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On smooth Fano fourfolds of Picard number two

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Abstract. We classify the smooth Fano 4-folds of Picard number two that have a general hypersurface Cox ring.

1. Introduction

By a Fano variety, we mean a normal projective complex variety with an ample anticanonical divisor. Our aim is to contribute to the explicit classification of smooth Fano varieties. In dimension two, these are the well known smooth del Pezzo surfaces. The smooth Fano threefolds have been classified by Iskovskih [26, 27] and Mori–Mukai [34, 35]. From dimension four on the classification problem is widely open in general, but there are trendsetting partial results, such as Batyrev's classification of the smooth toric Fano fourfoulds [3].

In the present article, we focus on the case of Picard number at most two. In this situation, all smooth Fano varieties coming with a torus action of complexity at most one are known [19] and in [21] one finds a natural extension to complexity two. As in [19, 21], our approach goes via the Cox ring. Recall that for any normal projective variety X with finitely generated divisor class group $\mathrm{Cl}(X)$, the Cox ring is defined as

$$\mathcal{R}(X) = \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)).$$

In case of a smooth Fano variety X, the Cox ring is known to be a finitely generated $\mathbb{C}$-algebra [6]. We restrict our attention to simply structured Cox rings: we say that a variety X with divisor class group $\mathrm{Cl}(X) = K$ has a *hypersurface Cox ring* if we have a K-graded presentation

$$\mathcal{R}(X) = R_g = \mathbb{C}[T_1, \dots, T_r]/\langle g \rangle,$$

where g is homogeneous of degree $\mu \in K$ and $T_1, \dots, T_r$ define a minimal system of K-homogeneous generators. In this situation, we call $\mathcal{R}(X)$ *spread* if each monomial of degree μ is a convex combination of monomials of g. Moreover, we call $\mathcal{R}(X)$ *general* (*smooth, Fano*) if g admits an open neighbourhood U in the vector space of all

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μ-homogeneous polynomials such that every $h \in U$ yields a hypersurface Cox ring R_h of a normal (smooth, Fano) variety X_h with divisor class group K; see also Definition 4.5.

Among the del Pezzo surfaces, there are no smooth ones with a hypersurface Cox ring, but in the singular case, we encounter many examples [14]. The first examples of smooth Fano varieties with a hypersurface Cox ring show up in dimension three; see Theorems 4.1 and 4.5 in [15], where, based on the classifications mentioned before, the Cox rings of the smooth Fano threefolds of Picard numbers one and two have been computed. For the smooth Fano fourfolds of Picard number one having a hypersurface Cox ring, we refer to numbers 2, 4, 5 and 8 in Küchle's list (Proposition 2.2.1 in [31]) in case of Fano index one, and in case of higher Fano index, to the numbers 14, 15, 18, 19, 20 and 22 in the list of Przyjalkowski and Shramov [36], p. 12.

We approach our main result, concerning Fano fourfolds of Picard number two. The notation is as follows. For any hypersurface Cox ring $\mathcal{R}(X) = R_g$ graded by $\mathrm{Cl}(X) = K$, we write $w_i = \deg(T_i) \in K$ for the generator degrees and $\mu = \deg(g) \in K$ for the degree of the relation. Moreover, in this setting, the anticanonical class of X is given by

$$-\mathcal{K} = w_1 + \cdots + w_r - \mu \in \mathrm{Cl}(X) = K.$$

If R_g is the Cox ring of a Fano variety X, then X can be reconstructed as the GIT quotient of the set of $(-\mathcal{K})$-semistable points of $\operatorname{Spec} R_g$ by the quasitorus $\operatorname{Spec} \mathbb{C}[K]$. In this setting, we refer to the Cox ring generator degrees $w_1, \dots, w_r \in K$ and the relation degree $\mu \in K$ as the *specifying data of the Fano variety* X. In the case of a smooth Fano fourfould X of Picard number two, $\mathrm{Cl}(X)$ equals $\mathbb{Z}^2$ and thus $\operatorname{Spec} \mathbb{C}[K]$ is a two-dimensional torus. Hence the hypersurface Cox ring R_g is of dimension six and has seven generators.

Theorem 1.1. *The following table lists the specifying data $w_1, \dots, w_7$ and μ in $\mathrm{Cl}(X) = \mathbb{Z}^2$, the anticanonical class $-\mathcal{K}$ and $\mathcal{K}^4$ for all smooth Fano fourfolds of Picard number two with a spread hypersurface Cox ring.*

No.	$[w_1, \dots, w_7]$	$\deg(g)$	$-\mathcal{K}$	$\mathcal{K}^4$
1	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	(1, 1)	(3, 2)	432
2		(2, 1)	(2, 2)	256
3		(3, 1)	(1, 2)	80
4		(1, 2)	(3, 1)	270
5		(2, 2)	(2, 1)	112
6		(3, 2)	(1, 1)	26
7	$\begin{bmatrix} 1&1&1&1&0&0&-1 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	(1, 1)	(2, 2)	416
8		(1, 2)	(2, 1)	163
9		(2, 1)	(1, 2)	224
10		(2, 2)	(1, 1)	52
11	$\begin{bmatrix} 1&1&1&1&0&0&-2 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	(1, 1)	(1, 2)	464
12		(1, 2)	(1, 1)	98
13	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	(1, 2)	(3, 2)	352
14		(2, 3)	(2, 1)	65
15	$\begin{bmatrix} 1&1&1&1&0&0&-1 \\ 0&0&0&1&1&1&1 \end{bmatrix}$	(1, 3)	(2, 1)	83
16	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ 0&0&0&0&1&1&1 \end{bmatrix}$	(2, 1)	(3, 2)	352
17		(3, 2)	(2, 1)	81
18	$\begin{bmatrix} 1&1&1&1&0&0&0 \\ -1&0&0&0&1&1&1 \end{bmatrix}$	(3, 1)	(1, 1)	38
19		(2, 1)	(2, 1)	192
20		(1, 1)	(3, 1)	432
21	$\begin{bmatrix} 1&1&1&1&1&0&0 \\ -1&0&0&0&1&1&1 \end{bmatrix}$	(3, 1)	(2, 1)	113

No.	$[w_1, \ldots, w_7]$	$\deg(g)$	$-\mathcal{K}$	$\mathcal{K}^4$
22	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(2,3)$	272
23		$(3,3)$	$(1,2)$	51
24	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}$	$(4,4)$	$(1,2)$	34
25	$\begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 & 1 & 1 \end{bmatrix}$	$(6,6)$	$(1,2)$	17
26	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(1,3)$	216
27	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,4)$	$(1,2)$	64
28	$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,6)$	$(1,1)$	8
29	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(2,2)$	192
30		$(3,3)$	$(1,1)$	18
31	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(4,2)$	$(1,2)$	48
32	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 1 & 1 \end{bmatrix}$	$(4,4)$	$(1,1)$	12
33	$\begin{bmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & 2 & 1 & 1 & 1 \end{bmatrix}$	$(4,6)$	$(1,3)$	50
34	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(3,4)$	378
35		$(3,3)$	$(2,3)$	144
36		$(4,4)$	$(1,2)$	20
37	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$	$(4,4)$	$(2,3)$	96
38	$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 1 & 1 & 1 & 3 & 1 & 1 \end{bmatrix}$	$(6,6)$	$(1,2)$	10
39	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 0 & 0 \\ 0 & 1 & 1 & 2 & 3 & 1 & 1 \end{bmatrix}$	$(6,6)$	$(2,3)$	48
40	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(2,4)$	352
41		$(3,3)$	$(1,3)$	99
42	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 & 1 & 1 & 1 \end{bmatrix}$	$(2,4)$	$(2,5)$	304
43		$(3,6)$	$(1,3)$	54
44	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}$	$(4,4)$	$(1,3)$	66
45	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 2 & 2 & 4 & 1 & 1 & 1 \end{bmatrix}$	$(4,8)$	$(1,3)$	36
46	$\begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 1 & 2 & 3 & 1 & 1 & 1 \end{bmatrix}$	$(6,6)$	$(1,3)$	33

No.	$[w_1, \ldots, w_7]$	$\deg(g)$	$-\mathcal{K}$	$\mathcal{K}^4$
47	$\begin{bmatrix} 1 & 1 & 2 & 3 & 0 & 0 & 0 \\ 0 & 2 & 4 & 6 & 1 & 1 & 1 \end{bmatrix}$	$(6,12)$	$(1,3)$	18
48	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}$	$(2,2)$	$(3,5)$	433
49	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 2 & 2 & 3 & 1 & 1 \end{bmatrix}$	$(3,6)$	$(2,5)$	145
50	$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 1 & 1 \end{bmatrix}$	$(2,4)$	$(2,3)$	144
51	$\begin{bmatrix} 1 & 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 3 & 1 & 1 & 1 \end{bmatrix}$	$(4,6)$	$(1,2)$	22
52	$\begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 1 & 3 & 2 & 1 & 1 \end{bmatrix}$	$(4,6)$	$(2,3)$	65
53	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(2,0)$	$(4,1)$	431
54		$(4,0)$	$(2,1)$	62
55	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(3,0)$	$(4,1)$	376
56	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 3 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(4,0)$	$(4,1)$	341
57	$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(6,0)$	$(2,1)$	31
58	$\begin{bmatrix} 1 & 1 & 1 & 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(6,0)$	$(1,2)$	16
59	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(6,0)$	$(2,2)$	64
60	$\begin{bmatrix} 1 & 1 & 1 & 2 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(6,0)$	$(3,2)$	80
61	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(4,0)$	$(2,2)$	128
62	$\begin{bmatrix} 1 & 1 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(4,0)$	$(3,2)$	160
63	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(3,0)$	$(2,2)$	192
64	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(3,0)$	$(3,2)$	240
65	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(2,0)$	$(3,2)$	432
66	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(2,0)$	$(4,2)$	480
67	$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$	$(2,0)$	$(5,2)$	624

Any two smooth Fano fourfolds of Picard number two with specifying data from distinct items of the table are not isomorphic to each other. Moreover, each of the items 1 *to* 67 *even defines a general smooth Fano hypersurface Cox ring and thus provides the specifying data for a whole family of smooth Fano fourfolds.*

Let us compare the result with existing classifications. Wiśniewski classified in [39] the smooth Fano fourfolds of Picard number and Fano index at least two, where the Fano index is the largest integer ι such that $-\mathcal{K} = \iota H$ holds with an ample divisor H.

Remark 1.2. In eight cases, the families listed in Theorem 1.1 consist of varieties of Fano index two and in all other cases, the varieties are of Fano index one. The conversion between Theorem 1.1 and Wiśniewski's results as presented in Table 12.7 of [37] is as follows:

Theorem 1.1	2	7	29	40	59	61	63	66
Table 12.7 in [37]	5	12	4	10	1	2	3	13

Theorem 1.1 has no overlap with Batyrev's classification [3] of smooth toric Fano fourfolds. Indeed, toric varieties have polynomial rings as Cox rings which are by definition no hypersurface Cox rings. However, there is some interaction with the case of torus actions of complexity one.

Remark 1.3. Eleven of the families of Theorem 1.1 admit small degenerations to smooth Fano fourfolds with an effective action of a three-dimensional torus. Here are these families and the corresponding varieties from Theorem 1.2 in [19].

Theorem 1.1	Theorem 1.2 in [19]
1	4.A: $m = 1, c = 0$
4	4.C: $m = 1$
7	2
13	5: $m = 1$
20	4.A: $m = 1, c = -1$
34	1
48	10: $m = 2$
53	7: $m = 1$
65	12: $m = 2, a = b = c = 0$
66	11: $m = 2, a_2 = 1$
67	11: $m = 2, a_2 = 2$

Moreover, observe that for the families 1, 20, 48, 53, 65, 66 and 67 of Theorem 1.1, the degeneration process gives a Fano smooth intrinsic quadric; compare [18], Theorem 1.3.

Remark 1.4. Coates, Kasprzyk and Prince classified in [10] the smooth Fano fourfolds that arise as complete intersections of ample divisors in smooth toric Fano varieties of dimension at most eight. Comparing anticanonical self-intersection numbers as well as the first six coefficients of the Hilbert series yields that at least the 17 families 14, 15, 24, 25, 28, 30, 32, 33, 38, 44, 45, 46, 47, 51, 52, 57 and 58 of Theorem 1.1 do not show up in [10].

In Sections 6 to 8, we investigate the geometry of the Fano varieties from Theorem 1.1. We take a look at the elementary contractions, see Proposition 6.2, we determine the Hodge numbers, see Propositions 7.1 and 7.2, and in many cases the infinitesimal deformations, see Corollary 8.2.

2. Factorial gradings

Here we provide the first part of the algebraic and combinatorial tools used in our classification. We recall the basic concepts on factorially graded algebras and, as a new result, prove Proposition 2.4, locating the relation degrees of a factorially graded complete intersection algebra. Moreover, we recall and discuss the GIT-fan of the quasitorus action associated with a graded affine algebra.

For the moment, $\mathbb{K}$ is any field. Let R be a *K-graded algebra*, which, in this article, means that K is a finitely generated abelian group and R is a $\mathbb{K}$-algebra coming with a direct sum decomposition into $\mathbb{K}$-vector subspaces

$$R = \bigoplus_{w \in K} R_w$$

such that $R_w R_{w'} \subseteq R_{w+w'}$ holds for all $w, w' \in R$. An element $f \in R$ is *homogeneous* if $f \in R_w$ holds for some $w \in K$; in that case, w is the *degree* of f, written $w = \deg(f)$. We say that R is *K-integral* if it has no homogeneous zero divisors.

Consider the rational vector space $K_{\mathbb{Q}} := K \otimes_{\mathbb{Z}} \mathbb{Q}$ associated with K. The *effective cone* of R is the convex cone generated by all degrees admitting a non-zero homogeneous element:

$$\mathrm{Eff}(R) := \mathrm{cone}(w \in K;\ R_w \neq 0) \subseteq K_{\mathbb{Q}}.$$

The K-grading of R is called *pointed* if $R_0 = \mathbb{K}$ holds and the effective cone $\mathrm{Eff}(R)$ contains no line. Note that $\mathrm{Eff}(R)$ is polyhedral, if the $\mathbb{K}$-algebra R is finitely generated.

Lemma 2.1. *Let R be a K-graded algebra. Assume that R is K-integral and every homogeneous unit of R is of degree zero.*

(i) *If $R_0 = \mathbb{K}$ holds, then the K-grading is pointed, and for every non-zero torsion element $w \in K$, we have $R_w = 0$.*

(ii) *The K-grading is pointed if and only if there is a homomorphism $\kappa\colon K \to \mathbb{Z}$ defining a pointed $\mathbb{Z}$-grading with effective cone $\mathbb{Q}_{\geq 0}$.*

Proof. We prove (i). It suffices to show that there is no non-zero $w \in K$ with $R_w \neq 0$ and $R_{-w} \neq 0$. Consider $f \in R_w$ and $f' \in R_{-w}$, both being non-zero. Then ff' is a non-zero element of R_0 and hence constant. Thus, f and f' are both units. By assumption, we have $w = 0$.

We prove (ii). If the K-grading is pointed, then we find a hyperplane $U \subseteq K_{\mathbb{Q}}$ intersecting $\mathrm{Eff}(X)$ precisely in the origin. Let $K_U \subseteq K$ be the subgroup consisting of all elements $w \in K$ with $w \otimes 1 \in U$. Then $K/K_U \cong \mathbb{Z}$ holds and we may assume that the projection $\kappa\colon K \to \mathbb{Z}$ sends the effective cone to the positive ray. Using (i), we see that for the induced $\mathbb{Z}$-grading all homogeneous elements of degree zero are constant. The reverse implication is clear according to (i). ∎

Let R be a K-integral algebra. A homogeneous non-zero non-unit $f \in R$ is *K-irreducible*, if admits no decomposition $f = f'f''$ with homogeneous non-zero non-units $f', f'' \in R$. A homogeneous non-zero non-unit $f \in R$ is *K-prime*, if for any two homogeneous $f', f'' \in R$ we have that $f \mid f'f''$ implies $f \mid f'$ or $f \mid f''$. Every K-prime element is K-irreducible. The algebra R is called *K-factorial*, or the K-grading just *factorial*, if R is K-integral and every homogeneous non-zero non-unit is a product of K-primes. In a K-factorial algebra, the K-prime elements are exactly the K-irreducible ones.

An ideal $\mathfrak{a} \subseteq R$ is *homogeneous* if it is generated by homogeneous elements. Moreover, an ideal $\mathfrak{a} \subseteq R$ is *K-prime* if for any two homogeneous $f, f' \in R$ we have that $ff' \in \mathfrak{a}$ implies $f \in \mathfrak{a}$ or $f' \in \mathfrak{a}$. A homogeneous ideal $\mathfrak{a} \subseteq R$ is K-prime if and only if $R/\mathfrak{a}$ is

K-integral. We say that homogeneous elements $g_1, \dots, g_s \in R$ *minimally generate* the K-homogeneous ideal $\mathfrak{a} \subseteq R$ if they generate $\mathfrak{a}$ and no proper subcollection of $g_1, \dots, g_s$ does so.

Lemma 2.2. *Let R be a K-graded algebra such that the grading is pointed, factorial and every homogeneous unit is of degree zero. If $g_1, \dots, g_s \in R$ minimally generate a K-prime ideal of R, then each g_i is a K-prime element of R.*

Proof. Assume that g_1 is not K-prime. Then g_1 is not K-irreducible and we can write $g_1 = g_1' g_1''$, with homogeneous non-zero non-units $g_1', g_1'' \in R$. As the ideal $\langle g_1, \dots, g_s \rangle \subseteq R$ is K-prime, it contains one of g_1' and g_1'', say g_1'. That means that

$$g_1' = h_1 g_1 + \dots + h_s g_s$$

holds with homogeneous elements $h_i \in R$. Take a coarsening $K \to \mathbb{Z}$ of the K-grading as provided by Lemma 2.1 (ii). Then the above representation of g_1' yields

$$\deg_{\mathbb{Z}}(g_1') = \deg_{\mathbb{Z}}(h_1) + \deg_{\mathbb{Z}}(g_1) = \dots = \deg_{\mathbb{Z}}(h_s) + \deg_{\mathbb{Z}}(g_s).$$

Consequently, $\deg_{\mathbb{Z}}(g_1') \geq \deg_{\mathbb{Z}}(g_1)$ or $h_1 = 0$. Since the $\mathbb{Z}$-grading of R is pointed, we have $\deg_{\mathbb{Z}}(g_1') < \deg_{\mathbb{Z}}(g_1') + \deg_{\mathbb{Z}}(g_1'') = \deg_{\mathbb{Z}}(g_1)$. Thus, $h_1 = 0$ holds. This implies $g_1 = g_1' g_1'' \in \langle g_2, \dots, g_s \rangle$. A contradiction. ∎

Given a finitely generated abelian group K and $w_1, \dots, w_r \in K$, there is a unique K-grading on the polynomial algebra $\mathbb{K}[T_1, \dots, T_r]$ satisfying $\deg(T_i) = w_i$ for $i = 1, \dots, r$. We call such grading a *linear* grading of $\mathbb{K}[T_1, \dots, T_r]$.

Lemma 2.3. *Consider a linear K-grading on $\mathbb{K}[T_1, \dots, T_r]$ and a K-homogeneous $g \in \mathbb{K}[T_1, \dots, T_r]$. Moreover, let $1 \leq i_1, \dots, i_q \leq r$ be pairwise distinct. Assume that T_{i_1} is not a monomial of g and that $g, T_{i_2}, \dots, T_{i_q}$ minimally generate a K-prime ideal in $\mathbb{K}[T_1, \dots, T_r]$. Then we have a presentation*

$$\deg(g) = \sum a_j \deg(T_j), \quad j \neq i_1, \dots, i_q, \ a_j \in \mathbb{Z}_{\geq 0}.$$

Proof. Suppose that $\deg(g)$ allows no representation as a positive combination over the $\deg(T_j)$ with $j \notin \{i_1, \dots, i_q\}$. Then each monomial of g must have a factor T_{i_j} for some $j = 1, \dots, q$. Write

$$g = g_1 T_{i_1} + g_2 T_{i_2} + \dots + g_q T_{i_q} = g_1 T_{i_1} + h,$$

with polynomials $g_j \in \mathbb{K}[T_1, \dots, T_r]$ such that g_1 depends on none of $T_{i_2}, \dots, T_{i_q}$. By assumption, $g_1 T_{i_1}$ is non-zero and we have a K-integral factor ring

$$\mathbb{K}[T_1, \dots, T_r] / \langle g, T_{i_2}, \dots, T_{i_q} \rangle \cong \mathbb{K}[T_j;\ j \neq i_2, \dots, i_r] / \langle g_1 T_{i_1} \rangle.$$

Consequently, $g_1 T_{i_1}$ is a K-prime polynomial. This implies $g_1 = c \in \mathbb{K}^*$ and thus we arrive at $g = c T_{i_1} + h$; a contradiction to the assumption that T_{i_1} is not a monomial of g. ∎

If R is a finitely generated K-graded algebra, then R admits homogeneous generators $f_1, \ldots, f_r$. Turning the polynomial ring $\mathbb{K}[T_1, \ldots, T_r]$ into a K-graded algebra via $\deg(T_i) := \deg(f_i)$, we obtain an epimorphism of K-graded algebras:

$$\pi \colon \mathbb{K}[T_1, \ldots, T_r] \to R, \quad T_i \mapsto f_i.$$

Together with a choice of K-homogeneous generators $g_1, \ldots, g_s$ for the ideal $\ker(\pi)$, we arrive at *K-graded presentation* of R by homogeneous generators and relations:

$$R = \mathbb{K}[T_1, \ldots, T_r] / \langle g_1, \ldots, g_s \rangle.$$

We call such presentation *irredundant* if $\ker(\pi)$ contains no elements of the form $T_i - h_i$ with $h_i \in \mathbb{K}[T_1, \ldots, T_r]$ not depending on T_i.

Proposition 2.4. *Let R be a finitely generated K-graded algebra such that the grading is pointed, factorial and every homogeneous unit is of degree zero. Let*

$$R = \mathbb{K}[T_1, \ldots, T_r] / \langle g_1, \ldots, g_s \rangle$$

be an irredundant K-graded presentation with $\dim(R) = r - s$ *such that $T_1, \ldots, T_r$ define K-prime elements in R. Then, for every $l = 1, \ldots, s$, we have*

$$\deg(g_l) \in \bigcap_{1 \le i < j \le r} \operatorname{cone}(\deg(T_k);\ k \neq i,\ k \neq j) \subseteq K_{\mathbb{Q}}.$$

Proof. It suffices to show that for any two $1 \le i < j \le r$, we can represent each $\deg(g_l)$ as a positive combination over the $\deg(T_k)$, where $k \neq i, j$. For $l = 1, \ldots, s$, set

$$g_{l,j} := g_l(T_1, \ldots, T_{j-1}, 0, T_{j+1}, \ldots, T_r) \in \mathbb{K}[T_1, \ldots, T_r].$$

Since T_j defines a K-prime element in R, the ideal $\langle T_j \rangle \subseteq R$ is K-prime and $\langle T_j \rangle$ lifts to a K-prime ideal

$$I_j := \langle g_1, \ldots, g_s, T_j \rangle = \langle g_{1,j}, \ldots, g_{s,j}, T_j \rangle \subseteq \mathbb{K}[T_1, \ldots, T_r].$$

Then $\mathbb{K}[T_1, \ldots, T_r]/I_j$ is isomorphic to $R/\langle T_j \rangle$. The latter algebra is of dimension $r - s - 1$ due to our assumptions. Thus, $g_{1,j}, \ldots, g_{s,j}, T_j$ minimally generate I_j. By Lemma 2.2, each $g_{l,j}$ is K-prime and hence defines a K-integral factor algebra

$$\mathbb{K}[T_m;\ m \neq j] / \langle g_{l,j} \rangle \cong \mathbb{K}[T_1, \ldots, T_r] / \langle g_l, T_j \rangle.$$

We conclude that g_l, T_j minimally generate a K-prime ideal in $\mathbb{K}[T_1, \ldots, T_r]$. Thus, we may apply Lemma 2.3 and obtain the assertion. ∎

We turn to the geometric point of view. So, $\mathbb{K}$ is now algebraically closed of characteristic zero and R an affine K-graded algebra, where affine means that R is finitely generated over $\mathbb{K}$ and has no nilpotent elements. Then we have the affine variety $\bar{X}$ with R as its algebra of global functions and the quasitorus H with K as its character group:

$$\bar{X} = \operatorname{Spec} R, \quad H = \operatorname{Spec} \mathbb{K}[K].$$

The K-grading of R defines an action of H on $\bar{X}$, which is uniquely determined by the property that each $f \in R_w$ satisfies $f(h \cdot x) = \chi^w(h) f(x)$ for all $x \in \bar{X}$ and $h \in H$, where χ^w is the character corresponding to $w \in K$. We take a look at the geometric invariant theory of the H-action on $\bar{X}$; see [2, 4]. The *orbit cone* $\omega_x \subseteq K_{\mathbb{Q}}$ associated with $x \in \bar{X}$ and the *GIT-cone* $\lambda_w \subseteq K_{\mathbb{Q}}$ associated with $w \in \mathrm{Eff}(R)$ are defined as

$$\omega_x = \mathrm{cone}(w \in K;\ f(x) \neq 0 \text{ for some } f \in R_w), \qquad \lambda_w := \bigcap_{x \in \bar{X}, w \in \omega_x} \omega_x.$$

Orbit cones as well as GIT-cones are convex polyhedral cones and there are only finitely many of them. The basic observation is that the GIT-cones form a fan $\Lambda(R)$ in $K_{\mathbb{Q}}$, the *GIT-fan*, having the effective cone $\mathrm{Eff}(R)$ as its support.

Remark 2.5. Let K be a finitely generated abelian group and R a K-integral affine algebra. Fix a K-graded presentation

$$R = \mathbb{K}[T_1, \dots, T_r] / \langle g_1, \dots, g_s \rangle.$$

This yields an H-equivariant closed embedding $\bar{X} = V(g_1, \dots, g_s) \subseteq \mathbb{K}^r$ of affine varieties. Moreover, we have a homomorphism

$$Q : \mathbb{Z}^r \to K, \quad \nu \mapsto \nu_1 \deg(T_1) + \dots + \nu_r \deg(T_r).$$

An $\bar{X}$*-face* is a face $\gamma_0 \preceq \gamma$ of the orthant $\gamma := \mathbb{Q}^r_{\geq 0}$ admitting a point $x \in \bar{X}$ such that one has

$$x_i \neq 0 \iff e_i \in \gamma_0$$

for the coordinates $x_1, \dots, x_r$ of x and the canonical basis vectors $e_1, \dots, e_r \in \mathbb{Z}^r$. Write $\mathfrak{S}(\bar{X})$ for the set of all $\bar{X}$-faces of $\gamma \subseteq \mathbb{Q}^r$. Then we have

$$\{Q(\gamma_0);\ \gamma_0 \in \mathfrak{S}(\bar{X})\} = \{\omega_x;\ x \in \bar{X}\}.$$

That means that the projected $\bar{X}$-faces are exactly the orbit cones. The $\bar{X}$-faces define a decomposition into locally closed subsets,

$$\bar{X} = \bigcup_{\gamma_0 \in \mathfrak{S}(X)} \bar{X}(\gamma_0), \qquad \bar{X}(\gamma_0) := \{x \in \bar{X};\ x_i \neq 0 \Leftrightarrow e_i \in \gamma_0\} \subseteq \bar{X}.$$

Definition 2.6. Let $I = \{i_1, \dots, i_k\}$ be a subset of $\{1, \dots, r\}$. Then the face γ_I of the orthant $\gamma = \mathbb{Q}^r_{\geq 0}$ associated with I is defined as

$$\gamma_I := \gamma_{i_1, \dots, i_k} := \mathrm{cone}(e_{i_1}, \dots, e_{i_k}).$$

Moreover, for a polynomial $g \in \mathbb{K}[T_1, \dots, T_r]$, the polynomial g_I associated with I is defined as

$$g_I := g(\tilde{T}_1, \dots, \tilde{T}_r), \qquad \tilde{T}_i := \begin{cases} T_i, & i \in I, \\ 0, & i \notin I. \end{cases}$$

Remark 2.7. In the setting of Remark 2.5, let $I = \{i_1, \dots, i_k\}$ be a subset of $\{1, \dots, r\}$.

(i) γ_I is an $\bar{X}$-face if and only if $\langle g_{1,I}, \dots, g_{s,I} \rangle$ contains no monomial.

(ii) If $\deg(g_j) \notin \mathrm{cone}(w_i;\, i \in I)$ holds for $j = 1, \ldots, s$, then γ_I is an $\bar{X}$-face.

(iii) If $(w_i;\, i \in I)$ is linearly independent in K, then γ_I is an $\bar{X}$-face if and only if none of $g_1, \ldots, g_s$ has a monomial $T_{i_1}^{l_1} \cdots T_{i_k}^{l_k}$ with $l_1, \ldots, l_k \in \mathbb{Z}_{\geq 0}$.

Proposition 2.8. *Let K be a finitely generated abelian group and R an affine algebra with a pointed K-grading. Consider a K-graded presentation*

$$R = \mathbb{K}[T_1, \ldots, T_r] / \langle g_1, \ldots, g_s \rangle$$

such that $T_1, \ldots, T_r$ define non-constant elements in R. Assume that there are a GIT-cone $\lambda \in \Lambda(R)$ of dimension at least two and an index i with $\deg(T_i) \in \lambda^\circ$.

(i) *There exists a j such that g_j has a monomial $T_i^{l_i}$ with $l_i \in \mathbb{Z}_{\geq 0}$.*

(ii) *There exists a j such that $\deg(g_j) = l_i \deg(T_i)$ holds with $l_i \in \mathbb{Z}_{\geq 0}$.*

(iii) *If $s = 1$ holds, then, $\deg(T_k)$ generates a ray of $\Lambda(R)$ whenever $k \neq i$.*

Proof. Because of $\deg(T_i) \in \lambda^\circ$, the ray τ generated by $\deg(T_i)$ is not an orbit cone. Thus, $\mathbb{Q}_{\geq 0} e_i$ is not an $\bar{X}$-face. This means that some g_j has a monomial $T_i^{l_i}$, which in particular proves (i) and (ii). To obtain (iii), first observe that $\deg(T_k) \in K_{\mathbb{Q}}$ is non-zero and thus lies in the relative interior of some GIT-cone $\varrho \in \Lambda(R)$ of positive dimension. Suppose that ϱ is not a ray. Then (i) yields that besides $T_i^{l_i}$ also $T_k^{l_k}$ is a monomial of the relation g_1. We conclude that $\gamma_{i,k}$ is an $\bar{X}$-face. Thus, $\deg(T_i)$ and $\deg(T_k)$ lie on a ray of $\Lambda(R)$. A contradiction. ■

3. Mori dream spaces

Mori dream spaces, introduced in [25], behave optimally with respect to the minimal model programme and are characterized as the normal projective varieties with finitely generated Cox ring. Well-known example classes are the projective toric or spherical varieties and, most important for the present article, the smooth Fano varieties. In this section, we provide a brief summary of the combinatorial approach [2,5,20] to Mori dream spaces, adapted to our needs. Moreover, as a new observation, we present Proposition 3.6, locating the relation degrees of a Cox ring inside the effective cone of a quasismooth Mori dream space.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero, R be a K-graded affine $\mathbb{K}$-algebra and consider the action of $H = \mathrm{Spec}\, \mathbb{K}[K]$ on variety $\bar{X} = \mathrm{Spec}\, R$. Mori dream spaces are obtained as quotients of the H-action. We briefly recall the general framework. Each cone $\lambda \in \Lambda(R)$ of the GIT-fan defines an H-invariant open set of *semistable points* and a *good quotient*:

$$\bar{X}^{ss}(\lambda) = \{x \in \bar{X};\ \lambda \subseteq \omega_x\} \subseteq \bar{X}, \qquad \bar{X}^{ss}(\lambda) \to \bar{X}^{ss}(\lambda) /\!\!/ H,$$

where $\omega_x \subseteq K_{\mathbb{Q}}$ denotes the orbit cone of $x \in \bar{X}$. Each of the quotient varieties $\bar{X}^{ss}(\lambda) /\!\!/ H$ is projective over $\mathrm{Spec}\, R_0$ and whenever $\lambda' \subseteq \lambda$ holds for two GIT-cones, then we have $\bar{X}^{ss}(\lambda) \subseteq \bar{X}^{ss}(\lambda')$ and thus an induced projective morphism $\bar{X}^{ss}(\lambda) /\!\!/ H \to \bar{X}^{ss}(\lambda') /\!\!/ H$ of the quotient spaces.

The K-grading of R is *almost free* if the (open) set $\bar{X}_0 \subseteq \bar{X}$ of points $x \in \bar{X}$ with trivial isotropy group $H_x \subseteq H$ has complement of codimension at least two in $\bar{X}$. Moreover, the *moving cone* of R is the convex cone $\mathrm{Mov}(R) \subseteq K_{\mathbb{Q}}$ obtained as the union over all $\lambda \in \Lambda(R)$, where $\bar{X}^{ss}(\lambda)$ has a complement of codimension at least two in $\bar{X}$.

Remark 3.1. Let R be a K-graded affine algebra such that the grading is factorial and any homogeneous unit is constant. Then R admits a system $f_1, \dots, f_r$ of pairwise non-associated K-prime generators. Moreover, if $f_1, \dots, f_r$ is such a system of generators for R, then the following holds.

(i) The K-grading is almost free if and only if any $r-1$ of $\deg(f_1), \dots, \deg(f_r)$ generate K as a group.

(ii) If the K-grading is almost free, then the orbit cones ω_x, where $x \in \bar{X}$, and the moving cone are given by

$$\omega_x = \mathrm{cone}(\deg(f_i);\ f_i(x) \neq 0), \quad \mathrm{Mov}(R) = \bigcap_{i=1}^{r} \mathrm{cone}(\deg(f_j);\ j \neq i).$$

We say that a K-graded affine $\mathbb{K}$-algebra R is an *abstract Cox ring* if it is integral, normal, has only constant homogeneous units, the K-grading is almost free, pointed, factorial and the moving cone $\mathrm{Mov}(R)$ is of full dimension in $K_{\mathbb{Q}}$.

Construction 3.2. Let R be an abstract Cox ring and consider the action of the quasitorus $H = \mathrm{Spec}\,\mathbb{K}[K]$ on the affine variety $\bar{X} = \mathrm{Spec}\,R$. For every GIT-cone $\lambda \in \Lambda(R)$ with $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$, we set

$$X(\lambda) := \bar{X}^{ss}(\lambda) /\!\!/ H.$$

The following proposition tells us in particular that Construction 3.2 delivers Mori dream spaces; see Theorem 3.2.14, Proposition 3.3.2.9 and Remark 3.3.4.2 in [2].

Proposition 3.3. *Let $X = X(\lambda)$ arise from Construction 3.2. Then X is normal, projective and of dimension* $\dim(R) - \dim(K_{\mathbb{Q}})$. *The divisor class group and the Cox ring of X are given as*

$$\mathrm{Cl}(X) = K, \quad \mathcal{R}(X) = \bigoplus_{\mathrm{Cl}(X)} \Gamma(X, \mathcal{O}_X(D)) = \bigoplus_{K} R_w = R.$$

Moreover, the cones of effective, movable, semiample and ample divisor classes of X are given in $\mathrm{Cl}_{\mathbb{Q}}(X) = K_{\mathbb{Q}}$ *as*

$$\mathrm{Eff}(X) = \mathrm{Eff}(R), \quad \mathrm{Mov}(X) = \mathrm{Mov}(R),$$
$$\mathrm{SAmple}(X) = \lambda, \quad \mathrm{Ample}(X) = \lambda^\circ.$$

By Corollary 3.2.1.11 in [2], all Mori dream space arise from Construction 3.2. For the subsequent work, we have to get more concrete, meaning that we will work in terms of generators and relations.

Construction 3.4. Let R be an abstract Cox ring and $X = X(\lambda)$ be as in Construction 3.2. Fix a K-graded presentation

$$R = \mathbb{K}[T_1, \dots, T_r] / \langle g_1, \dots, g_s \rangle$$

such that the variables $T_1, \dots, T_r$ define pairwise non-associated K-primes in R. Consider the orthant $\gamma = \mathbb{Q}^r_{\geq 0}$ and the projection

$$Q\colon \mathbb{Z}^r \to K, \quad e_i \mapsto w_i := \deg(T_i).$$

An X*-face* is an $\bar{X}$-face $\gamma_0 \preceq \gamma$ with $\lambda^\circ \subseteq Q(\gamma_0)^\circ$. Let $\mathrm{rlv}(X)$ be the set of all X-faces and $\pi\colon \bar{X}^{ss}(\lambda) \to X$ the quotient map. Then we have a decomposition

$$X = \bigcup_{\gamma_0 \in \mathrm{rlv}(X)} X(\gamma_0)$$

into pairwise disjoint locally closed sets $X(\gamma_0) := \pi(\bar{X}(\gamma_0))$, which we also call the *pieces* of X.

Recall that X is $\mathbb{Q}$*-factorial* if for every Weil divisor on X some non-zero multiple is locally principal. Moreover, X is *locally factorial* if every stalk $\mathcal{O}_x$, where $x \in X$ is a (closed) point, is a unique factorization domain. Finally, X is *quasismooth* if the open set $\bar{X}^{ss}(\lambda) \subseteq \bar{X}$ of semistable points is a smooth variety.

Proposition 3.5. *Consider the situation of Construction 3.4.*

(i) *The variety X is $\mathbb{Q}$-factorial, if and only if* $\dim(\lambda) = \dim(K_{\mathbb{Q}})$ *holds for* $\lambda = \mathrm{SAmple}(X)$.

(ii) *The variety X is locally factorial if and only if for every X-face $\gamma_0 \preceq \gamma$, the group K is generated by $Q(\gamma_0 \cap \mathbb{Z}^r)$.*

(iii) *The variety X is quasismooth if and only if every $\bar{X}(\gamma_0)$ consists of smooth points of $\bar{X}$ for every X-face $\gamma_0 \preceq \gamma$.*

(iv) *The variety X is smooth if and only if X is locally factorial and quasismooth.*

We refer to Corollaries 1.6.2.6, 3.3.1.8, and 3.3.1.9 in [2] for the above statements. Next we describe the impact of quasismoothness has an impact the position of the relation degrees.

Proposition 3.6. *In the situation of Construction 3.4, assume* $\dim(R) = r - s$ *and let X be quasismooth. Then, for every $j = 1, \dots, s$, we have*

$$\deg(g_j) \in \bigcap_{\gamma_0 \in \mathrm{rlv}(X)} \Big(Q(\gamma_0 \cap \mathbb{Z}^r) \cup \bigcup_{i=1}^{r} w_i + Q(\gamma_0 \cap \mathbb{Z}^r) \Big).$$

Proof. Consider any X-face γ_I, where $I \subseteq \{1, \dots, r\}$, and choose a point $x \in \bar{X}(\gamma_I)$. Then $x_i \neq 0$ holds if and only if $i \in I$. For any monomial T^ν, we have

$$\frac{\partial T^\nu}{\partial T_k}(x) \neq 0 \;\Rightarrow\; \nu \in \gamma_I \cup \gamma_I + e_k \;\Rightarrow\; \deg(T^\nu) = Q(\nu) \in Q(\gamma_I) \cup Q(\gamma_I) + w_k.$$

Now, since X is quasismooth, we have $\mathrm{grad}_{g_j}(x) \neq 0$ for all $j = 1, \dots, s$. Thus, every g_j must have a monomial T^{ν_j} with non-vanishing gradient at x. ∎

Finally, in case of a complete intersection Cox ring, we have an explicit description of the anticanonical class; see Proposition 3.3.3.2 in [2].

Proposition 3.7. *In the situation of Construction 3.4, assume that* $\dim(R) = r - s$ *holds. Then the anticanonical class of* X *is given in* $K = \mathrm{Cl}(X)$ *as*

$$-\mathcal{K}_X = \deg(T_1) + \cdots + \deg(T_r) - \deg(g_1) - \cdots - \deg(g_s).$$

4. General hypersurface Cox rings

First, we make our concept of a general hypersurface Cox ring precise. Then we present the toolbox to be used in the proof of Theorem 1.1 for verifying that given specifying data, that means a collection of the generator degrees and a relation degree, allow indeed a smooth general hypersurface Cox ring. We will have to deal with the following setting.

Construction 4.1. Consider a linear, pointed, almost free K-grading on the polynomial ring $S := \mathbb{K}[T_1, \dots, T_r]$ and the quasitorus action $H \times \bar{Z} \to \bar{Z}$, where

$$H := \mathrm{Spec}\,\mathbb{K}[K], \quad \bar{Z} := \mathrm{Spec}\,S = \mathbb{K}^r.$$

As earlier, we write $Q\colon \mathbb{Z}^r \to K$, $e_i \mapsto w_i := \deg(T_i)$ for the degree map. Assume that $\mathrm{Mov}(S) \subseteq K_{\mathbb{Q}}$ is of full dimension and fix $\tau \in \Lambda(S)$ with $\tau^\circ \subseteq \mathrm{Mov}(S)^\circ$. Set

$$\hat{Z} := \bar{Z}^{ss}(\tau), \quad Z := \hat{Z} /\!\!/ H.$$

Then Z is a projective toric variety with divisor class group $\mathrm{Cl}(Z) = K$ and Cox ring $\mathcal{R}(Z) = S$. Moreover, fix $0 \neq \mu \in K$, and for $g \in S_\mu$ set

$$R_g := S/\langle g \rangle, \quad \bar{X}_g := V(g) \subseteq \bar{Z}, \quad \hat{X}_g := \bar{X}_g \cap \hat{Z}, \quad X_g := \hat{X}_g /\!\!/ H \subseteq Z.$$

Then the factor algebra R_g inherits a K-grading from S and the quotient $X_g \subseteq Z$ is a closed subvariety. Moreover, we have

$$X_g \subseteq Z_g \subseteq Z,$$

where $Z_g \subseteq Z$ is the minimal ambient toric variety of X_g, that means the (unique) minimal open toric subvariety containing X_g.

Remark 4.2. In the situation of Construction 4.1, there is a (unique) GIT-cone $\lambda \in \Lambda(R_g)$ such that we have

$$\hat{X}_g = \bar{X}_g^{ss}(\lambda), \quad X_g = \bar{X}_g^{ss}(\lambda) /\!\!/ H.$$

Thus, if R_g is an abstract Cox ring and $T_1, \dots, T_r$ define pairwise non-associated K-primes in R_g, then X_g is as in Construction 3.4. In particular

$$\mathrm{Cl}(X) = K, \quad \mathcal{R}(X_g) = R_g$$

hold for the divisor class group and the Cox ring of X_g. Moreover, in $K_{\mathbb{Q}}$ we have the following

$$\tau^\circ = \mathrm{Ample}(Z) \subseteq \mathrm{Ample}(Z_g) = \mathrm{Ample}(X_g) = \lambda^\circ.$$

We are ready to formulate the precise definitions for our notions around hypersurface Cox rings.

Definition 4.3. Consider the situation of Construction 4.1.

(i) We call R_g a *hypersurface Cox ring* if $T_1, \dots, T_r$ define a minimal system of K-homogeneous generators for R_g.

(ii) We say that R_g is *spread* if every monomial $T^\nu \in \mathbb{K}[T_1, \dots, T_r]$ of degree $\mu = \deg(g) \in K$ is a convex combination of monomials of g.

Here, we tacitly identify a monomial $T^\nu = T_1^{\nu_1} \cdots T_r^{\nu_r}$ with its exponent vector $\nu = (\nu_1, \dots, \nu_r) \in \mathbb{Q}^r$ when we speak about convex combinations of monomials.

Remark 4.4. In the setting of Construction 4.1, assume that R_g is a hypersurface Cox ring.

(i) Since $T_1, \dots, T_r$ define a minimal system of K-homogeneous generators, R_g is not a polynomial ring.

(ii) As the K-grading is pointed, the T_i define pairwise non-associated K-prime elements in R_g.

(iii) R_g is spread if and only if the Newton polytope of g equals the convex hull over all monomials of degree $\mu = \deg(g) \in K$.

Definition 4.5. Consider the situation of Construction 4.1 and denote by $S_\mu \subseteq S = \mathbb{K}[T_1, \dots, T_r]$ the homogeneous component of degree $\mu \in K$.

(i) A *general hypersurface Cox ring* is a family R_g, where $g \in U$ with a non-empty open $U \subseteq S_\mu$, such that each R_g is a hypersurface Cox ring.

(ii) We say that a general hypersurface Cox ring R_g is *spread* if each R_g, where $g \in U$, is spread.

(iii) We say that a general hypersurface Cox ring R_g is *smooth* (*Fano*) if for some $\tau \in \Lambda(S)$ all the resulting X_g, where $g \in U$, are smooth (Fano).

We turn to the toolbox for verifying that given specifying data $w_1, \dots, w_r \in K$ and $\mu \in K$ as in Construction 4.1 lead to a smooth Fano general hypersurface Cox ring R_g in the above sense.

Remark 4.6. In the notation of Construction 4.1, a general hypersurface Cox ring R_g is Fano if and only if the generator and relation degrees satisfy

$$-\mathcal{K} = w_1 + \cdots + w_r - \mu \in \operatorname{Mov}(R_g)^\circ.$$

In this case, the unique cone $\tau \in \Lambda(S)$ with $-\mathcal{K} \in \tau^\circ$ defines Fano varieties X_g for all $g \in U$; see Proposition 3.7 and Remark 4.2.

In the notation of Construction 4.1, we denote by $U_\mu \subseteq S_\mu$ the non-empty open set of polynomials $f \in S$ of degree $\mu \in K$ such that each monomial of S_μ is a convex combination of monomials of f.

Remark 4.7. If R_g, where $g \in U$, is a general hypersurface Cox ring, then R_g, where $g \in U \cap U_\mu$, is a spread general hypersurface Cox ring. In particular, we can always assume a general hypersurface Cox ring to be spread.

Remark 4.8. In the situation of Construction 4.1, consider the rings R_g for $g \in U_\mu$. Then the following statements are equivalent.

(i) The variables $T_1, \dots, T_r$ form a minimal system of generators for all R_g, where $g \in U_\mu$.

(ii) The variables $T_1, \dots, T_r$ form a minimal system of generators for one R_g with $g \in U_\mu$.

(iii) We have $\mu \neq w_i$ for $i = 1, \dots, r$.

(iv) The polynomial $g \in U_\mu$ is not of the form $g = T_i + h$ with $h \in S_\mu$ not depending on T_i.

Lemma 4.9. *Consider a linear, pointed K-grading on $S := \mathbb{K}[T_1, \dots, T_r]$. Then, for any $0 \neq \mu \in K$, the irreducible polynomials $g \in S_\mu$ form an open subset of S_μ.*

Proof. Lemma 2.1 (ii) provides us with a coarsening homomorphism $\kappa \colon K \to \mathbb{Z}$ that turns S into a pointed $\mathbb{Z}$-graded algebra. Then S_μ is a vector subspace of the (finite dimensional) vector space $S_{\kappa(\mu)}$ of $\kappa(\mu)$-homogeneous polynomials and we may assume $K = \mathbb{Z}$ for the proof. Since the K-grading of S is pointed, we have $S^* = S_0 \setminus \{0\}$. Thus, a polynomial $g \in S_\mu$ is reducible if and only if it is a product of homogeneous polynomials of non-zero K-degree.

Now, let $u, v \in \mathbb{Z}$ with $u + v = \mu$ and $S_u \neq \{0\} \neq S_v$. Then the set of μ-homogeneous polynomials g admitting a factorization $g = fh$ with $f \in S_u$, $h \in S_u$ is exactly the affine cone over the image of the projectivized multiplication map,

$$\mathbb{P}(S_u) \times \mathbb{P}(S_v) \to \mathbb{P}(S_\mu), \qquad ([f],[h]) \mapsto [fh]$$

and thus is a closed subset of S_μ. As there are only finitely many such presentations $u + v = \mu$, the reducible $g \in S_\mu$ form a closed subset of S_μ. ∎

Proposition 4.10. *Consider the setting of Construction 4.1. For $1 \le i \le r$ denote by $U_i \subseteq S_\mu$ the set of all $g \in S_\mu$ such that g is prime in S and T_i is prime in R_g. Then $U_i \subseteq S_\mu$ is open. Moreover, U_i is non-empty if and only if there is a μ-homogeneous prime polynomial not depending on T_i.*

Proof. By Lemma 4.9, the $g \in S_\mu$ being prime in S form an open subset $U \subseteq S_\mu$. For any $g \in U$, the variable T_i defines a prime in R_g if and only if the polynomial $g_i := g(T_1, \dots, T_{i-1}, 0, T_{i+1}, \dots, T_n)$ is prime in $\mathbb{K}[T_j;\ j \neq i]$. Thus, using again Lemma 4.9, we see that the $g \in U$ with $T_i \in R_g$ prime form the desired open subset $U_i \subseteq U$. The supplement is clear. ∎

Checking the normality and K-factoriality of R_g amounts, in our situation, to proving factoriality. We will use Dolgachev's criterion, see Theorem 1.2 in [16] and [17], which tells us that a polynomial $g = \sum a_\nu T^\nu$ in $\mathbb{K}[T_1, \dots, T_r]$ defines a unique factorization domain if the Newton polytope $\Delta \subseteq \mathbb{Q}^r$ of g satisfies the following conditions:

(i) $\dim(\Delta) \ge 4$,

(ii) each coordinate hyperplane of $\mathbb{Q}^r$ intersects Δ non-trivially,

(iii) the dual cone of $\mathrm{cone}(\Delta - u;\ u \in \Delta_0)$ is regular for each one-dimensional face $\Delta_0 \preceq \Delta$,

(iv) for each face $\Delta_0 \preceq \Delta$ the zero locus of $\sum_{\nu \in \Delta_0} a_\nu T^\nu$ is smooth along the torus $\mathbb{T}^r = (\mathbb{K}^*)^r$.

We will call for short a convex polytope $\Delta \subseteq \mathbb{Q}^r_{\geq 0}$ with properties (i)-(ii) from above a *Dolgachev polytope*.

Proposition 4.11. *In the situation of Construction 4.1, suppose that one of the following conditions is fulfilled*:

(i) *K is of rank at most $r-4$ and torsion free, there is a $g \in S_\mu$ such that $T_1, \dots, T_r$ define primes in R_g, we have $\mu \in \tau^\circ$ and μ is base point free on Z.*

(ii) *The set* $\operatorname{conv}(\nu \in \mathbb{Z}^r_{\geq 0};\ Q(\nu) = \mu)$ *is a Dolgachev polytope.*

Then there is a non-empty open subset of polynomials $g \in S_\mu$ such that the ring R_g is factorial.

Proof. Assume that (i) is satisfied. If $\mu = \deg(T_i)$ holds for some i, then, as the grading is pointed, we have a non-empty open set of polynomials $g = T_i + h$ in S_μ with h not depending on T_i. The corresponding R_g are all factorial. Now assume $\mu \neq \deg(T_i)$ for all i. By Proposition 4.10, the set $U \subseteq S_\mu$ of all prime $g \in S_\mu$ such that $T_1, \dots, T_r$ define primes in R_g is open and, by assumption, $U \subseteq S_\mu$ is non-empty. Remark 4.8 yields that $T_1, \dots, T_r$ form a minimal system of generators for R_g. We conclude that for all $f \in U$, the complement of $\hat{X}_g$ in $\bar{X}_g$ is of codimension at least two. Since μ is base point free and ample on Z, we can apply Corollary 2.3 in [1], telling us that after suitably shrinking, U is still non-empty and R_g is the Cox ring of X_g for all $g \in U$. In particular, R_g is K-factorial. Since K is torsion free, R_g is a unique factorization domain.

Assume that (ii) holds. As $\Delta := \operatorname{conv}(\nu \in \mathbb{Z}^r_{\geq 0};\ Q(\nu) = \mu)$ is a Dolgachev polytope, we infer from Theorem 2 in §2 of [29] that there is a non-empty open subset of polynomials $g \in S_\mu$ with Newton polytope Δ satisfying the above conditions (i) to (iv). Thus, Dolgachev's criterion shows that R_g is a factorial ring. ∎

Proposition 4.12. *In the setting of Construction 4.1, assume that Z_g and $\hat{X}_g$ both are smooth. Then X_g is smooth.*

Proof. Consider the quotient map $p\colon \hat{Z} \to Z$. Since Z_g is smooth, H acts freely on $p^{-1}(Z_g)$. Thus, X_g inherits smoothness from $\hat{X}_g = p^{-1}(X_g)$. ∎

Lemma 4.13. *Consider a linear, pointed K-grading on $S := \mathbb{K}[T_1, \dots, T_r]$. Let $\lambda \in \Lambda(S)$ and set $W := (\mathbb{K}^r)^{ss}(\lambda)$. Then, for any $\mu \in K$, the polynomials $g \in S_\mu$ such that* $\operatorname{grad}(g)$ *has no zeroes in W form an open subset of S_μ.*

Proof. Consider the morphism $\varphi\colon S_\mu \times W \to \mathbb{K}^r$ sending (g, z) to $\operatorname{grad}_z(g)$ and the projection $\operatorname{pr}_1\colon S_\mu \times W \to S_\mu$ onto the first factor. Then our task is to show that $S_\mu \setminus \operatorname{pr}_1(\varphi^{-1}(0))$ is open in S_μ. We make use of the action of $H = \operatorname{Spec} \mathbb{K}[K]$ on W given by the K-grading and the commutative diagram

$$
\begin{array}{ccc}
S_\mu \times W & \longrightarrow & S_\mu \times W /\!\!/ H \\
& {\scriptstyle \operatorname{pr}_1} \searrow \quad \swarrow {\scriptstyle \operatorname{pr}_1} & \\
& S_\mu &
\end{array}
$$

where the horizontal arrow is the good quotient for H, acting trivially on S_μ and on W as indicated above. Since $\varphi^{-1}(0) \subseteq S_\mu \times W$ is invariant under the H-action, the image of $\varphi^{-1}(0)$ in $S_\mu \times W /\!\!/ H$ is closed. Since $W /\!\!/ H$ is projective, the image $\mathrm{pr}_1(\varphi^{-1}(0))$ is closed in S_μ. ∎

Proposition 4.14. *Consider the situation of Construction 4.1. Then the polynomials $g \in S_\mu$ such that $g \in S$ is prime and $\hat{X}_g$ is smooth form an open subset $U \subseteq S_\mu$. Moreover, U is non-empty if and only if there are $g_1, g_2 \in S_\mu$ such that $g_1 \in S$ is prime and $\mathrm{grad}(g_2)$ has no zeroes in $\hat{Z}$.*

Proof. By Lemma 4.9, the set V_1 of all prime polynomials of S_μ is open. Moreover, by Lemma 4.13, the set of all polynomials of S_μ such that $\mathrm{grad}(g)$ has no zeroes in $\hat{Z}$ is open. The assertion follows from $U = V_1 \cap V_2$. ∎

Corollary 4.15. *Let X be a variety with a general hypersurface Cox ring R. If X is smooth, then R is a smooth general hypersurface Cox ring.*

Proposition 4.16. *Consider the situation of Construction 4.1. If $\mu \in \mathrm{Cl}(Z)$ is base point free, then there is a non-empty open subset of $g \in S_\mu$ such that $X_g \cap Z^{\mathrm{reg}}$ is smooth.*

Proof. Observe that $\mathbb{P}(S_\mu)$ is the complete linear system associated with the divisor class $\mu \in \mathrm{Cl}(Z)$. If μ is a base point free class on Z, we can apply Bertini's first theorem (Theorem 4.1 in [30]) stating that there is a non-empty open subset $U \subseteq S_\mu$ such that for each $g \in U$ the singular locus of X_g is precisely $X_g \cap Z^{\mathrm{sing}}$. In particular, $X_g \cap Z^{\mathrm{reg}}$ is smooth for all $g \in U$. ∎

Remark 4.17. In the situation of Construction 4.1, let $N(g)$ be the Newton polytope of g. For $I \subseteq \{1, \ldots, r\}$, let $\gamma_I \preccurlyeq \gamma$ and $g_I \in \mathbb{K}[T_1, \ldots, T_r]$ be as in Definition 2.6 and assume $Z(\gamma_I) \neq \emptyset$. Then Proposition 3.1.1.12 in [2] yields the equivalence of the following statements.

(i) We have $X_g \cap Z(\gamma_I) \neq \emptyset$.

(ii) We have $\bar{X}_g \cap \bar{Z}(\gamma_I) \neq \emptyset$.

(iii) The polynomial g_I is not a monomial.

(iv) The number of vertices of $N(g)$ contained γ_I differs from one.

In particular, for the non-empty open subset $U_\mu \subseteq S_\mu$ of polynomials $f \in S$ of degree $\mu = \deg(g) \in K$ such that each monomial of S_μ is a convex combination of monomials of f, we obtain $Z_g = Z_{g'}$ for all $g, g' \in U_\mu$.

Definition 4.18. In the setting of Remark 4.17, we call $Z_\mu := Z_g$, where $g \in U_\mu$, the *μ-minimal ambient toric variety.*

Corollary 4.19. *In the setting of in Construction 4.1, assume $\mathrm{rank}(K) = 2$ and that $Z_\mu \subseteq Z$ is smooth. If $\mu \in \tau$ holds, then μ is base point free. Moreover, then there is a non-empty open subset of polynomials $g \in S_\mu$ such that X_g is smooth.*

Proof. According to Proposition 3.3.2.8 in [2], the class $\mu \in \mathrm{Cl}(Z)$ is base point free on Z if and only if the following holds:

$$\mu \in \bigcap_{\gamma_0 \in \mathrm{rlv}(Z)} Q(\gamma_0 \cap \mathbb{Z}^r).$$

To check the latter, let $\gamma_0 \in \mathrm{rlv}(Z)$. As $K_{\mathbb{Q}}$ is two-dimensional, we find $1 \le i, j \le r$ with $e_i, e_j \in \gamma_0$ and $\lambda^\circ \subseteq \mathrm{cone}(w_i, w_j)^\circ$. If w_i, w_j generate K as a group, then K is torsion-free, w_i, w_j form a Hilbert basis for $\mathrm{cone}(w_i, w_j)$ and thus μ is a positive combination of w_i, w_j. Otherwise, the toric orbit $Z(\gamma_{i,j})$ is not smooth, hence not contained in Z_μ. The latter means $V(g) \cap \bar{Z}(\gamma_{i,j}) = \emptyset$, which in turn shows that g has a monomial of the form $T_i^{l_i} T_j^{l_j}$ where $l_i + l_j > 0$. Thus, μ is a positive combination of w_i and w_j.

Knowing that μ is base point free, we obtain the supplement as a direct consequence of smoothness of Z_μ and Proposition 4.16. ■

5. Proof of Theorem 1.1

We work in the combinatorial framework for Mori dream spaces provided in the preceding sections. The ground field is now $\mathbb{K} = \mathbb{C}$, due to the references we use; see Remark 5.4. The major part of proving Theorem 1.1, is to figure out the candidates for specifying data of smooth general hypersurface Cox rings of Fano fourfolds of Picard number two. Having found the candidates, the remaining task is to verify them, that means to show that the given specifying data indeed define a smooth general hypersurface Cox ring of a Fano fourfold. The precise setting for the elaboration is the following.

Setting 5.1. Consider a K-graded algebra R and $X = X(\lambda)$, where $\lambda \in \Lambda(R)$ with $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$, as in Construction 3.2. Assume that $\dim(K_{\mathbb{Q}}) = 2$ holds and that we have an irredundant K-graded presentation

$$R = R_g = \mathbb{C}[T_1, \ldots, T_r] / \langle g \rangle$$

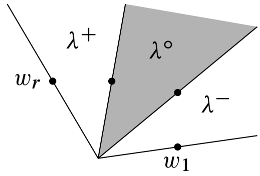
such that the T_i define pairwise nonassociated K-primes in R. Write $w_i := \deg(T_i)$, $\mu := \deg(g)$ for the degrees in K, also when regarded in $K_{\mathbb{Q}}$. Suitably numbering $w_1, \ldots, w_r$, we ensure counter-clockwise ordering, that means that we always have

$$i \le j \implies \det(w_i, w_j) \ge 0.$$

Note that each ray of $\Lambda(R)$ is of the form $\varrho_i = \mathrm{cone}(w_i)$, but not vice versa. We assume X to be $\mathbb{Q}$-factorial. According to Proposition 3.5 (i), this means $\dim(\lambda) = 2$. Then the effective cone of X is uniquely decomposed into three convex sets,

$$\mathrm{Eff}(X) = \lambda^- \cup \lambda^\circ \cup \lambda^+,$$

where λ^- and λ^+ are convex polyhedral cones not intersecting $\lambda^\circ = \mathrm{Ample}(X)$ and $\lambda^- \cap \lambda^+$ consists of the origin. By Remark 3.1 (ii) and Proposition 3.3, each of λ^- and λ^+ contains at least two of the degrees $w_1, \ldots, w_r$.



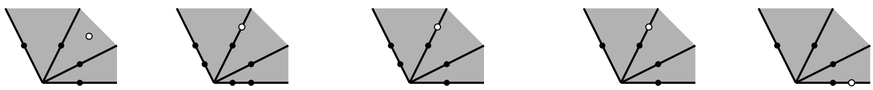
Note that λ^- as well as λ^+ might be one-dimensional. As a GIT-cone in $K_{\mathbb{Q}} \cong \mathbb{Q}^2$, the closure $\lambda = \mathrm{SAmple}(X)$ of $\lambda^\circ = \mathrm{Ample}(X)$ is the intersection of two projected $\bar{X}$-faces and thus we find at least one of the w_i on each of its bounding rays.

Remark 5.2. Setting 5.1 is respected by orientation preserving automorphisms of K. If we apply an orientation reversing automorphism of K, then we regain Setting 5.1 by reversing the numeration of $w_1, \ldots, w_r$. Moreover, we may interchange the numeration of T_i and T_j if w_i and w_j share a common ray without affecting Setting 5.1. We call these operations *admissible coordinate changes*.

Remark 5.3. In Setting 5.1, consider the rays $\varrho_i := \mathrm{cone}(w_i) \subseteq \mathbb{Q}^2$, where $i = 1, \ldots, r$, and the degree $\mu = \deg(g)$ of the relation. Set

$$\Gamma := \varrho_1 \cup \ldots \cup \varrho_r, \qquad \Gamma^\circ := \Gamma \cap \mathrm{Eff}(R)^\circ.$$

Then a suitable admissible coordinate change turns the setting into one of the following:



(I) $\mu \notin \Gamma$ | (IIa) $\mu \in \Gamma^\circ$, $\varrho_1 = \varrho_2$, $\varrho_{r-1} = \varrho_r$ | (IIb) $\mu \in \Gamma^\circ$, $\varrho_1 \neq \varrho_2$, $\varrho_{r-1} = \varrho_r$ | (IIc) $\mu \in \Gamma^\circ$, $\varrho_1 \neq \varrho_2$, $\varrho_{r-1} \neq \varrho_r$ | (III) $\mu \in \varrho_1$

where the figures exemplarily sketch the case $r = 5$, the black dots indicate the generator degrees, and the white dot stands for the relation degree.

Our proof of Theorem 1.1 will be split into Parts I, IIa, IIb, IIc and III according to the constellations of Remark 5.3. We exemplarily present Parts I, IIa and III. The remaining parts use analogous arguments and will be made available in [33]. The reason why we restrict Theorem 1.1 to the ground field $\mathbb{K} = \mathbb{C}$ is that we use the following references on complex Fano varieties.

Remark 5.4. Let X be a smooth complex Fano variety. Then the divisor class group $\mathrm{Cl}(X)$ of X is torsion free; see for instance Proposition 2.1.2 in [37]. Moreover, if $\dim(X) = 4$ holds, then Remark 3.6 in [8] tells us that any $\mathbb{Q}$-factorial projective variety being isomorphic in codimension one to X is smooth as well. In terms of Construction 3.2, the latter means that all varieties $X(\eta)$ are smooth, where $\eta \in \Lambda(R)$ is full-dimensional with $\eta^\circ \subseteq \mathrm{Mov}(R)^\circ$.

We treat Case 5.3 I that means that the degree of the defining relation is not proportional to any of the Cox ring generator degrees. Here are first constraints on the possible specifying data in this situation.

Proposition 5.5. *In Setting 5.1, assume that $r = 7$, $K \cong \mathbb{Z}^2$ holds, every two-dimensional $\lambda \in \Lambda(R)$ with $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$ defines a locally factorial $X(\lambda)$ and μ does not lie on any of the rays $\varrho_1, \dots, \varrho_7$. Then, after a suitable admissible coordinate change, we have $\mu \in \mathrm{cone}(w_4, w_5)^\circ$ and one of the following holds*:

(i)	$w_1 = w_2$ *and* $w_5 = w_6$,	(iv)	$w_2 = w_3$ *and* $w_6 = w_7$,
(ii)	$w_1 = w_2$ *and* $w_6 = w_7$,	(v)	$w_3 = w_4$ *and* $w_5 = w_6$,
(iii)	$w_2 = w_3$ *and* $w_5 = w_6$,	(vi)	$w_3 = w_4$ *and* $w_6 = w_7$.

Lemma 5.6. *Consider a locally factorial $X = X(\lambda)$ arising from Construction 3.4 with only one relation, i.e., $s = 1$. Let i, j with $\lambda \subseteq \mathrm{cone}(w_i, w_j)$. Then either w_i, w_j generate K as a group, or g_1 has precisely one monomial of the form $T_i^{l_i} T_j^{l_j}$, where $l_i + l_j > 0$.*

Proof. If $\gamma_{i,j}$ is an X-face, then Proposition 3.5 (ii) tells us that w_i and w_j generate K as a group. Now consider the case that $\gamma_{i,j}$ not an X-face. Then we must have $\lambda^\circ \not\subseteq Q(\gamma_{i,j})^\circ$ or $\gamma_{i,j}$ is not an $\bar{X}$-face. Proposition 3.5 (i) excludes the first possibility. Thus, the second one holds, which in turn means that g_1 has precisely one monomial of the form $T_i^{l_i} T_j^{l_j}$, where $l_i + l_j > 0$. ∎

Lemma 5.7. *Let $X = X(\lambda)$ be as in Setting 5.1 and let $1 \le i < j < k \le r$. If X is locally factorial, then w_i, w_j, w_k generate K as a group provided that one of the following holds*:

(i) $w_i, w_j \in \lambda^-$, $w_k \in \lambda^+$ *and g has no monomial of the form* $T_k^{l_k}$,

(ii) $w_i \in \lambda^-$, $w_j, w_k \in \lambda^+$ *and g has no monomial of the form* $T_i^{l_i}$.

Proof. Assume that (i) holds. If K is generated by w_i, w_k or by w_j, w_k, then we are done. Consider the case that none of the pairs w_i, w_k and w_j, w_k generates K. Applying Lemma 5.6 to each of the pairs shows that g has precisely one monomial of the form $T_i^{l_i} T_k^{l_k}$ with $l_i + l_k > 0$ and precisely one monomial of the form $T_j^{l_j} T_k^{l'_k}$ with $l_j + l'_k > 0$. By assumption, we must have $l_i, l_j > 0$. We conclude that $\gamma_{i,j,k}$ is an X-face. Since X is locally factorial, Proposition 3.5 (ii) yields that w_i, w_j, w_k generate K. If (ii) holds, then a suitable admissible coordinate change leads to (i). ∎

Lemma 5.8. *Assume u, w_1, w_2 generate the abelian group $\mathbb{Z}^2$. If $w_i = a_i w$ holds with a primitive $w \in \mathbb{Z}^2$ and $a_i \in \mathbb{Z}$, then (u, w) is a basis for $\mathbb{Z}^2$ and u is primitive.*

Lemma 5.9. *Let $w_1, \dots, w_4 \in \mathbb{Z}^2$ be such that $\det(w_1, w_3)$, $\det(w_1, w_4)$, $\det(w_2, w_3)$ and $\det(w_2, w_4)$ all equal one. Then $w_1 = w_2$ or $w_3 = w_4$ holds.*

Proof of Proposition 5.5. The assumption $\mu \notin \varrho_i$ yields $\varrho_i \in \Lambda(R)$ for $i = 1, \dots, 7$, see Remark 2.7 (ii). Proposition 2.4 gives $\mu \in \mathrm{cone}(w_3, w_5)$. The latter cone is the union of $\mathrm{cone}(w_3, w_4)$ and $\mathrm{cone}(w_4, w_5)$; both are GIT-cones, one of them is two-dimensional and hosts μ in its relative interior. A suitable admissible coordinate change yields $\mu \in \mathrm{cone}(w_4, w_5)^\circ$.

First we show that if $w_i \in \varrho_j$ holds for some $1 \le i < j \le 4$, then two of w_5, w_6, w_7 coincide. Consider the case $w_5, w_6 \in \varrho_5$. By assumption, $X(\lambda)$ is locally factorial for $\lambda = \mathrm{cone}(w_4, w_5)$. Thus, we can apply Lemma 5.7 to w_i, w_j, w_5 and also to w_i, w_j, w_6

and obtain that each of the triples generates K as a group. Lemma 5.8 yields that w_5 and w_6 are primitive and hence, lying on a common ray, coincide. Now, assume $w_6 \notin \varrho_5$. Then we consider $X = X(\lambda)$ for $\lambda = \mathrm{cone}(w_5, w_6)$. Using Lemma 5.7 as before, we see that w_i, w_j, w_6 as well as w_i, w_j, w_7 generate K as a group. For the primitive generator w of $\varrho_i = \varrho_j$, we infer $\det(w, w_6) = 1$ and $\det(w, w_7) = 1$ from Lemma 5.8. Moreover, $\gamma_{5,6}$ and $\gamma_{5,7}$ are X-faces due to Remark 2.7 (ii). Thus, Proposition 3.5 (ii) yields $\det(w_5, w_6) = 1$ and $\det(w_5, w_7) = 1$. Lemma 5.9 yields $w_6 = w_7$.

We conclude the proof by showing that at least two of $w_1, \ldots, w_4$ coincide. Consider the case $w_2 \in \varrho_3$. Then, by the first step, there are $5 \le i < j \le 7$ with $w_i = w_j$. Taking $X(\lambda)$ for $\lambda = \mathrm{cone}(w_4, w_5)$ and applying Lemma 5.7 to w_2, w_i, w_j as well as to w_3, w_i, w_j, we obtain that each of these triples generates K. Because of $w_i = w_j$, we directly see that w_2 and w_3, each being part of a $\mathbb{Z}$-basis, are primitive and hence coincide. We are left with the case that $\lambda' = \mathrm{cone}(w_2, w_3)$ is of dimension two. By assumption, the variety X' defined by λ' is locally factorial. Moreover, Remark 2.7 (ii) provides us with the X'-faces $\gamma_{1,3}, \gamma_{2,3}, \gamma_{1,4}$ and $\gamma_{2,4}$. By Proposition 3.5 (ii), all corresponding determinants $\det(w_k, w_m)$ equal one. Lemma 5.9 shows that at least two of $w_1, \ldots, w_4$ coincide. ∎

We are ready to enter Part I of the proof of Theorem 1.1. The task is to work out further the degree constellations left by Proposition 5.5. This leads to major multistage case distinctions. We demonstrate how to get through for two of the constellations of Proposition 5.5, chosen in a manner that basically all the necessary arguments of Part I of the proof show up. For the full elaboration of all cases we refer to [33].

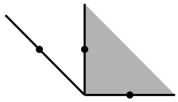
Proof of Theorem 1.1*, Part* I. In this part, we treat the case that $\mu = \deg(g)$ does not lie on any of the rays $\varrho_i = \mathrm{cone}(w_i)$. In particular, by Remark 2.7 (ii), all rays $\varrho_1, \ldots, \varrho_7$ belong to the GIT-fan $\Lambda(R)$. By Remark 5.4, every two-dimensional $\eta \in \Lambda(R)$ with $\eta^\circ \subseteq \mathrm{Mov}(R)^\circ$ produces a smooth variety $X(\eta)$. Thus, we can apply Proposition 5.5, which leaves us with $\mu \in \mathrm{cone}(w_4, w_5)^\circ$ and the six possible constellations for $w_1, \ldots, w_7$ given there. Again by Remark 5.4, the divisor class group of X is torsion free, that means that we have $K = \mathbb{Z}^2$.

Constellation 5.5 (i). We have $w_1 = w_2$ and $w_5 = w_6$. Lemma 5.7 applied to w_1, w_2, w_5 shows that w_1, w_5 form a basis of $\mathbb{Z}^2$. Thus, a suitable admissible coordinate change gives $w_1 = (1, 0)$ and $w_6 = (0, 1)$. Applying Lemma 5.7 also to w_1, w_2, w_7 and w_i, w_5, w_6, where $i = 1, \ldots, 4$, yields the first coordinate of $w_1, \ldots, w_4$ and the second coordinate of w_7 equal one. Thus, the degree matrix has the form

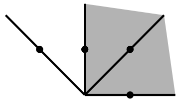
$$Q = [w_1, \ldots, w_7] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -a_7 \\ 0 & 0 & b_3 & b_4 & 1 & 1 & 1 \end{bmatrix}, \qquad b_3, b_4, a_7 \in \mathbb{Z}_{\ge 0}.$$

We determine the possible values of b_3 and b_4. If $b_3 > 0$ holds, then $\eta = \mathrm{cone}(w_2, w_3)$ is two-dimensional and satisfies $\eta^\circ \subseteq \mathrm{Mov}(R)^\circ$. Because of $\mu \in \mathrm{cone}(w_4, w_5)^\circ$, none of the monomials of g is of the form $T_1^{l_1} T_j^{l_j}$ with $j = 3, 4$. Lemma 5.6 applied to $X(\eta)$ gives $b_j = \det(w_1, w_j) = 1$ for $j = 3, 4$. If $b_3 = 0$ and $b_4 > 0$ hold, we argue similarly with

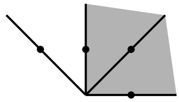
$\eta = \mathrm{cone}(w_2, w_4)$ and obtain $b_4 = 1$. Altogether, we arrive at the three cases:



5.5 (i-a): $b_3 = b_4 = 0$,



5.5 (i-b): $b_3 = 0, b_4 = 1$,



5.5 (i-c): $b_3 = b_4 = 1$.

Case 5.5 (i-a). Here, the semiample cone λ of $X = X(\lambda)$ must be the positive orthant. Thus, X being Fano just means that both coordinates of the anticanonical class $-\mathcal{K}_X \in K = \mathbb{Z}^2$ are strictly positive. According to Proposition 3.7, we have

$$-\mathcal{K}_X = (4 - a_7 - \mu_1,\ 3 - \mu_2).$$

We conclude $1 \le \mu_2 \le 2$ and $1 \le \mu_1 < 4 - a_7$, which implies in particular $0 \le a_7 \le 2$. Thus, the weights $w_1, \dots, w_7$ and the degree μ must be as in Theorem 1.1, Numbers 1 to 12. We exemplarily verify the candidate Number 12; the others are settled analogously. We have to deal with the specifying data

$$Q = [w_1, \dots, w_7] = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (1, 2).$$

We run Construction 4.1 with $\tau \in \Lambda$ such that $-\mathcal{K} = (1,1) \in \tau^\circ$, and show that the result is a smooth general hypersurface Cox ring R_g. First, one directly checks that the convex hull over the $\nu \in \mathbb{Z}^7$ with $Q(\nu) = \mu$ is Dolgachev polytope. Thus, Proposition 4.11 (i) delivers a non-empty open set $U \subseteq S_\mu$ such that R_g is factorial for all $g \in U$. Since $\mu \ne w_i$ holds for all i, Remark 4.8 ensures that $T_1, \dots, T_7$ are a minimal system of generators for R_g, whenever $g \in U$. For $i \ne 5, 6$, the degree w_i of T_i is indecomposable in the monoid $\mathrm{Eff}(R_g) \cap K$. We conclude that T_i is irreducible and thus prime in R_g, whenever $g \in U$. To see primality of T_5 and T_6, we use Proposition 4.10, where we can take $T_1 T_i^2 - T_2^5 T_7^2$ for $i = 6, 5$ as the required μ-homogeneous polynomial in both cases. The ambient toric variety Z is smooth due to Proposition 3.5 (iv). Thus, also Z_μ is smooth. Because of $\mu \in \tau^\circ$, Corollary 4.19 applies and, suitably shrinking U, we achieve that X_g is smooth for all $g \in U$.

Case 5.5 (i-b). Here, either $\lambda = \mathrm{cone}(w_3, w_4)$ or $\lambda = \mathrm{cone}(w_4, w_5)$ holds. In any case, the anticanonical class is given as

$$-\mathcal{K}_X = (4 - a_7 - \mu_1,\ 4 - \mu_2).$$

First assume that $\lambda = \mathrm{cone}(w_3, w_4)$ holds. Then, X being Fano, we have $-\mathcal{K}_X \in \lambda^\circ$. The latter is equivalent to the inequalities

$$4 - \mu_2 \ > \ 0, \qquad \mu_2 - \mu_1 - a_7 \ > \ 0.$$

Using $\mu \in \mathrm{cone}(w_4, w_5)^\circ$, we conclude $1 \le \mu_1 < \mu_2 \le 3$ and $0 \le a_7 \le 1$. Thus, we end up with

$$a_7 = 0 \ \text{ and } \ \mu = (1,2), (1,3), (2,3), \qquad a_7 = 1 \ \text{ and } \ \mu = (1,3).$$

Note that in all cases, $\gamma_{1,2,3,4}$ is an X-face according to Remark 2.7 (ii). Since X is quasismooth, Proposition 3.6 yields

$$\mu \in Q(\gamma_{1,2,3,4}) \cup w_7 + Q(\gamma_{1,2,3,4}).$$

This excludes $a_7 = 0$ and $\mu = (1,3)$. The remaining three cases are Numbers 13 to 15 of Theorem 1.1. All these candidates can be verified. Indeed, take $\tau = \mathrm{cone}(w_3, w_4)$ for all three cases and observe $-\mathcal{K} \in \mathrm{cone}(w_3, w_4)^\circ$. As in Case 5.5 (i-a), we find a non-empty open subset U of polynomials $g \in S_\mu$ such that R_g admits unique factorization, see that $T_1, \dots, T_7$ define pairwise non-associated primes in R_g and observe that Z and thus also Z_μ are smooth. For smoothness of X_g, it suffices to show that $\hat{X}_g$ is smooth; see Proposition 4.12. By Proposition 4.14, it suffices to find some $g \in S_\mu$ such that $\mathrm{grad}(g)$ has no zeroes in $\hat{Z}$, then shrinking U suitably yields that $\hat{X}_g$ is smooth for all $g \in U$. We just chose a random g of degree μ and verified this using [22]. For instance, for $a_7 = 0$ and $\mu = (1,2)$, that means Number 13, the following g does the job:

$$\begin{aligned} 8T_1T_5^2 &+ 7T_1T_5T_6 + 7T_1T_5T_7 + 6T_1T_6^2 + 4T_1T_6T_7 + T_1T_7^2 + 7T_2T_5^2 + 7T_2T_5T_6 \\ &+ 3T_2T_5T_7 + 8T_2T_6^2 + 5T_2T_6T_7 + 8T_2T_7^2 + 5T_3T_5^2 + 4T_3T_5T_6 + 9T_3T_5T_7 \\ &+ 2T_3T_6^2 + 9T_3T_6T_7 + T_3T_7^2 + 8T_4T_5 + 3T_4T_6 + 6T_4T_7. \end{aligned}$$

Now, assume that $\lambda = \mathrm{cone}(w_4, w_5)$ holds. The condition that $X = X(\lambda)$ is Fano means $-\mathcal{K}_X \in \lambda^\circ$, which translates into the inequalities $0 < 4 - a_7 - \mu_1 < 4 - \mu_2$. Moreover, $\mu \in \lambda^\circ$ implies $\mu_1 < \mu_2$ and we conclude

$$1 \le \mu_1 < \mu_2 < \mu_1 + a_7 \le 3.$$

This is only possible for $a_7 = 2$ and $\mu = (1,2)$. Then we have $w_4 = (1,1)$ and $w_7 = (-2,1)$. In particular, g admits no monomial of the form $T_4^{l_4}T_7^{l_7}$. Lemma 5.6 tells us that w_4 and w_7 generate $K = \mathbb{Z}^2$ as a group. A contradiction.

Case 5.5 (i-c). Remark 5.4 and Lemma 5.7 applied to $X(\eta)$ with $\eta = \mathrm{cone}(w_4, w_5)$ and w_3, w_4, w_7 yield $\det(w_4, w_7) = 1$. From this we infer $a_7 = 0$. Thus, either $\lambda = \mathrm{cone}(w_2, w_3)$ or $\lambda = \mathrm{cone}(w_4, w_5)$ holds. In any case, the anticanonical class is

$$-\mathcal{K}_X = (4 - \mu_1,\ 5 - \mu_2).$$

Assume $\lambda = \mathrm{cone}(w_2, w_3)$. Then the Fano condition $-\mathcal{K}_X \in \lambda^\circ$ implies $\mu_1 + 1 < \mu_2$. Remark 2.7 (ii) says that $\gamma_{1,2,3,4}$ is an X-face. As before, Proposition 3.6 gives

$$\mu \in Q(\gamma_{1,2,3,4}) \cup w_7 + Q(\gamma_{1,2,3,4}).$$

We conclude $\mu_1 + 1 \ge \mu_2$. A contradiction. Now, assume $\lambda = \mathrm{cone}(w_4, w_5)$. Then $-\mathcal{K}_X \in \lambda^\circ$ yields $\mu_1 \ge \mu_2$. But we have $\mu \in \mathrm{cone}(w_4, w_5)^\circ$, hence $\mu_1 < \mu_2$. A contradiction.

Constellation 5.5 (ii). We have $w_1 = w_2$ and $w_6 = w_7$. Lemma 5.7 applied to w_1, w_6, w_7 shows that w_1, w_7 generate $\mathbb{Z}^2$. Hence, a suitable admissible coordinate change yields $w_1 = w_2 = (1,0)$ and $w_6 = w_7 = (0,1)$. Now, applying Lemma 5.7 to w_3, w_6, w_7 and

w_4, w_6, w_7, we obtain that the first coordinates of w_3 and w_4 both equal one. Thus, the degree matrix has the form

$$Q = [w_1, \ldots, w_7] = \begin{bmatrix} 1 & 1 & 1 & 1 & a_5 & 0 & 0 \\ 0 & 0 & b_3 & b_4 & 1 & 1 & 1 \end{bmatrix}, \quad a_5, b_3, b_4 \in \mathbb{Z}_{\geq 0}.$$

By assumption w_4 and w_5 do not lie on a common ray. Consequently, $b_4 = 0$ or $a_5 = 0$ holds. If $a_5 = 0$ holds, then we are in Constellation 5.5 (i) just treated. So, assume $a_5 > 0$. Then $b_3 = b_4 = 0$ holds. Taking $X(\eta)$ for $\eta = \mathrm{cone}(w_5, w_6)$ and applying Lemma 5.6 to w_5, w_6 yields $a_5 = 1$. We arrive at the degree matrix

$$Q = [w_1, \ldots, w_7] = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Observe that either $\lambda = \mathrm{cone}(w_4, w_5)$ or $\lambda = \mathrm{cone}(w_5, w_6)$ holds. In any case, the anticanonical class of $X = X(\lambda)$ is given as

$$-\mathcal{K}_X = (5 - \mu_1,\ 3 - \mu_2).$$

First, assume $\lambda = \mathrm{cone}(w_4, w_5)$. Then X being Fano means $0 < 3 - \mu_2 < 5 - \mu_1$. We conclude $\mu_2 \leq 2$ and $\mu_1 \leq \mu_2 + 1$. Moreover, $\mu \in \mathrm{cone}(w_4, w_5)^\circ$ gives $0 < \mu_2 < \mu_1$. Thus, we have $\mu_1 = \mu_2 + 1$ and arrive at the possibilities $\mu = (2, 1), (3, 2)$, which are Numbers 16 and 17 in Theorem 1.1. Showing that these constellations indeed define smooth Fano varieties runs exactly as in Case 5.5 (i-a). Now, let $\lambda = \mathrm{cone}(w_5, w_6)$. Then X being Fano gives $0 < 5 - \mu_1 < 3 - \mu_2$. We conclude $\mu = (4, 1)$. Remark 2.7 (ii) provides us with the X-face $\gamma_{5,6,7}$. Proposition 3.6 says that μ should lie in $Q(\gamma_{5,6,7})$ or in $w_1 + Q(\gamma_{5,6,7})$. A contradiction. ∎

We treat Case 5.3 IIa, that means that the degree of the relation lies in the interior of the effective cone, is proportional to some Cox ring generator degree and $\varrho_1 = \varrho_2$, as well as $\varrho_{r-1} = \varrho_r$ hold.

Lemma 5.10. *In Setting 5.1, assume that* $\mathrm{Mov}(R) = \mathrm{Eff}(R)$ *and* $\mu \in \mathrm{Eff}(R)^\circ$ *hold. Let* Ω *denote the set of two-dimensional* $\lambda \in \Lambda(R)$ *with* $\lambda^\circ \subseteq \mathrm{Mov}(R)^\circ$.

- (i) *If* $X(\lambda)$ *is locally factorial for some* $\lambda \in \Omega$, *then* $\mathrm{Eff}(R)$ *is a regular cone and every* w_i *on the boundary of* $\mathrm{Eff}(R)$ *is primitive.*
- (ii) *If* $X(\lambda)$ *is locally factorial for each* $\lambda \in \Omega$, *then, for any* $w_i \in \mathrm{Eff}(R)^\circ$, *we have* $w_i = w_1 + w_r$ *or* g *has a monomial of the form* $T_i^{l_i}$.

Proof. We show (i). Let $w_i \in \varrho_r$. Due to $\mu \in \mathrm{Eff}(R)^\circ$, the relation g has no monomial of the form $T_i^{l_i}$. Thus, Lemmas 5.7 and 5.8 applied to the triple w_1, w_2, w_i show that w_i is primitive. Analogously, we see that any $w_i \in \varrho_1$ is primitive. In particular, we have $w_1 = w_2$. Thus, applying Lemma 5.7 to w_1, w_2, w_r, we obtain that $\mathrm{Eff}(R)$ is a regular cone.

We turn to (ii). By (i), we may assume $w_1 = w_2 = (1, 0)$ and $w_{r-1} = w_r = (0, 1)$. Consider $w_i \in \mathrm{Eff}(R)^\circ$ such that $T_i^{l_i}$ is not a monomial of g. Then we find GIT-cones $\lambda_1 \subseteq \mathrm{cone}(w_1, w_i)$ and $\lambda_2 \subseteq \mathrm{cone}(w_i, w_r)$ defining locally factorial varieties $X(\lambda_1)$ and $X(\lambda_2)$ respectively. Lemma 5.7, applied to w_1, w_2, w_i together with $X(\lambda_1)$ and to w_i, w_{r-1}, w_r together with $X(\lambda_2)$ shows $w_i = (1, 1) = w_1 + w_r$. ∎

Lemma 5.11. *In Setting 5.1, assume that $X = X(\lambda)$ is locally factorial and R_g is a spread hypersurface Cox ring.*

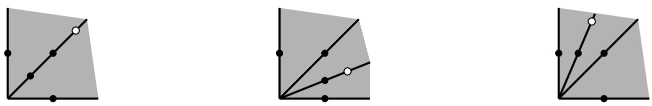
(i) *If w_i lies on the ray through μ, then g has a monomial of the form $T_i^{l_i}$, where $l_i \geq 2$.*

(ii) *If w_i, w_j, where $i \neq j$, lie on the ray through μ, then $\varrho_i = \varrho_j \in \Lambda(R_g)$ holds.*

Proof. We show (i). Suppose that g has no monomial of the form $T_i^{l_i}$ where $l_i \geq 2$. As R_g is a hypersurface Cox ring, also T_i is not a monomial of g. Then, on one of the extremal rays of $\mathrm{Eff}(R)$, we find a w_j such that $\gamma_{i,j}$ is a X-face; see Remark 2.7 (i). Proposition 3.5 (ii) yields that w_i, w_j generate $\mathbb{Z}^2$ as a group. In particular, w_i is primitive. Hence $\mu = k w_i$ holds for some $k \in \mathbb{Z}_{\geq 1}$. As R_g is spread, T_i^k must be a monomial of g. In addition, we obtain $k \geq 2$. A contradiction.

We prove (ii). Assertion (i) just proven and Remark 2.7 (i) tell us that $\gamma_{i,j}$ is an $\bar{X}$-face. Thus, being a ray, $Q(\gamma_{i,j}) = \varrho_i = \varrho_j$ belongs to the GIT-fan $\Lambda(R_g)$. ∎

Proof of Theorem 1.1, Part IIa. We deal with the specifying data of a smooth general hypersurface Cox ring R as in Remark 5.3 IIa defining a smooth Fano fourfould $X = X(\lambda)$. By Proposition 2.4, the relation degree μ lies on ϱ_3, ϱ_4 or ϱ_5. We claim that we cannot have $\varrho_3 = \varrho_4 = \varrho_5$. Otherwise Lemma 5.11 shows $\eta = \mathrm{cone}(w_1, w_3) \in \Lambda(R)$. Since $X(\eta)$ is smooth by Remark 5.4, we may apply Lemma 5.7 to the triple w_1, w_3, w_4. According to Lemma 5.8, we obtain $\det(w_1, v) = 1$, where v denotes the primitive generator of the ray ϱ_3. Analogous arguments yield $\det(v, w_7) = 1$. Using both determinantal equations, we conclude that v and $w_1 + w_7$ are collinear. In particular, $w_1 + w_7$ generates $\varrho_3 = \varrho_4 = \varrho_5$. Lemma 5.10 (i) tells us $w_1 = w_2$ and $w_6 = w_7$. As a result, Proposition 3.7 gives $-\mathcal{K}_X \in \varrho_3$. Moreover, Lemma 5.11 (ii) tells us $\varrho_3 \in \Lambda(R_g)$ and thus $\lambda = \varrho_3$, which contradicts $\mathbb{Q}$-factoriality, see Proposition 3.5 (i). A suitable admissible coordinate change yields $\mu \notin \varrho_5$ and we are left with the following three constellations:



(i) $\varrho_3 = \varrho_4$, $\mu \in \varrho_3$ (ii) $\varrho_3 \neq \varrho_4$, $\mu \in \varrho_3$ (iii) $\varrho_3 \neq \varrho_4$, $\mu \in \varrho_4$

By Lemma 5.10 (i), we can assume $w_1 = w_2 = (1, 0)$ and $w_6 = w_7 = (0, 1)$. We show $w_5 = (0, 1)$. Otherwise, by Lemma 5.10 (ii), we must have $w_5 = (1, 1)$. Consider $\lambda' = \mathrm{cone}(w_5, w_6)$. Then $\mu \notin \lambda'$ holds. Remark 2.7 (ii) tells us that $\gamma_{5,6}$ is an X'-face and hence λ' is a GIT-cone. The associated variety X' is smooth according to Remark 5.4. Thus, Proposition 3.6 yields $\mu \in w_i + \lambda'$ for some $1 \leq i \leq 7$. By the geometry of the possible degree constellations, only $i = 1, 2$ come into consideration. We conclude $\mu = (e + 1, e + f)$ with $e, f \in \mathbb{Z}_{\geq 0}$. Positive orientation of (μ, w_5) gives $f = 0$. Hence, μ is primitive. By Lemma 5.11 (i), this contradicts R_g being a spread hypersurface ring.

Constellation (i). Let $v = (v_1, v_2)$ be the primitive generator of $\varrho_3 = \varrho_4$. Thanks to Lemma 5.11 (ii), we have $\varrho_3 \in \Lambda(R_g)$ and thus also $\lambda' = \mathrm{cone}(w_3, w_7)$ is a GIT-cone. The associated variety X' is smooth by Remark 5.4. Applying Lemmas 5.7 and 5.8 to the triple w_3, w_4, w_7 yields $v_1 = 1$ and that the first coordinates of w_3, w_4 are coprime. Arguing similarly with w_1, w_3, w_4 gives $v_2 = 1$. So, the degree matrix has the form

$$Q = [w_1, \dots, w_7] = \begin{bmatrix} 1 & 1 & a & b & 0 & 0 & 0 \\ 0 & 0 & a & b & 1 & 1 & 1 \end{bmatrix}, \quad a, b \in \mathbb{Z}_{\geq 1}, \ \gcd(a, b) = 1.$$

We may assume $a \leq b$. By Lemma 5.11 (i), the relation g has monomials of the form $T_3^{l_3}$ and $T_4^{l_4}$. Since $\gcd(a, b) = 1$ holds, we conclude $\mu_1 = \mu_2 = dab$ with $d \in \mathbb{Z}_{\geq 1}$. In particular $\mu_1 \geq ab$ holds. By Proposition 3.7, the anticanonical class is given as

$$-\mathcal{K}_X = (2 + a + b - \mu_1, \ 3 + a + b - \mu_2).$$

From X being Fano we deduce $-\mathcal{K}_X \in \mathrm{Eff}(R)^\circ$, that means that each coordinate of $-\mathcal{K}_X$ is positive. Thus, we obtain

$$2 + a + b > dab \geq ab.$$

This implies $a = 1$ or $a = 2, b = 3$. In the case $a = 1$, using the inequality again leads to $3 + (1 - d)b > 0$ and we end up with possibilities

$$b = 1, \ d = 2, 3, \quad b = 2, \ d = 2,$$

leading to the specifying data of Numbers 22 to 24 of Theorem 1.1. The constellation $a = 2, b = 3$ immediately implies $d = 1$, which gives the specifying data of Number 25 of Theorem 1.1.

It remains to show that these specifying data yield Fano smooth general hypersurface Cox rings. We work in the setting of Construction 4.1 and treat exemplarily Number 25. From Remark 4.8 we infer that for all $g \in U_\mu$ the algebra R_g has $T_1, \dots, T_7$ as a minimal system of generators R_g. Moreover, Proposition 4.10 provides us with a non-empty open subset $U \subseteq S_\mu$ such that $T_1, \dots, T_7$ define primes in R_g for all $g \in U$, provided we deliver for each i a μ-homogeneous prime polynomial not depending on T_i. Here they are:

$$T_3^3 - T_4^2 \quad \text{for } i = 1, 2, 5, 6, 7, \qquad T_1^6 T_5^6 - T_2^6 T_6^5 T_7 \quad \text{for } i = 3, 4.$$

In Construction 4.1, consider $\lambda = \mathrm{cone}(w_3) \in \Lambda(R_g)$. Then $\lambda^\circ \subseteq \mathrm{Mov}(S)^\circ$ holds and we have $\mu \in \lambda^\circ$. One directly verifies that μ is basepoint free for Z. Thus, Proposition 4.11 (i) shows that after shrinking U suitably, R_g admits unique factorization in R_g for all $g \in U$. Since X_g should be a Fano variety, $-\mathcal{K} = (1, 2)$ has to be ample and thus we have to take $\lambda = \mathrm{cone}(w_4, w_5)$. Then Z_μ is smooth and $\mu \in \lambda$ holds. Thus, Proposition 4.19 shows that after possibly shrinking U again, X_g is smooth for all $g \in U$.

Constellation (ii). Here we obtain $w_4 = (0, 1)$ by the same arguments used for showing $w_5 = (0, 1)$. Write $w_3 = (a_3, b_3)$ and let k be the unique positive integer with $\mu = k w_3$. Then $k \geq 2$ as R_g is spread and $T_1, \dots, T_7$ form a minimal system of generators. By Proposition 3.7, the anticanonical class of $X = X(\lambda)$ is given as

$$-\mathcal{K}_X = (2 + (1 - k)a_3, \ 4 + (1 - k)b_3).$$

Moreover, we have $\varrho_3 \notin \Lambda(R_g)$ due to Lemma 5.11 (i) and Remark 2.7 (i), the defining GIT-cone λ of X is the positive orthant. Thus the Fano condition $-\mathcal{K}_X \in \lambda^\circ$ simply means that both coordinates of $-\mathcal{K}_X$ are positive. This leads to $a_3 = 1$, $k = 2$ and $b_3 \leq 3$. These are Numbers 26 to 28 of Theorem 1.1.

The verification of these candidates as specifying data of a general smooth Fano hypersurface Cox ring is done by the same arguments as in Case 5.5 (i-a) from Part I, except that for Numbers 27 and 28 one has to verify smoothness of Z_μ explicitly.

Constellation (iii). We obtain $w_3 = (1, 0)$ by analogous arguments as used for showing $w_5 = (0, 1)$ before. The degree $w_4 = (a_4, b_4)$ has to be determined. A suitable admissible coordinate change yields $a_4 \geq b_4$. By Proposition 3.7, the anticanonical class of $X = X(\lambda)$ is given as

$$-\mathcal{K}_X = (3 + (1-k)a_4,\ 3 + (1-k)b_4),$$

where $k \in \mathbb{Z}_{\geq 0}$ is defined via $\mu = kw_3$. As in the preceding constellation, we see that λ is the positive orthant. Thus, $X(\lambda)$ being Fano just means that both coordinates of $-\mathcal{K}_X$ are positive. We end up with the specifying data from Numbers 29 to 32 of Theorem 1.1.

In order to verify these candidates, one proceeds by the same arguments as used in Case 5.5 (i-a) from Part I, except that smoothness of Z_μ has to be checked explicitly. ■

We treat Case 5.3 III, that means that the degree μ of the relation lies in the bounding ray ϱ_1 of the effective cone.

Lemma 5.12. *Let $X = X(\lambda)$ be as in Setting 5.1, and let $1 \leq i < j \leq r$ be such that g neither depends on T_i nor on T_j. If X is quasismooth, then w_i, w_j lie either both in λ^- or both in λ^+.*

Proof. Otherwise, we may assume $w_i \in \lambda^-$ and $w_j \in \lambda^+$. Then $\gamma_{i,j}$ is an X-face and $\bar{X}(\gamma_{i,j})$ is a singular point of $\bar{X}$. According to Proposition 3.5 (iv), this contradicts quasismoothness of X. ■

Proof of Theorem 1.1, *Part* III. We may assume that the ray ϱ_1 is generated by the vector $(1, 0)$. Let m be the number with $w_1, \ldots, w_m \in \varrho_1$ and $w_{m+1}, \ldots, w_7 \notin \varrho_1$. Observe that due to $\mu \in \varrho_1$, the relation g only depends on $T_1, \ldots, T_m$.

The first step is to show that only for $m = 5$, the specifying data $w_1, \ldots, w_7$ and μ in $K = \mathbb{Z}^2$ allow a hypersurface Cox ring. Since $\mu \in \varrho_1$, Proposition 2.4 yields $m \geq 3$. As $\mathrm{Mov}(X)$ is of dimension two, we must have $m \leq 5$; see Setting 5.1. Lemma 5.12 shows $w_{m+1}, \ldots, w_r \in \lambda^+$. Applying Lemma 5.7 to triples w_1, w_2, w_i for $i \geq m + 1$, we obtain

$$\mu = (\mu_1, 0), \quad w_i = (a_i, 0),\ i = 1, \ldots, m, \quad w_i = (a_i, 1),\ i = m+1, \ldots, 7,$$

where, for any two $1 \leq i < j \leq m$, the numbers a_i and a_j are coprime and we may assume $a_7 = 0$. Moreover, we must have $a_{m+1} = \cdots = a_6$, because otherwise we obtain a GIT cone $\lambda \neq \eta \in \Lambda(R)$ with $\eta^\circ \in \mathrm{Mov}(R)^\circ$ and the associated variety $X(\eta)$ is not quasismooth by Lemma 5.12, contradicting Remark 5.4. Proposition 3.7 and the fact that X is Fano give us

$$(a_1 + \cdots + a_6 - \mu_1,\ 7 - m) = -\mathcal{K}_X \in \lambda^\circ = \mathrm{cone}((1, 0),\ (a_{m+1}, 1))^\circ.$$

Since $a_1, \ldots, a_m$ are pairwise coprime, the component μ_1 of the degree of the relation g is greater or equal to $a_1 \cdots a_m$. Using moreover $a_{m+1} = \cdots = a_6$, we derive from the above Fano condition

$$a_1 \cdots a_m \leq \mu_1 < a_1 + \cdots + a_m - a_{m+1},$$

where we may assume $a_1 \le \dots \le a_m$. We exclude $m = 3$: here, $g = g(T_1, T_2, T_3)$, the above inequality forces $a_1 = a_2 = 1$, hence $g(T_1, T_2, 0)$ is classically homogeneous and T_3 is not prime in R, a contradiction. Let us discuss $m = 4$. The above inequality and pairwise coprimeness of the a_i leave us with

$$a_1 = a_2 = a_3 = 1, \quad a_1 = a_2 = 1,\ a_3 = 2,\ a_4 = 3.$$

In the case $a_3 = 1$, we must have $\mu_1 = ka_4$ with some $k \in \mathbb{Z}_{\ge 2}$, because otherwise, the relation would be redundant or, seen similarly as above, one of T_1, T_2, T_3 would not be prime in R. The inequality gives $(k-1)a_4 < 3 - a_{m+1}$. We arrive at the following possibilities:

$$a_{m+1} = a_4 = 1,\ k = 2, \quad a_{m+1} = 0,\ a_1 = 1,\ k = 2, 3, \quad a_{m+1} = 0,\ a_1 = k = 2.$$

The first constellation implies that R is not factorial and hence is excluded. In the each of remaining ones, X is a product of $\mathbb{P}_2$ and a surface Y which must be smooth as X is so. Moreover, for the Picard numbers, we have

$$\rho(X) = \rho(\mathbb{P}_2) + \rho(Y).$$

Thus, $\rho(Y) = 1$. Finally, being a Mori fiber, Y is a del Pezzo surface. We arrive at $Y = \mathbb{P}_2$ and hence X is toric. A contradiction to X having a hypersurface Cox ring. We conclude that $m = 5$ is the only possibility. In this case, $\lambda = \operatorname{cone}(w_1, w_6)$ holds and our degree matrix is of the form

$$Q = [w_1, \dots, w_7] = \begin{bmatrix} a_1 & \dots & a_5 & a_6 & 0 \\ 0 & \dots & 0 & 1 & 1 \end{bmatrix}, \quad 1 \le a_1 \le \dots \le a_5,\ 0 \le a_6.$$

As mentioned before, g neither depends on T_6 nor on T_7. Consequently, we can write R as a polynomial ring over a K-graded subalgebra $R' \subseteq R$ as follows:

$$R = R'[T_6, T_7], \quad R' := \mathbb{C}[T_1, \dots, T_5]/\langle g \rangle.$$

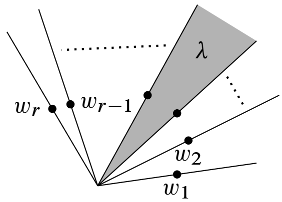
Moreover, R' is $\mathbb{Z}$-graded via $\deg(T_i) := a_i$. We claim that the $\mathbb{Z}$-graded algebra R' is a smooth Fano hypersurface Cox ring. if the K-graded algebra R is so. First observe that R' inherits the properties of an abstract Cox ring from R. Moreover, with $\bar{X}' = V(g) \subseteq \mathbb{C}^5$, we have $\bar{X} = \bar{X}' \times \mathbb{C}^2$. Now, the action of the one-dimensional torus $H' = \operatorname{Spec} \mathbb{C}[\mathbb{Z}]$ on $\bar{X}'$ admits a unique projective quotient in the sense of Construction 3.2, namely

$$X' = \hat{X}' /\!\!/ H', \quad \hat{X}' = \bar{X}' \setminus \{0\}.$$

Propositions 3.3 and 3.7 show that X' is a Fano variety. Observe that each X'-face of $\gamma_0' \preccurlyeq \gamma'$ of the orthant $\gamma' \subseteq \mathbb{Q}^5$ defines and X-face $\gamma_0 = \gamma_0' + \operatorname{cone}(e_6, e_7)$. In particular, using Proposition 3.5 (ii) and (iv), we see that X' is smooth if X is so. Moreover, R' is a smooth hypersurface Cox ring if R is so. The smooth Fano threefolds with hypersurface Cox ring are listed in Theorem 4.1 of [15], which gives us the possible values of $a_1, \dots, a_5$ and from the Fano condition on X, we infer $a_6 + \mu_1 < a_1 + \dots + a_5$. So, we end up with the specifying data as in Theorem 1.1 Numbers 58 to 67. To show that these data indeed produce smooth Fano general hypersurface Cox rings, one proceeds by using our toolbox in a similar way as in the previously presented parts of the proof. ■

6. Birational geometry

We begin with a look at the birational geometry of the Fano fourfolds from Theorem 1.1. Let us briefly recall the necessary background. Consider any $\mathbb{Q}$-factorial Mori dream space $X = X(\lambda)$ arising from an abstract Cox ring $R = \oplus_K R_w$ as in Construction 3.2. Assume that $K_{\mathbb{Q}} = \mathrm{Cl}_{\mathbb{Q}}(X)$ is of dimension two. Then the GIT-fan $\Lambda(R)$ looks as follows:



where, as in Setting 5.1, we order the generator degrees $w_1, \dots, w_r \in K$ of R counterclockwise. The moving cone $\mathrm{Mov}(X)$ is spanned by w_2 and w_{r-1}. If $w_2 \in \lambda$ holds, then with $\tau = \mathrm{cone}(w_2)$ we have

$$\bar{X}^{ss}(\lambda) \subseteq \bar{X}^{ss}(\tau),$$

which induces a morphism $\pi: X \to Y$ from $X = \bar{X}^{ss}(\lambda) /\!\!/ H$ onto $Y = \bar{X}^{ss}(\tau) /\!\!/ H$. Recall that π is an elementary contraction in the sense of [8]. In particular, we have the following two possibilities:

- If $w_2 \notin \mathrm{cone}(w_1)$ holds, then $\pi: X \to Y$ is birational and contracts the prime divisor $D_1 \subseteq X$ corresponding to the ray through w_1. In this case, we write $X \sim Y$ for the morphism π and denote by $C \subseteq Y$ the center of the contraction.
- $\pi: X \to Y$ is a proper fibration with $\dim(Y) < \dim(X)$. In this case, we write $X \to Y$ for the morphism π and denote by $F \subseteq X$ the general fiber.

Similarly, if $w_{r-2} \in \mathrm{Mov}(X)$ holds, we use the same notation. In general, λ need not to have common rays with $\mathrm{Mov}(X)$. However, given a ray $\varrho \subseteq \mathrm{Mov}(X)$, we find a small quasimodification $X \dashrightarrow X'$, where X' stems from a chamber $\lambda' \in \Lambda(R)$ sharing the ray ϱ with $\mathrm{Mov}(X)$. We then write $X' \sim Y$ or $X' \to Y$ etc. accordingly.

Remark 6.1. If X is as in Theorem 1.1, then X admits at least one elementary contraction and at most one small quasimodification $X \dashrightarrow X'$. If there is one, then X' is smooth due to Remark 5.4.

Now assume in addition that X has a hypersurface Cox ring and consider the toric embedding $X = X_g \subseteq Z$ from Construction 4.1. Given an elementary contraction of $\pi: X_g \to Y$, a suitable choice of the cone τ in Construction 4.1 leads to a commutative diagram

$$\begin{array}{ccc} X & \subseteq & Z \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi_Z} \\ Y & \subseteq & W \end{array}$$

where $\pi_Z: Z \to W$ is an elementary contraction of the ambient toric variety Z. In particular, we have in this setting that for every point $y \in Y$, the fiber $\pi^{-1}(y) \subseteq X$ is contained in the fiber $\pi_Z^{-1}(y) \subseteq Z$. This gives in particular a description for the general fiber $F \subseteq X$ as a subvariety of the general fiber $F_Z \subseteq Z$.

Let us fix the necessary notation to formulate the result. By $Y_{d;a_1^{k_1},\dots,a_n^{k_n}}$ we denote a (not necessarily general) hypersurface of degree d in the weighted projective space $\mathbb{P}_{a_1^{k_1},\dots,a_n^{k_n}}$, where, as usual, $a_i^{k_i}$ means that $a_i \in \mathbb{Z}_{\geq 1}$ is repeated k_i times. For a hypersurface of degree d in the classical projective space $\mathbb{P}_n$ we just write $Y_{d;n}$. In our situation, this notation applies to the target spaces $Y \subseteq W$ in case of a birational elementary contraction and to the general fiber $F \subseteq F_Z$ in case of a fibration.

Proposition 6.2. *The subsequent table lists the possible elementary contractions for X as in Theorem 1.1, where X is not a cartesian product; the notation Y^* in the context of a birational contraction indicates that the target space is singular.*

No.	Contraction 1	Contraction 2
1	$X \to \mathbb{P}_3$, $F = Y_{1;2}$	$X \to \mathbb{P}_2$, $F = Y_{1;3}$
2	$X \to \mathbb{P}_3$, $F = Y_{1;2}$	$X \to \mathbb{P}_2$, $F = Y_{2;3}$
3	$X \to \mathbb{P}_3$, $F = Y_{1;2}$	$X \to \mathbb{P}_2$, $F = Y_{3;3}$
4	$X \to \mathbb{P}_3$, $F = Y_{2;2}$	$X \to \mathbb{P}_2$, $F = Y_{1;3}$
5	$X \to \mathbb{P}_3$, $F = Y_{2;2}$	$X \to \mathbb{P}_2$, $F = Y_{2;3}$
6	$X \to \mathbb{P}_3$, $F = Y_{2;2}$	$X \to \mathbb{P}_2$, $F = Y_{3;3}$
7	$X \to \mathbb{P}_3$, $F = Y_{1;2}$	$X \sim Y_{2;5}$, $C = \mathbb{P}_1$
8	$X \to \mathbb{P}_3$, $F = Y_{2;2}$	$X \sim Y_{3;5}$, $C = \mathbb{P}_1$
9	$X \to \mathbb{P}_3$, $F = Y_{1;2}$	$X \sim Y^*_{3;5}$, $C = \mathbb{P}_1$
10	$X \to \mathbb{P}_3$, $F = Y_{2;2}$	$X \sim Y^*_{4;5}$, $C = \mathbb{P}_1$
11	$X \to \mathbb{P}_3$, $F = Y_{1;2}$	$X \sim Y^*_{3;1^4,2^2}$, $C = \mathbb{P}_1$
12	$X \to \mathbb{P}_3$, $F = Y_{2;2}$	$X \sim Y^*_{5;1^4,2^2}$, $C = \mathbb{P}_1$
13	$X \to \mathbb{P}_2$, $F = Y_{2;3}$	$X' \to \mathbb{P}_2$, $F = Y_{1;3}$
14	$X \to \mathbb{P}_2$, $F = Y_{3;3}$	$X' \to \mathbb{P}_2$, $F = Y_{2;3}$
15	$X \to \mathbb{P}_2$, $F = Y_{3;2}$	$X' \sim Y_{4;1^5,2}$, $C = \mathbb{P}_1$
16	$X \to \mathbb{P}_3$, $F = Y_{1;2}$	$X' \to \mathbb{P}_1$, $F = Y_{2;4}$
17	$X \to \mathbb{P}_3$, $F = Y_{2;2}$	$X' \to \mathbb{P}_1$, $F = Y_{3;4}$
18	$X \sim Y_{4;5}$, $C = \mathbb{P}_2$	$X \to \mathbb{P}_2$, $F = Y_{3;3}$
19	$X \sim Y_{3;5}$, $C = \mathbb{P}_2$	$X \to \mathbb{P}_2$, $F = Y_{2;3}$
20	$X \sim Y_{2;5}$, $C = \mathbb{P}_2$	$X \to \mathbb{P}_2$, $F = Y_{1;3}$
21	$X \sim Y_{4;1^5,2}$, $C = \mathbb{P}_2$	$X \to \mathbb{P}_1$, $F = Y_{3;4}$
22	$X' \to \mathbb{P}_1$, $F = Y_{2;4}$	$X \to \mathbb{P}_2$, $F = Y_{2;3}$
23	$X' \to \mathbb{P}_1$, $F = Y_{3;4}$	$X \to \mathbb{P}_2$, $F = Y_{3;3}$
24	$X' \to \mathbb{P}_1$, $F = Y_{4;1^4,2}$	$X \to \mathbb{P}_2$, $F = Y_{4;1^3,2}$
25	$X' \to \mathbb{P}_1$, $F = Y_{6;1^3,2,3}$	$X \to \mathbb{P}_2$, $F = Y_{6;1^2,2,3}$
26	$X \to \mathbb{P}_1$, $F = Y_{2;4}$	$X \to \mathbb{P}_3$, $F = Y_{2;2}$

No.	Contraction 1	Contraction 2
27	$X \to \mathbb{P}_1$ $F = Y_{4;1^4,2}$	$X \to \mathbb{P}_3$ $F = Y_{2;2}$
28	$X \to \mathbb{P}_1$ $F = Y_{6;1^4,3}$	$X \to \mathbb{P}_3$ $F = Y_{2;2}$
29	$X \to \mathbb{P}_2$ $F = Y_{2;3}$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
30	$X \to \mathbb{P}_2$ $F = Y_{3;3}$	$X \to \mathbb{P}_2$ $F = Y_{3;3}$
31	$X \to \mathbb{P}_2$ $F = Y_{2;3}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
32	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
33	$X' \sim Y^*_{6;1^4,2,3}$ $C = \{\text{pt}\}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
34	$X \sim Y_{2;5}$ $C = \mathbb{P}_1 \times \mathbb{P}_1$	$X \to \mathbb{P}_1$ $F = Y_{2;4}$
35	$X \sim Y_{3;5}$ $C = Y_{3;3}$	$X \to \mathbb{P}_1$ $F = Y_{3;4}$
36	$X \sim Y_{4;5}$ $C = Y_{4;3}$	$X \to \mathbb{P}_1$ $F = Y_{4;4}$
37	$X \sim Y_{4;1^5,2}$ $C = Y_{4;1^3,2}$	$X \to \mathbb{P}_1$ $F = Y_{4;1^4,2}$
38	$X \sim Y_{6;1^5,3}$ $C = Y_{6;1^3,3}$	$X \to \mathbb{P}_1$ $F = Y_{6;1^4,3}$
39	$X \sim Y_{6;1^4,2,3}$ $C = Y_{6;1^2,2,3}$	$X \to \mathbb{P}_1$ $F = Y_{6;1^3,2,3}$
40	$X \sim Y_{2;5}$ $C = \mathbb{P}_1$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
41	$X \sim Y_{3;5}$ $C = Y_{3;2}$	$X \to \mathbb{P}_2$ $F = Y_{3;3}$
42	$X \sim Y^*_{4;1^3,2^3}$ $C = \mathbb{P}_1$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
43	$X \sim Y^*_{6;1^2,2^3}$ $C = Y_{3;2}$	$X \to \mathbb{P}_2$ $F = Y_{3;3}$
44	$X \sim Y_{4;1^5,2}$ $C = Y_{4;1^2,2}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$

No.	Contraction 1	Contraction 2
45	$X \sim Y^*_{8;1^3,2^2,4}$ $C = Y_{4;1^2,2}$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
46	$X \sim Y_{6;1^4,2,3}$ $C = Y_{6;1,2,3}$	$X \to \mathbb{P}_2$ $F = Y_{6;1^2,2,3}$
47	$X \sim Y^*_{12;1^3,2,4,6}$ $C = Y_{6;1,2,3}$	$X \to \mathbb{P}_2$ $F = Y_{6;1^2,2,3}$
48	$X \sim \mathbb{P}_4$ $C = \mathbb{P}_1$	$X' \to \mathbb{P}_1$ $F = Y_{2;4}$
49	$X \sim Y^*_{6;1^2,2^3,3}$ $C = Y_{6;2^3,4}$	$X' \to \mathbb{P}_1$ $F = Y_{3;4}$
50	$X \sim Y^*_{4;1^5,2}$ $C = \mathbb{P}_1$	$X \to \mathbb{P}_2$ $F = Y_{2;3}$
51	$X \sim Y^*_{6;1^5,3}$ $C = \mathbb{P}_1$	$X \to \mathbb{P}_2$ $F = Y_{4;1^3,2}$
52	$X \sim Y^*_{6;1^4,2,3}$ $C = \mathbb{P}_1$	$X \to \mathbb{P}_1$ $F = Y_{4;1^4,2}$
53	$X \sim \mathbb{P}_4$ $C = \mathbb{P}_1 \times \mathbb{P}_1$	$X \sim Q_4$ $C = \{\text{pt}\}$
54	$X \sim Y_{4;1^5,2}$ $C = Y_{4;3}$	$X \sim Y_4$ $C = \{\text{pt}\}$
55	$X \sim \mathbb{P}_4$ $C = Y_{3;3}$	$X \sim Y^*_{3;1^5,2}$ $C = \{\text{pt}\}$
56	$X \sim \mathbb{P}_4$ $C = Y_{4;3}$	$X \sim Y_{3;1^5,3}$ $C = \{\text{pt}\}$
57	$X \sim Y_{6;1^4,2,3}$ $C = Y_{4;1^3,3}$	$X \sim Y_{6;1^5,3}$ $C = \{\text{pt}\}$
60	$X \to Y_{6;1^3,2,3}$ $F = \mathbb{P}_1$	$X \sim Y^*_{6;1^4,2,3}$ $C = \{\text{pt}\}$
62	$X \to Y_{4;1^4,2}$ $F = \mathbb{P}_1$	$X \sim Y^*_{4;1^5,2}$ $C = \{\text{pt}\}$
64	$X \to Y_{3;4}$ $F = \mathbb{P}_1$	$X \sim Y^*_{3;5}$ $C = \{\text{pt}\}$
66	$X \to Y_{2;4}$ $F = \mathbb{P}_1$	$X \sim Y^*_{2;5}$ $C = \{\text{pt}\}$
67	$X \to Y_{2;4}$ $F = \mathbb{P}_1$	$X \sim Y^*_{2;1^5,2}$ $C = \{\text{pt}\}$

The remaining families of Theorem 1.1 consist of cartesian products $Y \times \mathbb{P}_1$ where the first factor Y is a smooth threedimensional Fano hypersurface of Picard number one as displayed in the following table.

No.	58	59	61	63	65
Y	$Y_{6;1^4,3}$	$Y_{6;1^3,2,3}$	$Y_{4;1^4,2}$	$Y_{3;4}$	$Y_{2;4}$

The proof of this proposition is basically a case by case analysis of the contraction maps in coordinates. We restrict ourselves to perform this in the subsequent remark for one case, where we even go a bit deeper into the matter and specify also the singular fibers of the fibration.

Remark 6.3. We take a closer look at the varieties X from No. 9 of Theorem 1.1. In this case the specifying data, that means the degree matrix Q and the degree μ of the relation g, are given by

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \qquad \mu = (2, 1).$$

Due to $-\mathcal{K} = (1, 2)$, we have $\lambda = \operatorname{cone}(w_1, w_5)$. Observe that $\operatorname{Mov}(R)$ and λ share the rays ϱ_1 and ϱ_5. Thus X admits two elementary contractions $\pi_1 \colon X \to Y_1$ and $\pi_2 \colon X \to Y_2$ associated to ϱ_1 resp. ϱ_5. To study π_1 and π_2 we make use of the toric embedding $X = X_g \subseteq Z$ from Construction 4.1.

First, we discuss π_1. Since $w_2 \in \varrho_1$ holds, the morphism π_1 is a fibration. Moreover, π_1 is the restriction of the corresponding ambient toric elementary contraction $\pi_{1,Z}$ of Z, which in turn is explicitly given as follows:

$$\begin{array}{ccccc} \bar{X} & \subseteq & \mathbb{K}^7 & \xrightarrow{(z_1,\ldots,z_7) \mapsto (z_1,\ldots,z_4)} & \mathbb{K}^4 \\ \downarrow & & \downarrow & & \downarrow \\ X & \subseteq & Z & \xrightarrow{\quad \pi_{1,Z} \quad} & \mathbb{P}_3 \end{array}$$

Suitably sorting the terms of g yields a presentation $g = q_1 T_5 + q_2 T_6 + f T_7$ where $q_1, q_2 \in \mathbb{K}[T_1, \ldots, T_4]$ both are quadrics and $f \in \mathbb{K}[T_1, \ldots, T_4]$ is a cubic, each of which is general. Note that $V(g) \subseteq \mathbb{K}^7$ projects onto $\mathbb{K}^4$ thus $Y_1 = \mathbb{P}_3$. For any point $y = [y_1, \ldots, y_4] \in \mathbb{P}_3$ the fiber $\pi_{1,Z}^{-1}(y)$ of the ambient toric variety is given by the equations

$$y_2 T_1 - y_1 T_2 = y_3 T_2 - y_2 T_3 = y_4 T_3 - y_3 T_4 = 0.$$

Besides we have $y_i \neq 0$ for some i. Taking this into account one directly checks $\pi_{1,Z}^{-1}(y) \cong \mathbb{P}_2$. Being homogeneous g is compatible with this isomorphism, thereby we obtain

$$\pi_1^{-1}(y) \;\cong\; V(y_i q_1(y) T_0 + y_i q_2(y) T_1 + f(y) T_2) \subseteq \mathbb{P}_2.$$

We conclude that the general fiber $\pi_1^{-1}(y)$ is isomorphic to $\mathbb{P}_1$. In addition, $V(q_1, q_2, f) \subseteq \mathbb{P}_3$ consists of precisely 12 points $p_1, \ldots, p_{12}$, each of which has fiber $\pi_1^{-1}(p_i) \cong \mathbb{P}_2$.

We turn to π_2. From $w_7 \notin \varrho_5$ follows that π_2 is a birational morphism contracting the prime divisor $V(T_7) \subseteq X$. The according elementary contraction $\pi_{2,Z}$ of the ambient toric variety Z is the blow-up of $\mathbb{P}_5$ along $C = V_{\mathbb{P}_5}(T_0, \ldots, T_3) \cong \mathbb{P}_1$. The situation is as in the subsequent diagram:

$$\begin{array}{ccccc} \bar{X} & \subseteq & \mathbb{K}^7 & \xrightarrow{(z_1,\ldots,z_7)\mapsto(z_1z_7,\ldots,z_4z_7,z_5,z_6)} & \mathbb{K}^6 \\ \downarrow & & \downarrow & & \downarrow \\ X & \subseteq & Z & \xrightarrow{\pi_{2,Z}} & \mathbb{P}_5 \end{array}$$

The target variety $Y_2 \subseteq \mathbb{P}_5$ of π_2 is $V(g') \subseteq \mathbb{P}_5$ where $g' = g(T_0, \ldots, T_6, 1)$. From this we infer $C \subseteq Y_2$, so C is the center of π_2 as well. In particular π_2 is the blow-up of Y_2 along C. Moreover, the polynomial g' is an irreducible cubic living in $\langle T_0, \ldots, T_3\rangle^2$. Consequently, Y_2 is singular at every point of C.

7. Hodge numbers

Here we determine the Hodge numbers of the Fano fourfolds from Theorem 1.1. First, we note the following simple observation.

Proposition 7.1. *Let X be a smooth projective Fano fourfold of Picard rank 2. Then the Hodge diamond of X is the following*:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 2 & & 0 & & \\ & 0 & & h^{1,2} & & h^{2,1} & & 0 & \\ 0 & & h^{1,3} & & h^{2,2} & & h^{3,1} & & 0 \\ & 0 & & h^{3,2} & & h^{2,3} & & 0 & \\ & & 0 & & 2 & & 0 & & \\ & & & 0 & & 0 & & & \\ & & & & 1 & & & & \end{array}$$

Proof. Ampleness of $-K_X$ and the Kawamata–Viehweg vanishing theorem give $h^{p,0}(X) = 0$ for any $p > 0$. Moreover, plugging $H^i(X, \mathcal{O}) = 0$ for $i = 1, 2$ into the cohomology sequence associated with the exponential sequence yields $H^2(X, \mathbb{C}) \cong \mathbb{C}^2$. The Hodge decomposition together with $h^{1,0}(X) = h^{0,1}(X) = 0$ shows $h^{1,1}(X) = 2$. ■

By symmetry, we are left with computing the Hodge numbers $h^{2,1}$, $h^{3,1}$ and $h^{2,2}$. Here comes our result.

Proposition 7.2. *The subsequent table lists the Hodge numbers $h^{2,1}$, $h^{3,1}$ and $h^{2,2}$ for X as in Theorem 1.1.*

No.	$h^{2,1}$	$h^{3,1}$	$h^{2,2}$
1	0	0	3
2	0	0	10
3	0	0	29
4	0	0	3
5	0	3	40
6	0	30	185
7	0	0	4
8	0	1	23
9	0	0	14
10	0	18	126
11	0	0	5
12	0	12	95
13	0	0	4
14	0	6	65
15	0	5	55
16	0	0	6
17	0	9	77
18	0	21	143
19	0	1	22
20	0	0	3
21	0	5	53
22	0	0	10
23	0	13	103
24	0	35	218
25	0	114	591
26	0	0	10
27	0	20	138
28	0	112	570
29	0	1	22
30	0	45	255
31	0	10	94
32	0	100	508
33	0	24	162
34	0	0	4
35	0	1	28
36	0	22	162
37	0	5	60
38	0	71	402
39	0	24	170
40	0	0	4
41	1	1	23
42	0	0	10
43	1	19	131
44	1	5	54
45	1	50	288
46	1	24	163
47	1	159	793
48	0	0	3
49	1	2	31
50	0	3	40
51	0	65	356
52	0	20	139
53	0	0	3
54	0	6	72
55	0	0	8
56	0	1	21
57	0	25	181
58	52	0	2
59	21	0	2
60	21	0	2
61	10	0	2
62	10	0	2
63	5	0	2
64	5	0	2
65	0	0	2
66	0	0	2
67	0	0	2

Proof. We consider the toric embedding $X = X_g \subseteq Z_g$ as provided by Construction 4.1. The five-dimensional toric ambient variety Z_g is smooth and the decomposition

$$X = \bigcup_{\gamma_0 \in \mathrm{rlv}(X)} X(\gamma_0)$$

from Construction 3.4 is obtained by cutting down the toric orbit decomposition of Z_g. Now the idea is to compute the Hodge numbers in question via the Hodge–Deligne polynomial, being defined for any variety Y as

$$e(Y) := \sum_{p,q} e^{p,q}(Y) x^p \bar{x}^q \in \mathbb{Z}[x, \bar{x}],$$

with $e^{p,q}(Y)$ as in [13], p. 280. We also write $e^{p,q}$ instead of $e^{p,q}(Y)$. Recall that $e^{p,q} = e^{q,p}$ holds. Moreover, in case that Y is smooth and projective, the $e^{p,q}$ are related to the Hodge numbers as follows:

$$e^{p,q}(Y) = (-1)^{p+q}\, h^{p,q}(Y).$$

The Hodge–Deligne polynomial is additive on disjoint unions, multiplicative on cartesian products. We list the necessary steps for computing it in low dimensions. On $Y = \mathbb{C}^*$, it evaluates to $x\bar{x} - 1$. For a hypersurface $Y \subseteq (\mathbb{C}^*)^n$ with no torus factors, one has the Lefschetz type formula

$$e^{p,q}(Y) = e^{p+1,q+1}((\mathbb{C}^*)^n), \quad \text{for } p + q > n - 1,$$

see [13], p. 290. Moreover, according to [13], p. 291, with the Newton polytope Δ of the defining equation of Y, one has the following identity:

$$\sum_{q\geq 0} e^{p,q}(Y) = (-1)^{p+n-1}\binom{n}{p+1} + (-1)^{n-1}\varphi_{n-p}(\Delta),$$

where, denoting by $l^*(B)$ the number of interior points of a polytope B, the function φ_i is defined as

$$\varphi_0(\Delta) := 0, \quad \varphi_i(\Delta) := \sum_{j=1}^{i}(-1)^{i+j}\binom{n+1}{i-j} l^*(j\Delta),$$

This leads to an explicit formula for all $e^{p,0}(Y)$. Moreover, for $\dim(Y) \leq 3$, all the numbers $e^{p,q}$ are directly calculated using the above formulas. For $\dim(Y) = 4$, the values of $e^{1,1} + e^{1,2} + e^{1,3}$ and $e^{2,1} + e^{2,2}$ and $e^{3,1}$ can be directly computed using the above formulas. By the symmetry $e^{p,q} = e^{q,p}$ these sums involve just four numbers which thus can be expressed in terms of one of them, say $e^{1,2}$, plus known quantities. To determine the value of $e^{1,2}$ one passes to a smooth compactification Y' of Y for which

$$e^{1,2}(Y') = -h^{1,2}(Y') = -h^{3,2}(Y') = e^{3,2}(Y')$$

holds by Serre's duality and then observes that $e^{3,2}$ can be computed for all the strata via the Lefschetz formula. Now, we apply these principles to the strata $Y = X(\gamma_0)$ that have no torus factor and compute the desired $e^{p,q}$. If $Y = X(\gamma_0)$ has a torus factor, then we use multiplicativity of the Hodge–Deligne polynomial and again the above principles. ∎

Finally, we extend the discussion of the varieties X from Number 9 of Theorem 1.1 started in Remark 6.3 by some topological aspects.

Remark 7.3. Let X be as in Theorem 1.1, No. 9. Recall that we have a fibration $X \to \mathbb{P}_3$ with general fiber $F = \mathbb{P}_1$ and precisely 12 special fibers $F_1, \ldots, F_{12}$, lying over $p_1, \ldots, p_{12} \in \mathbb{P}_3$, each of the F_i being isomorphic to $\mathbb{P}_2$. We claim

$$F_i^2 = 1 \quad \text{for } i = 1, \ldots 12, \qquad F_i \cdot F_j = 0 \quad \text{for } 1 \leq i < j \leq 12.$$

The second part is clear because of F_i and F_j do not intersect for $i < j$. In order to establish the first part, we show $F_1^2 = 1$, where we may assume $p_1 = [1, 0, 0, 0]$. Consider the zero sets $L_1, L_2 \subseteq X$ of two general polynomials in the variables T_2, T_3, T_4. By definition $L_1 \cap L_2 = F$ and $L_1 \sim L_2$, that is the two surfaces are rationally equivalent. Thus $L_i \sim F + S_i$ for some surface S_i. Observe that we have

$$F \cdot L_i = 0, \quad S_i \cdot L_i = 0$$

because L_i is rationally equivalent to a complete intersection of two general polynomials in $T_1, \ldots, T_4$, which has empty intersection with L_i. We deduce

$$F^2 = -F \cdot S_1 = S_1 \cdot S_1 = S_1 \cdot S_2,$$

using $S_1 \sim S_2$ in the last step. For computing the last intersection number, we may assume $L_1 = V(T_2, T_3, g)$ and $L_2 = V(T_2, T_4, g)$. Then $S_1 = V(T_2, T_3, h_1)$ with

$$h_1 = T_4^{-1}(q_1(T_1, 0, 0, T_4)T_5 + q_2(T_1, 0, 0, T_4)T_6 + f(T_1, 0, 0, T_4)T_7),$$

where the division by T_4 can be performed because by hypothesis q_1, q_2 and f do not contain a pure power of T_1. Similarly $S_2 = V(T_2, T_4, h_2)$, where

$$h_2 = T_3^{-1}(q_1(T_1, 0, T_3, 0)T_5 + q_2(T_1, 0, T_3, 0)T_6 + f(T_1, 0, T_3, 0)T_7).$$

It follows that

$$\begin{aligned} S_1 \cap S_2 &= V(T_2, T_3, T_4, \alpha_1 T_1 T_5 + \alpha_2 T_1 T_6 + \alpha_3 T_1^2 T_7, \beta_1 T_1 T_5 + \beta_2 T_1 T_6 + \beta_3 T_1^2 T_7) \\ &= V(T_2, T_3, T_4, \alpha_1 T_5 + \alpha_2 T_6 + \alpha_3 T_1 T_7, \beta_1 T_5 + \beta_2 T_6 + \beta_3 T_1 T_7). \end{aligned}$$

Now one directly checks that $S_1 \cap S_2$ is a point and the intersection is transverse. Thus, we arrive at $S_1 \cdot S_2 = 1$, proving the $F_1^2 = 1$. Now, fix two general linear forms $\ell_1, \ell_2 \in \mathbb{C}[T_1, \dots, T_4]$ and set

$$E := V(T_6, T_7, g) \subseteq X, \quad L := V(\ell_1, \ell_2, g) \subseteq X.$$

We claim that the classes of $E, L, F_1, \dots, F_{12}$ in $H^{2,2}(X) \cap H^4(X, \mathbb{Q})$ are linearly independent. First observe that $F_1, \dots, F_{12}$ are linearly independent: passing to the self-intersection, $\sum_i a_i F_i \sim 0$ turns into $\sum_i a_i^2 = 0$ and thus, being rational numbers, all a_i vanish. Now, by definition of L one has $L^2 = L \cdot F_i = 0$ for any i, in particular the class of L cannot be in the linear span of the classes of the 12 fibers. The statement then follows from $E \cdot L = 2$, which in turn holds due to

$$E \cap L = V(\ell_1, \ell_2, T_6, T_7, g) = V(\ell_1, \ell_2, T_6, T_7, q_1 T_5) = V(\ell_1, \ell_2, T_6, T_7, q_1).$$

Combining linear independence of $E, L, F_1, \dots, F_{12} \in H^{2,2}(X) \cap H^4(X, \mathbb{Q})$ with $h^{2,2}(X) = 14$ as provided by Proposition 7.2, we retrieve that the varieties X from Number 9 of Theorem 1.1 satisfy the Hodge conjecture; which, in this case, is known to hold also by [11] and [38], proof of Lemma 15.2.

8. Deformations and automorphisms

We take a look at the deformations of the varieties from Theorem 1.1. For any variety X, we denote by $\mathcal{T}_X$ its tangent sheaf. If X is Fano, then it is unobstructed and thus its versal deformation space is of dimension $h^1(X, \mathcal{T}_X)$. The following observation makes precise how the problem of determining $h^1(X, \mathcal{T}_X)$ is connected with determining the automorphisms in our setting.

Proposition 8.1. *Let X be a smooth Fano variety X with a general hypersurface Cox ring $\mathcal{R}(X) = \mathbb{C}[T_1, \dots, T_r]/\langle g \rangle$ and associated minimal toric embedding $X \subseteq Z$. Assume that $\mu = \deg(g) \in \mathrm{Cl}(Z)$ is base point free and no $w_i = \deg(T_i) \in \mathrm{Cl}(Z)$ lies in $\mu + \mathbb{Z}_{\geq 0} w_1 +$*

$\cdots + \mathbb{Z}_{\geq 0} w_r$. *Then we have*

$$h^1(X, \mathcal{T}_X) = \dim(\mathcal{R}(Z)_\mu) - 1 + \operatorname{rank}(\operatorname{Cl}(Z)) - \sum_{i=1}^{r} \dim(\mathcal{R}(Z)_{w_i}) + h^0(X, \mathcal{T}_X)$$
$$= -1 + \dim(\mathcal{R}(Z)_\mu) - \dim(\operatorname{Aut}(Z)) + \dim(\operatorname{Aut}(X)).$$

Proof. First look at $0 \to \mathcal{T}_X \to \imath^*\mathcal{T}_Z \to \mathcal{N}_X \to 0$, the normal sheaf sequence for the inclusion $\imath\colon X \subseteq Z$. By assumption, $\mu - \mathcal{K}_X$ is ample and thus we obtain

$$h^1(X, \mathcal{T}_X) - h^0(X, \mathcal{T}_X) = -h^0(X, \imath^*\mathcal{T}_Z) + h^0(X, \mathcal{N}_X) + h^1(X, \imath^*\mathcal{T}_Z),$$

according to the Kawamata–Viehweg vanishing theorem. The task is to evaluate the right hand side. First, note that we have

$$h^0(X, \mathcal{N}_X) = \dim(\mathcal{R}(X)_\mu) = \dim(\mathcal{R}(Z)_\mu) - 1.$$

For the remaining two terms, we use the Euler sequence of Z restricted to X which in our setting is given by

$$0 \longrightarrow \mathcal{O}_X \otimes \operatorname{Cl}(Z) \longrightarrow \bigoplus_{i=1}^{r} \mathcal{O}_X(D_i) \longrightarrow \imath^*\mathcal{T}_Z \longrightarrow 0,$$

where $D_i \subseteq X$ denotes the prime divisor defined by the Cox ring generator T_i. Since X is Fano, $h^i(X, \mathcal{O}_X)$ vanishes for all $i > 0$. As a first consequence, we obtain

$$h^0(X, \imath^*\mathcal{T}_Z) = \sum_{i=1}^{r} \dim(\mathcal{R}(X)_{w_i}) - \operatorname{rank}(\operatorname{Cl}(Z)) = \sum_{i=1}^{r} \dim(\mathcal{R}(Z)_{w_i}) - \operatorname{rank}(\operatorname{Cl}(Z)),$$

using $\mathcal{R}(X)_{w_i} \cong H^0(X, D_i)$ and $\mathcal{R}(X)_{w_i} = \mathcal{R}(Z)_{w_i}$, where the latter holds by assumption. Moreover, we can conclude

$$h^1(X, \imath^*\mathcal{T}_Z) = \sum_{i=1}^{r} h^1(X, D_i).$$

We evaluate the right hand side. Since X has a general hypersurface Cox ring, Z is smooth (Proposition 3.3.1.12 in [2]) and μ is base point free, we can infer smoothness of

$$D_i = V(g) \cap V(T_i) \subseteq Z$$

from Bertini's theorem. Now choose $\varepsilon > 0$ such that $\varepsilon D_i - \mathcal{K}_X$ is nef and big. Then, using once more the Kawamata–Viehweg vanishing theorem, we obtain

$$h^1(X, D_i) = h^1(X,\ \mathcal{K}_X + (\varepsilon D_i - \mathcal{K}_X) + (1 - \varepsilon) D_i) = 0.$$

Consequently, $h^1(X, \imath^*\mathcal{T}_Z)$ vanishes. This gives the first equality of the assertion. The second one follow from Theorem 4.2 in [12] and Lemma 3.4 in [32]. ∎

Observe that Proposition 8.1 applies in particular to all smooth Fano non-degenerate toric hypersurfaces in the sense of Khovanskii ([29] and Definition 4.1 in [24]), where Lemma 3.3 (v) of the latter guarantees base point freeness of $\mu \in \mathrm{Cl}(Z)$. Concerning the varieties from Theorem 1.1, we can say the following.

Corollary 8.2. *For each of the Fano varieties X listed in Theorem 1.1, except possibly numbers 13, 14, 15, 33 and 67, we have*

$$h^1(X, \mathcal{T}_X) = -1 + \dim(\mathcal{R}(Z)_\mu) - \dim(\mathrm{Aut}(Z)) + \dim(\mathrm{Aut}(X)).$$

Proof. Using Proposition 3.3.2.8 in [2], one directly checks that $\mu \in \mathrm{Cl}(X)$ and hence also $\mu \in \mathrm{Cl}(Z)$ are base point free in all cases except the Numbers 13, 14, 15 and 33. Number 67 violates the assumption on the generator degrees. ∎

The only serious task left open by Proposition 8.1 for explicitly computing $h^1(X, \mathcal{T}_X)$ is to determine the dimension of $\mathrm{Aut}(X)$. As general tools, we mention Theorem 4.4 in [23], the algorithms presented thereafter and their implementation provided by [28]. The subsequent example discussions indicate how one might proceed in concrete cases.

Example 8.3. The variety X from No. 65 is a product of the smooth projective quadric $Q_4 \subseteq \mathbb{P}_4$ and a projective line. So, X is known to be infinitesimally rigid. Via Proposition 8.1, this is seen as follows:

$$\begin{aligned} h^1(X, \mathcal{T}_X) &= -1 + \dim(\mathcal{R}(Z)_\mu) - \dim(\mathrm{Aut}(Z)) + \dim(\mathrm{Aut}(X)) \\ &= -1 + 15 - 27 + 13 = 0. \end{aligned}$$

All ingredients are classical: first, by Corollary I.2 in [7] the unit component of the automorphism group of a product is the product of the unit components of the respective automorphism groups. Second, the group $\mathrm{Aut}(Q_n) = \mathrm{O}(n)$ is of dimension $n(n-1)/2$.

Example 8.4. For the varieties X from No. 1, the algorithm [28] is feasible and tells us that $\mathrm{Aut}(X)$ is of dimension 12. In particular, we see that also these varieties are infinitesimally rigid:

$$\begin{aligned} h^1(X, \mathcal{T}_X) &= -1 + \dim(\mathcal{R}(Z)_\mu) - \dim(\mathrm{Aut}(Z)) + \dim(\mathrm{Aut}(X)) \\ &= -1 + 12 - 23 + 12 = 0. \end{aligned}$$

In suitable linear coordinates respecting the grading, $g = T_1T_5 + T_2T_6 + T_3T_7$ holds and the automorphisms on X are induced by the five-dimensional diagonally acting torus respecting g and the group $\mathrm{GL}(3)$ acting on $\mathcal{R}(X)_{w_1} \oplus \mathcal{R}(X)_{w_5}$ via

$$A \cdot (T_1, T_2, T_3, T_4; T_5, T_6, T_7) := (A \cdot (T_1, T_2, T_3), T_4; (A^{-1})^t \cdot (T_5, T_6, T_7)).$$

The two previous examples fit into the class of *intrinsic quadrics*, that means varieties having a hypersurface Cox ring with a quadric as defining relation. The ideas just observed lead to the following general observation.

Corollary 8.5. *Let X be a variety satisfying all the assumptions of Proposition 8.1 and assume that $\mathrm{Aut}_H(\bar{Z})$ acts almost transitively on $\mathcal{R}(Z)_\mu$.*

(i) *The variety X is infinitesimally rigid and the dimension of its automorphism group is given by*

$$\dim(\mathrm{Aut}(X)) = \dim(\mathrm{Aut}_H(\bar{Z})) - (\dim(\mathcal{R}(Z)_\mu) - 1) - \mathrm{rank}(\mathrm{Cl}(Z)).$$

(ii) *If X is an intrinsic quadric, then $\mathrm{Aut}_H(\bar{Z})$ acts almost transitively on $\mathcal{R}(Z)_\mu$ and thus the statements from* (i) *hold for X.*

Proof. We take $X \subseteq Z$ as in Construction 4.1. According to Theorem 4.4 (iv) in [23], the unit component $\mathrm{Aut}(X)^0$ equals the stabilizer $\mathrm{Aut}(Z)^0_X$ of $X \subseteq Z$ under the action of $\mathrm{Aut}(Z)^0$ on Z. Thus, using Theorem 4.2.4.1 in [2], we obtain

$$\begin{aligned}\dim(\mathrm{Aut}(X)) = \dim(\mathrm{Aut}(Z)^0_X) &= \dim(\mathrm{Aut}_H(\bar{Z})^0_{\bar{X}}) - \dim(H)\\ &= \dim(\mathrm{Aut}_H(\bar{Z})^0) - (\dim(\mathcal{R}(Z)_\mu) - 1) - \mathrm{rank}(\mathrm{Cl}(Z)),\end{aligned}$$

where $\mathcal{R}(Z)_\mu$ is the space of defining equations and "-1" pops up as we are looking for only the zero sets of these equations. Thus, Proposition 8.1 gives the first statement. For the second one, note that $\mathrm{Aut}_H(\bar{Z})$ acts almost transitively on $\mathcal{R}(Z)_\mu$ due to Proposition 2.1 in [18]. ■

Let us take up once more the geometric discussion of the varieties from No. 9 of Theorem 1.1 started in Remarks 6.3 and 7.3. Using geometric properties observed so far, we see $\mathrm{Aut}(X)$ is trivial.

Remark 8.6. Let X be as in Theorem 1.1, No. 9. We claim that $\mathrm{Aut}(X)$ is finite in this case. As a consequence, we obtain

$$\begin{aligned}h^1(X, \mathcal{T}_X) &= \dim(\mathcal{R}(Z)_\mu) - 1 + \mathrm{rank}(\mathrm{Cl}(Z)) - \sum_{i=1}^{r} \dim(\mathcal{R}(Z)_{w_i})\\ &= 40 - 1 + 2 - 29 = 12.\end{aligned}$$

Look at the fibration $\pi_1 \colon X \to Y_1 = \mathbb{P}_3$ from Remark 6.3. By Proposition I.1 in [7], there is an induced action of the unit component $\mathrm{Aut}(X)^0$ on Y_1 turning π_1 into an equivariant map. This means in particular that the induced action permutes the image points of the 12 singular fibers of π_1. By the generality assumption, these 12 points do not lie in a common hyperplane and thus induced action of $\mathrm{Aut}(X)^0$ on Y_1 must be trivial. Recall that any point of the fiber π_1 over $[y] = [y_1, \ldots, y_4]$ has Cox coordinates

$$[y, x, z] = [y_1, \ldots, y_4, x_1, x_2, z], \quad \text{where } q_1(y)x_1 + q_2(y)x_2 + f(y)z = 0,$$

with general quadrics q_1, q_2 and a general cubic f in the first four variables. Let us see in these terms what it means that the π_1-fibers are invariant under $\mathrm{Aut}(X)^0$. Consider the action of the characteristic quasitorus $H = \mathrm{Spec}\,\mathbb{C}[\mathrm{Cl}(Z)]$ on $\bar{Z} = \mathbb{C}^r$ given by the $\mathrm{Cl}(Z)$-grading of $\mathbb{C}[T_1, \ldots, T_r]$. The group $\mathrm{Aut}_H(\bar{Z})$ of H-equivariant automorphisms is concretely given as

$$G = \mathrm{GL}(4) \times \mathrm{GL}(2) \times \mathbb{K}^*.$$

According to Theorem 4.4 in [23], we obtain $\mathrm{Aut}(X)^0$ as a factor group of the unit component of the subgroup $\mathrm{Aut}_H(\bar{X})$ of $\mathrm{Aut}_H(\bar{Z})$ stabilizing $\bar{X} \subseteq \bar{Z}$. We take a closer look at the action of an element $\gamma = \mathrm{diag}(A_1, A_2, \alpha_3)$ of $\mathrm{Aut}_H(\bar{X})$ on $\hat{X} \subseteq \bar{X}$. Given general $y \in \mathbb{C}^4$ and $x \in \mathbb{C}^2$, we find $z \in \mathbb{C}$ such that $[y, x, z]$ is a point of $\hat{X}$. In particular, $\gamma \cdot [y, x, z]$ belongs to the fiber of π_1 over $[y]$. The latter implies $A_1 \cdot y = \eta y$ with $\eta \in \mathbb{K}^*$ and for the matrix $A_2 = (a_{ij})$ it gives

$$\begin{aligned} 0 &= q_1(y)(a_{11}x_1 + a_{12}x_2) + q_2(y)(a_{21}x_1 + a_{22}x_2) + \alpha_3 f(y)z \\ &= q_1(y)((a_{11} - \alpha_3)x_1 + a_{12}x_2) + q_2(y)(a_{21}x_1 + (a_{22} - \alpha_3)x_2). \end{aligned}$$

Recall that this holds for any general choice of y and x. As a consequence, we arrive at $a_{11} - \alpha_3 = 0 = a_{12}$, because otherwise $q_1 q_2^{-1} \in \mathbb{C}(T_1, T_2)$ holds in $\mathbb{C}(X)$ which is impossible due to the general choice of q_1 and q_2. By the same argument, we see $a_{22} - \alpha_3 = 0 = a_{21}$. Thus, γ acts trivially on each fiber of π_1 and we conclude that $\mathrm{Aut}(X)$ is of dimension zero.

Proposition 8.1 suggests that the infinitesimal deformations of X can be obtained by varying the coefficients of the defining equation in the Cox ring. As a possible approach to turn this impression into a precise statement, we mention the comparison theorem of Christophersen and Kleppe (Theorem 6.2 in [9]), which relates in particular deformations of a variety to deformations of its Cox ring.

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