi \circ \psi$ where $\varphi\colon \mathcal{X} \to \mathbb{P}^1$ is a simple cover of degree m and $\psi\colon \mathcal{Y} \to \mathcal{X}$ is an unramified Galois cover of degree n with cyclic Galois group. They also characterized, for this case, the Galois group of $\varphi \circ \psi$, by studying the corresponding fields of meromorphic functions.

We will now extend the Biggers–Fried result to the case when $\psi\colon \mathcal{Y} \to \mathcal{X}$ is an unramified Galois cover of degree n with abelian Galois group. An interesting point is that we will give group-theoretical proofs for our results.

We start with the following auxiliary lemma.

Lemma 4.3. *Let G be a finite group, and let $H \leq G$ with $|G:H| = k$ and abelian core $K = H_G$. If $L \leq G$ is such that $L \cong \boldsymbol{S}_k$ and $L^g \nleq H$ for all $g \in G$, then $G = K \rtimes L$.*

Proof. By the action of G on the set of the right cosets of H in G, we have that $G/K \lesssim \mathbf{S}_k$. If $K \cap L = \{1\}$, then $KL/K \cong L \cong \mathbf{S}_k$ and $k! = |KL/K| \leq |G/K| \leq |\mathbf{S}_k| = k!$. Hence $G/K \cong KL/L \cong \mathbf{S}_k$ and $G = K \rtimes L$.

Let $U = K \cap L$ and suppose that $U \neq \{1\}$. Then U is a non-trivial abelian normal subgroup of $L \cong \mathbf{S}_k$, and hence $2 \leq k \leq 4$.

- If $k = 2$, then $\mathbf{S}_2 \cong L \nleq H$ and $U = \{1\}$, a contradiction.
- If $k = 3$, then $|U| = 3$ and $|KL/K| = |L/U| = 2$. Furthermore, since $|G:H| = 3$ and $|KL/K| = 2$, we have that $G/K \cong \mathbf{S}_3$. Hence H/K and KL/K are 2-Sylow subgroups of G/K. Thus $H/K = (KL/K)^g = KL^g/K$ for some $g \in G$. It follows that $H = KL^g$ and $L^g \leq H$, a contradiction.

• If $k = 4$, then $|U| = 4$ and $|KL/K| = |L/U| = 6$. Since $|G : H| = 4$ and $|KL/K| = 6$, we conclude that $G/K \cong \mathbf{S}_4$. Hence H/K and KL/K are normalizers of 3-Sylow subgroups of G/K. Therefore $H/K = (KL/K)^g = KL^g/K$ for some $g \in G$. In this way $H = KL^g$ and $L^g \leq H$, the final contradiction. □

Proposition 4.4. *If* $\varphi\colon \mathcal{Y} \to \mathcal{X}$ *is unramified with abelian Galois group, then the following properties hold:*

(1) $K = H_{\mathfrak{G}}$ *is abelian.*

(2) $\mathfrak{G} = K \rtimes L$, *with* $L \cong \mathbf{S}_m$. *Also,* $L \cap H \cong \mathbf{S}_{m-1}$, $L \cap H \leq N$ *and* $N = (L \cap H)(N \cap K)$.

(3) $H = NK$.

(4) $N \cap K \neq \{1\}$.

(5) $K = (N \cap K)(N^g \cap K)$ *for each* $g \notin H$.

Proof. (1) Let $\{x_1, x_2, \ldots, x_m\}$ be a right transversal of H in $\mathfrak{G}$. The result follows by considering the group monomorphism

$$\Phi : K \to (H/N)^{x_1} \times (H/N)^{x_2} \times \cdots \times (H/N)^{x_m}$$

defined by $\Phi(k) = (N^{x_1}k, N^{x_2}k, \ldots, N^{x_m}k)$.

(2) By Proposition 4.2, $\mathfrak{G}$ contains a subgroup L isomorphic to $\mathbf{S}_m$, which is generated by the elements

$$g_i = (i \quad i+1)(m+i \quad m+i+1)\cdots((n-1)m+i \quad (n-1)m+i+1),$$

with $1 \leq i < m$.

By Remark 3.2, H stabilizes the set $\Delta_1 = \{1, m+1, 2m+1, \ldots, (n-1)m+1\}$. Hence for all $g \in \mathfrak{G}$ there exists some i such that $g_i^g \notin H$. Therefore $L^g \not\leq H$, and $\mathfrak{G} = K \rtimes L$, according to Lemma 4.3.

We have $L \cap H = \langle g_2, g_3, \ldots, g_{m-1} \rangle \leq L \cap N \cong \mathbf{S}_{m-1}$, since $m = |\mathfrak{G} : H| = |L : L \cap H|$. Now $N = (N \cap K)(L \cap H)$, since $H = K(H \cap L)$.

(3) Since $\mathfrak{G} = K \rtimes L$ we have $\mathfrak{G}/K \cong \mathbf{S}_m$ and $H/K \cong \mathbf{S}_{m-1}$. Let $V \trianglelefteq \mathfrak{G}$ such that $V/K \cong \mathbf{A}_m$, the alternating group.

If $NK \neq H$, then $N \leq NK \leq H \cap V \leq V$, since $N \trianglelefteq H$. By Proposition 3.4 and Remark 3.5, the group $\mathfrak{G}$ has a geometric presentation given by

$$\mathfrak{G} = \Big\langle g_1, g_2, \ldots, g_r \ \ / \ \ \prod_{i=1}^{r} g_i = 1, \ \ g_i^2 = 1, \quad i = 1, 2, \ldots, r \ \text{ and } \ \mathcal{R} \Big\rangle,$$

where each generator g_i is an element of some conjugate of N. Then

$$\mathfrak{G} = \langle g_1, g_2, \ldots, g_r \rangle \leq \langle N^g \ / \ g \in \mathfrak{G} \rangle \leq V,$$

a contradiction. Hence $NK = H$.

(4) If $N \cap K = \{1\}$, then the commutator subgroup $[N, K] = \{1\}$, since $N \trianglelefteq H$ and $K \trianglelefteq H$. Therefore, K commutes with N and similarly with all conjugates of N. Hence K commutes with L and $L \trianglelefteq \mathfrak{G}$, a contradiction, according to Proposition 4.2.

(5) Let $g \notin H$. Since H/N is abelian, $N(N^g \cap K) \trianglelefteq H$. So $(N \cap K)(N^g \cap K) \trianglelefteq H$. Similarly, $(N \cap K)(N^g \cap K) \trianglelefteq H^g$. Hence $(N \cap K)(N^g \cap K) \trianglelefteq \langle H, H^g \rangle = \mathfrak{G}$, since H is a maximal subgroup of $\mathfrak{G}$.

By item (2), $\mathfrak{G} = KL$ with $L \cong \mathbf{S}_m$ and $N = (N \cap K)(H \cap L) \leq (N \cap K)(N^g \cap K)L$. Let $x \in \mathfrak{G}$ and write $x = yz$ with $x \in K$ and $z \in L$. Then $N^x = N^{yz} = N^z \leq ((N \cap K)(N^g \cap K)L)^z \leq (N \cap K)(N^g \cap K)L$. Hence $(N \cap K)(N^g \cap K)L = \mathfrak{G}$, according to Remark 3.5. Therefore

$$K = K \cap (N \cap K)(N^g \cap K)L = (N \cap K)(N^g \cap K)(L \cap K) = (N \cap K)(N^g \cap K). \quad \square$$

Proposition 4.5. *If $\varphi\colon \mathcal{Y} \to \mathcal{X}$ is an unramified Galois cover with abelian Galois group, then $K \cong (H/N)^{m-1}$.*

Proof. Assume $H/N \cong \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s}$ and consider $M_j \leq H$ such that $M_j/N \cong \mathbb{Z}_{n_j}$ and $M_j \cap M_i = N$ for all $1 \leq j,\ i \leq s$ and $j \neq i$.

Let $g \notin H$ and $N_2 = N^g$. For each $1 \leq j \leq s$, we have

$$M_j \cap K = (M_j \cap K) \cap K = (M_j \cap K) \cap (N \cap K)(N_2 \cap K) = (N \cap K)(M_j \cap N_2 \cap K)$$

and

$$M_j = N(K \cap M_j) = N(N \cap K)(M_j \cap N_2 \cap K) = N(M_j \cap N_2 \cap K) = N\langle x_j \rangle,$$

with $x_j \in M_j \cap N_2 \cap K$.

Using the permutational representation described in Remark 3.2, we may identify the elements (renumbering if necessary)

$$\begin{aligned} x_1 = {} & (1,\ m+1,\ 2m+1, \ldots, (n_1-1)m+1) \\ & (n_1 m+1,\ (n_1+1)m+1,\ (n_1+2)m+1, \ldots, (2n_1-1)m+1) \\ & (2n_1 m+1,\ (2n_1+1)m+1,\ (2n_1+2)m+1, \ldots, (3n_1-1)m+1) \cdots \\ & ((q_1-1)n_1 m+1,\ ((q_1-1)n_1+1)m+1,\ ((q_1-1)n_1+2)m+1, \\ & \qquad \ldots, (q_1 n_1 - 1)m+1)\, \mathcal{C}_{13} \cdots \mathcal{C}_{1m} \end{aligned}$$

and for $2 \leq j \leq s$,

$$\begin{aligned} x_j = {} & (1,\ t_j m+1,\ 2t_j m+1, \ldots, (n_j-1)t_j m+1) \\ & (m+1,\ (t_j+1)m+1,\ (2t_j+1)m+1, \ldots, ((n_j-1)t_j+1)m+1) \\ & (2m+1,\ (t_j+2)m+1,\ (2t_j+2)m+1, \ldots, ((n_j-1)t_j+2)m+1) \cdots \\ & ((q_j-1)m+1,\ (q_j+t_j-1)m+1,\ (2t_j+q_j-1)m+1, \\ & \qquad \ldots, ((n_j-1)t_j+q_j-1)m+1)\, \mathcal{C}_{j3} \cdots \mathcal{C}_{jm}, \end{aligned}$$

where $q_j = |H : M_j|$, $t_j = |M_1 M_2 \cdots M_{j-1} : N|$ and $\mathcal{C}_{ji}$ is a permutation of Δ_i for each $3 \leq i \leq m$.

For each $1 \leq i \leq m$, $2 \leq j \leq s$ and $1 \leq r \leq m-1$, consider the elements

$$\begin{aligned} P_{1i} = (1,\ & m+i,\ 2m+i, \ldots, (n_1-1)m+i) \\ & (n_1 m+i,\ (n_1+1)m+i,\ (n_1+2)m+i, \ldots, ((2n_1-1)m+i) \\ & (2n_1 m+i,\ (2n_1+1)m+i,\ (2n_1+2)m+i, \ldots, ((3n_1-1)m+i) \cdots \\ & ((q_1-1)n_1 m+i,\ ((q_1-1)n_1+1)m+i,\ ((q_1-1)n_1+2)m+i, \\ & \qquad \ldots, (q_1 n_1-1)m+i) \\ P_{ji} = (1,\ & t_j m+i,\ 2t_j m+i, \ldots, (n_j-1)t_j m+i) \\ & (m+i,\ (t_j+1)m+i,\ (2t_j+1)m+i, \ldots, ((n_j-1)t_j+1)m+i) \\ & (2m+i,\ (t_j+2)m+i,\ (2t_j+2)m+i, \ldots, ((n_j-1)t_j+2)m+i) \cdots \\ & ((q_j-1)m+i,\ (q_j+t_j-1)m+i,\ (2t_j+q_j-1)m+i, \\ & \qquad \ldots, ((n_j-1)t_j+q_j-1)m+i) \end{aligned}$$

and

$$\begin{aligned} g_r = {} & (r, r+1)(m+r, m+r+1)(2m+r, 2m+r+1) \\ & \ldots((n-1)m+r, (n-1)m+r+1). \end{aligned}$$

According to Proposition 4.4 we have $\mathfrak{G} = KL$ with $L = \langle g_1, g_2, \ldots, g_{m-1}\rangle$.

But $a_{j1} = x_j g_1 {x_j}^{-1} g_1 = P_{j1} P_{j2}^{-1} \in K$, and also for $T = (g_1\ g_2 \cdots g_{m-1})^{-1}$ we have $a_{j2} = T a_{j1} T^{-1} = P_{j2} P_{j3}^{-1}$, $a_{j3} = T^2 a_{j2} T^{-2} = P_{j3} P_{j4}^{-1}, \ldots, a_{jm} = T^{m-1} a_{jm-1} T^{-(m-1)} = P_{jm} P_{j1}^{-1}$. Hence, $a_{j1} a_{j2} a_{j3} \cdots a_{j\,m} = 1$, and the normal subgroup $\mathcal{Q}_j = \langle a_{j1}, a_{j2}, a_{j3}, \ldots, a_{j\,m-1}\rangle$ has order n_j^{m-1}. Applying the same arguments for all $1 \leq j \leq s$, we obtain a normal subgroup $\mathcal{Q} = \mathcal{Q}_1 \mathcal{Q}_2 \cdots \mathcal{Q}_s \leq K$ of order $(n_1 n_2 \cdots n_s)^{m-1} = n^{m-1}$.

Let $\{N_1, N_2, \ldots, N_m\}$ be the set of the m different conjugates of N in $\mathfrak{G}$. For the group homomorphism

$$\phi : K \to K/(N_1 \cap K) \times K/(N_2 \cap K) \times \cdots \times K/(N_m \cap K)$$

defined by $\phi(k) = (k(N_1 \cap K), k(N_2 \cap K), \ldots, k(N_m \cap K))$, we have

$$\ker(\phi) = N_1 \cap K \cap \cdots \cap N_m \cap K = N_1 \cap \cdots \cap N_m = \{1\},$$

and thus $K \cong \mathrm{Im}(\phi)$.

Suppose that

$$\begin{aligned} & (a_{11}(N_1 \cap K), (N_2 \cap K), \ldots, (N_m \cap K)) \\ & \qquad = \phi(k) = (k(N_1 \cap K), \ldots, k(N_m \cap K)) \in \mathrm{Im}(\phi). \end{aligned}$$

Then $k \in N_2 \cap \cdots \cap N_m$ and hence $k = C_1$ is a permutation of the set Δ_1 and $a_{11}(N_1 \cap K) = P_{11} P_{12}^{-1}(N_1 \cap K) = C_1(N_1 \cap K)$. Therefore $C_1 = P_{11}$ and $P_{12} \in N_2 \cap K$. In this way we obtain $\mathcal{P}_1 = \langle P_{11}, P_{12}, P_{13}, \ldots, P_{1m}\rangle \leq K$ and $\mathcal{P}_1 = \mathcal{Q}_1 \langle P_{11}\rangle$.

Consider, renumbering if necessary, $\{a_{11}, a_{21}, \ldots, a_{u1}\}$ such that $\mathcal{P}_j \leq K$ for $1 \leq j \leq u$ and $\{a_{u+11}, \ldots, a_{s1}\}$ such that $(a_{i1}(N_1 \cap K), (N_2 \cap K), \ldots, (N_m \cap K)) \notin \mathrm{Im}(\phi)$ for $u+1 \leq i \leq s$. Hence

$$\begin{aligned} K &= \mathcal{P}_1\langle P_{11}\rangle \mathcal{P}_2\langle P_{21}\rangle \cdots \mathcal{P}_u\langle P_{u1}\rangle \mathcal{Q}_{u+1}\cdots \mathcal{Q}_s \\ &= P_{11}P_{21}\cdots P_{u1}\mathcal{Q}_1\mathcal{Q}_2\cdots\mathcal{Q}_s = P_{11}P_{21}\cdots P_{u1}\mathcal{Q}. \end{aligned}$$

Since ${P_{j1}}^{-r} g_1 P_{j1}^r = {P_{j1}}^{-r}{P_{j2}}^r g_1 \in \langle P_{j1}{P_{j2}}^{-1}, g_1\rangle$, we have $\langle g_1\rangle^{\mathfrak{G}} = \langle g_1\rangle^{KL} = \langle g_1\rangle^{P_{11}P_{21}\cdots P_{u1}\mathcal{Q}L} \leq \mathcal{Q}L$, a contradiction. Therefore, for all j, i we have $(a_{ji}(N_1 \cap K), (N_2 \cap K), \ldots, (N_m \cap K)) \notin \mathrm{Im}(\phi)$. Therefore $K = \mathcal{Q}$ and $K \cong (H/N)^{m-1}$. □

We summarize the above results in the following:

Theorem 4.6. *Let $\mathfrak{G}$ be the Galois group of a factorized cover $\varphi\circ\psi$ with $\varphi\colon \mathcal{X} \to \mathbb{P}^1$ a simple cover of degree m and $\psi\colon \mathcal{Y} \to \mathcal{X}$ an unramified Galois cover of degree n, with abelian Galois group of type $(n_1, n_2, \ldots, n_s)$. Then*

$$\mathfrak{G} = (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_s})^{m-1} \rtimes \boldsymbol{S}_m.$$

Corollary 4.7. *Let $\mathcal{Z}$ be the Riemann surface associated to the Galois cover of $\varphi \circ \psi$. Then a geometric system of generators for the action of $\mathfrak{G}$ on $\mathcal{Z}$, as a transitive subgroup of $\boldsymbol{S}_{mn}$, is given by*

$$\Big\{g_1, g_1, \ldots, g_{m-1}, g_{m-1}, g_m, g_m, g_{m+1}, g_{m+1}, \ldots, g_{m+s-1}, g_{m+s-1}, \underbrace{g_1, g_1, \ldots, g_1, g_1}_{2(\mathrm{g}(\mathcal{X})-s)}\Big\}$$

where, for $1 \leq i \leq m-1$ and $2 \leq j \leq s$,

$$\begin{aligned} g_i &= (i,\ i+1)\,(m+i,\ m+i+1)\,(2m+i,\ 2m+i+1)(3m+i,\ 3m+i+1)\cdots \\ &\quad ((n_1n_2\cdots n_s-2)m+i,\ (n_1n_2\cdots n_s-2)m+i+1) \\ &\quad ((n_1n_2\cdots n_s-1)m+i,\ (n_1n_2\cdots n_s-1)m+i+1), \\ g_m &= x_1^{-1} g_1\, x_1, \\ g_{m+j-1} &= x_j^{-1} g_1\, x_j, \end{aligned}$$

with

$$\begin{aligned} x_1 = {} & (1, m+1, \ldots, (n_1-1)m+1)(n_1m+1, (n_1+1)m+1, \ldots, (2n_1-1)m+1)\cdots \\ & ((n_2\cdots n_s-2)n_1m+1, ((n_2\cdots n_s-2)n_1+1)m+1, \\ & \qquad \ldots, ((n_2\cdots n_s-1)n_1-1)m+1) \\ & ((n_2\cdots n_s-1)n_1m+1, ((n_2\cdots n_s-1)n_1+1)m+1, \cdots (n_1n_2\cdots n_s-1)m+1), \end{aligned}$$

and

$$\begin{aligned} x_j = {} & (1,\ t_jm+1,\ 2t_jm+1, \ldots, (n_j-1)t_jm+1) \\ & (m+1,\ (t_j+1)m+1,\ (2t_j+1)m+1, \ldots, ((n_j-1)t_j+1)m+1) \\ & (2m+1,\ (t_j+2)m+1,\ (2t_j+2)m+1, \ldots, ((n_j-1)t_j+2)m+1)\cdots \\ & ((q_j-1)m+1,\ (q_j+t_j-1)m+1,\ (2t_j+q_j-1)m+1, \\ & \qquad \ldots, ((n_j-1)t_j+q_j-1)m+1). \end{aligned}$$

Proof. Following the proof of Proposition 4.5, we have that the given set is a generator system and obviously satisfies the Riemann–Hurwitz equation (2.1) with branching data $(0;\underbrace{2,2,\dots,2,2}_{2(\mathrm{g}(\mathcal{X})+m-1)})$. Also, we have

$$\mathrm{g}(\mathcal{Z}) = \frac{n^{m-1}m!\,(\mathrm{g}(\mathcal{X})+m-3)}{2}+1.$$

Considering $N = \langle g_2,\dots,g_{m-1},g_m,g_{m+1},\dots,g_{m+s-1}\rangle$ and $H = N_{\mathfrak{G}}(N)$, we obtain $|\mathfrak{G}:H| = m$, $|H:N| = n$ and H/N is abelian of type $(n_1,n_2,\dots,n_s)$.

For each $1 \le i \le m+s-1$ let $G_i =< g_i >$. Since

$$|H:N|\,|H\backslash\mathfrak{G}/G_i| = |N\backslash\mathfrak{G}/G_i|,$$

we have that $\mathcal{Z}_N \to \mathcal{Z}_H$ is an unramified Galois cover, according to Proposition 3.3. Also, since

$$|H\backslash\mathfrak{G}/G_i| = m-1 \quad\text{and}\quad |N\backslash\mathfrak{G}/G_i| = n(m-1)$$

we have $\mathrm{g}(\mathcal{Z}_H) = \mathrm{g}(\mathcal{X})$ and $\mathrm{g}(\mathcal{Z}_N) = |H:N|(\mathrm{g}(\mathcal{X})-1) = n(\mathrm{g}(\mathcal{X})-1)$. □

References

[1] Allcock, D. and Hall, Ch.: Monodromy groups of Hurwitz–type problems. *Adv. Math.* **225** (2010), no. 1, 69–80.

[2] Arenas-Carmona, L. and Rojas, A. M.: Unramified prime covers of hyperelliptic curves and pairs of p-gonal curves. In *In the tradition of Ahlfors–Bers, V: The triennial Ahlfors–Bers Colloquium (Rutgers University, 2008),* 35–47. Contemp. Math. 510, American Mathematical Society, 2010.

[3] Artebani, M. and Pirola, G. P.: Algebraic functions with even monodromy. *Proc. Amer. Math. Soc.* **133** (2005), no. 2, 331–341

[4] Biggers, R. and Fried, M.: Irreducibility of moduli spaces of cyclic unramified covers of genus g curves. *Trans. Amer. Math. Soc.* **295** (1986), no. 1, 59–70.

[5] Broughton, S. A.: Classifying finite group actions on surfaces of low genus. *J. Pure Appl. Algebra* **69** (1991), no. 3, 233–270.

[6] Carocca, A., Lange, H., Rodríguez, R. E. and Rojas, A. M.: Products of Jacobians as Prym–Tyurin varieties. *Geom. Dedicata* **139** (2009), 219–231.

[7] Coxeter, H. S. M.: The groups determined by the relations $S^l = T^m = (S^{-1}T^{-1}ST)^p = 1$. Part I. *Duke Math. J.* **2** (1936), no. 1, 61–73.

[8] Diaz, S., Donagi, R. and Harbater, D.: Every curve is a Hurwitz space. *Duke Math. J.* **59** (1989), no. 3, 737–746.

[9] Fried, M.: Combinatorial computation of moduli dimension of Nielsen classes of covers. In *Graphs and algorithms (Boulder, CO, 1987),* 61–79. Contemp. Math. 89, Amer. Math. Soc., Providence, RI, 1989.

[10] Guralnick, R. M. and Neubauer, M. G.: Monodromy groups of branched coverings: the generic case. In *Recent developments in the inverse Galois problem (Seattle, WA, 1993),* 325–352. Contemp. Math. 186, Amer. Math. Soc., Providence, RI, 1995.

[11] Guralnick, R. M. and Shareshian, J.: *Symmetric and alternating groups as monodromy groups of Riemann surfaces. I. Generic covers and covers with many branch points.* Mem. Amer. Math. Soc. **189** (2007), no. 886.

[12] Kanev, V.: Hurwitz spaces of Galois coverings of $\mathbb{P}^1$, whose Galois groups are Weyl groups. *J. Algebra* **305** (2006), no. 1, 442–456.

[13] Kerber, A.: *Applied finite group actions.* Second edition. Algorithms and Combinatorics 19, Springer-Verlag, Berlin, 1999.

[14] Magaard, K. and Völklein, H.: The monodromy group of a function on a general curve. *Israel J. Math.* **141** (2004), 355–368.

[15] Recillas, S.: The Jacobian of the Galois extension of a trigonal curve. In *XXVIth National Congress of the Mexican Mathematical Society (Morelia, 1993),* 159–167. Aportaciones Mat. 14, Soc. Mat. Mexicana, México, 1994.

[16] Rojas, A. M.: Group actions on Jacobian varieties. *Rev. Mat. Iberoam.* **23** (2007), no. 2, 397–420.

[17] Vetro, F.: Irreducibility of Hurwitz spaces of coverings with monodromy groups Weyl groups of type $W(B_d)$. *Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8)* **10** (2007), no. 2, 405–431.

[18] Zariski, O.: *Collected Papers. Vol. III. Topology of curves and surfaces, and special topics in the theory of algebraic varieties.* Edited and with an introduction by M. Artin and B. Mazur. Mathematicians of Our Time, The MIT Press, Cambridge, Mass.-London, 1978.

Received August 30, 2016. Published online December 6, 2018.

Angel Carocca: Departamento de Matemática, Universidad de La Frontera, Casilla 54-D, 4780000 Temuco, Chile.
E-mail: angel.carocca@ufrontera.cl

Martha Romero Rojas: Departamento de Matemáticas, Universidad del Cauca, Calle 5 No. 4 - 70, Popayán Cauca 190003, Colombia.
E-mail: mjromero@unicauca.edu.co

The first author was supported by Fondecyt grant 1130445 and CONICYT PIA ACT 1415. The second author would like to thank Universidad del Cauca for support through Proyecto VRI 4188.