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Corrigendum: A coordinate-free proof of the finiteness principle for Whitney's extension problem

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The purpose of this note is to draw attention to a misleading remark in the introduction of [1]. In our discussion of Theorem 1.2 we make the following claim: "one may check that the constants λ_1 and λ_2 in Theorem 1.2 are harmless polynomial functions of D". Although we believe this to be true, the statement does not follow from the arguments of the paper. Several modifications are needed to obtain the claim, which we will now describe.

The first issue relates to the ineffective constant R_0 in Lemma 3.13 which arises due to the use of a compactness argument in the proof. This issue can be resolved, but the proofs are not as straightforward as we had once thought. In a forthcoming paper by the first three named authors, we give a direct geometric proof of Lemma 3.13 with the constant $R_0 = O(\exp(\text{poly}(D)))$. This is sufficient to obtain the polynomial dependence of λ_1 and λ_2, as claimed.

The second issue relates to an unfortunate typo appearing in Section 8 of the paper: in Lemma 8.7, the constant C should be replaced by $C \cdot C_{\text{old}}$; here, $C_{\text{old}} = C^{\#}(K-1)$, and C is a constant determined by m and n. When accounting for the missing factor of C_{old}, we find that a number of constants in Sections 8 and 9 which are claimed to depend only on m, n, actually depend also on the inductive parameter K. In Section 9.1 we claim that $C^{\#}(K)$ and $\ell^{\#}(K)$ have the form

$$C^{\#}(K) = C^K, \quad \ell^{\#}(K) = \chi \cdot K, \tag{0.1}$$

where C and χ are constants determined by m and n. The scaling (0.1) is responsible for the scaling $C^{\#} = \exp(\lambda_1 \mathcal{C}(E))$ and $k^{\#} = \exp(\lambda_2 \mathcal{C}(E))$ in Theorem 1.2; see Remark 5.7.

In the rest of this note, we will demonstrate that the scaling (0.1) is not ruined when we properly account for the factor of C_{old} in Lemma 8.7.

We first state an amended form of Lemma 8.1, which is the main result of Section 8 of [1]. We then explain why the amended Lemma 8.1 is sufficient to obtain the scaling (0.1). We finally discuss the proof of the amended Lemma 8.1.

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Lemma 1 (Amended Lemma 8.1 of [1]). *There exist constants $\overline{\chi} \geq 1$ and $\overline{C} \geq 1$, determined by m and n, such that the following holds. Let $\overline{\ell} = \ell_{\text{old}} + \overline{\chi}$. Suppose the parameter $\ell^{\#}$ in the Main Decomposition Lemma (Lemma 7.1 of [1]) is chosen so that $\ell^{\#} \geq \overline{\ell}$. Let $(P_B)_{B \in \mathcal{W}}$ be as in the outcome of the Main Decomposition Lemma. Then $P_B - P_{B'} \in \overline{C} \cdot C_{\text{old}} \cdot \mathcal{B}_{z_B, \text{diam}(B)}$ for any $B, B' \in \mathcal{W}_0$ with $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$.*

Comparing Lemma 1 to Lemma 8.1 of [1], the key difference is the factor of C_{old} in the estimates on $P_B - P_{B'}$. We additionally amend the statement that $\overline{\ell} - \ell_{\text{old}} = \overline{\chi}$ is a universal constant, rather than $\overline{\ell}$ itself. Recall here that $\ell_{\text{old}} = \ell^{\#}(K-1)$.

When we apply Lemma 1 in place of Lemma 8.1 in Section 9 of the paper, one change is required: in (9.2), the constant $\overline{C}$ is replaced by $\overline{C} \cdot C_{\text{old}}$. As before, condition (9.1) and this amended form of (9.2) imply that the $C^{m-1,1}$-seminorm of our interpolant $F = \sum_B F_B \theta_B$ is at most CC_{old}. This completes the proof of the Main lemma for K with $C^{\#} = C \cdot C_{\text{old}}$ and $\ell^{\#} = \ell_{\text{old}} + \overline{\chi}$. Since C and $\overline{\chi}$ depend only on m and n, we obtain (0.1) by induction on K.

We now give the details of the proof of Lemma 1. As we shall see, the proof is nearly identical to the proof of Lemma 8.1 from the original text.

We begin by stating the amended form of Lemma 8.7 with the corrected factor of C_{old}. We then state the corresponding amended form of Lemma 8.8. For details on the role of the constant A, see the definition of the keystone balls in Section 8.2.1. The proof of Lemma 2 below is identical to the proof of Lemma 8.7.

Lemma 2 (Amended Lemma 8.7 of [1]). *Let $B^{\#} \in \mathcal{W}$ be a keystone ball such that $AB^{\#} \subset 2B_0$. Let $\chi = \lceil \log(D \cdot (180A)^n + 1)/\log(D+1) \rceil$. There exists a constant $C_0 \geq 1$ determined by m and n, so that if $\ell \geq \ell_{\text{old}} + \chi$ then*

$$\sigma_\ell(x) \subset C_0 C_{\text{old}} \sigma(x, E \cap AB^{\#}) \quad \textit{for any } x \in AB^{\#}.$$

Lemma 3 (Amended Lemma 8.8 of [1]). *For an appropriate choice of A determined by m and n, for every keystone ball $B^{\#} \in \mathcal{W}$ with $AB^{\#} \subset 2B_0$,*

$$\sigma_\ell(z_{B^{\#}}) \cap V \subset C C_{\text{old}} \mathcal{B}_{z_{B^{\#}}, \text{diam}(B^{\#})},$$

for all $\ell \geq \ell_{\text{old}} + \chi$ as in Lemma 2; here, $\chi, C \geq 1$ are determined by m and n.

Proof. We follow the proof of Lemma 8.8, but we insert a factor of $\widetilde{C}_{\text{old}} := C_0 C_{\text{old}}$ on the ball $\mathcal{B}_{z_{B^{\#}}, A\,\text{diam}(B^{\#})}$ in equation (8.3). By Lemma 2, and Lemma 2.9 of [1],

$$\begin{aligned} \sigma_\ell(z_{B^{\#}}) \cap \widetilde{C}_{\text{old}} \mathcal{B}_{z_{B^{\#}}, A\,\text{diam}(B^{\#})} &\subset \widetilde{C}_{\text{old}} (\sigma(z_{B^{\#}}, E \cap AB^{\#}) \cap \mathcal{B}_{z_{B^{\#}}, A\,\text{diam}(B^{\#})}) \\ &\subset \widehat{C} \widetilde{C}_{\text{old}} \cdot \sigma(z_{B^{\#}}) \quad (\ell \geq \ell_{\text{old}} + \chi). \end{aligned} \tag{0.2}$$

Intersecting with V in (0.2) and applying the inclusion $\sigma(z_{B^{\#}}) \cap V \subset \widehat{R} \mathcal{B}_{z_{B^{\#}}, \text{diam}(B^{\#})}$ for $\widehat{R} = \widehat{R}(m,n)$ (see property (c) of the Main Decomposition Lemma),

$$\sigma_\ell(z_{B^{\#}}) \cap V \cap (\widetilde{C}_{\text{old}} \mathcal{B}_{z_{B^{\#}}, A\,\text{diam}(B^{\#})}) \subset \widehat{C} \widetilde{C}_{\text{old}} \widehat{R} \mathcal{B}_{z_{B^{\#}}, \text{diam}(B^{\#})}. \tag{0.3}$$

As $A\mathcal{B}_{z_{B^{\#}}, \text{diam}(B^{\#})} \subset \mathcal{B}_{z_{B^{\#}}, A\,\text{diam}(B^{\#})}$ for $A \geq 1$, if $A > \widehat{C}\widehat{R}$ then (0.3) yields $\sigma_\ell(z_{B^{\#}}) \cap V \subset \widehat{C} \widetilde{C}_{\text{old}} \widehat{R} \cdot \mathcal{B}_{z_{B^{\#}}, \text{diam}(B^{\#})}$. We take $A = 2\widehat{C}\widehat{R}$ to complete the proof. ∎

Rather than to give the amended Lemma 8.2, we will simply derive the required upper inclusion on $\sigma_\ell(x)$ that is needed for the proof of Lemma 1. The inclusion (0.4) below is valid under the following *small ball assumption*: there exists a ball $\widehat{B} \in \mathcal{W}_0$ with $\operatorname{diam}(\widehat{B}) \leq \epsilon^* \operatorname{diam}(B_0)$ (for $\epsilon^* = 1/3A^2$). This is the main case in the proof of Lemma 1; as before, the proof of Lemma 1 is easy in the complementary case. Under the *small ball assumption*, we will demonstrate that

$$\left\{\sigma_{\ell+1}(x) + \mathcal{B}_{z_B, \operatorname{diam}(B)}\right\} \cap V \subset C\, C_{\text{old}} \mathcal{B}_{z_B, \operatorname{diam}(B)} \tag{0.4}$$

for $x \in 3B$, $B \in \mathcal{W}_0$, $\ell \geq \ell_{\text{old}} + \chi$. Here, C is a constant determined by m and n.

Fix $B \in \mathcal{W}_0$ and $x \in 3B$. Recall that we associate to B a keystone ball $B^\# = \kappa(B)$ satisfying the geometric relations $B^\# \subset CB$, $AB^\# \subset 2B_0$, and $\operatorname{diam}(B^\#) \leq \operatorname{diam}(B)$. From these relations, since $z_B \in \frac{6}{5}B$ and $z_{B^\#} \in \frac{6}{5}B^\#$ (see the Main Decomposition Lemma), we conclude that $|z_B - z_{B^\#}| \leq C' \operatorname{diam}(B)$.

To prove (0.4) we follow the proof of Lemma 8.2 of [1] (see Section 8.2.2), but we use Lemma 3 in place of Lemma 8.8. Thus, in place of equation (8.4) we obtain

$$\sigma_\ell(z_{B^\#}) \cap V \subset C\, C_{\text{old}} \mathcal{B}_{z_{B^\#}, \operatorname{diam}(B)}. \tag{0.5}$$

We shall demonstrate that the additional factor of $C_{\text{old}} = C^\#(K-1)$ in the inclusion (0.5) is effectively harmless to the constants in our proofs.

In addition to (0.5), we state the matching lower inclusion on $\sigma_\ell(z_{B^\#})$. By an outcome of the Main Decomposition Lemma (see condition (c)), and using that $\sigma(z_{B^\#}) \subset \sigma_\ell(z_{B^\#})$,

$$\mathcal{B}_{z_{B^\#}, \operatorname{diam}(B)}/V \subset \widehat{R} \cdot (\sigma_\ell(z_{B^\#}) \cap \mathcal{B}_{z_{B^\#}, \operatorname{diam}(B)})/V, \quad \widehat{R} = \widehat{R}(m,n). \tag{0.6}$$

We will now shift the inclusions (0.5) and (0.6) from the basepoint $z_{B^\#}$ to the point $x \in 3B$. As in the original argument, we apply Lemma 2.6 of [1] to show

$$\sigma_{\ell+1}(x) + \mathcal{B}_{z_{B^\#}, \operatorname{diam}(B)} \subset \sigma_\ell(z_{B^\#}) + \widetilde{C} \cdot \mathcal{B}_{z_{B^\#}, \operatorname{diam}(B)}, \quad \widetilde{C} = \widetilde{C}(m,n). \tag{0.7}$$

In our original argument, we apply Lemma 3.3 (the "Stability I" estimates) to show that that the inclusions (0.5) and (0.6) are stable with respect to forming the Minkowski sum of $\sigma_\ell(z_{B^\#})$ with the ball $\mathcal{B}_{z_{B^\#}, \operatorname{diam}(B)}$ in $\mathcal{P}$. Unfortunately, a naïve application of this lemma degrades the constants in (0.5) and (0.6) to a quadratic dependence on C_{old}. We avoid this issue by use of the following variant of Lemma 3.3 which allows us to obtain improved estimates for the stability of inclusions with asymmetric constants.

Lemma 4 (Stability I'). *If Ω is a symmetric convex set in a Hilbert space X, and $V \subset X$ is a subspace, satisfying* (i) $\Omega \cap V \subset Z\mathcal{B}$, *and* (ii) $R^{-1}\mathcal{B}/V \subset (\Omega \cap \mathcal{B})/V$, *for constants $R, Z \geq 1$, then for any $\lambda \geq 1$,*

$$(\Omega + \lambda \mathcal{B}) \cap V \subset Z \cdot (3R\lambda + 1)\mathcal{B}.$$

Proof. We copy the proof of Lemma 3.3 in [1], with the obvious changes to account for the appearance of distinct constants Z and R in (i) and (ii). □

We apply the above lemma to the Hilbert space $X = (\mathcal{P}, \langle \cdot, \cdot \rangle_{z_{B\#}, \mathrm{diam}(B)})$. From the inclusions (0.5) and (0.6), we thus deduce that

$$\big(\sigma_\ell(z_{B\#}) + \widetilde{C}\mathcal{B}_{z_{B\#}, \mathrm{diam}(B)}\big) \cap V \subset C\, C_{\mathrm{old}} \cdot (3\widehat{R}\widetilde{C} + 1)\, \mathcal{B}_{z_{B\#}, \mathrm{diam}(B)}.$$

From (0.7), we then obtain

$$\big(\sigma_{\ell+1}(x) \cap \mathcal{B}_{z_{B\#}, \mathrm{diam}(B)}\big) \cap V \subset \widehat{C}\, C_{\mathrm{old}} \cdot \mathcal{B}_{z_{B\#}, \mathrm{diam}(B)}, \quad \widehat{C} = \widehat{C}(m,n).$$

Since $\mathcal{B}_{z_{B\#}, \mathrm{diam}(B)}$ and $\mathcal{B}_{z_B, \mathrm{diam}(B)}$ are comparable up to constant factors (recall that $|z_B - z_{B\#}| \le C\,\mathrm{diam}(B)$; see equation (2.4) of [1]), we obtain (0.4).

0.1. Proof of Lemma 1

We adapt the proof of Lemma 8.1 of [1], but we replace the use of Lemma 8.2 by an application of (0.4). As in the previous argument, without loss of generality we can make the *small ball assumption* (as before, the proof in the complementary case is essentially trivial). Let $B, B' \in \mathcal{W}_0$ with $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$. As an outcome of the Main Decomposition Lemma we obtain that

$$P_B - P_{B'} \in C'' \cdot (\sigma_{\ell^\# - 2}(z_B) + \mathcal{B}_{z_B, \mathrm{diam}(B)}) \cap V. \tag{0.8}$$

(See the fourth and fifth inline equations in the proof of Lemma 8.1.) Next we apply (0.4) with $\ell = \ell_{\mathrm{old}} + \chi$ and $x = z_B$. If $\ell^\# - 2 \ge \ell + 1$ then $\sigma_{\ell^\# - 2}(z_B) \subset \sigma_{\ell+1}(z_B)$, so (0.4) and (0.8) imply that

$$P_B - P_{B'} \in C'' \cdot C\, C_{\mathrm{old}}\, \mathcal{B}_{z_B, \mathrm{diam}(B)}. \tag{0.9}$$

Thus, $P_B - P_{B'} \in \overline{C}\, C_{\mathrm{old}} \mathcal{B}_{z_B, \mathrm{diam}(B)}$ for $\ell^\# \ge \ell_{\mathrm{old}} + \overline{\chi}$, where the constants $\overline{\chi} = \chi + 3$ and $\overline{C}$ depend only on m and n. This finishes the proof of Lemma 1.

References

[1] Carruth, J., Frei-Pearson, A., Israel, A. and Klartag, B.: A coordinate-free proof of the finiteness principle for Whitney's extension problem. *Rev. Mat. Iberoam.* **36** (2020), no. 7, 1917–1956.

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