



Harmonic quasiconformal mappings between \mathcal{C}^1 smooth Jordan domains

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Abstract. We prove the following result. If f is a harmonic quasiconformal mapping between two Jordan domains D and Ω having \mathcal{C}^1 boundaries, then the function f is globally Hölder continuous for every $\alpha < 1$ but it is not necessarily Lipschitz in general. This result extends and improves a classical theorem of S. Warschawski for conformal mappings.

1. Introduction

Let U and V be two domains in the complex plane \mathbb{C} . We say that a twice differentiable mapping $f = u + iv: U \rightarrow V$ is harmonic if $\Delta f := \Delta u + i \Delta v = 0$ in U . Any harmonic homeomorphism is by Lewy's theorem a diffeomorphism. If its Jacobian $J_f = |f_z|^2 - |f_{\bar{z}}|^2$ is positive, then it is sense-preserving.

We say that a function $u: D \rightarrow \mathbb{R}$ is ACL (absolutely continuous on lines) in the region D if for every closed rectangle $R \subset D$ with sides parallel to the x and y -axes, u is absolutely continuous on a.e. horizontal and a.e. vertical lines in \mathbb{R} . Such a function has partial derivatives u_x, u_y a.e. in D .

A sense-preserving homeomorphism $w: D \rightarrow \Omega$, where D and Ω are subdomains of the complex plane \mathbb{C} , is said to be K -quasiconformal (K -q.c), with $K \geq 1$, if w is ACL in D in the sense that its real and imaginary part are ACL in D , and

$$(1.1) \quad |Dw| \leq Kl(Dw) \quad \text{a.e. on } D,$$

(cf. [1], pp. 23–24). Here $A = D(w)$ is the formal differential matrix defined by

$$A = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

and

$$|A| = \max_{|h|=1} |Ah|, \quad l(A) = \min_{|h|=1} |Ah|,$$

where $|\cdot|$ is the Euclidean norm.

Notice that the condition (1.1) can be written as

$$|w_{\bar{z}}| \leq k|w_z| \quad \text{a.e. on } D, \quad \text{where } k = \frac{K-1}{K+1}, \quad \text{i.e., } K = \frac{1+k}{1-k}.$$

The class of quasiconformal harmonic mappings was firstly considered by O. Martio in [23]. The class of q.c. harmonic mappings contains conformal mappings, and this is why the class has shown a large interest for experts in geometric function theory.

We should mention here the following result of Pavlović [28], which states that a harmonic quasiconformal mapping of the unit disk \mathbb{D} onto itself is bi-Lipschitz continuous. In order to explain the importance of his result, let us state the following two separate results. If we assume that the mapping $f: \mathbb{D} \rightarrow \mathbb{D}$ is merely quasiconformal, then it is only Hölder continuous with the Hölder coefficient $\alpha = (1-k)/(1+k)$. This is the celebrated Mori theorem. On the other hand, if $f: \mathbb{D} \rightarrow \mathbb{D}$ is merely a harmonic diffeomorphism, then by a result of Hengartner and Schober it has a continuous extension up to the boundary (see Theorem 4.3 in [9] or Section 3.3 in [8]). However, in view of the Radó–Kneser–Choquet theorem, this is the best regularity that such a mapping can have at the boundary.

We define the Poisson kernel by

$$P(z, \theta) = \frac{1}{2\pi} \frac{1 - |z|^2}{|z - e^{i\theta}|^2}, \quad |z| < 1, \theta \in [0, 2\pi).$$

For a mapping $f \in L^1(\mathbb{T})$, where \mathbb{T} is the unit circle, the Poisson integral is defined by

$$w(z) = P[f](z) = \int_0^{2\pi} P(z, \theta) f(e^{i\theta}) d\theta.$$

The well-known Radó–Kneser–Choquet theorem states the following: if f is a homeomorphism of the unit circle onto a convex Jordan curve γ , then its Poisson integral is a harmonic diffeomorphism of the unit disk \mathbb{D} onto the Jordan domain Ω bounded by γ .

A special case is when $\gamma = \mathbb{T}$. E. Heinz has proved that, if f is a harmonic diffeomorphism of the unit disk onto itself, then the Hilbert–Schmidt norm of its derivative,

$$(1.2) \quad \|Df\|^2 = |f_x|^2 + |f_y|^2 \geq c,$$

where $c > 0$, depends only on $f(0)$. It follows from (1.2) that the inverse of a quasiconformal harmonic mapping of the unit disk onto itself is Lipschitz continuous. So the main achievement of Pavlović in [28] (see also [30]) was to prove that a harmonic quasiconformal mapping of the unit disk onto itself is Lipschitz continuous on the closure of the domain.

In order to formulate some additional results in this topic, recall that a rectifiable Jordan curve is \mathcal{C} , Dini smooth, $\mathcal{C}^{1,\alpha}$, for $\alpha \in (0, 1]$, if its arch-length parametrisation $g: [0, |\gamma|] \rightarrow \gamma$ is \mathcal{C}^1 , Dini smooth and $\mathcal{C}^{1,\alpha}$, respectively. Here $|\gamma|$ is the length of γ .

In [11], the author proved that every quasiconformal harmonic mapping between Jordan domains with $\mathcal{C}^{1,\alpha}$ boundaries is Lipschitz continuous on the closure of domain. Later, this result has been extended to Jordan domains with only Dini smooth boundaries [13].

A bi-Lipschitz property for harmonic quasiconformal mappings of the half-plane onto itself has been established by the author and Pavlović in [16].

Further, it has been shown (see [12]) that a quasiconformal harmonic mappings between $\mathcal{C}^{1,1}$ (not necessarily convex) Jordan domains is bi-Lipschitz continuous. The same conclusion is obtained in [5] by Božin and Mateljević for the case of $\mathcal{C}^{1,\alpha}$ Jordan domains. Further results in the two dimensional case can be found in [17]. Some results concerning the several-dimensional case can be found in [3], [18] and [25]. For a different setting concerning the class of quasiconformal harmonic mappings, we refer to the papers [7], [22] and [27]. For example, the article [22] deals with the following problem of the class of quasiconformal harmonic mappings.

The quasi-hyperbolic metric d_h in a domain D of complex plane is defined as follows. For each $z_1, z_2 \in D$,

$$d_h(z_1, z_2) = \inf \int_{\gamma} d(z, \partial D)^{-1} |dz|,$$

where the infimum is taken over all rectifiable arcs γ joining x_1 and x_2 in D . V. Manojlović in [22] proved the following theorem: if $f: D \rightarrow D'$ is a quasiconformal and harmonic mapping, then it is bi-Lipschitz with respect to quasihyperbolic metrics on D and D' .

In order to formulate the main theorem of this paper, let us define the chord-arc curves. A rectifiable Jordan curve γ is a B -chord-arc curve if $L_{\gamma}(z_1, z_2) \leq B|z_1 - z_2|$ for all $z_1, z_2 \in \gamma$, where $L_{\gamma}(z_1, z_2)$ denotes the length of the shortest arc of γ joining z_1 and z_2 . Here $B \geq 1$.

Theorem 1.1. *Let D and Ω be Jordan domains having \mathcal{C}^1 boundaries and assume that $a \in D$ and $b \in \Omega$. Assume that ω_D (ω_{Ω}) is the modulus of continuity of the derivative of arc-length parametrisation of ∂D ($\partial\Omega$). Assume further that ∂D and $\partial\Omega$ satisfy the B -arc-chord condition for some $B \geq 1$. Then for every $\alpha \in (0, 1)$ and $k \in [0, 1)$, there is a constant $M_{\alpha} = M_{\alpha}(a, b, k, B, \omega_D, \omega_{\Omega})$ so that every harmonic $K = (1+k)/(1-k)$ -quasiconformal mapping $f = g + \bar{h}$ of D onto Ω so that $f(a) = b$ satisfies the condition*

$$|f(z) - f(w)| \leq M_{\alpha}|z - w|^{\alpha}, \quad z, w \in D.$$

Moreover, for every $p > 0$, there is a constant B_p , that depends on the same parameters as M_{α} , so that

$$(1.3) \quad \int_D |Df(z)|^p d\lambda(z) \leq B_p^p,$$

where $|Df(z)| = |f_z| + |f_{\bar{z}}| = |g'| + |h'|$. In other words, g' and h' belong to the Bergman space \mathcal{A}^p for every $p > 0$. Here λ is the Lebesgue measure in the plane.

Remark 1.2. In Theorem 1.1 we consider the mappings between Jordan domains. The same conclusion can be made for multiply-connected domain bounded by finite number of \mathcal{C}^1 Jordan curves. We also expect that a similar conclusion can be made for non-bounded domains, but we did not pursue this question seriously.

1.1. The organization of the paper

We continue this section with some immediate corollaries of the main result. We prove first that a K -quasiconformal mapping between \mathcal{C}^1 domains is β -Hölder continuous for

every $\beta < 1/K$. In particular, we prove that a conformal mapping is β -Hölder continuous for every $\beta < 1$. In the second section we prove a variation of the main result which will be needed to prove Theorem 1.1 in full generality. The proof of Theorem 1.1 is presented in the last section. The proof depends on a two-side connection between the α -Hölder constant and the so-called α -Bloch type norm of the holomorphic function defined on the unit disk expressed in Lemma 1.4. By using this connection, and by a subtle application of \mathcal{C}^1 smoothness of the boundary curve of the image domain, we first find an a priori estimate of the α -Hölder constant of a harmonic quasiconformal mapping of the unit disk onto a \mathcal{C}^1 Jordan domain having \mathcal{C}^1 extension up to the boundary. Then we use an approximation argument to get an estimate of α -Hölder constant for a harmonic q.c. mapping which has not necessary smooth extension up to the boundary. To deal with the mappings whose domain is not the unit disk is a simple matter having proved the results from the second section.

1.2. Some immediate consequences

Corollary 1.3. *If f is a univalent conformal mapping between two Jordan domains D and Ω with \mathcal{C}^1 boundaries, then f is α -Hölder continuous for every $0 < \alpha < 1$. Moreover, if ∂D and $\partial\Omega$ satisfy the B -arc-chord condition for some $B \geq 1$, then for every $\alpha \in (0, 1)$ and every $a \in D$ and $b = f(a) \in \Omega$, there exists $M = M(\alpha, a, b, B, \omega_D, \omega_\Omega)$ so that*

$$\frac{1}{M}|z - w|^{1/\alpha} \leq |f(z) - f(w)| \leq M|z - w|^\alpha$$

for every $z, w \in D$.

Proof of Corollary 1.3. Let a be a univalent conformal mapping of the unit disk \mathbb{D} onto D and let b be a univalent conformal mapping of the unit disk onto Ω . Then, in view of Theorem 1.1, b and a^{-1} are $\sqrt{\alpha}$ -Hölder continuous. Then $f = b \circ a^{-1}$ is α -Hölder continuous. ■

Now we prove the following theorem, which deals with Hölder continuity of quasiconformal mappings between smooth domains.

Theorem 1.4. *Assume that D and Ω are two Jordan domains with \mathcal{C}^1 boundaries and assume that $a \in D$ and $b \in \Omega$. Assume further that ∂D and $\partial\Omega$ satisfy the B -arc-chord condition for some $B \geq 1$. Let $K \geq 1$. Then for every $\beta < 1/K$, there is a constant $M_\beta = M(\beta, a, b, \omega_D, \omega_\Omega, B, K)$ so that if $f: D \rightarrow \Omega$ is K -quasiconformal with $f(a) = b$, then*

$$|f(z) - f(w)| \leq M_\beta |z - w|^\beta, \quad z, w \in D.$$

In connection to Theorem 1.4, we want to mention that some more general results are known under some more general conditions on the domains, but they do not cover this result. For example, O. Martio and R. Näkki in [24] showed that if f induces a boundary mapping which belongs to $\text{Lip}_\alpha(\partial D)$, then f is in $\text{Lip}_\beta(D)$, where

$$\beta = \min\{\alpha, 1/K\};$$

the exponent β is sharp. We also want to refer to the papers [19] and [26], which also consider the global Hölder continuity of quasiconformal mappings. Concerning the integrabi-

lity of the derivative of a quasiconformal mapping and its connection to the global Hölder continuity, we refer to the paper by Astala and Koskela [2].

Proof of Theorem 1.4. Let $\phi: \mathbb{D} \rightarrow D$ and $\psi: \Omega \rightarrow \mathbb{D}$ be conformal diffeomorphisms so that $\phi(0) = a$ and $\psi(b) = 0$. Then $f_0 = \psi \circ f \circ \phi$ is a K -quasiconformal mapping of the unit disk onto itself so that $f_0(0) = 0$. Thus by Mori's theorem,

$$|f_0(z) - f_0(w)| \leq 16|z - w|^{1/K}.$$

Now, if $\beta < 1/K$, then there are two constants $\alpha_1 < 1$ and $\alpha_2 < 1$ so that $\alpha_1 \cdot \alpha_2/K = \beta$. Since $f = \psi^{-1} \circ f_0 \circ \phi^{-1}$, by making use of Corollary 1.3, we get and ψ^{-1} is α_1 -Hölder continuous and ϕ^{-1} is α_2 -Hölder continuous. By having in mind the fact that f_0 is $1/K$ -Hölder continuous, it follows that f is β -Hölder continuous, as claimed. ■

Remark 1.5. Similar results can be shown for multiply connected domains in the complex plane having a \mathcal{C}^1 boundary. If f a conformal mapping of the unit disk onto a Jordan domain with merely \mathcal{C}^1 boundary, then f is not necessarily Lipschitz continuous. See an example given by Lesley and Warschawski in [21], as well as the example $f_0(z) = 2z + (1 - z) \log(1 - z)$ given in Pommerenke's book [31], which is a conformal diffeomorphism of the unit disk onto a Jordan domain with merely \mathcal{C}^1 boundary. Then $|f'_0(z)|$ is not bounded and thus f_0 is not Lipschitz continuous. The content of Corollary 1.3 is not new (see for example [20]). See also Warschawski [34], Corollary, p. 255, for a related result. We should also mention the paper by Brennan [6], where the famous Brannen conjecture comes from. Theorem 3 of that paper contains a short proof of the special case of (1.3) for $\Omega = \mathbb{D}$ and f being conformal.

2. Auxiliary results

The starting point of this section is the theorem of Warschawski for conformal mappings which states the following. Assume that f is a conformal mapping of the unit disk onto a Jordan domain Ω with a \mathcal{C}^1 boundary γ . Assume that g is the arc-length parametrisation of γ , and assume that $\omega = \omega_{g'}$ is the modulus of continuity of g' . Assume also that γ satisfies the B -chord-arc condition for some constant $B > 1$. Then for every $p \in \mathbb{R}$, there is a constant A_p , depending only on Ω , ω , B , p and $f(0)$, so that

$$(2.1) \quad \int_{\mathbb{T}} |f'(z)|^p |dz| \leq E_p^p.$$

We first give an extension of (1.3), and prove a variation of the main result needed in the sequel.

Theorem 2.1. *If $f = g + \bar{h}$ is a K -q.c. harmonic mapping of the unit disk \mathbb{D} onto a domain Ω with \mathcal{C}^1 boundary, so that h has holomorphic extension beyond the boundary of the unit disk, then $g', 1/g' \in H^p(\mathbb{D})$ for every $p > 0$. Moreover,*

$$\int_{\mathbb{T}} |g'(z)|^p |dz| \leq F_p^p,$$

where F_p is a constant that depends on the same parameters as E_p in (2.1) as well as on k .

Now recall the Morrey inequality.

Proposition 2.2 (Morrey's inequality). *Assume that $2 < p \leq \infty$ and assume that U is a bounded domain in \mathbb{R}^2 with \mathcal{C}^1 boundary. Then there exists a constant C depending only on p and U so that*

$$(2.2) \quad \|u\|_{\mathcal{C}^{0,\alpha}(U)} \leq C \|u\|_{W^{1,p}(U)}$$

for every $u \in \mathcal{C}^1(U) \cap L^p(U)$, where

$$\|u\|_{\mathcal{C}^{0,\alpha}(U)} = \sup_{z \neq w} \frac{|u(z) - u(w)|}{|z - w|^\alpha}, \quad \alpha = 1 - \frac{2}{p},$$

and

$$\|u\|_{W^{1,p}(U)} = \|u\|_{L^p(U)} + \|Du\|_{L^p(U)}.$$

Here $W^{1,p}(U)$ is the Sobolev space.

Corollary 2.3. *Under the conditions of the previous theorem, for every $\alpha < 1$, f and f^{-1} are α -Hölder continuous. The result is optimal since f is not necessarily Lipschitz in general.*

Remark 2.4. If $h \equiv 0$, then Theorem 2.1 reduces to the classical result of Warschawski [33], see also a similar result by Smirnov [32] and Goluzin [10], Theorem 7, p. 415. We include the proof of Theorem 2.1 for the completeness of the argument.

Proof of Corollary 2.3. Let $\alpha < 1$ and let us prove that f is α -Hölder continuous. We have

$$|f(e^{it}) - f(e^{is})| = \int_s^t |\partial_\tau f(e^{i\tau})| d\tau \leq \left(\int_s^t |\partial_\tau f(e^{i\tau})|^p d\tau \right)^{1/p} \left(\int_s^t d\tau \right)^{1/q}.$$

Therefore for $\alpha = 1 - 1/p = 1/q$ we get

$$|f(e^{it}) - f(e^{is})| \leq \|\partial_\tau f\|_p |s - t|^\alpha.$$

As h is smooth in $\overline{\mathbb{D}}$, it follows that g is α -Hölder continuous in \mathbb{T} . By using the well-known Hardy–Littlewood theorem ([10], Theorem 4, p. 413), we get that g is α -Hölder continuous on \mathbb{D} . Thus f is α -Hölder continuous on \mathbb{D} .

To prove that f^{-1} is α -Hölder continuous, observe that for $w = f(z)$,

$$\partial_w f^{-1}(w) = \frac{\overline{f_z}}{J_f} = \frac{\overline{g'(z)}}{|g'(z)|^2 - |h'(z)|^2}.$$

Thus

$$\begin{aligned} \int_\Omega |\partial_w f^{-1}(w)|^p d\lambda(w) &= \int_{\mathbb{D}} \left(\frac{|g'(z)|}{|g'(z)|^2 - |h'(z)|^2} \right)^p J_f d\lambda(z) \\ &\leq \int_{\mathbb{D}} \frac{|g'(z)|^{p+2}}{|g'(z)|^{2p}} \frac{1+k^2}{(1-k^2)^p} d\lambda(z) \\ &= \frac{1+k^2}{(1-k^2)^p} \int_{\mathbb{D}} |g'(z)|^{2-p} d\lambda(z), \quad k = (K-1)/(K+1). \end{aligned}$$

Here λ is the Lebesgue measure in the plane. Therefore by using the isoperimetric inequality for holomorphic functions we get

$$\begin{aligned} \int_{\Omega} |Df^{-1}(w)|^p d\lambda(w) &\leq \frac{(1+k^2)(1+k^p)}{(1-k^2)^p} \int_{\mathbb{D}} |g'(z)|^{2-p} d\lambda(z) \\ &\leq \frac{(1+k^2)(1+k^p)}{4\pi(1-k^2)^p} \left(\int_{\mathbb{T}} |g'(z)|^{1-p/2} |dz| \right)^2 < \infty. \end{aligned}$$

From (2.2) we infer that $u = f^{-1}$ is α -Hölder continuous and the corollary is proved. ■

In the proof of Theorem 2.1, we use the following proposition.

Proposition 2.5 ([14]). *If $f(z) = \mathcal{P}[f^*](z)$ is a quasiconformal harmonic mapping of the unit disk onto a Jordan domain bounded by a curve γ , then the function*

$$U(z) := \arg \left(\frac{1}{z} \frac{\partial}{\partial \varphi} f(z) \right)$$

is well defined and smooth in $\mathbb{D}^* := \mathbb{D} \setminus \{0\}$ and has a continuous extension to \mathbb{T} if and only if $\gamma \in \mathcal{C}^1$. Furthermore, there holds

$$U(e^{i\varphi}) = \beta(\varphi) - \varphi,$$

where $\beta(\varphi)$ is the tangent angle of γ at $f^*(e^{i\varphi})$.

Proof of Theorem 2.1. By the assumption we have that $h(z) = \sum_{j=0}^{\infty} b_j z^j$ for $|z| < \rho$, where ρ is a certain constant bigger than 1.

Therefore, the mapping

$$h_1(z) = \frac{1}{z} \overline{h' \left(\frac{1}{\bar{z}} \right)} = \sum_{j=0}^{\infty} \frac{j \bar{b}_j}{z^j}$$

is a well defined holomorphic function in the domain $D_1 = \{z : |z| > 1/\rho\}$.

Since $\Gamma = \partial\Omega$ is rectifiable, for $z = re^{it}$, we have that

$$F(z) = \partial_t f(re^{it}) = izg'(z) - i\overline{zh'(z)} \in h^1(\mathbb{D}),$$

(see e.g. [15], [28]). Therefore, by having in mind the quasiconformality, we get that $g', h' \in H^1(\mathbb{D})$. In particular, there exist non-tangential limits of those functions almost everywhere on \mathbb{T} . We recall that $h^1(\mathbb{D})$ and $H^1(\mathbb{D})$ are the Hardy classes of harmonic and holomorphic functions, respectively, defined in the unit disk \mathbb{D} .

Let

$$H(z) = i \left(zg'(z) - \frac{1}{z} h_1(z) \right), \quad 1/\rho < |z| < 1.$$

Then, for almost every $t \in [-\pi, \pi]$, we have

$$\lim_{r \rightarrow 1} H(re^{it}) = \lim_{r \rightarrow 1} F(re^{it}).$$

Then there is a set of points $0 < \varphi_1 < \varphi_2 < \varphi_3 < \varphi_4 < 2\pi$ so that

$$(2.3) \quad \lim_{r \rightarrow 1} H(re^{i\varphi_j}) = H(e^{i\varphi_j}),$$

exist for every $j = 1, 2, 3, 4$.

Let $1 < R < \rho$ and let $S_1 = \{z = re^{i\phi}; \phi \in (\varphi_1, \varphi_4), r \in (1/R, 1)\}$, $S_2 = \{z = re^{i\phi}; \phi \in (\varphi_3, 2\pi + \varphi_2), r \in (1/R, 1)\}$ and let $w = \Phi_j(z)$ be a conformal mapping of the unit disk onto the region S_j so that

$$(2.4) \quad \Phi_1(0) = \frac{1}{2} \left(\frac{1}{R} + 1 \right) e^{i/2(\varphi_1 + \varphi_4)}, \quad \Phi_2(0) = -\frac{1}{2} \left(\frac{1}{R} + 1 \right) e^{i/2(\varphi_2 + \varphi_3)}.$$

Let $s_1, s_2, s_3, s_4 \in [0, 2\pi]$ so that $\varphi_1 < s_1 < s_2 < \varphi_2$, and $\varphi_3 < s_3 < s_4 < \varphi_4$. Then

$$\{e^{is} : s \in (s_1, s_4) \cup (s_3, 2\pi + s_2)\} = \mathbb{T}.$$

Observe that $\mathbb{T} \subset D_1$.

Define the holomorphic mapping $K_j(z) = H(\Phi_j(z))$, $z \in \mathbb{D}$, $j = 1, 2$. In view of (2.3), we have that H is bounded on the boundary arcs $I_j = [1/R, 1]e^{i\varphi_j}$, $j = 1, 4$, of S . Also it is clear that it is bounded in the inner arc. Therefore K_j is a non-vanishing bounded analytic function defined in the unit disk. Let $L_j(z) = \log K_j(z)$. Then for $j = 1, 2$,

$$v_j(z) = \Im L_j(z) = \arg(K_j(z))$$

is a bounded harmonic function, so that $\lim_{r \rightarrow 1} v_j(re^{it}) = v_j(e^{it})$ is a continuous function on the unit circle.

To show that v is a bounded well-defined function, observe that

$$H(z) = zg' \left(1 - \frac{h_1(z)}{z^2} \right),$$

and so

$$\arg H(z) = \arg(zg') + \arg \left(1 - \frac{h_1(z)}{z^2 g'(z)} \right).$$

First of all for $|z|$ close to 1, the function

$$\Re \left(1 - \frac{h_1(z)}{z^2 g'(z)} \right)$$

is bigger than $1 - (1 + k)/2$, where k is the constant of quasiconformality. On the other hand, in view of Proposition 2.5, $i(g' - \overline{zh'}/z) = f_t(e^{it})/z$ has a continuous argument in the punctured disk $0 < |z| \leq 1$. Since $\Re(1 - \overline{zh'}/(zg')) > 0$, we obtain that $\arg(g')$ is well-defined and bounded function close to the boundary of the unit disk.

We can also choose R close enough to 1 so that the variation of the argument satisfies

$$(2.5) \quad \Delta_{\mathbb{T}} \arg K_j(e^{it}) \leq 1 + \Delta_{\mathbb{T}} \arg H_j(e^{is}).$$

Assume that $\epsilon > 0$ so that $\epsilon|p| < \pi/2$, and let

$$P_j(t) = a_{j,0} + \sum_{m=1}^n c_m \cos mt + d_m \sin mt$$

be a trigonometric polynomial so that

$$|v_j(e^{it}) - P_j(t)| \leq \epsilon \quad \text{for } t \in [0, 2\pi].$$

Let Ψ be the holomorphic function so that $\Im(\Psi(e^{it})) = P_j(t)$ and $\Psi_j(0) = a_{j,0}$. Observe that

$$a_{j,0} = \frac{1}{2\pi} \int_{-\pi}^{\pi} P_j(t) dt$$

and

$$(2.6) \quad |a_{j,0}| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |P_j(t)| dt \leq \epsilon + \frac{1}{2\pi} \int_{-\pi}^{\pi} |v_j(t)| dt.$$

Then for every $r \in (0, 1)$ we have

$$\int_0^{2\pi} e^{p(L_j(re^{it}) - \Psi(re^{it}))} \frac{dt}{2\pi} = e^{p(\Psi(0) - L_j(0))}.$$

So by taking the real part and letting $r \rightarrow 1$ we get

$$\int_0^{2\pi} e^{p\Re(L_j(e^{it}) - \Psi(e^{it}))} \cos\{p\Im[L_j(e^{it}) - \Psi(e^{it})]\} \frac{dt}{2\pi} = \Re e^{p(L_j(0) - \Psi(0))}.$$

Thus

$$\int_0^{2\pi} e^{p\Re(L_j(e^{it}) - \Psi(e^{it}))} \frac{dt}{2\pi} \leq \frac{|\Re e^{p(L_j(0) - \Psi(0))}|}{\cos p\epsilon}.$$

Therefore,

$$\int_0^{2\pi} e^{p\Re(L_j(e^{it}))} \frac{dt}{2\pi} \leq \max_{t \in [0, 2\pi]} e^{p\Re(\Psi(e^{it}))} \frac{|\Re e^{p(L_j(0) - \Psi(0))}|}{\cos p\epsilon} = G_p.$$

The constant G_p depends on the same parameters as the constant E_p from (2.1) together with the constant of quasiconformality k , and this follows from the fact that $\Psi(0) = a_{j,0}$, (2.4), (2.6), (2.5) and a Cauchy type inequality for $H(z)$ in the annulus $1/R < |z| < 1$, where $1/R = (1/\rho + 1)/2$. Here ρ is a given constant bigger than 1 as in the begging of the proof.

Since $p\Re L_j(z) = p \log |K_j(z)|$, it follows that $\exp(p \log |K_j(z)|) = |K_j(z)|^p$. Therefore $K_j \in H^p$. Now we have

$$\begin{aligned} \int_{\mathbb{T}} |H(e^{is})|^p ds &\leq \int_{\{e^{is}: s_1 \leq s \leq s_4\}} |H(e^{is})|^p ds + \int_{\{e^{is}: s_3 \leq s \leq s_2 + 2\pi\}} |H(e^{is})|^p ds \\ &= \int_{\{e^{it}: t_1 \leq t \leq t_4\}} |H(\Phi_1(e^{it}))|^p |\Phi_1'(e^{it})| dt \\ &\quad + \int_{\{e^{it}: t_3 \leq t \leq t_2 + 2\pi\}} |H(\Phi_2(e^{it}))|^p |\Phi_2'(e^{it})| dt, \end{aligned}$$

where $\Phi_1(t_i) = s_i, i = 1, 4$, and $\Phi_2(t_i) = s_i, i = 2, 3$. Moreover $|\Phi_1'(e^{it})|$ is bounded on $\{e^{it} : t_1 \leq t \leq t_4\}$ and $|\Phi_2'(e^{it})|$ on $\{e^{it} : t_3 \leq t \leq t_2 + 2\pi\}$. Therefore,

$$\begin{aligned} \int_{\mathbb{T}} |H(e^{is})|^p ds &\leq C \int_{\{e^{it}: t_1 \leq t \leq t_4\}} |H(\Phi_1(e^{it}))|^p dt + C \int_{\{e^{it}: t_3 \leq t \leq t_2 + 2\pi\}} |H(\Phi_2(e^{it}))|^p dt \\ &\leq C(\|K_1\|_p^p + \|K_2\|_p^p) \leq L_p^p < \infty. \end{aligned}$$

The constant L_p depends on the same parameters as E_p from (2.1) and the quasiconformal constant k .

Thus $H \in H^p(\mathbb{D})$, and so $f_t \in h^p(\mathbb{D})$. Since f is quasiconformal, it follows that $g' \in H^p$. ■

Lemma 2.6. *Let $\alpha \in (0, 1)$. Then there is a positive constant $C(\alpha) > 1$ satisfying the following property. If f is a holomorphic function defined in the unit disk with continuous extension up to the boundary, and if*

$$X = \sup_{e^{it} \neq e^{is}} \frac{|f(e^{it}) - f(e^{is})|}{|e^{it} - e^{is}|^\alpha} \quad \text{and} \quad Y = \sup_{|z| < 1} (1 - |z|)^{1-\alpha} |f'(z)|,$$

then

$$(2.7) \quad \frac{1}{C(\alpha)} X \leq Y \leq C(\alpha) X.$$

Remark 2.7. We want to mention that a result similar to Lemma 2.6 is probably valid for the more general classes of mappings such as, real harmonic functions, or quasiconformal harmonic mappings, but we do not need such results (see e.g. [29]).

Proof of Lemma 2.6. First we have for $z = re^{i\theta}$ that

$$f'(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f(e^{it})e^{it}}{(e^{it} - z)^2} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(f(e^{it}) - f(e^{i\theta}))e^{it}}{(e^{it} - z)^2} dt.$$

Therefore

$$|f'(z)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|f(e^{i(t+\theta)}) - f(e^{i\theta})|}{1 + r^2 - 2r \cos t} dt.$$

By using the inequality $|e^{it} - 1| \leq |t|$, and introducing the change of variables $\varphi = 2t\sqrt{r}/\pi$, it follows that

$$\begin{aligned} |f'(z)| &\leq \frac{X}{\pi} \int_0^\pi \frac{t^\alpha}{(1-r)^2 + \frac{4r}{\pi^2} t^2} dt = \frac{X\pi^\alpha}{2^{\alpha+1}} \frac{r^{-(1+\alpha)/2}}{(1-r)^{1-\alpha}} \int_0^{\frac{2\sqrt{r}}{1-r}} \frac{\varphi^\alpha}{1+\varphi^2} d\varphi \\ &\leq \frac{X\pi^\alpha}{2^{\alpha+1}} \frac{r^{-(1+\alpha)/2}}{(1-r)^{1-\alpha}} \int_0^\infty \frac{\varphi^\alpha}{1+\varphi^2} d\varphi. \end{aligned}$$

So for $r > 1/2$ we have

$$(1 - |z|)^{1-\alpha} |f'(z)| \leq X \frac{\pi^\alpha}{2^{\alpha+1}} 2^{(1+\alpha)/2} \int_0^\infty \frac{\varphi^\alpha}{1 + \varphi^2} ds.$$

Thus, after a long but elementary calculation, we get that

$$(1 - |z|)^{1-\alpha} |f'(z)| \leq X \frac{\pi^{1+\alpha}}{2^{(1+3\alpha)/2}} \sec \left[\frac{\pi\alpha}{2} \right].$$

For $r < 1/2$ we have

$$\begin{aligned} (1 - |z|)^{1-\alpha} |f'(z)| &\leq \frac{X(1-r)^\alpha}{\pi} \int_0^\pi \frac{t^\alpha}{(1-r)^2 + \frac{4r}{\pi^2} t^2} dt \\ &\leq \frac{X(1-r)^\alpha}{\pi} \int_0^\pi \frac{t^\alpha}{(1-r)^2} ds \leq X 2^{2-\alpha} \frac{\pi^\alpha}{\alpha + 1}. \end{aligned}$$

Conversely, by using the proof of the Hardy–Littlewood theorem (see [10], Theorem 3 on p. 411), if

$$(1 - |z|)^{1-\alpha} |f'(z)| \leq Y,$$

then for $|s - t| \leq 1$ we get

$$|f(e^{it}) - f(e^{is})| \leq Y(2/\alpha + 1) |t - s|^\alpha.$$

Therefore for $t, s \in [-\pi, \pi]$, by noticing that $e^{it} = e^{it+2\pi i}$, for the case $|t - s| > 1$ or for the case $|2\pi - (t - s)| > 1$ we get

$$\begin{aligned} |f(e^{it}) - f(e^{is})| &\leq \sum_{j=1}^4 |f(e^{it_j}) - f(e^{it_{j-1}})| \\ &\leq \sum_{j=1}^4 Y(2/\alpha + 1) |t_j - t_{j-1}|^\alpha \leq 4Y(2/\alpha + 1) |t - s|^\alpha. \end{aligned}$$

So (2.7) is satisfied for

$$C(\alpha) = \max \left\{ 2^{2-\alpha} \frac{\pi^{\alpha+1}}{\alpha + 1}, \frac{\pi^{1+\alpha}}{2^{(1+3\alpha)/2}} \sec \left[\frac{\pi\alpha}{2} \right], 4 \left(\frac{2}{\alpha} + 1 \right) \right\}. \quad \blacksquare$$

3. Proof of the main result (Theorem 1.1)

We divide the proof into two cases.

- (a) D is the unit disk \mathbb{D} ,
- (b) D is a general Jordan domain with a \mathcal{C}^1 boundary.

Case (a). Since $\gamma \in \mathcal{C}^1$, γ has the following property. For every point $p \in \gamma$ there are complex numbers $|a| = 1$ and b so that the parametrisation of the curve

$$(3.1) \quad \gamma_p = a \cdot (\gamma - p)$$

above the point 0 has the form $\eta_p(x) = (x, \varphi_p(x))$, so that $\varphi_p(0) = \varphi'_p(0) = 0$.

Further, for every p and every $\epsilon > 0$, there is $\delta_0 = \delta_0(\epsilon)$ so that

$$|\varphi_p(x) - \varphi_p(0) - \varphi'_p(0)x| \leq \epsilon|x|,$$

for $|x| \leq \delta$. Moreover, δ_0 can be chosen to be independent on p ; that is, it depends on ϵ and γ only.

Let $x(t) = \Re(f(e^{it}))$. Then locally $y(t) = \Im(f(e^{it})) = \varphi(x(t))$. Assume also that $x(0) = 0$ and $f(1) = (0, 0)$. For fixed $\epsilon > 0$, because of Theorem 1.4 there is $\delta > 0$ ($\delta < 1$) so that $|t| \leq \delta$ implies $|x(t)| \leq \delta_0$ and so that

$$(3.2) \quad |\varphi_p(x(t)) - \varphi_p(0) - \varphi'_p(0)x(t)| \leq \epsilon|x(t)|.$$

Since $\varphi_p(0) = \varphi'_p(0) = 0$ we get

$$|\varphi_p(x(t))| \leq \epsilon|x(t)|, \quad |t| \leq \delta.$$

Let

$$v(z) = \Im f(z) = \Im(g + \bar{h}) = \Re(i(h(z) - g(z)))$$

and

$$u(z) = \Re f(z) = \Re(g(z) + h(z)).$$

Then by the Schwarz formula we get

$$i(h(z) - g(z)) = i\Im(h(0) - g(0)) + \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{is} + z}{e^{is} - z} \tilde{v}(s) ds$$

where

$$\tilde{v}(s) = \Re(i(h(e^{is}) - g(e^{is}))).$$

Thus

$$(3.3) \quad i(h'(z) - g'(z)) = \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\tilde{v}(s) - \tilde{v}(0)}{(e^{is} - z)^2} ds.$$

From now on we divide the proof into two steps.

3.1. Assume that f is smooth up to the boundary

If f has a smooth extension up to the boundary, then g' and h' have continuous extension to the boundary. Let $\alpha \in (0, 1)$ be arbitrary. Define

$$A = \max_{|z| < 1} (1 - |z|)^{1-\alpha} |i(h'(z) - g'(z))|.$$

We can assume that $A = (1 - \rho)^{1-\alpha} |i(h'(\rho) - g'(\rho))|$ for some $\rho \in [0, 1)$. Then we get

$$B = \max_{|z| < 1} (1 - |z|)^{1-\alpha} (|h'(z)| + |g'(z)|) \leq KA,$$

where K is the constant of the quasiconformality. In particular, from Lemma 2.6, h and g are α -Hölder's continuous on the boundary \mathbb{T} . More precisely,

$$|h(e^{it}) - h(e^{is})| \leq KAC(\alpha) |e^{it} - e^{is}|^\alpha$$

and

$$|g(e^{it}) - g(e^{is})| \leq KAC(\alpha) |e^{it} - e^{is}|^\alpha.$$

Therefore

$$|f(e^{it}) - f(e^{is})| \leq 2KAC(\alpha) |e^{it} - e^{is}|^\alpha.$$

In particular for $\tilde{u}(s) = \Re(f(e^{is})) = \Re(g(e^{it}) + h(e^{it}))$ we have

$$|\tilde{u}(s) - \tilde{u}(0)| \leq 2KAC(\alpha) |s|^\alpha.$$

Then, having in mind that for $t \in (-\delta, \delta)$, $\tilde{v}(t) = \varphi(\tilde{u}(t))$, from (3.3), (3.2) and the proof of Lemma 2.6, we get

$$\begin{aligned} |i(h'(\rho) - g'(\rho))(1 - \rho)^{1-\alpha} &\leq (1 - \rho)^{1-\alpha} \int_{-\pi}^{\pi} \frac{|\tilde{v}(s) - \tilde{v}(0)|}{\rho^2 - 2\rho \cos s + 1} \frac{ds}{\pi} \\ &= (1 - \rho)^{1-\alpha} \int_{[-\delta, \delta]} \frac{|\tilde{v}(s) - \tilde{v}(0)|}{\rho^2 - 2\rho \cos s + 1} \frac{ds}{\pi} \\ &\quad + (1 - \rho)^{1-\alpha} \int_{[-\pi, \pi] \setminus [-\delta, \delta]} \frac{|\tilde{v}(s) - \tilde{v}(0)|}{\rho^2 - 2\rho \cos s + 1} \frac{ds}{\pi} \\ &\leq 2\epsilon KAC(\alpha) \int_{[-\delta, \delta]} \frac{(1 - \rho)^{1-\alpha} |s|^\alpha}{\rho^2 - 2\rho \cos s + 1} \frac{ds}{\pi} + Z \\ &\leq 2\epsilon KAC^2(\alpha) + Z, \end{aligned}$$

where

$$Z = (1 - \rho)^{1-\alpha} \int_{[-\pi, \pi] \setminus [-\delta, \delta]} \frac{|\tilde{v}(s) - \tilde{v}(0)|}{\rho^2 - 2\rho \cos s + 1} \frac{ds}{\pi}.$$

Further,

$$Z \leq \text{diam}(\Omega) \frac{2\pi}{\pi} \frac{1}{1 + \cos^2 \delta - 2 \cos \delta \cdot \cos \delta} = \frac{2 \text{diam}(\Omega)}{\sin^2 \delta}.$$

So

$$A \leq 2\epsilon KAC^2(\alpha) + X \leq 2\epsilon KAC^2(\alpha) + \frac{2 \text{diam}(\Omega)}{\sin^2 \delta}.$$

By choosing $\epsilon > 0$ so that

$$2\epsilon KAC^2(\alpha) < A/2,$$

we get

$$(3.4) \quad A \leq \frac{4 \text{diam}(\Omega)}{\sin^2 \delta}.$$

Observe that δ , and so A depends on K , γ , α and modulus of continuity of f at the boundary, but not on a specific point $z \in \mathbb{D}$.

3.2. Approximation argument

If $p \in \partial\Omega = \gamma$ and $\gamma \in \mathcal{C}^1$, then, after possible rotation and translation of Ω (similarly as in (3.1)), which preserves the harmonicity and the quasiconformal constant of the corresponding mapping, we can assume that $p = 0$ and the unit normal vector is $N_p = (1, 0)$.

So we can find a sub-arc of γ containing p at its interior which is the graphic of a function defined as follows:

$$\gamma_p(\eta) = \{(x, \phi(x)) : x \in (-\eta, \eta)\}.$$

We also can assume that $\eta > 0$ is a positive constant that depends only on γ but not on the specific point p . Then we have $\phi'(0) = 0$. Let $\Omega_p \subset \Omega$ be a Jordan domain bounded by a \mathcal{C}^1 Jordan curve Γ_p consisted of $\gamma_p(\eta/2)$ and an interior part, which we denote by $\chi_p(\eta)$, which is subset of Ω and assume that $a_p \in \Omega_p$ be a fixed point. Then for small enough $\sigma = \sigma(\gamma) > 0$, the domain $\Omega_p(\kappa) = \Omega_p - \kappa N_p$ is a subset of Ω , for every $\kappa \in [0, \sigma]$.

Let $\Phi_{p,\kappa} : \mathbb{D} \rightarrow f^{-1}(\Omega_p(\kappa))$ be a conformal mapping so that

$$\Phi_{p,\kappa}(0) = f^{-1}(a_p - \kappa N_p).$$

Since \mathbb{T} is compact, there is a finite family of Jordan domains Ω_{p_j} , $j = 1, \dots, n$, so that $T_j := f^{-1}(\partial\Omega \cap \partial\Omega_{p_j})$, $j = 1, \dots, n$, covers \mathbb{T} . Moreover, $f \circ \Phi_{p_j,\kappa} : \mathbb{D} \rightarrow \Omega_{p_j}$ is α -Hölder continuous in \mathbb{D} . Further, there is a constant A_{p_j} , which depends only on Ω_{p_j} , so that

$$|f \circ \Phi_{p_j,\kappa}(e^{it}) - f \circ \Phi_{p_j,\kappa}(e^{is})| \leq A_{p_j} |e^{is} - e^{it}|^\alpha.$$

Note that A_{p_j} also depends on the modulus of continuity of $f \circ \Phi_{p_j,\kappa}$, where $\kappa \in [0, \sigma]$, but this family is uniformly continuous, and we can choose modulus of continuity that does not depend on κ , so A_{p_j} will not depend on κ either. Namely, the K -quasiconformal mappings $G_\kappa := \kappa + f \circ \Phi_{p_j,\kappa}$, $\kappa \in [0, \sigma]$, map the unit disk onto $\Omega_{p_j} \in \mathcal{C}^1$ and satisfy the condition $G_\kappa(0) = a_{p_j}$. By letting $\kappa \rightarrow 0$, we get

$$|f \circ \Phi_{p_j,0}(e^{it}) - f \circ \Phi_{p_j,0}(e^{is})| \leq A_{p_j} |e^{is} - e^{it}|^\alpha.$$

Therefore, by having in mind the fact that $\Phi_{p_j,0}^{-1}$ is smooth on T_j , we conclude that f is α -Hölder continuous in $T'_j \subset T_j$, where T'_j is a little bit smaller arc, but so that $\mathbb{T} \subset \bigcup_{j=1}^n T'_j$. Thus, f is α -Hölder continuous in \mathbb{T} . By standard arguments, we now obtain that f is α -Hölder continuous in \mathbb{D} , concluding the case (a).

If we want to get a more explicit estimate of A , then we repeat the procedure in the previous subsection, but with

$$A = \sup_{|z|<1} (1 - |z|)^{1-\alpha} |i(g'(z) - h'(z))|,$$

and thus we get the estimate

$$A - \varepsilon \leq \frac{4 \operatorname{diam}(\Omega)}{\sin^2 \delta}$$

instead of (3.4) for arbitrary $\varepsilon > 0$, and thus (3.4) is valid also in this case. Further,

$$|Df(z)| = (|g'(z)| + |h'(z)|) \leq K(|g'(z) - h'(z)|) \leq KA(1 - |z|)^{1-\alpha},$$

and so, for $(1 - \alpha)p < 1$,

$$\begin{aligned} (3.5) \quad \int_{\mathbb{D}} |Df(z)|^p d\lambda(z) &\leq K^p \int_{\mathbb{D}} A^p (1 - |z|)^{(1-\alpha)p} d\lambda(z) \\ &= \frac{2\pi K^p A^p}{2 - 3(1 - \alpha)p + (1 - \alpha)^2 p^2} = C_{p,\alpha}^p. \end{aligned}$$

For example, by choosing $\alpha = 1 - 1/(2p)$, we get

$$C_p^p = \frac{8}{3} \pi K^p A^p.$$

Case (b). The Hölder continuity follows from the case (a) and Theorem 1.4. To deal with the integral, we use the change of variables. Namely, let $\phi: \mathbb{D} \rightarrow D$ be a biholomorphism so that $\phi(0) = a$. Then by using Hölder's inequality, the isoperimetric inequality and the relations (2.1) and (3.5), we get

$$\begin{aligned} \int_D |Df(z)|^p \, d\lambda(z) &= \int_{\mathbb{D}} (|Df(\phi(\zeta))| \cdot |\phi'(\zeta)|)^p |\phi'(\zeta)|^{2-p} \, d\lambda(\zeta) \\ &= \int_{\mathbb{D}} (|Df(\phi(\zeta))| \cdot |\phi'(\zeta)|)^p |\phi'(\zeta)|^{2-p} \, d\lambda(\zeta) \\ &\leq \left(\int_{\mathbb{D}} (|Df(\phi(\zeta))| \cdot |\phi'(\zeta)|)^q \, d\lambda(\zeta) \right)^{p/q} \left(\int_{\mathbb{D}} |\phi'(\zeta)|^{(2-p)q'} \, d\lambda(\zeta) \right)^{1/q'} \\ &\leq C_q^p \cdot \frac{1}{(4\pi)^{1/q'}} \left(\int_{\mathbb{T}} |\phi'(\zeta)|^{(1-p/2)q'} \, d\lambda(\zeta) \right)^{2/q'} \\ &\leq C_q^p \cdot \frac{1}{(4\pi)^{1/q'}} (E_{(1-p/2)q'})^{2-p} = B_p^p, \end{aligned}$$

where $1/q + 1/q' = 1$, and $q = p + 1$.

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