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Hypersurfaces with prescribed curvatures in the de Sitter space

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Abstract. In this paper, we establish a relationship between spacelike hypersurfaces in the de Sitter space $\mathbb{S}_1^{n+1}$ and conformal metrics on the sphere $\mathbb{S}^n$. As a consequence of this relation and some deep results in conformal geometry, we classify spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ satisfying certain global conditions and a very general Weingarten relation $W(k_1, \dots, k_n) = 0$ of elliptic type, where $k_1, \dots, k_n$ are its principal curvatures. Also, we particularize to some important cases as the rth mean curvatures.

1. Introduction

The study of hypersurfaces with constant mean curvature in Lorentzian spaces has a lot of interest, from both the physical and the mathematical point of view.

From the physical point of view, that interest became clear when Lichnerowicz [36] showed in 1944 that the Cauchy problem for the Einstein equations, with initial conditions on a spacelike hypersurface with vanishing mean curvature, has a nice form, reducing it to a linear differential system of first-order and to a non-linear second-order elliptic differential equation (see also [7, 19, 39]).

From a mathematical point of view, spacelike hypersurfaces with constant mean curvature appear as critical points of variational problems associated with certain geometric functionals, such as the volume functional [13, 22] and the energy functional [20]. For example, analogously to the Euclidean context, these hypersurfaces are critical points of the classical isoperimetric problem in the Lorentzian context, which lies in finding among all the compact hypersurfaces that enclose a given volume, those with the greatest area [7, 10, 11]. Moreover, the spacelike hypersurfaces are also interesting because of their nice Bernstein-type properties. Many authors obtained results about the solution to the corresponding Bernstein problem for spacelike hypersurfaces in the Lorentz–Minkowski space $\mathbb{L}^{n+1}$, see [1, 15, 18, 27, 30, 42, 47, 49].

A basic problem in the theory of submanifolds, in a Riemannian or Lorentzian ambient space, is to classify the totally umbilical hypersurfaces as the unique ones satisfying a

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certain relation among its curvatures, under some global hypothesis on the submanifold. Though very different methods have been used for these classifications in the Riemannian case, we can emphasize the celebrated Alexandrov reflection method, developed by Alexandrov [5] in 1962 for proving that any compact embedded constant mean curvature surface in $\mathbb{R}^3$ must be a round sphere. However, this method, as other classical tools in the Riemannian case, cannot be used in the de Sitter space, which has needed the development of new techniques.

Goddard [24] conjectured in 1977 that the only complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature H should be the totally umbilical ones. Although this conjecture is false in its original statement, it has motivated a lot of works trying to find some positive answer under appropriate additional hypotheses. For instance, Akutagawa [2] showed that Goddard's conjecture is true when $0 \leq H^2 \leq 1$ in the case $n = 2$, and when $0 \leq H^2 < 4(n - 1/n^2)$ in the case $n \geq 3$. Later, Montiel [40] solved Goddard's problem in the compact case proving that the only compact spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature are the totally umbilical round spheres (see also [43] for an alternative proof of both facts in the two-dimensional case). Also, Montiel [41] and Aquino and de Lima [8] got uniqueness theorems about complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant mean curvature.

On the other hand, some other authors, such as Zheng [51, 52], Li [35] or Cheng and Ishikawa [17], have also obtained interesting results related to the characterization of the totally umbilical round spheres as the only compact spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with constant scalar curvature. When the spacelike hypersurface is complete, Camargo, Chaves and Sousa [16] gave also results of this type.

The natural generalization of mean and scalar curvatures for a spacelike hypersurface in de Sitter space are the rth mean curvatures H_r for $r = 1, \ldots, n$. Actually, H_1 is the mean curvature and H_2 is, up to a constant, the scalar curvature of the hypersurface. Aledo, Alías and Romero [3] developed some integral formulas for compact spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ and applied them in order to characterize the totally umbilical round spheres as the only compact spacelike hypersurfaces with constant higher order mean curvature which are contained in the chronological future (or past) of an equator of $\mathbb{S}_1^{n+1}$. Shu [46] studied complete spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ with non-zero constant rth mean curvature and two distinct principal curvatures, and gave some characterizations of Riemannian product structures. We refer the reader to [6, 31, 38] and references therein for related matters.

For linear Weingarten spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ satisfying $H_2 = aH + b$, for some $a, b \in \mathbb{R}$, Hou and Yang [28] gave a classification according to the sectional curvature or the length of the second fundamental form. Later, de Lima and Velásquez [37] studied the geometry of these hypersurfaces and established new characterizations of the hyperbolic cylinders of $\mathbb{S}_1^{n+1}$, using as main analytical tool a suitable maximum principle for complete noncompact Riemannian manifolds. In the compact case, they obtained a rigidity result according to the length of its second fundamental form. Also, Gomes, de Lima, dos Santos and Vélasquez in [25] proved that a complete linear Weingarten spacelike hypersurface in $\mathbb{S}_1^{n+1}$ satisfying $H_2 = aH + b$, for some $a, b \in \mathbb{R}$, and having two distinct principal curvatures must be isometric to a certain isoparametric hypersurface in $\mathbb{S}_1^{n+1}$, under suitable restrictions on the values of the mean curvature and of the norm of the traceless part of its second fundamental form. Their approach was based on the use of a Simons type formula related to an appropriated Cheng–Yau modified operator jointly with some generalized maximum principles.

Another classical problem is the following: given a Riemannian manifold (M, g_0), a smooth functional $f(x_1, \dots, x_n)$ and a constant c, does there exist a conformal metric $g = e^{2\rho} g_0$ on M such that the eigenvalues λ_i of its Schouten tensor satisfy $f(\lambda_1, \dots, \lambda_n) = c$ on M?

Recall that on a Riemannian manifold $(M, g), n > 2$, the Riemann curvature tensor can be decomposed as

$$\mathrm{Riem}_g = W_g + \mathrm{Sch}_g \odot g,$$

where W_g is the Weyl tensor, $\odot$ is the Kulkarni–Nomizu product, and

$$\mathrm{Sch}_g := \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{S(g)}{2(n-1)} g\Big) \tag{1.1}$$

is the Schouten tensor, where Ric_g and $S(g)$ stand for the Ricci and scalar curvatures of g, respectively. As the Weyl tensor is conformally invariant, the Schouten tensor encodes all the information on how curvature varies by a conformal change of metric. So, it is the main object of study in conformal geometry.

Note that when $f(x_1, \dots, x_n) = x_1 + \cdots + x_n$, we have the famous *Yamabe problem*, which is to find a conformal metric $g = e^{2\rho} g_0$ such that the scalar curvature of (M, g) is constant, where (M, g_0) is a compact Riemannian manifold without boundary and $\rho \in \mathcal{C}^\infty(M)$. This problem got the attention of the mathematical community and it was studied by authors as Yamabe himself [50], Trudinger [48], Aubin [9] and Schoen [44], who gave a complete answer to the problem.

Observe also that, when $f(\lambda) = \sigma_k(\lambda)^{1/k}$, $\lambda = (\lambda_1, \dots, \lambda_n)$, where $\sigma_k(\lambda)$ is the kth elementary symmetric polynomial of its arguments $\lambda_1, \dots, \lambda_n$ and $\sigma_k(\lambda) =$ constant, one has the so-called σ_k-Yamabe problem. This is an active research topic and has interactions with other fields, such as mathematical general relativity [12, 26].

Espinar, Gálvez and Mira [21] provided a geometric back-and-forth procedure which relates the theory of hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$ and the theory of conformal metrics on the sphere $\mathbb{S}^n$. This yields a new way of applying methods from geometric PDEs to investigate hypersurfaces in $\mathbb{H}^{n+1}$, and vice versa. They used this bridge between both theories in order to give an explicit equivalence of the Christoffel problem in $\mathbb{H}^{n+1}$ with the famous problem of prescribing scalar curvature on $\mathbb{S}^n$ for conformal metrics, posed by Nirenberg and Kazdan–Warner.

In this paper, we establish a relationship between spacelike hypersurfaces in the de Sitter space $\mathbb{S}^{n+1}_1$ and conformal metrics on the sphere $\mathbb{S}^n$, inspired by the above correspondence. As a consequence of this relation and some deep results in conformal geometry, we will classify spacelike hypersurfaces in $\mathbb{S}^{n+1}_1$ satisfying certain global conditions and a very general Weingarten relation $W(k_1, \dots, k_n) = 0$ of elliptic type, where $k_1, \dots, k_n$ are its principal curvatures. Also, we will particularize to some important cases as the rth mean curvatures.

This work is organized as follows. After some preliminaries, in Section 3 we will establish the correspondence between spacelike hypersurfaces in the de Sitter space $\mathbb{S}^{n+1}_1$ and conformal metrics on the sphere $\mathbb{S}^n$. Let $\phi\colon M \to \mathbb{S}^{n+1}_1$ be an oriented spacelike hypersurface with unit normal $\eta\colon M \to \mathbb{H}^{n+1}$; we will associate to it a map in the light cone $\psi := \phi - \eta\colon M \to \mathbb{N}^{n+1}$ and a map $G := [\psi]\colon M \to \mathbb{S}^n \equiv \mathbb{N}^{n+1}/\mathbb{R}^*$, that we will call the *de Sitter Gauss map*, which will encode geometric information about our initial hypersurface M. We will demonstrate that ψ is a Riemannian immersion or G is a local diffeomorphism if and only if all the principal curvatures of M are different from -1. We will define *weakly convex* hypersurfaces in $\mathbb{S}_1^{n+1}$ as those having all principal curvatures $k_i(p) > -1$, for all p. So, its de Sitter Gauss map is a local diffeomorphism. In such a case, we can consider the locally conformally flat Riemannian metric $g_\infty = \psi^*(\langle,\rangle) = e^{2\rho}G^*(\langle,\rangle_{\mathbb{S}^n})$, which will be called the *light cone metric* of ϕ. Moreover, $e^\rho\colon M \to \mathbb{R}^+$ will be also called the *support function* of the hypersurface. We will show that any oriented spacelike hypersurface M in $\mathbb{S}_1^{n+1}$, such that its de Sitter Gauss map G is a local diffeomorphism, can be explicitly recovered in terms of G and the light cone metric g_∞, which can be considered as a conformal metric on $\mathbb{S}^n$ (Theorem 3.4).

In addition, we will relate the first and second fundamental forms of our hypersurface with the conformal metric g_∞ and its Schouten tensor (Theorem 3.8). In particular, we will establish an explicit relationship between the principal curvatures of the spacelike hypersurface M and the eigenvalues of the Schouten tensor of g_∞ (Corollary 3.9).

With all this, we will obtain a fundamental correspondence to study geometric problems of spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$ in terms of conformal metrics on the sphere and vice versa, which will be exploited in Section 4, where we will determine conditions to classify spacelike hypersurfaces in the de Sitter space satisfying a non trivial relation between its principal curvatures $W(k_1, \ldots, k_n) = 0$, that is, the so-called Weingarten hypersurfaces. We will use the previous relation between hypersurfaces in $\mathbb{S}_1^{n+1}$ and conformal metrics on the sphere, and deep results of A. Li and Y.Y. Li in [33] for the compact case (Theorem 4.1) and [34] for the non-compact case (Theorem 4.4). As examples, we will particularize to some important cases such as the positive constant rth mean curvatures, for the compact (Corollary 4.2) and the non-compact case (Corollary 4.5).

These results constitute a relevant advance in what refers to classification theorems for Weingarten hypersurfaces, since the family of Weingarten functionals for which the result holds is extremely large.

In order to use these results for the non-compact case, we need to prove that the de Sitter Gauss map G of the hypersurface is a diffeomorphism. As it will be shown in Section 5, in Theorem 5.5 and Theorem 5.8, this happens under different natural conditions on the immersion.

2. Preliminaries

Let $\mathbb{L}^{n+2}$ be the $(n+2)$-dimensional Lorentz–Minkowski space endowed with canonical linear coordinates $x = (x_0, x_1, \ldots, x_{n+1})$ and the scalar product $\langle,\rangle$ given by the quadratic form $-dx_0^2 + dx_1^2 + \cdots + dx_{n+1}^2$.

The $(n+1)$-dimensional de Sitter space can be seen as the Lorentzian submanifold

$$\mathbb{S}_1^{n+1} = \{x \in \mathbb{L}^{n+2} : \langle x, x\rangle = 1\}.$$

Moreover, the light cone is given as the non-vanishing vectors in $\mathbb{L}^{n+2}$ with vanishing norm, that is,

$$\mathbb{N}^{n+1} = \{x \in \mathbb{L}^{n+2} : \langle x, x\rangle = 0,\ x \neq 0\}.$$

We will study spacelike hypersurfaces in the de Sitter space, that is, immersions from a connected n-dimensional manifold M into $\mathbb{S}_1^{n+1}$ such that its induced metric is Riemannian.

Observe that given a point p of the unit sphere $\mathbb{S}^n$ of the Euclidean space $\mathbb{R}^{n+1}$, the curve $\gamma_p(t) = \cosh t(0, p) + \sinh t(1, 0)$ is a timelike geodesic of $\mathbb{S}_1^{n+1}$. These curves γ_p give a foliation of $\mathbb{S}_1^{n+1}$, and so if $\phi\colon M \to \mathbb{S}_1^{n+1}$ is a spacelike hypersurface these curves must be transverse. In particular, M must be orientable.

Thus, for any spacelike hypersurface $\phi\colon M \to \mathbb{S}_1^{n+1}$ there exists a globally defined normal η from M into the two sheeted hyperbolic space

$$\mathbb{H}^{n+1} = \{x \in \mathbb{L}^{n+2} : \langle x, x\rangle = -1\}.$$

Moreover, the projection $\pi_1\colon \mathbb{S}_1^{n+1} \to \mathbb{S}^n$ defined as $\pi_1(x_0, q) = q/\sqrt{1 + x_0^2}$ is a local diffeomorphism, where $\pi_1(x_0, q)$ is the unique point $p \in \mathbb{S}^n$ such that (x_0, q) belongs to the image of γ_p. Therefore, if M is compact then M is diffeomorphic to the sphere $\mathbb{S}^n$ (see also [40]).

2.1. Totally umbilical spacelike hypersurfaces in $\mathbb{S}_1^{n+1}$

Let us fix a vector $a \in \mathbb{L}^{n+2}$ with $\varepsilon := \langle a, a\rangle \in \{-1, 0, 1\}$, and consider

$$\mathcal{H} = \left\{p \in \mathbb{S}_1^{n+1} : \langle p, a\rangle = b\right\}, \tag{2.1}$$

for a real constant b such that $b^2 > \varepsilon$. Then, one has (see [40]):

1) If $\varepsilon = -1$ (intersection of a spacelike hyperplane with $\mathbb{S}_1^{n+1}$), then $\mathcal{H}$ is a compact totally umbilical spacelike hypersurface isometric to an n-dimensional sphere of radius $\sqrt{b^2 + 1}$. A unit normal is given by $\eta(p) = \frac{1}{\sqrt{1+b^2}}\,(a - b\,p)$ and its shape operator A is

$$Av := -\overline{\nabla}_v \eta = -D_v \eta = \frac{b}{\sqrt{1 + b^2}}\, v$$

for every tangent vector v, where $\overline{\nabla}$ and D denote the Levi-Civita connection of $\mathbb{S}_1^{n+1}$ and $\mathbb{L}^{n+2}$, respectively. In particular, $\mathcal{H}$ has constant principal curvatures for η equal to $b/\sqrt{1 + b^2} \in (-1, 1)$.

2) If $\varepsilon = 0$ (intersection of a lightlike hyperplane with $\mathbb{S}_1^{n+1}$), then $\mathcal{H}$ is a totally umbilical spacelike hypersurface isometric to the n-dimensional Euclidean space $\mathbb{R}^n$. A unit normal is given by $\eta(p) = \frac{1}{b}a - p$ and its shape operator satisfies $Av = v$ for all tangent vectors v. Thus, $\mathcal{H}$ has constant principal curvatures for η equal to 1.

3) If $\varepsilon = 1$ (intersection of a timelike hyperplane with $\mathbb{S}_1^{n+1}$), then $\mathcal{H}$ is a totally umbilical spacelike hypersurface isometric to an n-dimensional two sheeted hyperbolic space of radius $\sqrt{b^2 - 1}$. A unit normal is given by $\eta(p) = \frac{1}{\sqrt{b^2-1}}\,(a - b\,p)$ and its corresponding shape operator satisfies $Av = \frac{b}{\sqrt{b^2-1}} v$. So, $\mathcal{H}$ has constant principal curvatures for η equal to $b/\sqrt{b^2 - 1} \in \mathbb{R}\setminus[-1, 1]$.

We remark that every totally umbilical spacelike hypersurface in $\mathbb{S}_1^{n+1}$ must be a piece of one of the previous examples. Moreover, they have constant sectional curvature.

2.2. Klein model of $\mathbb{S}_1^{n+1}$

Consider the strip or subset of the Euclidean unit sphere $\mathbb{S}^{n+1}$ given by

$$B = \left\{p = (p_0, \dots, p_{n+1}) \in \mathbb{R}^{n+2} : |p|_e = 1, -\frac{1}{\sqrt{2}} < p_0 < \frac{1}{\sqrt{2}}\right\} \subset \mathbb{S}^{n+1},$$

where $|\cdot|_e$ denotes the usual Euclidean norm. The boundary of B is given by

$$\partial B = \partial B^+ \cup \partial B^-,$$

where

$$\partial B^+ = \left\{p = (p_0, \dots, p_{n+1}) \in \mathbb{R}^{n+2} : |p|_e = 1, p_0 = \frac{1}{\sqrt{2}}\right\} \subset \mathbb{S}^{n+1},$$
$$\partial B^- = \left\{p = (p_0, \dots, p_{n+1}) \in \mathbb{R}^{n+2} : |p|_e = 1, p_0 = -\frac{1}{\sqrt{2}}\right\} \subset \mathbb{S}^{n+1}.$$

The transformation $T\colon \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2} \to B \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ defined as

$$T(p) = \frac{p}{|p|_e}$$

is a diffeomorphism. If we induce the metric of $\mathbb{S}_1^{n+1}$ in B, via T, we have that T is an isometry. So, the Klein model of the de Sitter space $\mathbb{S}_1^{n+1}$ is the subset $B \subset \mathbb{S}^{n+1}$ endowed with this induced metric $T^*(\langle , \rangle)$.

If we consider a spacelike hypersurface $\phi\colon M \to \mathbb{S}_1^{n+1}$, we can view it in the Lorentz–Minkowski space $\mathbb{L}^{n+2}$ and in the Klein model of $\mathbb{S}_1^{n+1}$ as follows:

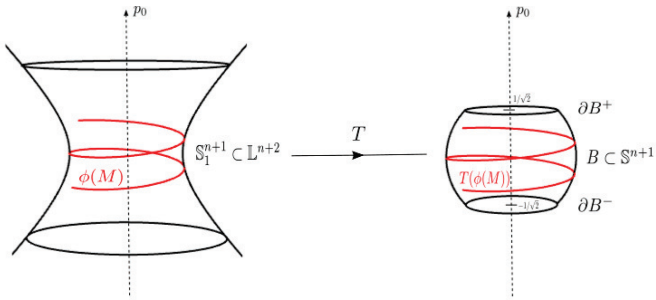

Figure 1. Lorentz–Minkowski space $\mathbb{L}^{n+2}$ and Klein model of $\mathbb{S}_1^{n+1}$.

Since the geodesics of the de Sitter space $\mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$ are the intersection of vectorial planes of $\mathbb{L}^{n+2}$ with $\mathbb{S}_1^{n+1}$, the geodesics of the Klein model are the intersection of these planes with B. Therefore, the geodesics of the Klein model of $\mathbb{S}_1^{n+1}$ are nothing but the intersection of B with the Euclidean geodesics of $\mathbb{S}^{n+1}$ with its usual metric.

There exist three types of geodesics of $\mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2}$: spacelike, lightlike and timelike geodesics. In the Klein model, the spacelike geodesics do not intersect the boundary of B at infinity, while both the lightlike and timelike geodesics intersect ∂B at infinity at two different points, one on ∂B^+ and the other on ∂B^-.

2.3. The de Sitter Gauss map

The projective quotient $\mathbb{S}^n_\infty = \mathbb{N}^{n+1}/(\mathbb{R}\backslash\{0\})$ is naturally identified with the unit sphere using the projection $\pi\colon \mathbb{N}^{n+1} \to \mathbb{S}^n \equiv \mathbb{S}^n_\infty$ given by

$$\pi(q) = \frac{1}{q_0}\,\overline{q}, \tag{2.2}$$

where $q = (q_0, \overline{q}) = (q_0, q_1, \ldots, q_{n+1}) \in \mathbb{N}^{n+1}$.

Thus, if $\phi\colon M \to \mathbb{S}^{n+1}_1$ is an oriented spacelike hypersurface with unit normal η, then we can define an associated light cone map $\psi = \phi - \eta\colon M \to \mathbb{N}^{n+1}$ from M into the light cone. Its quotient map $G = [\psi]\colon M \to \mathbb{S}^n \equiv \mathbb{S}^n_\infty$ will be called its *de Sitter Gauss map*.

There is a simple geometric interpretation for G in the Klein model: if $\phi\colon M \to \mathbb{S}^{n+1}_1$ is an oriented spacelike hypersurface and we choose $p \in M$ with unit normal $\eta(p)$ at $\phi(p)$, then we can take the unique geodesic $\gamma(t)$, $t \geq 0$, of $\mathbb{S}^{n+1}_1$ that starts at $\phi(p)$ with initial speed $-\eta(p)$. Now, if we consider the projection T to the Klein model, which is an isometry, we have that $T \circ \gamma$ is a geodesic of the Klein model (i.e., the intersection of a geodesic of the sphere $\mathbb{S}^{n+1}$ with B) that starts at $T(\phi(p))$ and such that $q = \lim_{t\to\infty} T(\gamma(t))$ belongs to the boundary of B. So, the de Sitter Gauss map $G(p) = [q]$. Observe that, if $\eta \in \mathbb{H}^{n+1}_-$ (i.e., $\eta_0 < 0$) then $q \in \partial B^+$, and if $\eta \in \mathbb{H}^{n+1}_+$ (i.e., $\eta_0 > 0$) then $q \in \partial B^-$.

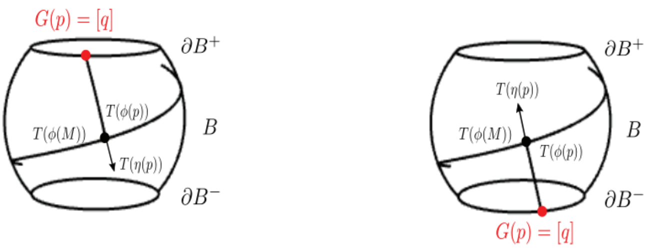

Figure 2. Geometric interpretation for the de Sitter Gauss map G in the Klein model. The picture on the left corresponds to the case $\eta \in \mathbb{H}^{n+1}_-$ and the one on the right to the case $\eta \in \mathbb{H}^{n+1}_+$.

If we replace η by $-\eta$ for the spacelike immersion ϕ, we obtain a different de Sitter Gauss map $G^+ = [\phi + \eta] : M \to \mathbb{S}^n_\infty$, with different properties from the previous map G.

We will prove in the next section that G (or G^+) can be used in order to recover the immersion ϕ itself.

For a totally umbilical spacelike hypersurface $\mathcal{H}$ given by (2.1), its de Sitter Gauss map G is a local diffeomorphism when $\varepsilon = \pm 1$, as it can be deduced from Lemma 3.1 in Section 3. Analogously, G^+ is also a local diffeomorphism in this case.

When $\varepsilon = 0$, the de Sitter Gauss map G of the hypersurface $\mathcal{H}$, with associated unit normal $\eta(p) = \frac{1}{b}\,a - p$, is given by

$$G(p) = \frac{1}{2p_0 - \frac{1}{b}a_0}\left(2\overline{p} - \frac{1}{b}\,\overline{a}\right), \tag{2.3}$$

where $a = (a_0, \overline{a}) = (a_0, a_1, \ldots, a_{n+1}) \in \mathbb{N}^{n+1}$ and $p = (p_0, \overline{p}) = (p_0, p_1, \ldots, p_{n+1}) \in \mathbb{S}_1^{n+1}$. Thus, G is a global diffeomorphism onto $\mathbb{S}^n \setminus \{\frac{1}{a_0}\,\overline{a}\}$. However, $G^+(p) = \frac{1}{a_0}\,\overline{a}$ is constant.

3. A representation formula for spacelike hypersurfaces

Let $\phi\colon M \to \mathbb{S}_1^{n+1}$ be an oriented spacelike hypersurface with unit normal η. The associated light cone map $\psi = \phi - \eta\colon M \to \mathbb{N}^{n+1}$ can be written as $\psi = \psi_0(1, G)$, where G is the de Sitter Gauss map of the immersion.

Then,

$$\langle d\psi, d\psi \rangle = \psi_0^2 \langle dG, dG \rangle_{\mathbb{S}^n}, \tag{3.1}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{S}^n}$ denotes the usual Euclidean metric of $\mathbb{S}^n$.

On the other hand, chosing an orthonormal basis of principal directions $\{e_1, \ldots, e_n\}$ of ϕ at a point $p \in M$ with associated principal curvatures $\{k_1, \ldots, k_n\}$, one has

$$\langle d\psi(e_i), d\psi(e_j) \rangle = (1 + k_i)^2\, \delta_{ij}, \tag{3.2}$$

where δ_{ij} denotes the Kronecker delta.

As a consequence, one obtains:

Lemma 3.1. *Let $\phi\colon M \to \mathbb{S}_1^{n+1}$ be an oriented spacelike hypersurface. The following conditions are equivalent at $p \in M$:*

(1) *The de Sitter Gauss map G is a local diffeomorphism at p.*

(2) *The associated light cone map ψ is a spacelike immersion into $\mathbb{N}^{n+1}$.*

(3) *All principal curvatures of ϕ are different from -1.*

Using this result, we introduce a general notion of convexity based on the regularity of the de Sitter Gauss map.

Definition 3.2. A spacelike hypersurface $\phi\colon M \to \mathbb{S}_1^{n+1}$ is said to be weakly convex if there exists a unit normal η such that all principal curvatures of ϕ satisfy simultaneously $k_i(p) > -1$ for all $p \in M$.

It is important to observe that the family of weakly convex spacelike hypersurfaces contains every "convex" spacelike hypersurface in $\mathbb{S}_1^{n+1}$. That is, each spacelike hypersurface with principal curvatures $k_i \geq 0$ at every point is weakly convex.

Definition 3.3. Let $\phi\colon M \to \mathbb{S}_1^{n+1}$ be an oriented spacelike hypersurface with unit normal η and associated light cone map $\psi = \phi - \eta\colon M \to \mathbb{N}^{n+1}$. If its de Sitter Gauss map G is a local diffeomorphism, from (3.1), we can consider the locally conformally flat Riemannian metric

$$g_\infty = \psi^*(\langle , \rangle) = e^{2\rho} G^*(\langle , \rangle_{\mathbb{S}^n}),$$

which will be called the light cone metric of ϕ. Moreover, e^ρ will be also called the support function of the hypersurface.

If the de Sitter Gauss map G of the hypersurface ϕ is a local diffeomorphism, then we can use G in order to parameterize the immersion locally. That is, we can assume that locally $\phi\colon U \subseteq \mathbb{S}^n \to \mathbb{S}^{n+1}_1$ for a certain subset U of the sphere and $G(x) = x$. This allows us to recover the immersion in terms of the de Sitter Gauss map and its support function.

Theorem 3.4. *Let $\phi\colon U \subseteq \mathbb{S}^n \to \mathbb{S}^{n+1}_1$ be an oriented spacelike hypersurface with de Sitter Gauss map $G(x) = x$, and support function $e^\rho\colon U \to (0, \infty)$. Then*

$$\phi = \pm\Big(\frac{e^\rho}{2}\left(1 - e^{-2\rho}(1 + \|\nabla^{g_0}\rho\|^2_{g_0})\right)(1, x) - e^{-\rho}(0, -x + \nabla^{g_0}\rho)\Big), \tag{3.3}$$

where g_0 is the restriction to U of the canonical metric on $\mathbb{S}^n$ and $\nabla^{g_0}\rho$ is the gradient of ρ with respect to g_0. Here, the plus-minus sign $\pm$ is plus if the first coordinate η_0 of the unit normal η is negative, and minus if $\eta_0 > 0$.

Proof. First, let us prove that the immersion ϕ can be recovered in terms of its associated light cone map ψ and its light cone metric g_∞.

Claim 3.5. *Let $S(g_\infty)$ be the scalar curvature of the Riemannian metric g_∞, and let $\triangle^{g_\infty}$ be its Laplacian. Then,*

$$\phi = -\frac{1}{n}\triangle^{g_\infty}\psi + \frac{1}{2}\Big(1 - \frac{S(g_\infty)}{n(n-1)}\Big)\psi. \tag{3.4}$$

Proof. Let $\{e_1, \dots, e_n\}$ be an orthonormal basis of principal directions of ϕ at a point $p \in M$ with associated principal curvatures $\{k_1, \dots, k_n\}$. It is clear from (3.2) that if we choose

$$b_i = \frac{1}{1 + k_i}\, e_i,$$

then $\langle d\psi(b_i), d\psi(b_j)\rangle = \delta_{ij}$, that is, $\{b_1, \dots, b_n\}$ is an orthonormal basis for the immersion ψ.

Hence, if we consider $\psi\colon M \to \mathbb{N}^{n+1} \subset \mathbb{L}^{n+2}$ as a spacelike codimension-2 submanifold of $\mathbb{L}^{n+2}$, then $\{\phi, \eta\}$ spans its normal space, and its second fundamental form II is given by

$$\mathrm{II}(b_i, b_j) = -\Big(\frac{1}{1 + k_i}\,\phi + \frac{k_i}{1 + k_i}\,\eta\Big)\,\delta_{ij}.$$

Thus, if we denote by $K(b_i, b_j)$ the sectional curvature of the plane spanned by the unit vectors b_i and b_j, we have from the Gauss equation that

$$K(b_i, b_j) = \langle \mathrm{II}(b_i, b_i), \mathrm{II}(b_j, b_j)\rangle - \|\mathrm{II}(b_i, b_j)\|^2 = -1 + \frac{1}{1 + k_i} + \frac{1}{1 + k_j}\cdot$$

Hence, the scalar curvature $S(g_\infty)$ of the Riemannian metric g_∞ is given by

$$S(g_\infty) = -n(n-1) + 2(n-1)\sum_{i=1}^{n}\frac{1}{1 + k_i}\cdot \tag{3.5}$$

On the other hand, it is well known that $\triangle^{g_\infty}\psi = n\vec{H}$, where the mean curvature vector field $\vec{H}$ of the immersion ψ is given by

$$\vec{H} = \frac{1}{n}\sum_{i=1}^{n} \mathrm{II}(b_i, b_i) = -\frac{1}{n}\sum_{i=1}^{n}\Big(\frac{1}{1+k_i}\phi + \frac{k_i}{1+k_i}\eta\Big)$$
$$= -\phi + \frac{1}{n}\Big(\sum_{i=1}^{n}\frac{k_i}{1+k_i}\Big)\psi = -\phi + \frac{1}{n}\Big(n - \sum_{i=1}^{n}\frac{1}{1+k_i}\Big)\psi.$$

Therefore,

$$\phi = -\frac{1}{n}\triangle^{g_\infty}\psi + \frac{1}{n}\Big(n - \sum_{i=1}^{n}\frac{1}{1+k_i}\Big)\psi = -\frac{1}{n}\triangle^{g_\infty}\psi + \frac{1}{2}\Big(1 - \frac{S(g_\infty)}{n(n-1)}\Big)\psi,$$

as we wanted to show. ∎

Now, we compute $\triangle^{g_\infty}\psi$ in terms of the support function and the canonical metric g_0 of the sphere $\mathbb{S}^n$. More concretely:

Claim 3.6. *The Laplacian of the associated light cone map ψ can be computed as*

$$\triangle^{g_\infty}\psi = \pm e^{-\rho}\left(\left(\triangle^{g_0}\rho + (n-1)\|\nabla^{g_0}\rho\|^2_{g_0}\right)(1,x) + n\,(0, -x + \nabla^{g_0}\rho)\right), \tag{3.6}$$

where the plus-minus sign $\pm$ is plus if $\eta_0 < 0$ and minus if $\eta_0 > 0$.

Proof. Recall that $\psi = \psi_0(1,G) = \psi_0(1,x)$, since we are assuming $G(x) = x$. Moreover, $\psi_0^2 = e^{2\rho}$, so we have $\psi_0 = \pm e^{\rho}$, where the sign $\pm$ depends on the sign of ψ_0. Note that the immersion ψ in the light cone $\mathbb{N}^{n+1}$ lies in $\mathbb{N}^{n+1}_+ = \{x \in \mathbb{L}^{n+2} : \langle x,x\rangle = 0,\ x_0 > 0\}$ if and only if the unit normal η lies in $\mathbb{H}^{n+1}_- = \{x \in \mathbb{L}^{n+2} : \langle x,x\rangle = -1,\ x_0 < 0\}$.

Thereby, $\psi = e^{\rho}(1,x)$ if η lies in $\mathbb{H}^{n+1}_-$, and $\psi = -e^{\rho}(1,x)$ if η lies in $\mathbb{H}^{n+1}_+ = \{x \in \mathbb{L}^{n+2} : \langle x,x\rangle = -1,\ x_0 > 0\}$. We now prove Claim 3.6 when $\eta_0 < 0$; analogously, it can be proved when $\eta_0 > 0$.

Consider a fixed point $x \in U \subseteq \mathbb{S}^n$ and an orthonormal basis $\{v_1, \ldots, v_n\}$ of $(\mathbb{S}^n, g_0)$ in a neighborhood of x such that $\nabla^{g_0}_{v_i} v_j = 0$ at the point x. Denote by $\{e_0, \ldots, e_{n+1}\}$ the canonical basis of $\mathbb{L}^{n+2}$ and let us write $\psi = (\psi_0, \ldots, \psi_{n+1})$ in these coordinates. Let us also write the point $x \in \mathbb{S}^n \subseteq \mathbb{R}^{n+1} \equiv \{v \in \mathbb{L}^{n+2} :\ v_0 = 0\}$ as $x = \sum_{k=1}^{n+1} x_k e_k$. Then the Laplacian $\triangle^{g_0}\psi$ at x can be computed as

$$\triangle^{g_0}\psi = \Big(\triangle^{g_0}(e^{\rho}), \triangle^{g_0}(e^{\rho})x + e^{\rho}\triangle^{g_0}x + 2e^{\rho}\sum_{k=1}^{n+1} g_0(\nabla^{g_0}x_k, \nabla^{g_0}\rho)e_k\Big)$$
$$= (\triangle^{g_0}(e^{\rho}), \triangle^{g_0}(e^{\rho})x + e^{\rho}\triangle^{g_0}x + 2e^{\rho}\nabla^{g_0}\rho)$$
$$= e^{-\rho}\triangle^{g_0}(e^{\rho})\psi + (0, e^{\rho}\triangle^{g_0}x + 2e^{\rho}\nabla^{g_0}\rho).$$

Using that $\triangle^{g_0}x = -nx$ and $\triangle^{g_0}(e^{\rho}) = e^{\rho}(\triangle^{g_0}\rho + \|\nabla^{g_0}\rho\|^2_{g_0})$, we can rewrite the previous Laplacian as

$$\triangle^{g_0}\psi = (\triangle^{g_0}\rho + \|\nabla^{g_0}\rho\|^2_{g_0})\psi + e^{\rho}(0, -nx + 2\nabla^{g_0}\rho). \tag{3.7}$$

Introduce the following notation: for any $X \in \mathfrak{X}(\mathbb{S}^n)$,

$$g_0(\nabla^{g_0}\psi, X) := \sum_{k=0}^{n+1} g_0(\nabla^{g_0}\psi_k, X)e_k = \sum_{k=0}^{n+1} X(\psi_k)e_k = X(\psi) \in \mathfrak{X}(\psi) \equiv \mathfrak{X}(\phi).$$

Then,

$$\begin{aligned} g_0(\nabla^{g_0}\psi, \nabla^{g_0}\rho) &= (\nabla^{g_0}\rho)(\psi) \\ &= \sum_{i=1}^{n} v_i(\rho)v_i(\psi) = \sum_{i=1}^{n} v_i(\rho)(e^\rho(v_i(\rho))(1,x) + e^\rho(0,v_i)) \\ &= e^\rho\Big(\sum_{i=1}^{n} v_i^2(\rho)\Big)(1,x) + e^\rho\Big(0, \sum_{i=1}^{n} v_i(\rho)v_i\Big) = \|\nabla^{g_0}\rho\|_{g_0}^2\psi + e^\rho(0,\nabla^{g_0}\rho). \end{aligned} \tag{3.8}$$

Since $g_\infty = e^{2\rho}g_0$, we can use the usual relation between the Laplacians of conformal metrics in order to deduce that

$$\triangle^{g_\infty}\psi = e^{-2\rho}(\triangle^{g_0}\psi + (n-2)g_0(\nabla^{g_0}\rho, \nabla^{g_0}\psi)).$$

Thus, from the previous equation and using (3.7) and (3.8), we obtain

$$\triangle^{g_\infty}\psi = e^{-2\rho}(\triangle^{g_0}\rho + (n-1)\|\nabla^{g_0}\rho\|_{g_0}^2)\psi + ne^{-\rho}(0, -x + \nabla^{g_0}\rho),$$

as wanted. ∎

Using again that the light cone metric g_∞ and the usual metric of the sphere g_0 are conformal, their scalar curvatures satisfy the relation

$$S(g_\infty) = e^{-2\rho}(S(g_0) - 2(n-1)\triangle^{g_0}\rho - (n-2)(n-1)\|\nabla^{g_0}\rho\|_{g_0}^2), \tag{3.9}$$

where $S(g_0) = n(n-1)$.

Bearing in mind that $\psi = \pm e^\rho(1,x)$, where the sign $\pm$ is plus if $\eta_0 < 0$ and minus if $\eta > 0$, and substituting (3.6) and (3.9) into (3.4), one has

$$\phi = \pm\Big(\frac{e^\rho}{2}\left(1 - e^{-2\rho}(1 + \|\nabla^{g_0}\rho\|_{g_0}^2)\right)(1,x) - e^{-\rho}(0, -x + \nabla^{g_0}\rho)\Big).$$

This proves Theorem 3.4. ∎

We have showed that a spacelike hypersurface in the de Sitter space $\mathbb{S}_1^{n+1}$, with principal curvatures different from -1, can be recovered in terms of its associated light cone map. Moreover, the light cone map is clearly recovered in terms of its induced metric which is conformal to the usual spherical metric. This gives us the possibility of posing a problem on spacelike hypersurfaces in terms of conformal geometry and vice versa.

Recall that on a general Riemannian manifold (M,g), $n > 2$, the Riemann curvature tensor can be decomposed as

$$\mathrm{Riem}_g = W_g + \mathrm{Sch}_g \odot g,$$

where W_g is the Weyl tensor, $\odot$ is the Kulkarni–Nomizu product, and

$$\mathrm{Sch}_g := \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{S(g)}{2(n-1)}g\Big) \tag{3.10}$$

is the Schouten tensor, where Ric_g and $S(g)$ are the Ricci tensor and the scalar curvature of g, respectively.

The Weyl tensor is conformally invariant, so the Schouten tensor encodes all the information on how curvature varies by a conformal change of metric. This makes the Schouten tensor the main object of study in conformal geometry.

For the sphere $(\mathbb{S}^n, g_0)$, $n > 2$, it is clear from (3.10) that the Schouten tensor is simply given by $\mathrm{Sch}_{g_0} = \frac{1}{2}g_0$. Moreover, given two conformal Riemannian metrics $\tilde{g} = e^{2u}g$, it is well known that

$$\mathrm{Sch}_{\tilde{g}} = \mathrm{Sch}_g - \nabla^{2,g}u + du \otimes du - \frac{1}{2}\,\|\nabla^g u\|_g^2\, g,$$

where $\nabla^g u$ and $\nabla^{2,g}u$ denote respectively the gradient and the Hessian of u with respect to g. Hence, the Schouten tensor of the light cone metric, $g_\infty = e^{2\rho}g_0$, can be computed as

$$\mathrm{Sch}_{g_\infty} = -\nabla^{2,g_0}\rho + d\rho \otimes d\rho - \frac{1}{2}(-1 + \|\nabla^{g_0}\rho\|_{g_0}^2)\, g_0. \tag{3.11}$$

The Schouten tensor (3.10) is not defined for 2-dimensional metrics. However, the formula (3.11) makes sense also for $n = 2$, so we may naturally define the Schouten tensor for conformal metrics on $\mathbb{S}^2$ in a unifying way:

Definition 3.7. Let $g = e^{2\rho}g_0$ denote a conformal metric on $\mathbb{S}^2$. Its Schouten tensor is defined as the symmetric $(0,2)$-type tensor given by

$$\mathrm{Sch}_g = -\nabla^{2,g_0}\rho + d\rho \otimes d\rho - \frac{1}{2}(-1 + \|\nabla^{g_0}\rho\|_{g_0}^2)\, g_0.$$

We can now relate geometric objects of our spacelike hypersurface in $\mathbb{S}_1^{n+1}$ and geometric objects of its associated light cone metric.

Theorem 3.8. *Let $\phi\colon U \subseteq \mathbb{S}^n \to \mathbb{S}_1^{n+1}$ be an oriented spacelike hypersurface with de Sitter Gauss map $G(x) = x$, and support function $e^\rho\colon U \to (0,\infty)$. Then the first fundamental form* I *and second fundamental form* II *of ϕ at $x_0 \in U \subset \mathbb{S}^n$ are given by*

$$\mathrm{I}(v_i, v_j) = \frac{e^{-2\rho}}{4}\,\big(g_\infty(v_i,v_j) + 2\,\mathrm{Sch}_{g_\infty}(v_i,v_j)\big)^2, \tag{3.12}$$

$$\mathrm{II} = -I + \frac{1}{2}g_\infty + \mathrm{Sch}_{g_\infty}. \tag{3.13}$$

Here $\{v_1,\dots,v_n\} \in T_{x_0}\mathbb{S}^n$ is an orthonormal basis with respect to g_0 at x_0.

Proof. Let simply denote $\{v_1,\dots,v_n\}$ an orthonormal frame with respect to g_0 that agrees with the previous orthonormal basis at x_0 and such that $(\nabla^{g_0}_{v_i}v_j)(x_0) = 0$ for every i, j. By Theorem 3.4 we can write (3.3) as

$$\pm\phi = f(1,x) - e^{-\rho}\xi \tag{3.14}$$

where

$$f := \frac{e^{\rho}}{2}(1 - e^{-2\rho}(1 + \|\nabla^{g_0}\rho\|^2_{g_0})) \quad \text{and} \quad \xi := (0, -x + \nabla^{g_0}\rho).$$

As $(\nabla^{g_0}_{v_i} v_j)(x_0) = 0$, we have $v_i(v_j) = -\delta_{ij} x$ at x_0, and at this point,

$$v_i(\xi) = \Big(0, -v_i(x) + v_i\Big(\sum_{k=1}^{n} v_k(\rho) v_k\Big)\Big) = (0, -v_i - v_i(\rho)x + \sum_{k=1}^{n} v_i(v_k(\rho)) v_k). \tag{3.15}$$

Now, using (3.14) and (3.15),

(3.16)

$$\pm v_i(\phi) = v_i(f)(1, x) + (f + e^{-\rho})(0, v_i) + e^{-\rho}(0, \sum_{k=1}^{n}\{v_i(\rho)v_k(\rho) - v_i(v_k(\rho))\}v_k).$$

Since $\langle(1, x), (1, x)\rangle = 0$ and $\langle(1, x), (0, v_i)\rangle = 0$, (3.16) yields $\langle(1, x), v_i(\phi)\rangle = 0$ and

$$\begin{aligned}\langle v_i(\phi), v_j(\phi)\rangle &= (f + e^{-\rho})^2 \delta_{ij} + 2(f + e^{-\rho})\, e^{-\rho}(v_i(\rho)v_j(\rho) - v_i(v_j(\rho))) \\ &\quad + e^{-2\rho}\sum_{k=1}^{n}(v_i(\rho)\, v_k(\rho) - v_i(v_k(\rho)))\,(v_j(\rho)\, v_k(\rho) - v_j(v_k(\rho))).\end{aligned}$$

If we consider the $n \times n$ matrices A and B given by

$$A = \big(e^{-\rho}(v_i(\rho)\, v_j(\rho) - v_i(v_j(\rho))\big) \quad \text{and} \quad B := (f + e^{-\rho})\,\mathrm{Id}_n,$$

then one can easily check that $\langle v_i(\phi), v_j(\phi)\rangle$ is the (i, j) entry of the matrix $(A + B)^2$. Moreover, A is the matrix expression of $e^{-\rho}(d\rho \otimes d\rho - \nabla^{2,g_0}\rho)$ with respect to the basis $\{v_1, \ldots, v_n\}$. Hence, $A + B$ is just

$$\frac{e^{\rho}}{2}\, g_0 + e^{-\rho}\left(-\nabla^{2,g_0}\rho + d\rho \otimes d\rho - \frac{1}{2}(-1 + \|\nabla^{g_0}\rho\|^2_{g_0})\, g_0\right).$$

That is, from (3.11), $A + B = e^{-\rho}(g_\infty/2 + \mathrm{Sch}_{g_\infty})$.

Therefore,

$$\langle v_i(\phi), v_j(\phi)\rangle = \frac{e^{-2\rho}}{4}\,(g_\infty(v_i, v_j) + 2\,\mathrm{Sch}_{g_\infty}(v_i, v_j))^2,$$

which proves (3.12).

On the other hand, observe that

$$\pm v_i(\psi) = e^{\rho}\, v_i(\rho)(1, x) + e^{\rho}(0, v_i).$$

Since, $\mathrm{II} := -\langle d\phi, d\eta\rangle = -\langle d\phi, d\phi\rangle + \langle d\phi, d\psi\rangle$, and from (3.16) and (3.11), $\langle d\phi, d\psi\rangle$ can be computed as

$$\langle d\phi, d\psi\rangle = \frac{g_\infty}{2} + \mathrm{Sch}_{g_\infty}, \tag{3.17}$$

and we obtain (3.13). ∎

Consider a conformal metric $g = e^{2\rho} g_0$ on an open set $U \subseteq \mathbb{S}^n$. Given a point $x \in U$, we can consider the eigenvectors $\{v_1, \dots, v_n\} \in T_x\mathbb{S}^n$ and the eigenvalues $\{\lambda_1, \dots, \lambda_n\}$ of the Schouten tensor Sch_g with respect to g. Thus, $g(v_i, v_j) = \delta_{ij}$ and $\mathrm{Sch}_g(v_i, v_j) = \lambda_i \delta_{ij}$. Then we have:

Corollary 3.9. *Let $\phi: U \subset \mathbb{S}^n \to \mathbb{S}_1^{n+1}$ be a spacelike hypersurface with de Sitter Gauss map $G(x) = x$. Let us denote by $\{k_1, \dots, k_n\}$ the principal curvatures of the immersion at a point $x \in U$. Then the eigenvalues of the Schouten tensor of its light cone metric g_∞ at x are given by*

$$\lambda_i = \frac{1 - k_i}{2(1 + k_i)}, \quad i = 1, \dots, n. \tag{3.18}$$

Moreover, the eigendirections of Sch_{g_∞} at x coincide with the principal directions of ϕ at x.

Proof. Let $\{e_i, \dots, e_n\}$ be an orthonormal basis of principal directions of ϕ at x, and define $v_i := \frac{1}{1+k_i} e_i$. Then $g_\infty(v_i, v_j) = \delta_{ij}$ and

$$\langle d\phi(v_i), d\psi(v_j)\rangle = \frac{1}{1 + k_i}\, \delta_{ij}.$$

Now, using (3.17),

$$\langle d\phi(v_i), d\psi(v_j)\rangle = \frac{1}{2}\delta_{ij} + \mathrm{Sch}_{g_\infty}(v_i, v_j).$$

Therefore, the eigendirections of Sch_{g_∞} at x agree with the principal directions of ϕ at x, and

$$\lambda_i = \mathrm{Sch}_{g_\infty}(v_i, v_i) = \frac{1}{1 + k_i} - \frac{1}{2} = \frac{1 - k_i}{2(1 + k_i)}. \qquad \blacksquare$$

4. Uniqueness of spacelike hypersurfaces with prescribed curvatures

In this section we will study spacelike hypersurfaces in the de Sitter space satisfying a very general Weingarten relation $W(k_1, \dots, k_n) = 0$ of elliptic type, and particularize to some important cases such as the rth mean curvatures. For this, we will make use of the previous relation with conformal metrics on the sphere.

As usual, we say that a spacelike hypersurface is a Weingarten hypersurface if its principal curvatures $k_1, \dots, k_n$ satisfy a non trivial relation

$$W(k_1, \dots, k_n) = 0.$$

Here, $W(x_1, \dots, x_n)$ is a $\mathcal{C}^1$ symmetric function in x_i and x_j for all i, j.

An important problem in hypersurface theory is to determine under which global conditions we can classify all the solutions to the Weingarten functional W. And, in particular, which Weingarten functionals have the totally umbilical hypersurfaces as the unique global solutions.

For $n=2$ this can be done for compact Weingarten surfaces satisfying an elliptic relation $W(k_1,k_2)=0$ as a consequence of Poincaré's index theorem (see for instance [4, 23]). In particular, it is known that every compact Weingarten surface in $\mathbb{S}^3_1$ satisfying an elliptic relation $W(k_1,k_2)=0$ must be totally umbilical.

We will focus on general elliptic Weingaten hypersurfaces when $n\geq 3$. Different results are known in this case for the rth mean curvatures and for some linear Weingarten functionals (see [3, 8, 17, 25, 35, 37, 38, 40, 41, 46, 51, 52] and references therein).

We recall that a Weingarten functional $W(x_1,\dots,x_n)$ is said to be elliptic on a domain $U\subset\mathbb{R}^n$ if
$$\frac{\partial W}{\partial x_i}>0,\quad i=1,\dots,n,$$
at every point $x\in U$.

Theorem 4.1. *Let U be a symmetric domain of $\mathbb{R}^n$, $n\geq 3$, let W be a symmetric, elliptic functional on U, and let $\phi\colon M\to\mathbb{S}^{n+1}_1$ be a compact weakly convex spacelike hypersurface satisfying*
$$W(k_1,\dots,k_n)=0,$$
with $(k_1,\dots,k_n)\in U$.

Assume that the domain U satisfies

$$\text{(4.1)}\qquad \begin{array}{l} \textit{if } x=(x_1,\dots,x_n),\ y=(y_1,\dots,y_n)\in U \textit{ with } x_i<y_i,\\ \textit{then } [x_1,y_1]\times\cdots\times[x_n,y_n]\subset U.\end{array}$$

Then, ϕ must be a totally umbilical immersion.

Proof. Associated with ϕ we consider its associated light cone immersion $\psi=\phi-\eta$, which is in fact an immersion from Lemma 3.1. This lemma also asserts that the de Sitter Gauss map $G=[\phi-\eta]\colon M\to\mathbb{S}^n$ is a local diffeomorphism, and since M is compact and $\mathbb{S}^n$ is simply connected, we obtain that G is a global diffeomorphism. Hence, we will identify M and $\mathbb{S}^n$ in order to use Corollary 3.9.

Next, we consider the symmetric function
$$f(y_1,\dots,y_n)=1-W\Big(\frac{1-2y_1}{1+2y_1},\dots,\frac{1-2y_n}{1+2y_n}\Big),$$
which is well defined in the symmetric domain
$$V=\Big\{(y_1,\dots,y_n)\in\mathbb{R}^n:\ y_i>-\frac{1}{2}\ \text{and}\ \Big(\frac{1-2y_1}{1+2y_1},\dots,\frac{1-2y_n}{1+2y_n}\Big)\in U\Big\}.$$

The light cone metric g_∞ of ϕ is conformal to the standard metric of the sphere and given by $e^{2\rho}\langle,\rangle_{\mathbb{S}^n}$. Thus, from Corollary 3.9, if $\lambda_1,\dots,\lambda_n$ are the eigenvalues of its Schouten tensor, then
$$f(\lambda_1,\dots,\lambda_n)=1-W\Big(\frac{1-2\lambda_1}{1+2\lambda_1},\dots,\frac{1-2\lambda_n}{1+2\lambda_n}\Big)=1-W(k_1,\dots,k_n)=1,$$
where $k_1,\dots,k_n$ are the principal curvatures of ϕ.

Now, in order to prove that ϕ is totally umbilical, we will associate to f a new functional F defined for symmetric matrices which is in the conditions of Corollary 1.6 in [33]. If this is the case, this will show that $\lambda_1 = \cdots = \lambda_n$, or equivalently, $k_1 = \cdots = k_n$, as we want to prove.

Thus, denote by $\mathcal{S}^{n\times n}$ the set of real symmetric matrices of order n, and let $\mathcal{U}$ be the set of symmetric matrices whose eigenvalues belong to V. It is a standard fact that $\mathcal{U}$ is a domain in $\mathcal{S}^{n\times n}$, which is invariant under orthogonal conjugation, i.e.,

$$O\,\mathcal{U}\,O^{-1} = \mathcal{U} \tag{4.2}$$

for all $O \in \mathcal{O}(n)$, where $\mathcal{O}(n)$ denotes the set of orthogonal matrices.

We define $F(A) = f(\lambda_1, \dots, \lambda_n)$ for any $A \in \mathcal{U}$, where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of the symmetric matrix A. Since f is symmetric, F has the same regularity as f and

$$F(OAO^{-1}) = F(A) \tag{4.3}$$

for all $A \in \mathcal{U}$, $O \in \mathcal{O}(n)$.

Moreover,

$$\frac{\partial f}{\partial y_i} = \frac{\partial W}{\partial x_i}\,\frac{4}{(1+2y_i)^2} > 0.$$

This implies that F is an elliptic functional (see [14]), i.e.,

$$\Big(\frac{\partial F}{\partial a_{ij}}(A)\Big) \text{ is positive definite,} \tag{4.4}$$

where $A = (a_{ij}) \in \mathcal{U}$.

On the other hand, from the definition of V, the fact that the function $g(y) = \frac{1-2y}{1+2y}$ is strictly decreasing for $y > -1/2$ and (4.1), it is clear that if $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n) \in V$ with $x_i < y_i$ then $[x_1, y_1] \times \cdots \times [x_n, y_n] \subset V$. Observe, in addition, that if $A, B \in \mathcal{S}^{n\times n}$, with B positive definite, and $(x_1, \dots, x_n), (y_1, \dots, y_n)$ are the eigenvalues of A and $A + B$ respectively, then they can be obviously ordered such that $x_i < y_i$.

With all of this, we obtain that

$$\mathcal{U} \cap \{A + tB : t > 0\} \text{ is convex, for all } A \in \mathcal{S}^{n\times n}, B \in \mathcal{S}_+^{n\times n}, \tag{4.5}$$

where $\mathcal{S}_+^{n\times n}$ denotes the set of positive definite symmetric matrices of order n. This can be easily proved from the previous facts because if $A + t_1 B, A + t_2 B \in \mathcal{U}$, with $t_1 < t_2$, then their eigenvalues can be ordered as $(x_1, \dots, x_n), (y_1, \dots, y_n)$ with $x_i < y_i$. Thus, the eigenvalues of any matrix $A + tB$, with $t_1 < t < t_2$, belong to $[x_1, y_1] \times \cdots \times [x_n, y_n] \subset V$, and so $A + tB \in \mathcal{U}$.

Therefore, conditions (4.2), (4.3), (4.4) and (4.5) allow us to apply Corollary 1.6 in [33] to the Schouten tensor Sch_{g_∞} of the light cone metric, and to obtain that

$$e^{\rho} = a\,|J_\varphi|^{1/n},$$

where a is a positive constant and J_φ is the Jacobian of a conformal diffeomorphism $\varphi\colon \mathbb{S}^n \to \mathbb{S}^n$.

In particular, $\lambda_1 = \cdots = \lambda_n$ at any point, or equivalently, $k_1 = \cdots = k_n$ as we wanted to show. ∎

We apply this result to the rth mean curvatures of a compact spacelike hypersurface in $\mathbb{S}_1^{n+1}$. Recall that the rth mean curvature H_r of a spacelike hypersurface is defined as

$$\binom{n}{r} H_r = \sum_{i_1<\cdots<i_r} k_{i_1} \cdots k_{i_r}, \quad \text{for } 1 \leq r \leq n.$$

Associated with H_r, we denote by Γ_r the connected component of

$$\Big\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i_1<\cdots<i_r} x_{i_1} \cdots x_{i_r} > 0\Big\}$$

containing the positive cone $\{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_1, \ldots, x_n > 0\}$. It is known that Γ_r is a convex cone with vertex at the origin and $W_r(x_1, \ldots, x_n) = \sum_{i_1<\cdots<i_r} x_{i_1} \cdots x_{i_r}$ satisfies

$$\frac{\partial W_r}{\partial x_i} > 0 \quad \text{in } \Gamma_r, \quad 1 \leq i \leq n, \tag{4.6}$$

for all $r = 1, \ldots, n$ (see for instance [14]).

Corollary 4.2. *Let $\phi\colon M \to \mathbb{S}_1^{n+1}$, $n \geq 3$, be a compact weakly convex spacelike hypersurface satisfying $H_r = c_0$ for a positive constant c_0, and assume there exists $p \in M$ such that $(k_1(p), \ldots, k_n(p)) \in \Gamma_r$. Then ϕ must be totally umbilical.*

Proof. This is simply a consequence of Theorem 4.1, choosing $U = \Gamma_r$ and

$$W = \binom{n}{r}^{-1} W_r - c_0.$$

Observe that (4.6) guarantees the ellipticity of W, and (4.1) is satisfied because Γ_r is a convex cone. ∎

Remark 4.3. Given an oriented compact spacelike hypersurface $\phi\colon M \to \mathbb{S}_1^{n+1}$ with unit normal η, it is well known that there exists a point $p \in M$ such that $k_i(p) > -1$ for all $i = 1, \ldots, n$. This can be checked since the compact spacelike hypersurface given by $x_0 = k$, for $k \in \mathbb{R}$, is a totally umbilical spacelike hypersurface with constant principal curvatures $k_i \in (-1, 1)$. Thus, the point $p \in M$ can be chosen as the highest or lowest point for the function x_0 for ϕ, by comparison with the totally umbilical surfaces $x_0 = k$.

We would like to observe that the hypothesis on M of being weakly convex is directly satisfied in many occasions. For instance, the Weingarten functional

$$W(k_1, \ldots, k_n) = (k_1 + 1) \cdots (k_n + 1) - c_0, \tag{4.7}$$

for $c_0 \neq 0$, is clearly elliptic in the domain $U = (-1, \infty) \times \cdots \times (-1, \infty)$. Thus, if $\phi\colon M \to \mathbb{S}_1^{n+1}$ satisfies $W(k_1, \ldots, k_n) = 0$ then, from Remark 4.3, there exists a point $p \in M$ with $(k_1(p), \ldots, k_n(p)) \in U$, and so $(k_1(q), \ldots, k_n(q)) \in U$ for all $q \in M$ because $W_{|\partial U} = -c_0 \neq 0$. Therefore, from Theorem 4.1, the unique compact spacelike hypersurfaces satisfying (4.7) must be totally umbilical.

This is true not only for the previous functional (4.7) but for many functionals of the form

$$W(k_1, \ldots, k_n) = f\Big(\frac{1}{k_1 + 1}, \ldots, \frac{1}{k_n + 1}\Big),$$

where $f(x_1, \ldots, x_n)$ is a symmetric functional.

Finally, we will show a uniqueness result for non compact Weingarten hypersurfaces, but, as expected, some additional hypothesis on the immersion are necessary.

Theorem 4.4. *Let U be a symmetric domain of $\mathbb{R}^n$, $n \geq 3$, let W be a symmetric, elliptic functional on U, and let $\phi\colon M \to \mathbb{S}_1^{n+1}$ be a weakly convex spacelike hypersurface satisfying*

$$W(k_1, \ldots, k_n) = 0,$$

with $(k_1, \ldots, k_n) \in U$.

Assume that the de Sitter Gauss map $G\colon M \to \mathbb{S}^n$ is a diffeomorphism into $\mathbb{S}^n \backslash \{p_0\}$, for some $p_0 \in \mathbb{S}^n$, and that the principal curvatures satisfy

$$\sum_i \frac{1}{k_i + 1} \geq \frac{n}{2}. \tag{4.8}$$

If the domain U satisfies (4.1), then ϕ must be a totally umbilical immersion.

Proof. We can proceed as in the proof of Theorem 4.1 in order to show that (4.2), (4.3), (4.4) and (4.5) are satisfied.

Moreover, we can identify M with $\mathbb{R}^n \equiv \mathbb{S}^n \backslash \{p_0\}$ since G is a diffeomorphism. Thus, as the light cone metric g_∞ is conformal to the usual metric g_0 on the sphere and, using the stereographic projection, g_0 is conformal to the canonical flat metric g_e of $\mathbb{R}^n$, we can write $g_\infty = u^{4/(n-2)} g_e$, for a certain positive function u.

On the other hand, from (3.18), we obtain that (4.8) is equivalent to the condition $\sum_i \lambda_i \geq 0$, where λ_i are the eigenvalues of the Schouten tensor of g_∞. Then, since

$$-\frac{2}{n-2} u^{-\frac{n+2}{n-2}} \Delta u = \sum_i \lambda_i,$$

(see, for instance, [33]) we have that u is superharmonic, and we can use Theorem 1.3 in [34] for the case $p = (n+2)/(n-2)$ in order to determine u, or equivalently, to determine the cone light metric g_∞. Again, this implies that $\lambda_1 = \cdots = \lambda_n$, or equivalently, $k_1 = \cdots = k_n$, i.e., ϕ is totally umbilical. ∎

We remark that there are different natural geometric conditions on the immersion ϕ which imply that the de Sitter Gauss map G is a diffeomorphism into $\mathbb{S}^n$ minus a point, as we will see in the next section.

The additional hypothesis in Theorem 4.4 are necessary to ensure that the immersion is totally umbilical, since there are non totally umbilical complete Weingarten hypersurfaces for many different functionals (see, for instance, [25, 37, 38, 46]).

Finally, as a consequence of Theorem 4.4, we obtain the following corollary for the rth mean curvatures, whose proof follows as in Corollary 4.2.

Corollary 4.5. *Let $\phi\colon M \to \mathbb{S}_1^{n+1}$, $n \geq 3$, be a weakly convex spacelike hypersurface satisfying $H_r = c_0$ for a positive constant c_0, with $p \in M$ such that $(k_1(p), \ldots, k_n(p)) \in \Gamma_r$.*

Assume that the de Sitter Gauss map $G\colon M \to \mathbb{S}^n$ is a diffeomorphism into $\mathbb{S}^n \backslash \{p_0\}$, for some $p_0 \in \mathbb{S}^n$, and that the principal curvatures satisfy (4.8). Then ϕ must be totally umbilical.

5. Injectivity of the de Sitter Gauss map

We will assume throughout this section that the dimension $n \geq 3$. We know that the de Sitter Gauss map $G\colon M \to \mathbb{S}^n$ of a compact weakly convex spacelike hypersurface $\phi\colon M \to \mathbb{S}_1^{n+1}$ is always injective; however, the de Sitter Gauss map of a non-compact weakly convex spacelike hypersurface, in general, may not be injective.

On the other hand, the de Sitter Gauss map G of a weakly convex spacelike hypersurface M in the de Sitter space $\mathbb{S}_1^{n+1}$ is naturally a development map from M equipped with the light cone metric g_∞ into the sphere $\mathbb{S}^n$. Therefore, due to Kulkarni and Pinkall [32], we obtain:

Lemma 5.1. *Let $\phi\colon M \to \mathbb{S}_1^{n+1}$ be a weakly convex spacelike hypersurface such that its associated light cone metric g_∞ is complete on M. Then the de Sitter Gauss map $G\colon M \to \mathbb{S}^n$ is a covering onto its image in the sphere. Thus, if $G(M) \subset \mathbb{S}^n$ is simply connected, then G is injective.*

In order to study the image of the de Sitter Gauss map in $\mathbb{S}^n$, we give the following definitions for a non-compact hypersurface in $\mathbb{S}_1^{n+1}$.

Definition 5.2. Let $\phi\colon M \to \mathbb{S}_1^{n+1}$ be a properly immersed spacelike hypersurface and let $T\colon \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2} \to B \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ be the projection of the de Sitter space $\mathbb{S}_1^{n+1}$ to the Klein model B. We define the boundary at infinity $\partial_\infty\phi(M)$ of ϕ as the collection of points $q \in \partial B$ such that there is a sequence $q_i = T(\phi(p_i))$, for $p_i \in M$, that converges to q in $\overline{B}$.

Definition 5.3. Let $\phi\colon M \to \mathbb{S}_1^{n+1}$ be a properly immersed spacelike hypersurface. We say that the de Sitter Gauss map G is regular at infinity if, for each $q \in \partial_\infty\phi(M) \subset \partial B$ and each sequence $p_i \in M$ such that $T(\phi(p_i)) \to q$,

$$\lim_{i\to\infty} G(p_i) = \pi(q).$$

Here, π is the projection of the light cone $\mathbb{N}^{n+1}$ to the sphere $\mathbb{S}^n$, given by (2.2).

An immediate consequence of such regularity is the following.

Lemma 5.4. *Let $\phi\colon M \to \mathbb{S}_1^{n+1}$ be a properly immersed, weakly convex, spacelike hypersurface such that its de Sitter Gauss map $G\colon M \to \mathbb{S}^n$ is regular at infinity. Then*

$$\partial G(M) \subset \pi(\partial_\infty\phi(M)).$$

Proof. Consider $q \notin \pi(\partial_\infty\phi(M))$. We would like to show that $q \notin \partial G(M)$. Suppose that $q \in \partial G(M)$. Since ϕ is weakly convex, we have that G is a local diffeomorphism (in particular, $G(M)$ is open). So, $q \notin G(M)$. Moreover, there is a sequence $q_i \in G(M)$ such that $q_i \to q$ in $\mathbb{S}^n$. Let $p_i \in M$ be such that $G(p_i) = q_i$. Since $\overline{B}$ is compact, at least for a subsequence, we know that $T(\phi(p_i)) \to x \in \overline{B}$.

As ϕ is proper, if $x \in B$, then $x = T(\phi(p))$ for some $p \in M$. Now, since G is a local diffeomorphism (in particular, it is continuous), we get $q = G(p)$, which contradicts that $q \notin G(M)$.

On the other hand, if $x \in \partial B$, by the definition of boundary at infinity, then $x \in \partial_\infty\phi(M)$. From the regularity at infinity of the de Sitter Gauss map, one concludes that $G(p_i) \to \pi(x)$, which contradicts that $q \notin \pi(\partial_\infty\phi(M))$. ∎

Now, we are able to establish the following injectivity theorem.

Theorem 5.5. *Let $\phi: M \to \mathbb{S}_1^{n+1}$ be a properly immersed, weakly convex, spacelike hypersurface such that its associated light cone metric g_∞ is complete and its de Sitter Gauss map G is regular at infinity. Suppose that the Hausdorff dimension of $\pi(\partial_\infty \phi(M)) \subset \mathbb{S}^n$ is less than $n-2$. Then the de Sitter Gauss map is injective and*

$$\partial G(M) = \pi(\partial_\infty \phi(M)).$$

Proof. By Lemma 5.4, we have that

$$\partial G(M) \subset \pi(\partial_\infty \phi(M))$$

and, by hypothesis, the Hausdorff dimension of $\partial G(M)$ is less than $n-2$.

From Hurewicz and Wallman [29], Chapter VII.4, the Hausdorff dimension of a metric space is greater than or equal to its topological dimension. So, we obtain that the topological dimension of $\partial G(M)$ is less than $n-2$.

Also, from Hurewicz and Wallman [29], Theorem IV.4, if a subset D of $\mathbb{S}^n$ has topological dimension less than or equal to $n-2$, then the complement in $\mathbb{S}^n$ of D, denoted by $\mathbb{S}^n \setminus D$, is connected. Hence, we can apply this result to the subset $\partial G(M) \subset \mathbb{S}^n$ and get that $\mathbb{S}^n \setminus \partial G(M)$ is connected, i.e., $\mathbb{S}^n \setminus \partial G(M) = G(M)$. Now, since $n \geq 3$, we can deform any loop in $G(M)$ into a point in $\mathbb{S}^n$ without leaving $G(M)$, and conclude that $G(M)$ is simply connected in $\mathbb{S}^n$. So, by Lemma 5.1, one obtains that the de Sitter Gauss map is injective. Hence, if $q \in \pi(\partial_\infty \phi(M)) \subset \mathbb{S}^n$, then $q \notin G(M)$, and consequently, $q \in \partial G(M)$, i.e.,

$$\pi(\partial_\infty \phi(M)) \subset \partial G(M). \qquad \blacksquare$$

In order to clarify the above theorem, we will see two simple and illustrative types of spacelike hypersurfaces in the de Sitter space.

Example 5.6. Let $\mathcal{H}$ be the complete totally umbilical spacelike hypersurface in $\mathbb{S}_1^{n+1}$ given by

$$\mathcal{H} = \{p \in \mathbb{S}_1^{n+1} : \langle p, a \rangle = b\}, \tag{5.1}$$

where $a = (a_0, \overline{a}) = (a_0, a_1, a_2, \dots, a_{n+1}) \in \mathbb{N}^{n+1}$, $p = (p_0, p_1, p_2, \dots, p_{n+1})$ and $b \in \mathbb{R}\setminus\{0\}$. Up to isometries, we can take $a = (1, 1, 0, \dots, 0)$.

First, it follows from (2.3) that G is a global diffeomorphism onto $\mathbb{S}^n \setminus \{\frac{1}{a_0}\, \overline{a}\}$. So,

$$\partial G(\mathcal{H}) = \frac{1}{a_0}\, \overline{a} = (1, 0, \dots, 0).$$

Secondly, let us calculate $\pi(\partial_\infty \phi(\mathcal{H}))$. By (5.1), we have that

$$\phi(p_2, \dots, p_{n+1}) = \Big(\frac{1}{2b}(b^2 + 1 - p_2^2 - \dots - p_{n+1}^2) - b, \\ \frac{1}{2b}(b^2 + 1 - p_2^2 - \dots - p_{n+1}^2), p_2, \dots, p_{n+1} \Big).$$

If $T\colon \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2} \to B \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ is the projection of the de Sitter space $\mathbb{S}_1^{n+1}$ to the Klein model B and $c := p_2^2 + \cdots + p_{n+1}^2$, we obtain

$$\begin{aligned} T(\phi(p_2,\ldots,p_{n+1})) &= \frac{\phi(p_2,\ldots,p_{n+1})}{|\phi(p_2,\ldots,p_{n+1})|_e} \\ &= \frac{\left(\frac{1}{2b}(b^2+1-c)-b, \frac{1}{2b}(b^2+1-c), p_2,\ldots,p_{n+1}\right)}{\sqrt{(\frac{1}{2b}(b^2+1-c)-b)^2 + (\frac{1}{2b}(b^2+1-c))^2 + c}}. \end{aligned}$$

Now, if $\phi_0 \to \pm\infty$, then $p_i^2 \to \infty$, for some $i = 2,\ldots,n+1$, and $c \to \infty$. Hence,

$$T(\phi(p_2,\ldots,p_{n+1})) \to \left(\frac{\frac{-1}{2b}}{\frac{1}{\sqrt{2}|b|}}, \frac{\frac{-1}{2b}}{\frac{1}{\sqrt{2}|b|}}, 0,\ldots,0\right) = -\frac{1}{\sqrt{2}}\operatorname{sign}(b)a,$$

i.e., $\partial_\infty\phi(\mathcal{H}) = -\frac{1}{\sqrt{2}}\operatorname{sign}(b)a \in \partial B$, and

$$\pi(\partial_\infty\phi(\mathcal{H})) = (1,0,\ldots,0).$$

Thus, we get

$$\partial G(\mathcal{H}) = \pi(\partial_\infty\phi(\mathcal{H})).$$

Example 5.7. Let $\mathcal{C}$ be the analogous of the "hyperbolic cylinder" in $\mathbb{S}_1^{n+1}$, given by

$$\mathcal{C} = \{p \in \mathbb{S}_1^{n+1} : -p_0^2 + p_1^2 = -\sinh^2(r), p_0 > 0\}, \tag{5.2}$$

where $r \in \mathbb{R}^+$ and $p = (p_0, p_1, p_2, \ldots, p_{n+1})$. Then, it is easy to see that

$$\eta(p) = \frac{1}{\sinh(r)\cosh(r)}(p_0, p_1, 0, \ldots, 0) + \tanh(r)p$$

is a unit normal field for $\mathcal{C}$. So, we can verify that its associated Weingarten endomorphism A has two eigenvalues, $\coth(r)$ and $\tanh(r)$, with multiplicities 1 and $n-1$, respectively. Note that, since $r \in \mathbb{R}^+$, all the eigenvalues are greater than -1. So, $\mathcal{C}$ is weakly convex.

First, we will calculate $\partial G(\mathcal{C})$. The de Sitter Gauss map G of the hypersurface $\mathcal{C}$, associated with η, is given by

$$G(p) = \pi\left((1-\coth(r))q_1, (1-\tanh(r))q_2\right),$$

where $q_1 := (p_0, p_1)$ and $q_2 := (p_2, \ldots, p_{n+1})$. Given $\pi(g_1, g_2) \in \mathbb{S}^n$ with $g_1 = (1, x_1)$ and $g_2 = (x_2, \ldots, x_{n+1})$, we want to determine $p = (q_1, q_2) \in \mathcal{C}$ such that $G(p) = \pi(g_1, g_2)$, i.e, $p = (q_1, q_2) \in \mathcal{C}$ satisfying

$$(1-\coth(r))q_1 = \lambda g_1 \tag{5.3}$$

and

$$(1-\tanh(r))q_2 = \lambda g_2, \tag{5.4}$$

for $\lambda \in \mathbb{R}^*$. Since $p = (q_1, q_2) \in \mathcal{C}$, we have that $\langle q_2, q_2\rangle_e = 1 + \sinh^2(r)$. So, by (5.4),

$$(1 - \tanh(r))^2(1 + \sinh^2(r)) = \lambda^2 \langle g_2, g_2\rangle,$$

or equivalently,

$$\lambda = \pm\sqrt{\frac{(1 - \tanh(r))^2(1 + \sinh^2(r))}{\langle g_2, g_2\rangle}}.$$

Now, since $p_0 > 0$ and (5.3), one obtains that

$$\lambda = -\sqrt{\frac{(1 - \tanh(r))^2(1 + \sinh^2(r))}{\langle g_2, g_2\rangle}}.$$

Hence, the equations (5.3) and (5.4) have a unique solution provided that $g_2 \neq 0$. Note that $\{\pi((g_1, g_2)) \in \mathbb{S}^n : g_2 = 0\} = \{\pi(1, x_1, 0, \dots, 0) \in \mathbb{S}^n\} = \{(\pm 1, 0, \dots, 0)\}$. Consequently, G is a global diffeomorphism into $\mathbb{S}^n \backslash \{(\pm 1, 0, \dots, 0)\}$, and so,

$$\partial G(\mathcal{C}) = \{(\pm 1, 0, \dots, 0)\}.$$

Secondly, let us determine $\pi(\partial_\infty \phi(\mathcal{C}))$. By (5.2), we have that

$$\phi(p_1, p_2, \dots, p_n) = \Big(\sqrt{p_1^2 + \sinh^2(r)}, p_1, p_2, \dots, p_n, \pm\sqrt{1 + \sinh^2(r) - p_2^2 - \dots - p_n^2}\Big).$$

If $T\colon \mathbb{S}_1^{n+1} \subset \mathbb{L}^{n+2} \to B \subset \mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ is the projection of the de Sitter space $\mathbb{S}_1^{n+1}$ to the Klein model B,

$$\begin{aligned} &T(\phi(p_1, p_2, \dots, p_n)) \\ &\quad = \frac{\big(\sqrt{p_1^2 + \sinh^2(r)}, p_1, p_2, \dots, p_n, \pm\sqrt{1 + \sinh^2(r) - p_2^2 - \dots - p_n^2}\big)}{\sqrt{2p_1^2 + 2\sinh^2(r) + 1}}. \end{aligned}$$

Now, if $\phi_0 \to \infty$, then $p_1 \to \pm\infty$. Hence,

$$T(\phi(p_1, p_2, \dots, p_n)) \to \Big(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0, \dots, 0\Big),$$

i.e., $\partial_\infty \phi(\mathcal{C}) = (\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, 0, \dots, 0) \in \partial B$, and

$$\pi(\partial_\infty \phi(\mathcal{C})) = (\pm 1, 0, \dots, 0).$$

Thus, we get

$$\partial G(\mathcal{C}) = \pi(\partial_\infty \phi(\mathcal{C}))$$

and finish with Example 5.7.

Finally, as a consequence of the celebrated injectivity result of Schoen and Yau [45], we will also get the injectivity of the de Sitter Gauss map imposing some curvature condition on the hypersurface.

Theorem 5.8. *Let $\phi\colon M \to \mathbb{S}_1^{n+1}$ be a weakly convex spacelike hypersurface such that its associated light cone metric g_∞ is complete and its principal curvatures satisfy*

$$\sum_i \frac{1}{1+k_i} \geq \frac{n}{2}. \tag{5.5}$$

Then the de Sitter Gauss map is injective.

Proof. First, by (3.5) and the hypothesis (5.5), we have that

$$S(g_\infty) = 2(n-1)\sum_{i=1}^{n} \frac{1}{1+k_i} - n(n-1) \geq 0,$$

where $S(g_\infty)$ denotes the scalar curvature of g_∞. Second, the fact that ϕ is weakly convex implies that G is a local diffeomorphism and that $g_\infty = e^{2\rho} G^* g_{\mathbb{S}^n}$. So, $G\colon M \to \mathbb{S}^n$ is a conformal map.

Schoen and Yau [45], Theorem 3.5, proved that if (M, g) is a complete Riemannian manifold with nonnegative scalar curvature and $\Phi\colon M \to \mathbb{S}^n$ is a conformal map, then Φ is injective.

Thus, we can apply this result to (M, g_∞) and G, and conclude that the de Sitter Gauss map G is injective. ∎

Remark 5.9. Observe that if $\phi\colon M \to \mathbb{S}_1^{n+1}$ is a complete spacelike hypersurface such that all the principal curvatures $k_i \geq k_0 > -1$, for some number k_0 at all points in M, then, by (3.2), the light cone metric g_∞ is complete.

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