Free groups as end homogeneity groups of 3-manifolds

Dennis J. Garity and Dušan D. Repovš

Abstract. For every finitely generated free group F, we construct an irreducible open 3-manifold M_F whose end set is homeomorphic to a Cantor set, and with the end homogeneity group of M_F isomorphic to F. The end homogeneity group is the group of all self-homeomorphisms of the end set that extend to homeomorphisms of the entire 3-manifold. This extends an earlier result that constructs, for each finitely generated abelian group G, an irreducible open 3-manifold M_G with end homogeneity group G. The method used in the proof of our main result also shows that if G is a group with a Cayley graph in $\mathbb{R}^3$ such that the graph automorphisms have certain nice extension properties, then there is an irreducible open 3-manifold M_G with end homogeneity group G.

Introduction

Each compact set K in S^3 (or in $\mathbb{R}^3$) has for its complement an open 3-manifold M^3. There is often a close relation between properties of the embedding of K in S^3 and properties of this 3-manifold complement. Consider, for example, embeddings of knots and the study of knot complements. See also [21] for a construction of a Cantor set with hyperbolic complement.

If the compact set K is a Cantor set C in S^3, then the complement of C is an open 3-manifold M^3 with end set C. The embedding of the Cantor set gives rise to properties of the corresponding complementary 3-manifold M^3 and of the end set C of M^3. See [8], [10], [21] for examples of this. There has been extensive study of embeddings of Cantor sets in S^3 or $\mathbb{R}^3$. The space of embeddings is extremely complicated. See for example [15].

In [9], the authors showed that for every finitely generated abelian group G, there is an irreducible open 3-manifold M_G with end homogeneity group isomorphic to G. The end homogeneity group is the group of all end homeomorphisms that extend to the entire 3-manifold. This 3-manifold is the complement of a carefully constructed Cantor set C_G associated with G in S^3.

For further examples of properties of embeddings related to properties of the containing 3-manifold or of the 3-manifold arising as the complement, see [16] for a relation of

certain group presentations to certain wild embeddings of arcs and [6] for a construction of certain wild Cantor sets that are Julia sets. See also [11] for another example of the use of embeddings to construct certain 3-manifolds.

In this paper, the authors show that for any finitely generated free group F, there is an irreducible open 3-manifold M_F with the end homogeneity group F. The proof leads to a technique which shows that if a group G has a Cayley graph embeddable in S^3 in a particularly nice way, then there is an irreducible open 3-manifold M_G with the end homogeneity group isomorphic to G.

In Section 1 we discuss concepts and results needed in the paper. This includes the concepts of geometric index, local genus, embedding homogeneity groups, end homogeneity groups, Antoine Cantor sets, and rigidly embedded Cantor sets. In Section 2, we carefully construct a specific Cayley graph in $\mathbb{R}^3$ for the free group on N generators, F_N. In Section 3 we construct a specific Cantor set in S^3 related to the Cayley graph constructed in Section 2. In Section 4 we prove the main theorems. We end with some open questions.

1. Preliminaries

1.1. Background information

We refer to [13], [14], [12], [10] for a discussion of Cantor sets in general and of rigid Cantor sets, and to [24] for results about local genus of points in Cantor sets and defining sequences for Cantor sets. The bibliographies in these papers contain additional references to results about Cantor sets. Two Cantor sets X and Y in $\mathbb{R}^3$ (or S^3) are said to be *equivalent* or *equivalently embedded* if there is a self-homeomorphism of $\mathbb{R}^3$ (or S^3) taking X to Y. If there is no such homeomorphism, the Cantor sets are said to be *inequivalent*, or *inequivalently embedded*. A Cantor set C is said to be *rigidly embedded* in $\mathbb{R}^3$ (or S^3) if the only self-homeomorphism of C that extends to a homeomorphism of $\mathbb{R}^3$ (or S^3) is the identity.

1.2. Geometric index

We list the results which we will need on geometric index. See Schubert [17], [12] for more details. If K is a link in the interior of a solid torus T, the *geometric index* of K in T, denoted by $\mathrm{N}(K, T)$, is the minimum of $|K \cap D|$ over all meridional disks D of T intersecting K transversely. If T is a solid torus and M is a finite union of disjoint solid tori so that $M \subset \mathrm{Int}\,T$, then the geometric index $\mathrm{N}(M, T)$ of M in T is $\mathrm{N}(K, T)$ where K is a core of M.

Note: An *unknotted solid torus* is a solid torus $B^2 \times S^1$ with center line $\{0\} \times S^1$ an unknotted circle in S^3.

Theorem 1.1 ([17], [12], Theorem 3.1). *Let T_0 and T_1 be unknotted solid tori in S^3 with $T_0 \subset \mathrm{Int}(T_1)$ and $\mathrm{N}(T_0, T_1) = 1$. Then ∂T_0 and ∂T_1 are parallel; i.e., the 3-manifold $T_1 - \mathrm{Int}(T_0)$ is homeomorphic to $\partial T_0 \times I$, where I is the closed unit interval $[0, 1]$.*

Theorem 1.2 ([17], [12], Theorem 3.2). *Let T_0 be a finite union of disjoint solid tori. Let T_1 and T_2 be solid tori so that $T_0 \subset \mathrm{Int}(T_1)$ and $T_1 \subset \mathrm{Int}(T_2)$. Then $\mathrm{N}(T_0, T_2) = \mathrm{N}(T_0, T_1) \cdot \mathrm{N}(T_1, T_2)$.*

There is one additional result we will need.

Theorem 1.3 ([17], [12], Theorem 3.3). *Let T be a solid torus in S^3 and let $T_1, \ldots, T_n$ be unknotted pairwise disjoint solid tori in T, each of geometric index 0 in T. Then the geometric index of $\bigcup_{i=1}^{n} T_i$ in T is even.*

1.3. Defining sequences and local genus

We review the definition and some basic facts from [24] about the local genus of points in a Cantor set. Also see [24] for a discussion of defining sequences.

A *defining sequence* for a Cantor set X in $\mathbb{R}^3$ is a properly nested sequence of handlebodies (M_i) such that $\bigcap M_i = X$. Let $\mathcal{D}(X)$ be the set of all defining sequences for a Cantor set $X \subset S^3$. Let $(M_i) \in \mathcal{D}(X)$ be one of these defining sequences. For $A \subset X$, denote by M_i^A the union of those components of M_i which intersect A. Define

$$g_A(X; (M_i)) = \sup\{g(M_i^A);\ i \geq 0\} \quad \text{and}$$
$$g_A(X) = \inf\{g_A(X; (M_i));\ (M_i) \in \mathcal{D}(X)\},$$

where $g(M_i^A)$ is the genus of M_i^A. The number $g_A(X)$ is called *the genus of the Cantor set X with respect to the subset A*. For $A = \{x\}$ we call the number $g_{\{x\}}(X)$ *the local genus of the Cantor set X at the point x* and denote it by $g_x(X)$.

Remark 1.4. The genus measures the minimum genus of handlebodies M_i needed in any defining sequence for the Cantor set. The standard Cantor set has genus 0 at each point since it can be defined as an intersection of genus 0 balls. An Antoine Cantor set (illustrated in Figure 1) has genus 1 at each point since it is defined by genus 1 manifolds (solid tori), but cannot be defined by genus 0 manifolds (balls).

Remark 1.5. Let x be an arbitrary point of a Cantor set X and let $h \colon S^3 \to S^3$ be a homeomorphism. Then the local genus $g_x(X)$ is the same as the local genus $g_{h(x)}(h(X))$. Also note that if $x \in C \subset C'$, then the local genus of x in C is less than or equal to the local genus of x in C'. See Theorem 2.4 in [24].

The following result from [24] is needed to show that certain points in our examples have local genus > 1.

Theorem 1.6 ([24]). *Let $X, Y \subset S^3$ be Cantor sets and let $p \in X \cap Y$. Suppose there exist a 3-ball B and a 2-disc $D \subset B$ such that*

(1) *$p \in \mathrm{Int} B$, $\partial D = D \cap \partial B$, $D \cap (X \cup Y) = \{p\}$; and*

(2) *$X \cap B \subset B_X \cup \{p\}$ and $Y \cap B \subset B_Y \cup \{p\}$, where B_X and B_Y are the components of $B - D$.*

Then $g_p(X \cup Y) = g_p(X) + g_p(Y)$.

1.4. Embedding homogeneity groups and end homogeneity groups

For background on Freudenthal compactifications and theory of ends, see [4], [7], [20]. For an alternate proof using defining sequences of the result that every homeomorphism of the open 3-manifold extends to a homeomorphism of its Freudenthal compactification, see [8].

Each Cantor set C in S^3 has for its complement an open 3-manifold M^3 with end set C. The embedding of the Cantor set gives rise to properties of the corresponding complementary 3-manifold M^3. See [8], [10], [21] for examples of this.

We investigate possible group actions on the end set C of the open 3-manifold M^3 in the following sense: the *homogeneity group of the end set* is the group of homeomorphisms of the end set C that extend to homeomorphisms of the open 3-manifold M^3. Referring specifically to the embedding of the Cantor set, this group can also be called the *embedding homogeneity group of the Cantor set*. See [5], [22] for a discussion and overview of some other types of homogeneity.

The standardly embedded Cantor set is at one extreme here. The embedding homogeneity group is the full group of self-homeomorphisms of the Cantor set, an extremely rich group (there is such a homeomorphism taking any countable dense subset to any other). Cantor sets with this full embedding homogeneity group are called *strongly homogeneously embedded*. See Daverman [3] for an example of a non-standard Cantor set with this property.

At the other extreme are *rigidly embedded* Cantor sets, i.e., those Cantor sets for which only the identity homeomorphism extends. Shilepsky [19] constructed Antoine type [2] rigid Cantor sets. Their rigidity is a consequence of Sher's result [18] that if two Antoine Cantor sets are equivalently embedded, then the stages of their defining sequences must match up exactly. Newer examples [14], [12] of non-standard Cantor sets were constructed that are both rigidly embedded and have simply connected complement. See [23] for additional examples of rigidity.

These examples naturally lead to the question of which types of groups can arise as end homogeneity groups between the two extremes mentioned above. In this paper we show that for each finitely generated free group G, there is an irreducible open 3-manifold with the end set homeomorphic to a Cantor set and with the end homogeneity group isomorphic to G.

Remark 1.7. The Cantor sets produced are *unsplittable* as are the set of ends, in the sense that no 2-sphere separates points of the Cantor set (respectively, points of the end set). Correspondingly, the complements of the Cantor sets produced are irreducible in the sense that every 2-sphere in the complement bounds a 3-ball.

1.5. Antoine Cantor sets

An Antoine Cantor set is described as follows. Let M_0 be an unknotted solid torus in S^3. Let M_1 be a chain of at least four linked, pairwise disjoint, unknotted solid tori in M_0 as in Figure 1. Inductively, M_i consists of pairwise disjoint solid tori in S^3 and M_{i+1} is obtained from M_i by placing a chain of at least four linked, pairwise disjoint, unknotted solid tori in each component of M_i. If the diameter of the components goes to 0, the Antoine Cantor set is $C = \bigcap_{i=0}^{\infty} M_i$.
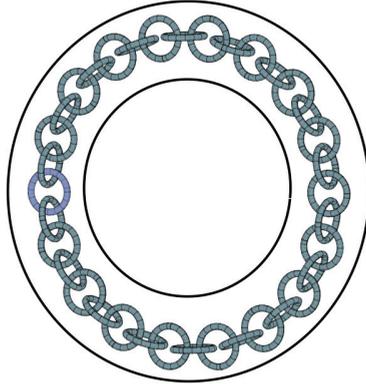
Figure 1. Two stages for an Antoine construction.

We refer to Sher's paper [18] for basic results and description of Antoine Cantor sets. The key result we will need is the following.

Theorem 1.8 (Theorems 1 and 2 in [18]). *Suppose C and D are Antoine Cantor sets in S^3 with defining sequences $\{M_i\}$ and $\{N_i\}$, respectively. Then C and D are equivalently embedded if and only if there is a self-homeomorphism h of S^3 with $h(M_i) = N_i$ for each i.*

In particular, the number and adjacency of links in the chains must match up at each stage.

Remark 1.9. A standard argument shows that Antoine Cantor set cannot be separated by any 2-sphere.

Remark 1.10. Also note that the homeomorphism in Theorem 1.8 can be realized as the final stage of an ambient isotopy in S^3 since each of the homeomorphisms in the argument can be realized by an ambient isotopy.

2. A Cayley graph for F_N

Let F_N be the free group on generators $a_1, \ldots, a_N$. Denote the inverse of a_i by $\overline{a_i}$ and let $g(F_N)$, the full generating set of F_N, be $\{a_1, \ldots, a_N, \overline{a_1}, \ldots, \overline{a_N}\}$. For economy of notation, let $\beta_j = \alpha_j$ for $1 \leq j \leq N$ and let $\beta_j = \overline{\alpha_{j-N}}$ for $N + 1 \leq j \leq 2N$. Note that $\overline{\beta_k} = \beta_{k+n \pmod{2N}}$. Elements of F_N correspond to reduced words using elements of $g(F_N)$ as letters. Group multiplication corresponds to concatenation and reduction of words. We first describe a specific embedding for the Cayley graph $G(F_N)$ associated with this presentation of F_N.

The identity is represented by the empty word and corresponds to a vertex $v(e)$ at the origin. There are $\sigma(n) = 2N \cdot (N - 1)^{n-1}$ reduced words of length n. These will be represented by $\sigma(n)$ equally spaced vertices on the circle of radius n centered at the origin,

at radial angles

$$\left\{ n(k) \equiv \frac{2\pi}{\sigma(n)} k - \frac{2\pi}{2\sigma(n)} \mid 1 \le k \le \sigma(n) \right\}.$$

Each of these points, $p(n, k) \equiv n \cdot e^{n_k \cdot i}$ for $1 \le k \le \sigma(n)$, on the circle of radius n will be joined by edges to $N - 1$ vertices on the circle of radius $n + 1$. Specifically, $p(n, k)$ will be joined to the points

$$\{ p(n + 1, j) \mid (k - 1)(N - 1) < j \le (k)(N - 1) \}.$$

Figure 2 show the first two stages of the Cayley graph for F_2 and F_3. The colored directed edges correspond to generators and will be described in more detail in the next subsection.

(a) F_2 Stage 2.

(b) F_2 Stage 3.

(c) F_3 Stage 2.

(d) F_3 Stage 3.
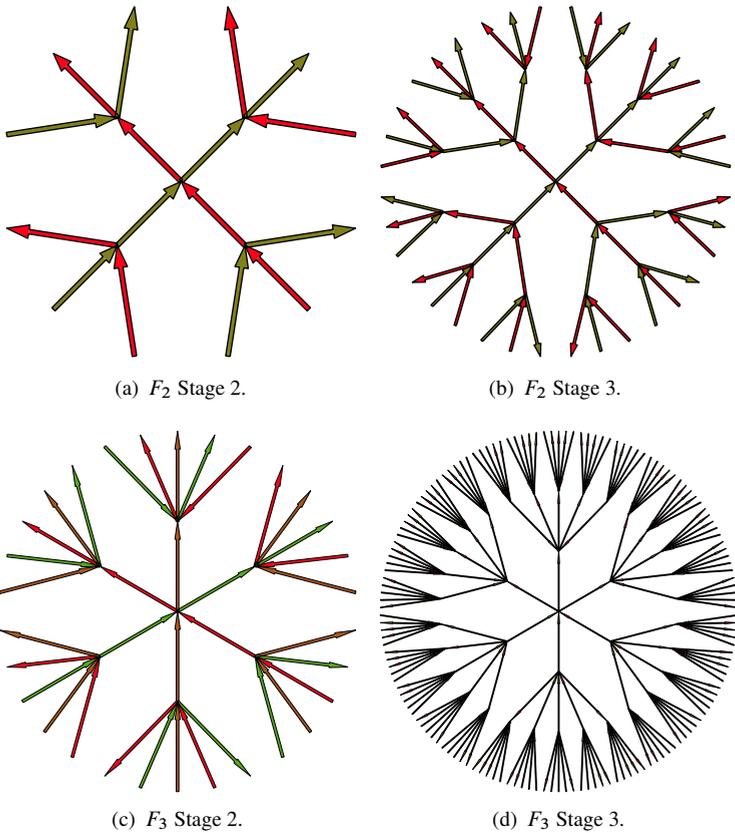
Figure 2. Cayley graphs for F_2 and F_3.

2.1. Labeling edges and vertices

We inductively describe how to label the vertices and how to label and orient the edges of the graph described above.

Remark 2.1. Note that an edge oriented from v to w labeled with the generator a_i is regarded the same as that edge oriented from w to v and labeled with the generator $\overline{a_i}$.

Begin by labeling the vertex at the origin corresponding to the identity as $v(e)$. Label the vertex at $p(1, j)$ for $1 \leq j \leq 2N$ as $v(\beta_j)$.

Label the edge from $v(e)$ to $v(\beta_j)$, $1 \leq j \leq 2N$ with the generator β_j and orient it from $v(e)$ to $v(\beta_j)$. Note that this is equivalent to labeling the edge from $v(\beta_j)$ to $v(e)$ with the generator $\overline{\beta_j}$ and orienting it from $v(\beta_j)$ to $v(e)$. For notation, we refer to this edge as $E(v(e), \beta_j)$ or $E(v(\beta_j), \overline{\beta_j})$ to emphasize the initial vertex and generator needed to get to the other vertex. To emphasize the initial and final vertices, we can use the notation $E(v(e), v(\beta_j)) = E(v(\beta_j), v(e))$.

Note that starting at any point on the circle of radius $1/2$ centered at the origin and traveling in the counterclockwise direction around this circle, one encounters the labeled edges in the cyclic order $E(v(e), \beta_1), \ldots, E(v(e), \beta_{2N})$, or equivalently, by Remark 2.1, in the cyclic order

$$E(v(e), \beta_1), \ldots, E(v(e), \beta_N), E(v(\beta_{N+1}), \beta_1), \ldots, E(v(\beta_{2N}), \beta_N).$$

Inductively assume the vertices at distance m from the origin, $m \leq n$, have been labeled as $v(g_i)$, where g_i is a reduced word of length m. Also assume that edges from vertices $v(g_i)$ at distance m from the origin ($m \leq n-1$) to $v(g_\ell)$ at distance $m + 1$ from the origin have been labeled $E(v(g_i), \beta_k)$ where $g_\ell = g_i \circ \beta_k$.

Consider $v(g)$ for some g of reduced length n. Then $g = g' \circ \beta_k$ for some reduced word g' of length $n - 1$ and some $\beta_k \in g(F_N)$. The edge joining $v(g')$ to $v(g)$ is labeled $E(v(g'), \beta_k)$, or equivalently $E(v(g), \overline{\beta_k}) = E(v(g), \beta_{k+N(\text{mod } 2N)})$. Let $j = k + n(\text{mod 2N})$.

Form a circle of radius $1/2$ about $v(g)$. Starting at the edge $E(v(g), \beta_j)$, proceed around the circle in the counterclockwise direction labeling the edges encountered as

$$E(v(g), \beta_{j+1}), \ldots, E(v(g), \beta_{j+2N-1}),$$

all subscripts mod $(2N)$.

Label the vertices at distance $n + 1$ from the origin as follows. If v is such a vertex, then there is an edge $E(v(g), \beta_j)$ from a vertex $v(g)$ at distance n from the origin to v. Label the vertex v as $v(g \circ \beta_j)$.

The inductive assumption is easily checked for the newly labeled vertices and edges. See Figure 3 for an illustration, keeping in mind Remark 2.1.

2.2. Cayley graph automorphisms for $G(F_N)$

For each element g of F_N we define a graph automorphism h_g of $G(F_N)$ as follows. For a vertex $v(g_i)$ of $G(F_N)$ corresponding to the element g_i of F_N, $h_g(v(g_i))$ is defined to be $v(g \circ g_i)$. The edge from $v(g_i)$ to $v(g_i \circ \beta_k)$ can be labeled $E(v(g_i), \beta_k)$ when oriented
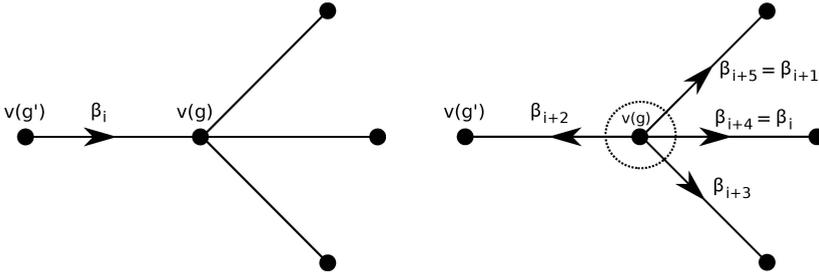
Figure 3. Inductively labelling edges.

from $v(g_i)$ to $v(g_i \circ \beta_k)$. Define $h_g(E(v(g_i), \beta_k)$ to be the linear homeomorphism from $E(v(g_i), \beta_k)$ to $E(v(g \circ g_i), \beta_k)$.

Remark 2.2. Note that these graph automorphisms define homeomorphisms from $G(F_N)$ to itself. Note also that $h_{g_1} \circ h_{g_2} = h_{g_1 g_2}$.

2.3. Extending graph automorphisms to homeomorphisms of $\mathbb{R}^3$

The graph automorphisms h_{β_i}, for $1 \leq i \leq N$, can be extended to homeomorphisms h'_{β_i} from $\mathbb{R}^2$ to itself. Note that $\mathbb{R}^2 - G(F_N)$ consists of a collections of unbounded regions with boundary in $G(F_N)$. Adding in the point at infinity, each region is a closed disc with boundary consisting of edges in $G(F_N)$ together with this added point. The graph automorphism h_{β_i} takes the boundary of one of these disc regions to the boundary of another, and can thus be extended to the entire region. There are many choices for these extension homeomorphisms. They coincide up to proper isotopy.

Piecing together these extensions, one gets an extension of h_{β_i} to a self homeomorphism of $\mathbb{R}^2$, h'_{β_i} for $1 \leq i \leq N$. Define h'_{β_i} for $N + 1 \leq i \leq 2N$ to be $(h'_{\beta_{i-N}})^{-1}$. Since $h_{\beta_i} = (h_{\beta_{i-N}})^{-1}$ for $N + 1 \leq i \leq 2N$ as graph automorphisms, it follows that h'_{β_i} is an extension of h_{β_i} for $N + 1 \leq i \leq 2N$.

For a general $g \in F_N$, g has a unique reduced representation $g = \beta_{i1} \circ \beta_{i2} \circ \cdots \circ \beta_{iM}$ where the β_{ij} are in $g(F_N)$. Define $h'(g)$ to be

$$h'_g = h'_{\beta_{i1}} \circ h'_{\beta_{i2}} \circ \cdots \circ h'_{\beta_{iM}}.$$

The homeomorphism $h'_g : \mathbb{R}^2 \to \mathbb{R}^2$ can then be extended to a homeomorphism $H_g : \mathbb{R}^3 \to \mathbb{R}^3$ by crossing with the identity.

Remark 2.3. Note that $h'_{g_1} \circ h'_{g_2} = h'_{g_1 \circ g_2}$. This follows from the uniqueness of reduced representations of elements of F_N. The definition of H_g as the product of h'_g with the identity shows that it is also true that $H_{g_1} \circ H_{g_2} = H_{g_1 \circ g_2}$.

3. A Cantor set $C(F_N)$ with embedding homogeneity group F_N

3.1. A collection of balls and tubes in $\mathbb{R}^3$ containing $G(F_N)$

In $\mathbb{R}^3$, choose a 3-ball $B(v(e))$ around the origin $= v(e)$ so that the homeomorphic copies $H_{\beta_i}(B(v(e)))$, $1 \le i \le 2N$, together with $B(v(e))$ are all pairwise disjoint. Let $B(v(\beta_i))$ denote the 3-ball $H_{\beta_i}(B(v(e)))$. The original ball $B(v(e))$ can be chosen so that it is contained in a ball of radius $1/5$ about $v(e)$, and so that each $B(v(\beta_i))$ is contained in a ball of radius $1/5$ about $v(\beta_i)$.

For a general $g \in F_N$, use the unique reduced representation of g as $\beta_{i1} \circ \beta_{i2} \circ \cdots \circ \beta_{iM}$, where the β_{ij} are in $g(F_N)$. The ball $B(v(g))$ is then defined to be to be

$$H_{\beta_{i1}} \circ H_{\beta_{i2}} \circ \cdots \circ H_{\beta_{iM}}(B(v(e))) = H_g(B(v(e))).$$

We similarly define a collection of tubes. For each i, $1 \le i \le N$, choose a thin tube $D(v(e), \beta_i)$ joining $B(v(e))$ to $B(v(\beta_i))$ so that $B(v(e)) \cup D(v(e), \beta_i) \cup B(v(\beta_i))$ contains $E(v(e), \beta_i)$. Choose the tubes $D(v(e), \beta_1), \ldots, D(v(e), \beta_N)$, so as to be pairwise disjoint. For each i, $N + 1 \le i \le 2N$, let $D(v(e), \beta_i)$ be $H_{\beta_i}(D(v(e), \beta_{i-N})) = H_{\beta_i}(D(v(e), \overline{\beta_i}))$. See Figure 4 for an illustration for F_2.

Remark 3.1. Note that the relationship $D(v(e), \beta_i) = H_{\beta_i}(D(v(e), \overline{\beta_i}))$ then holds for all i, $1 \le i \le 2N$.
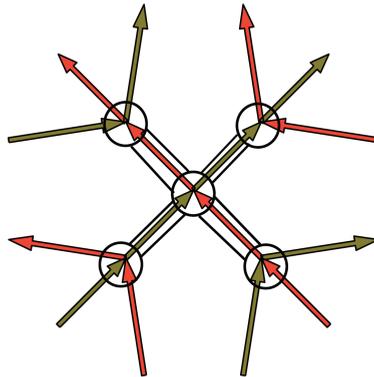
Figure 4. Balls, tubes around $G(F_2)$.

3.2. Rigid Cantor sets using $G(F_N)$ as a guide

We will now define rigid Antoine Cantor sets associated with each vertex in $G(F_N)$, and a chain of three linked rigid Cantor sets for each edge in $G(F_n)$. See Figure 5 for an illustration of the outer tori in defining sequences for these rigid Cantor sets in the case of F_2.
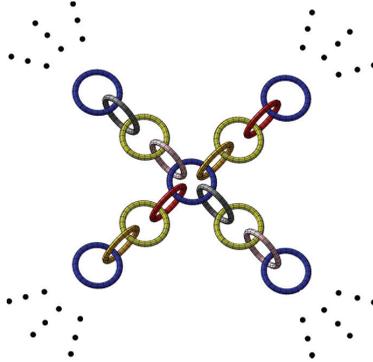
Figure 5. Construction of rigid Cantor sets placement for $G(F_2)$.

To begin, construct a rigid Antoine Cantor set $C(v(e))$ so the outer torus, $T(C(v(e)))$, of the construction is in $B(v(e))$. This $T(C(v(e)))$ corresponds to the central torus in Figure 5 in the case of F_2. Let $T'(C(v(e)))$ be the union of the second stage tori in the construction of $C(v(e))$. See Figure 1 for an illustration of the first and second stages of an Antoine construction.

For a general $g \in F_N$, use the unique reduced representation of g as $\beta_{i1} \circ \beta_{i2} \circ \cdots \circ \beta_{iM}$ where the β_{ij} are in $g(F_N)$. Define a rigid Cantor set $C(v(g)) \subset B(v(g))$ to be

$$H_{\beta_{i1}} \circ H_{\beta_{i2}} \circ \cdots \circ H_{\beta_{iM}}(C(v(e))) = H_g(C(v(e))).$$

Remark 3.2. Note that $C(v(e))$ and any $C(v(g))$ are equivalently embedded.

Similarly, let

$$T(C(v(g))) = H_g(T(C(v(e)))), \quad \text{and let} \quad T'(C(v(g))) = H_g(T'(C(v(e)))).$$

The four outer tori in Figure 5 correspond to $T(C(a_1)), T(C(a_2)), T(C(\overline{a_1}))$, and $TC(\overline{a_2})$ in the case of F_2.

We next construct rigid Cantor sets $C_1(v(e), \beta_k), C_2(v(e), \beta_k)$ and $C_3(v(e), \beta_k)$ for each k, $1 \le k \le N$. Choose a series of three linked tori, $T_1(v(e), \beta_k), T_2(v(e), \beta_k)$ and $T_3(v(e), \beta_k)$ joining $T(C(v(e)))$ and $T(C(\beta_k))$, for $1 \le k \le N$, and contained in $B(v(e)) \cup D(v(e), \beta_k) \cup B(v(\beta_k))$ as in Figure 5. Label the torus linked to $T(C(v(e)))$ as $T_1(v(e), \beta_k)$, and use it as the outer torus for a rigid Antoine Cantor set $C_1(v(e), \beta_k)$. Label the torus linked to $T(C(\beta_k))$ as $T_3(v(e), \beta_k)$, and use it as the outer torus for a rigid Antoine Cantor set $C_3(v(e), \beta_k)$. Label the remaining torus as $T_2(v(e), \beta_k)$ and use it as the outer torus for a rigid Antoine Cantor set $C_2(v(e), \beta_k)$.

Remark 3.3. Choose these $3N$ rigid Cantor sets $\{T_i(v(e), \beta_k), 1 \le 1 \le 3, 1 \le k \le N\}$ so that they are all inequivalently embedded, and so that they are all inequivalent to $C(v(e))$.

For $N + 1 \le k \le 2N$, and $1 \le j \le 3$, let $C_j(v(e), \beta_k) = H_{\beta_k}(C_j(v(e), \overline{\beta_k}))$, and let $T(C_j(v(e), \beta_k)) = H_{\beta_k}(T(C_j(v(e), \overline{\beta_k})))$. This completes the definition of the $3 \cdot 2N$

rigid Cantor sets along the edges $E(v(e), \beta_k)$. See Figure 5 for outer tori corresponding to these Cantor sets in the case $F(2)$.

For $g \in F_N$, and for $1 \leq i \leq 3$ and $1 \leq k \leq 2N$, let $C_i(v(g), \beta_k) = H_g(C_i(v(e), \beta_k))$ and let $T(C_i(v(g), \beta_k)) = H_g(T(C_i(v(e), \beta_k)))$.

For any rigid Antoine Cantor set C in this construction, $T(C)$ represents the outer torus in the Antoine construction and $T'(C)$ represents the union of the second stage tori in the construction.

Remark 3.4. Because $E(v(g), \beta_k)$ and $E(v(g \circ \beta_k), \overline{\beta_k})$ are identified, it appears that there are two possible definitions for the C_i on this edge: either $H_g(C_i(v(e), \beta_k))$ or $H_{g \circ \beta_k}(C_i(v(e), \overline{\beta_k}))$. However,

$$H_{g \circ \beta_k}(C_i(v(e), \overline{\beta_k})) = H_g(H_{\beta_k}(H_{\overline{\beta_k}}(C_i(v(e), \beta_{k_i})))) = H_g(C_i(v(e), \beta_k)).$$

Remark 3.5. Note that none of the $C_i(v(g), \beta_k)$ are equivalent to any $C(v(g))$, and that $C_i(v(g), \beta_k)$ is equivalent to $C_j(v(g'), \beta_\ell)$ if and only if one of the following two conditions holds:

$$i = j \text{ and } \beta_k = \beta_\ell, \quad \text{or} \quad i = j \text{ and } \beta_k = \overline{\beta_\ell}.$$

The union of the Cantor sets $C_i(g, \beta_k)$ together with the union of the Cantor sets $C(v(g))$ is not a Cantor set since it is not compact. To remedy this, we add the point at infinity, denoted by p, to $\mathbb{R}^3$ to get S^3. We then define the Cantor set $\mathcal{C}$ associated with $G(F_N)$ to be

$$\mathcal{C} = C(G(F_N)) = \{C_i(v(g), \beta_k), g \in F_N, 1 \leq k \leq 2N\} \cup \{C(v(g)), g \in F_N\} \cup \{p\}.$$

4. Main theorems

We now have $\mathcal{C} \subset S^3$. One can show that $\mathcal{C}$ cannot be separated by any 2-sphere, by an argument similar to that referred to in Remark 1.9. Let $\mathcal{M} = M(F_N)$ be the 3-manifold $S^3 - \mathcal{C}$. By Remark 1.7, $\mathcal{M}$ is irreducible.

We will show that the embedding homogeneity group of $\mathcal{C} \subset S^3$ is F_N, and correspondingly, that the end homogeneity group of $\mathcal{M}$ is F_N.

We use the following three technical lemmas in the proof of Theorem 4.6 below. In reading the first lemma, think of T_1 and T_2 as two linked tori in a stage of an Antoine construction as in Figure 1. The second lemma shows that homeomorphisms of S^3 taking $\mathcal{C}$ to itself necessarily fix the point p. The third lemma shows that homeomorphisms of S^3 taking some $C(v(g))$ to some $C(v(g'))$, when restricted to $C(v(g))$, agree with some H_{g_1} restricted to $C(v(g))$.

Lemma 4.1. *Let C_1, C_2 and C_3 be rigid Antoine Cantor sets in $\mathbb{R}^3$. Let T_i and T_i' be the first and second stages in the construction of C_i. Assume that T_1 and T_2 are linked (and thus are disjoint). Let $h: \mathbb{R}^3 \to \mathbb{R}^3$ be a homeomorphism such that*

$(\star)$ $h(C_1) \cap C_3 \neq \emptyset$.

Assume in addition that

a) $h(\partial(T_1 \cup T_2) \cup \partial(T_1' \cup T_2')) \cap (\partial(T_3) \cup \partial(T_3')) = \emptyset$,

b) $h(C_1) \cap T_3 \subset C_3$, and

c) $h(C_2) \cap T_3 \subset C_3$.

If $h(T_1) \subset T_3$, then

I) *either the geometric index of $h(T_1)$ in T_3 is 0, $h(T_2)$ is also contained in T_3, and the geometric index of $h(T_2)$ in T_3 is also 0,*

II) *or the geometric index of $h(T_1)$ in T_3 is 1 and $C_3 \subset h(T_1)$.*

Proof. We have $h(T_1) \subset T_3$ and we consider the geometric index of $h(T_1)$ in T_3.

Index 0: Assume that the geometric index is 0. Then $h(T_1)$ is contained in a cell B in T_3, and so contracts in T_3.

If $\partial(h(T_2))$ does not intersect $\partial(B)$, then either $h(T_2) \subset B$, $B \subset h(T_2)$ or $h(T_2)$ misses B. The last case cannot occur because of the linking of $h(T_2)$ and $h(T_1)$. In the first case, $h(T_2) \subset T_3$ and the geometric index of $h(T_2)$ in T_3 is 0. So condition I) holds. The second case cannot occur because $h(T_1)$ and $h(T_2)$ are disjoint.

If $\partial(h(T_2))$ does intersect $\partial(B)$, then $h(T_2) \subset T_3$ by (a) above. If the geometric index of $h(T_2)$ in T_3 is ≥ 1, then $h(T_2)$ cannot be contained in any component of T_3' by Theorem 1.2. Thus, by (a), $h(T_2)$ contains any component of T_3' that it intersects. By (c), $h(C_2) \cap C_3 \neq \emptyset$. So $h(T_2)$ intersects and thus contains some component D_1 of T_3'. The geometric index of D_1 in $h(T_2)$ must be 0 by Theorem 1.2. Thus D_1 is contained in a cell in $h(T_2)$ and contracts in $h(T_2)$.

Let $D_1, D_2, \ldots, D_m$ be the components of T_3', listed so that D_i links D_{i+1}. Since D_1 and D_2 are linked, the Antoine construction guarantees that any contraction of D_1 intersects $C_3 \cap D_2$. So $h(T_2)$ intersects and thus contains D_2 and D_2 has geometric index 0 in $h(T_2)$. Continuing inductively, all of the D_i are contained in $h(T(2))$ and thus $h(T_2)$ contains C_3.

Since $h(T_1)$ is disjoint from $h(T_2)$, this contradicts condition ($\star$).

Index greater than 1: If the geometric index of $h(T_1)$ in T_3 is greater than 1, then by Theorem 1.2, $h(T_1)$ cannot be contained in any component of T_3'. So $h(T_1)$ contains each component of T_3' that it intersects. Let $D_1, D_2, \ldots, D_m$ be the components of T_3', listed so that D_i links D_{i+1}. By condition ($\star$), $h(T_1)$ intersects and thus contains some component of T_3', say D_1. The geometric index of D_1 in $h(T_1)$ is 0 by Theorem 1.2.

By an argument similar to that in the Index 0 case above, $h(T_1)$ contains each D_i, and each D_i is of index 0 in $h(T_1)$. By Theorem 1.3, the geometric index of T_3' in $h(T_1)$ is even. Since the geometric index of T_3' in T_3 is two [1], the geometric index of T_3' in $h(T_1)$ cannot be 0 and so is at least two. Theorem 1.2 now implies that the geometric index of T_3' in T_3 is at least 4, which is a contradiction. So the the geometric index of $h(T_1)$ in T_3 cannot be greater than 1.

Index equal to 1: If the geometric index of $h(T_1)$ in T_3 is 1, then by Theorem 1.2, $h(T_1)$ cannot be contained in any component of T_3'. An argument similar to that in the previous two cases shows that $h(T_1)$ contains each component of T_3'. Thus $C_3 \subset h(T_1)$. So condition II) holds. ∎

Lemma 4.2. *Let $\Gamma: \mathcal{C} \to \mathcal{C}$ be a homeomorphism that extends to a homeomorphism of S^3. Then $\Gamma(p) = p$.*

Proof. The local genus of any point in $\mathcal{C}$ other than p is less than or equal to one because these points are defined by nested sequences of tori which are of genus one. If we can show the local genus of p is at least 2, then the results mentioned in Remark 1.5 show that $\Gamma(p) = p$.

Remark 4.3. The local genus of p in $\mathcal{C}$ cannot be 0 because otherwise, there would be arbitrarily small 3-balls containing p with boundary missing $\mathcal{C}$. These boundary 2-spheres in the complement of $\mathcal{C}$ would separate linked stages of some Antoine construction. This cannot happen.
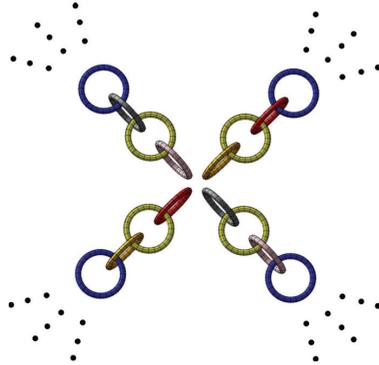
Figure 6. Start of construction for $\mathcal{C}_1$ and $\mathcal{C}_2$ in the case F(2).

To see that the local genus of p in $\mathcal{C}$ is at least 2, consider 2 subsets of $\mathcal{C}$ that meet in p. These subsets are obtained by dividing $\mathcal{C} - C(v(e))$ into two subsets that meet in p.

For a nonidentity $g \in F_N$, with unique reduced representation as $\beta_{i1} \circ \beta_{i2} \circ \cdots \circ \beta_{iM}$, where the β_{ij} are in $g(F_N)$, let $g_1 = \beta_{i1}$. Let

$$\mathcal{C}_1 = \bigcup_{k=1}^{N} \left(\{C(v(g)), g_1 = \beta_k\} \cup \bigcup_{i=1}^{3} \{C_i(v(g), \beta_j), g_1 = \beta_k\} \right) \cup \{p\},$$

and let

$$\mathcal{C}_2 = \bigcup_{k=N+1}^{2N} \left(\{C(v(g)), g_1 = \beta_k\} \cup \bigcup_{i=1}^{3} \{C_i(v(g), \beta_j), g_1 = \beta_k\} \right) \cup \{p\}.$$

Figure 6 illustrates the start of the described construction in the case $N = 2$. The Cantor set $\mathcal{C}_1$ arises from tori in the upper half of the figure, and the Cantor set set $\mathcal{C}_2$ arises from tori in the lower half of the figure.

Note that $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect in p. The local genus of p in each of these 2 Cantor sets is at least 1 by reasoning similar to that in Remark 4.3.

The $\mathcal{C}_i$ satisfy the conditions of Theorem 1.6. It follows that the genus of p in $\mathcal{C}$ is ≥ 2. *Note:* In fact, the local genus of p in $\mathcal{C}$ is ∞.   ∎

Lemma 4.4. *If Γ is a self homeomorphism of S^3 and $\Gamma(C(v(g))) = C(v(g'))$ for some $g, g' \in F_N$ with $g' = g \circ g_1$, then $\Gamma|_{C(v(g))} = H_{g_1}|_{C(v(g))}$. In particular, if $\Gamma(C(v(e))) = C(v(g))$ for some $g \in F_N$, then $\Gamma|_{C(v(e))} = H_g|_{C(v(e))}$.*

Proof. We have

$$H_{g_1}(C(v(g))) = H_{g_1}\big(H_g(C(v(e)))\big) = H_{g_1 \circ g}(C(v(e))) = H_{g'}(C(v(e))) = C(v(g')).$$

So both H_{g_1} and Γ take $C(v(e))$ onto $C(v(g'))$. Thus $H_{g_1} \circ \Gamma^{-1}$ takes $C(v(e))$ onto $C(v(e))$. The rigidity of $C(v(e))$ implies that $H_{g_1}|_{C(v(e))} = \Gamma|_{C(v(e))}$. ∎

Theorem 4.5 (Main theorem). $\mathcal{M}$ *is an irreducible* 3-*manifold with end homogeneity group* F_N.

This follows immediately from the next theorem.

Theorem 4.6. $\mathcal{C} \subset S^3$ *is unsplittable and the embedding homogeneity group of* $\mathcal{C} \subset S^3$ *is* F_N.

Proof. The unsplittability was addressed above. Let $\mathcal{H}$ be the embedding homogeneity group of $\mathcal{C}$. Note that the homeomorphisms $H_g \colon \mathbb{R}^3 \to \mathbb{R}^3$, $g \in F_N$, extend to homeomorphisms $H'_g \colon S^3 \to S^3$, $g \in F_N$, by fixing the point at infinity.

For each $g \in F_N$, let $\eta_g = H'_g|_{\mathcal{C}}$. Then each η_g takes $\mathcal{C}$ onto $\mathcal{C}$. This follows directly from the definition of $C(v(g)) = H_g(C(v(e)))$ and $C_i(v(g), \beta_k) = H_g(C_i(v(e), \beta_k))$. The elements $\{\eta_g, g \in F_N\}$ form a subgroup of $\mathcal{H}$ since $H'_{g_1} \circ H'_{g_2} = H'_{g_1 \circ g_2}$, by Remark 2.2.

Let γ be any self homeomorphism of $\mathcal{C}$ that extends to a homeomorphism Γ of S^3. We will show that $\gamma = \eta(g)$ for some $g \in F_N$.

Step 1. We first show that $\Gamma(C(v(e))) = C(v(g))$ for some $g \in F_N$. If this is true, then the definition of $C(v(g))$ as $H_g(C(v(e)))$ and Lemma 4.4 shows that $\Gamma|_{C(v(e))} = \eta_g|_{C(v(e))}$. Let

$$\Lambda_N = \{T(C(v(g))) \mid \|v(g)\| \leq N\} \bigcup \{T(C_j(v(g), \beta_k)) \mid \|v(g)\| \leq N\},$$
$$\Delta_N = \{C(v(g)) \mid \|v(g)\| \leq N\} \bigcup \{C_j(v(g), \beta_k) \mid \|v(g)\| \leq N\}, \quad \text{and}$$
$$\Lambda'_N = \{T'(C(v(g))) \mid \|\, v(g)\,\| \leq N\} \bigcup \{T'(C_j(v(g), \beta_k)) \mid \|v(g)\| \leq N\}.$$

$\Gamma(C(v(e)))$ does not contain p by Lemma 4.2. So there is a positive integer N_1 such that $\Gamma(C(v(e))) \subset \Delta_{N_1}$. Similarly, there is a positive integer $N_2 > N_1$ such that $\Gamma^{-1}(\Delta_{N_1}) \subset \Delta_{N_2}$.

Using techniques similar to those in [18], or [10], choose a homeomorphism k of S^3 to itself, fixed on $\mathcal{C}$, so that

$$(4.1) \qquad k\big(\Gamma(\partial(\Lambda_{N_2+1}) \cup \partial(\Lambda'_{N_2+1}))\big) \cap (\partial(\Lambda_{N_2+1}) \cup \partial(\Lambda'_{N_2+1})) = \emptyset.$$

Let $\Gamma' = k \circ \Gamma$. Let c be a point of $C(v(e))$ and let $\Gamma'(c) = \Gamma(c) = q \in \widetilde{C}$, where $\widetilde{C} = C(v(g))$ (or $\widetilde{C} = C_i(v(g), \beta_k)$) for some g, and where $T = T(C(v(g)))$ (or $T = T(C_i(v(g), \beta_k))$). We will show that in fact, $\Gamma'(C(v(e))) = \Gamma(C(v(e))) = C(v(g))$ for some g, and that the second case above cannot occur. Observe that, by equation (4.1) above, either $\Gamma'(T(C(v(e)))) \subset \mathrm{Int}\,T$ or $\mathrm{Int}(\Gamma'(T(C(v(e))))) \supset T$.

Step 1a. Assume that $\Gamma'(T(C(v(e)))) \subset \operatorname{Int}(T)$. In this case, Lemma 4.1 applies with $T(C(v(e))) = T_1$, any $T_1(v(e), \beta_i) = T_2$, $T = T_3$. If $\Gamma'(T(C(v(e))))$ has geometric index 0 in T_3, choose a linked chain of tori in the construction of $C(G(F_N))$ joining $T(C(v(e)))$ to a $T(C(v(g')))$, where $\|g'\| = N_2 + 1$. It follows, by repeated applications of Lemma 4.1, that $\Gamma'(T(C(v(g')))) \subset \Delta_{N_1}$, which is a contradiction. So $\Gamma'(T(C(v(g))))$ has geometric index greater than or equal to 1 in T. By Lemma 4.1, this geometric index is 1 and $\widetilde{C} \subset \Gamma'(T(C(v(g))))$.

Since $\Gamma(\mathcal{C}) = \mathcal{C}$, we have $\widetilde{C} = \Gamma(C(v(e)))$. Since $C(v(e))$ in not equivalent to any $C_i(v(g), \beta_k)$, by Remarks 3.3 and 3.5 we get $\widetilde{C} = C(v(g))$ for some g. It follows from Lemma 4.4 that $\Gamma|_{C(v(e))} = \eta_g|_{C(v(e))}$.

Step 1b. Assume that $\operatorname{Int}(\Gamma'(T(C(v(e))))) \supset \operatorname{Int} T$. Then $(\Gamma')^{-1}(T) \subset \operatorname{Int}(T(C(v(e))))$. The argument from Case I can now be repeated replacing Γ' by $(\Gamma')^{-1}$ and interchanging T and $T(C(v(e)))$. It follows that $(\Gamma')^{-1}(\widetilde{C}) = C(v(e))$ and so $\Gamma'(C(v(e)) = C(v(g))$ for some g as claimed. Again it follows from Lemma 4.4 that $\Gamma|_{C(v(e))} = \eta_g|_{C(v(e))}$.

Step 2. By an argument similar to that in Step 1, for each $g_i \in F_N$, $\Gamma(C(v(g_i)))$ $= C(g_{m(i)} \circ g_i)$ for some $g_{m(i)} \in F_N$. Also for each i, j and k,

$$\Gamma(C_j(v(g_i)\beta_k)) = C_j(v(g_{m(i,k)} \circ g_i), \beta_k) \quad \text{for some } g_{m(i,k)} \in F_N.$$

Working inductively outward from $C(v(e))$, and using the fact that if T_1 and T_2 are linked tori, then $\Gamma(T_1)$ and $\Gamma(T_2)$ are linked, one sees that each $g_{m(i)} = g$ and each $g_{m(i,k)} = g$.

By Lemma 4.4, it follows that each $\Gamma|_{C_j(v(g_i)\beta_k)} = \eta_g|_{C_j(v(g_i)\beta_k)}$, and also that each $\Gamma|_{C(v(g_i))} = \eta_g|_{C(v(g_i))}$. It now follows that $\Gamma = \eta(g)$ for the g from Steps 1 and 2. This shows that $\{\eta(g), g \in F_N\}$ is in fact, the group $\mathcal{H}$ and that $\mathcal{H}$ is isomorphic to F_N. ∎

The method used in the construction of $\mathcal{C}$ can be used for any Cayley graph of a finitely generated group F for which the graph automorphisms have certain extension properties. This observation is given in the next theorem.

Theorem 4.7. *Suppose F is a finitely presented group with a Cayley graph $G(F)$ in S^3. Suppose the graph automorphisms h_g for $g \in F$ can be extended to homeomorphisms H_g of S^3. Suppose also that for every pair of group elements g_1, g_2, $H_{g_1} \circ H_{g_2} = H_{g_1 \circ g_2}$. Then there is an open irreducible 3-manifold M_F with embedding homogeneity group isomorphic to F.*

Proof. The construction of a collection of Cantor sets modeled on the graph, and the proof analogous to the proof of Theorem 4.6 go through in this case. ∎

Questions

In [9], the authors showed that for every finitely generated abelian group G, there is an irreducible open 3-manifold M_G with end homogeneity group isomorphic to G. In this paper, we show that for any finitely generated free group F_N, there is an irreducible open 3-manifold M_{F_N} with end homogeneity group F_N.

Question 1. *Which non-abelian finitely generated groups can arise as end homogeneity groups of (irreducible) open 3-manifolds?*

In light of Theorem 4.7, this leads to the following question.

Question 2. *Which non-abelian finitely generated groups have Cayley graphs embeddable in S^3 with the properties listed in Theorem 4.7?*

A related question is the following.

Question 3. *If not all finitely generated non-abelian groups have Cayley graphs with the properties listed in Theorem 4.7, is there a way to characterize which ones do have this property?*

Appendix

This section lists some of the important notation used in the paper.

- F_N is the free group on N generators.
- $g(F_N) = \{a_1, \ldots, a_N, \overline{a_1}, \ldots, \overline{a_N}\}$, is the full set of generators for F_N.
- $\beta_j = \alpha_j$ for $1 \leq j \leq N$ and $\beta_j = \overline{\alpha_{j-N}}$ for $N + 1 \leq j \leq 2N$.
- $G(F_N)$ is the Cayley graph constructed in $\mathbb{R}^3$ for F_N.
- $v(e), v(\beta_j), v(g)$ are vertices of $G(F_N)$ associated with group elements.
- $E(v(g), \beta_k) = E((v(g), v(g \circ \beta_k)))$ is the oriented edge labeled β_k from $v(g)$ to $v(g \circ \beta_k)$.
  This edge is the regarded the same as $E(v(g \circ \beta_k), \overline{\beta_k}) = E(v(g \circ \beta_k), v(g))$.
- h_g is the automorphism of $G(F_N)$ given by $h_g(v(g_i)) = v(g \circ g_i)$,
  $h_g(E(v(g_i), \beta_k)) = E(v(g \circ g_i), \beta_k)$.
- H_g is the extension of h_g to a homeomorphism of $\mathbb{R}^3$.
- $B(v(g)) = H_g(B(v(e)))$ is a 3-ball about the vertex $v(g)$.
- $D(g, \beta_k) = H_g(D(e, \beta_k))$ is a tube joining $B(v(g))$ to $B(v(g \circ \beta - k))$.
- $C(v(g)) = H_g(C(v(e))$ is a rigid Cantor set in $B(v(g))$.
- $C_1(v(g), \beta_k), C_2(v(g), \beta_k), and\, C_3(v(g), \beta_k)$, form a linked chain of rigid Cantor sets joining $C(v(g))$ to $C(v(g \circ \beta_k))$. $C_i(v(g), \beta_k) = H_g(C_i(v(e)), \beta_k)$.
- $C(G(F_N)) = \mathcal{C}$. $M_{F_N} = \mathcal{M}$.

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Dennis J. Garity
Mathematics Department, Oregon State University, Corvallis, OR 97331, USA;
garity@oregonstate.edu

Dušan D. Repovš
Faculty of Education, and Faculty of Mathematics and Physics, University of Ljubljana;
and Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia;
dusan.repovs@guest.arnes.si