Rev. Mat. Iberoam. 38 (2022), no. 1, 257–267
DOI 10.4171/RMI/1278

Nash blowups in prime characteristic

Daniel Duarte and Luis Núñez-Betancourt

Abstract. We initiate the study of Nash blowups in prime characteristic. First, we show that a normal variety is non-singular if and only if its Nash blowup is an isomorphism, extending a theorem by A. Nobile. We also study higher Nash blowups, as defined by T. Yasuda. Specifically, we give a characteristic-free proof of a higher version of Nobile's theorem for quotient varieties and hypersurfaces. We also prove a weaker version for F-pure varieties.

1. Introduction

The Nash blowup is a natural modification of an algebraic variety that replaces singular points by limits of tangent spaces at non-singular points. The main open problem in this topic is whether the iteration of the Nash blowup solves the singularities of the variety. This question is usually attributed to J. Nash [20] but it also appears in the work of J. G. Semple [24]. If true, it would give a canonical way to resolve singularities. This problem has been an object of intense study [2, 10, 12–15, 18, 20, 22, 26].

In order to be able to achieve a resolution of singularities using Nash blowups, it is needed that this process always modifies a singular variety. One of the first results that appeared in the theory of Nash blowups is Nobile's theorem [20]. It states that, for equidimensional varieties over $\mathbb{C}$, the Nash blowup is an isomorphism if and only if the variety is non-singular. In additon to being of central interest for the theory of Nash blowups, Nobile's theorem has other applications. For instance, it appears in the study of link theoretic characterization of smoothness [8].

Unfortunately, Nobile's theorem fails over fields of prime characteristic. There are examples of singular curves over fields of prime characteristic whose Nash blowup is an isomorphism [20]. Since the main goal of this theory is to resolve singularities, these examples discouraged a further study of Nash blowups in prime characteristic. One of the main purposes of this paper is to provide evidence that the classical Nash blowup in prime characteristic behaves as expected after adding mild hypotheses. In our first main result we provide a version of Nobile's theorem in prime characteristic for normal varieties.

Main Theorem (see Theorem 3.10). *Let X be a normal irreducible variety. If $\mathrm{Nash}_1(X) \cong X$, then X is a non-singular variety.*

2020 Mathematics Subject Classification: Primary 14B05, 14E15, 16S32, 13A35; Secondary 14J70.
Keywords: Nash blowups, normal varieties, differential operators, methods in prime characteristic.

We stress that the hypothesis of X being normal is frequently assumed for many results in characteristic zero. For instance, M. Spivakovsky [26] showed that a sequence of normalized Nash blowups eventually gives a resolution of singularities for surfaces. Our main theorem implies that the original question regarding the Nash blowup and resolution of singularities can be reconsidered regardless of the characteristic by iterating the normalized Nash blowup.

More recently, T. Yasuda [30] introduced a higher-order version of the Nash blowup replacing tangent spaces by infinitesimal neighborhoods of order n. This is denoted by $\mathrm{Nash}_n(X)$. The main goal for this generalization was to investigate whether $\mathrm{Nash}_n(X)$ would give a one-step resolution of singularities for $n \gg 0$. This question has been settled recently for varieties over $\mathbb{C}$: it has an affirmative answer for curves [30], but it is false in general [29]. Higher versions of Nobile's theorem have been proved for some families of varieties [5, 9, 11].

Furthering the ideas in the proof of Main Theorem, we obtain a weaker version of Nobile's theorem for F-pure varieties and the higher Nash blowup. Specifically, we show that if X is F-pure and $\mathrm{Nash}_n(X) \cong X$ for some $n \geq 1$, then X is a strongly F-regular variety (see Theorem 3.14). As a consequence, we obtain that, if $\mathrm{Nash}_1(X) \cong X$, then X is a non-singular variety (see Corollary 3.15). It is worth mentioning that strongly F-regular varieties have mild singularities, for instance, they are normal and Cohen–Macaulay.

We also provide examples of higher-order versions of Nobile's theorem that work in prime characteristic. We show this property for quotient varieties (see Theorem 4.1), and for normal hypersurfaces (see Theorem 4.2). We point out that the proofs of these two results are characteristic-free.

We end this introduction with a few comments about the techniques used in this manuscript. B. Teissier [27] pioneered the use of derivations to study the Nash blowup in characteristic zero. We further this line of research by using rings of differential operators and modules of principal parts in any characteristic. These techniques played a key role in the results presented in this paper. In particular, we use new developments in this line focused on singularities [4] and homological methods [3, 7]. Furthermore, we combine this approach with the use of Frobenius map to detect regularity [19] and certain type of singularities [25] to prove our main theorem.

Convention: Throughout this paper, $\mathbb{K}$ denotes an algebraically closed field and all varieties are assumed to be irreducible. In particular, X always denotes an irreducible variety over $\mathbb{K}$. We denote as $\mathbb{N}$ the set of non-negative integers and $\mathbb{Z}^+$ the set of positive integers. By a local $\mathbb{K}$-algebra $(R, \mathfrak{m}, \mathbb{K})$, we mean a local ring R with maximal ideal $\mathfrak{m}$ such that $\mathbb{K} \subseteq R$ and the map $\mathbb{K} \hookrightarrow R \twoheadrightarrow R/\mathfrak{m}$ is an isomorphism.

2. Nash blowups and Nobile's theorem

In this section we recall the definition of Nash blowups of algebraic varieties. Then we discuss a classical theorem of A. Nobile [20] in the theory of Nash blowups that characterizes smoothness in terms of these blowups.

Let X be an irreducible algebraic variety of dimension d over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic. Let $\mathcal{I}_X$ be the sheaf of ideals defining the diagonal $\Delta \hookrightarrow$

$X \times X$. Let $\mathcal{P}_X^n := \mathcal{O}_{X\times X}/\mathcal{I}_X^{n+1}$ be the sheaf of principal parts of order n of X. Denote as $\mathrm{Grass}_{\binom{n+d}{d}}(\mathcal{P}_X^n)$ the Grassmannian of locally free quotients of $\mathcal{P}_X^n$ of rank $\binom{n+d}{d}$, and let $G_n\colon \mathrm{Grass}_{\binom{n+d}{d}}(\mathcal{P}_X^n) \to X$ be the structural morphism.

The Grassmannian satisfies the following universal property [17]. Let $h\colon Y \to X$ be a morphism. There is a bijective correspondence between locally free quotients of rank $\binom{n+d}{d}$ of $h^*\mathcal{P}_X^n$ and morphisms $h'\colon Y \to \mathrm{Grass}_{\binom{n+d}{d}}(\mathcal{P}_X^n)$ such that the following diagram commutes:

$$\begin{array}{ccc} Y & \longrightarrow & \mathrm{Grass}_{\binom{n+d}{d}}(\mathcal{P}_X^n) \\ & {\scriptstyle h}\searrow & \downarrow {\scriptstyle G_n} \\ & & X. \end{array}$$

Let $U \subseteq X$ be the set of non-singular points of X, and let $i : U \hookrightarrow X$ the inclusion morphism. Since $i^*\mathcal{P}_X^n = \mathcal{P}_X^n|_U$ is locally free of rank $\binom{n+d}{d}$, we have that there exists a morphism $\sigma\colon U \to \mathrm{Grass}_{\binom{n+d}{d}}(\mathcal{P}_X^n)$ by the universal property of Grassmannians.

Definition 2.1 ([20, 21, 30]). Let $\mathrm{Nash}_n(X)$ denote the closure of the image $\sigma(U)$ in $\mathrm{Grass}_{\binom{n+d}{d}}(\mathcal{P}_X^n)$ with its reduced scheme structure, and let $\pi_n\colon \mathrm{Nash}_n(X) \to X$ be the restriction of G_n. We call $(\mathrm{Nash}_n(X), \pi_n)$ the *Nash blowup of order n of X*.

Remark 2.2. T. Yasuda defines the Nash blowup of order n of X using a different parameter space: the Hilbert scheme of points. Both definitions are equivalent [30], Proposition 1.8.

The following theorem is a classical result in the theory of the usual Nash blowup.

Theorem 2.3 (Nobile's theorem [20]). *If* $\mathrm{char}(\mathbb{K}) = 0$*, then* $\mathrm{Nash}_1(X) \cong X$ *if and only if* X *is non-singular.*

There are generalizations of this result for $n \geq 1$ in some cases [5, 9, 11]. On the other hand, it is well known that this result is not true if $\mathrm{char}(\mathbb{K}) > 0$. The classical counterexample is given by the cusp. If $X = \mathbf{V}(x^3 - y^2)$ and $\mathrm{char}(\mathbb{K}) = 2$, then $\mathrm{Nash}_n(X) \cong X$ for all $n \geq 1$ (for $n = 1$ this was proved by A. Nobile [20], and for $n \geq 1$ by T. Yasuda [30]).

We are interested in studying analogs of Theorem 2.3 for $n \geq 1$ in arbitrary characteristic. Because of the previous example, it is necessary to add extra conditions on the variety if $\mathrm{char}(\mathbb{K}) > 0$. We prove that Nobile's theorem, or weaker versions of it, hold for some families of varieties.

B. Teissier gave a different proof of Nobile's theorem [27] using the module of differentials and derivations in characteristic zero. A key part of his proof is that $\mathrm{Nash}_1(X) \cong X$ implies that the module of differentials have a free summand of maximal rank. We give an extension to this fact to the module of principal parts following the same ideas. We give a proof of this result to stress that it is characteristic-free.

Lemma 2.4 ([27]). *Let X be a variety of dimension d. Assume that* $\mathrm{Nash}_n(X) \cong X$ *is an isomorphism. Then,*

$$\mathcal{P}^n_{\mathcal{O}_{X,x}|\mathbb{K}} \cong \mathcal{O}_{X,x}^{\binom{n+d}{d}} \oplus T_x$$

for each $x \in X$, where $\mathcal{P}^n_{\mathcal{O}_{X,x}|\mathbb{K}}$ is the module of principal parts of $\mathcal{O}_{X,x}$ and T_x is its torsion module.

Proof. Since $\mathrm{Nash}_n(X) \cong X$, we have the following commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{\pi_n^{-1}} & \mathrm{Nash}_n(X) & \hookrightarrow & \mathrm{Grass}_{\binom{n+d}{d}}(\mathcal{P}^n_X) \\ & \searrow^{\mathrm{Id}} & \downarrow{\pi_n} & \swarrow_{G_n} & \\ & & X. & & \end{array}$$

By the universal property of the Grassmannian, $\mathrm{Id}^*\mathcal{P}^n_X = \mathcal{P}^n_X$ has a locally free quotient of rank $\binom{n+d}{d}$. Then, there exists a surjective morphism

$$\mathcal{P}^n_X \longrightarrow \mathcal{L} \longrightarrow 0\ ,$$

where $\mathcal{L}$ is a locally free $\mathcal{O}_X$-module of rank $\binom{n+d}{d}$. Therefore, for each $x \in X$ the previous exact sequence induces

$$\mathcal{P}^n_{\mathcal{O}_{X,x}|\mathbb{K}} \longrightarrow \mathcal{O}_{X,x}^{\binom{n+d}{d}} \longrightarrow 0\ . \tag{2.1}$$

Since $\mathcal{O}_{X,x}^{\binom{n+d}{d}}$ is free and $\mathrm{rank}(\mathcal{P}^n_{\mathcal{O}_{X,x}|\mathbb{K}}) = \binom{n+d}{d}$, we conclude that $\mathcal{P}^n_{\mathcal{O}_{X,x}|\mathbb{K}} \cong \mathcal{O}_{X,x}^{\binom{n+d}{d}} \oplus T_x$, where T_x is the torsion submodule of $\mathcal{P}^n_{\mathcal{O}_{X,x}|\mathbb{K}}$. ∎

Remark 2.5. For $n = 1$ and $\mathrm{char}(\mathbb{K}) = 0$, the existence of the surjective morphism (2.1) is used by B. Teissier to prove that $\mathcal{O}_{X,x}$ is a regular local ring, implying Nobile's theorem. The proof uses a result by O. Zariski regarding derivations which allows to apply induction on the dimension of the ring. Unfortunately, it is not clear how to extend Zariski's result for higher-order differential operators.

3. Analogues of Nobile's theorem for normal and F-pure varieties

We start by recalling definitions and properties regarding differential operators that are used to prove our main result (Theorem 3.10).

Definition 3.1 ([16]). Let R be a $\mathbb{K}$-algebra. The $\mathbb{K}$-*linear differential operators of R of order n*, $D^n_{R|\mathbb{K}} \subseteq \mathrm{Hom}_{\mathbb{K}}(R, R)$, are defined inductively as follows:

(i) $D^0_{R|\mathbb{K}} = \mathrm{Hom}_{\mathbb{K}}(R, R)$.

(ii) $D^n_{R|\mathbb{K}} = \{\delta \in \mathrm{Hom}_{\mathbb{K}}(R, R) \mid \delta r - r\delta \in D^{n-1}_{R|\mathbb{K}}\ \forall\ r \in R\}$.

The ring of $\mathbb{K}$-linear differential operators is defined by $D_{R|\mathbb{K}} = \bigcup_{n\in\mathbb{N}} D^n_{R|\mathbb{K}}$.

Definition 3.2. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local $\mathbb{K}$-algebra with $\mathbb{K}$ as a coefficient field. We define the nth differential powers [6] of $\mathfrak{m}$ by

$$\mathfrak{m}^{\langle n\rangle} = \left\{f \in R \mid \delta(f) \in \mathfrak{m} \text{ for all } \delta \in D^{n-1}_{R|\mathbb{K}}\right\}$$

for $n \in \mathbb{Z}^+$. The differential core of R [4] is defined by $\mathfrak{p}_{\mathrm{diff}}(R) = \bigcap_{n\in\mathbb{Z}^+} \mathfrak{m}^{\langle n\rangle}$.

Proposition 3.3 ([4], Proposition 4.15). *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local $\mathbb{K}$-algebra with $\mathbb{K}$ as a coefficient field. Then,*

$$\dim_{\mathbb{K}}(R/\mathfrak{m}^{\langle n+1\rangle}) = \operatorname{free.rank}(\mathcal{P}^n_{R|\mathbb{K}}),$$

where free. rank(M) *denotes the maximal rank of a free module that splits from M (that is,* free. rank$(M) = \max\{t \mid \exists\phi : M \to R^t$ *surjective*$\}$).

We now present a perfect pairing between differential operators and differential powers. This was implicitly introduced in previous work regarding convergence of differential signature, see [4], Section 8.

Lemma 3.4. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local $\mathbb{K}$-algebra with $\mathbb{K}$ as a coefficient field, and let $\mathcal{J}_{R|\mathbb{K}} = \{\delta \in D_{R|\mathbb{K}} \mid \delta(R) \subseteq \mathfrak{m}\}$. There exists a nondegenerate $\mathbb{K}$-bilinear function*

$$(\ ,\ ) : \left(D^{n-1}_{R|\mathbb{K}}/\left(\mathcal{J}_{R|\mathbb{K}} \cap D^{n-1}_{R|\mathbb{K}}\right)\right) \times R/\mathfrak{m}^{\langle n\rangle} \to R/\mathfrak{m}$$

defined by $(\overline{\delta}, \overline{r}) \mapsto \overline{\delta(r)}$.

Proof. By definition, $D^{n-1}_{R|\mathbb{K}}\mathfrak{m}^{\langle n\rangle} \subseteq \mathfrak{m}$ and $\mathcal{J}_{R|\mathbb{K}}R \subseteq \mathfrak{m}$. Then, $(\ ,\ )$ is a well defined function. Since $\mathbb{K}$-linearity in each entry is given by the definition of $(\ ,\ )$, we focus on non-degeneracy.

Let $\overline{\delta} \in D^{n-1}_{R|\mathbb{K}}/\left(\mathcal{J}_{R|\mathbb{K}} \cap D^{n-1}_{R|\mathbb{K}}\right)$ be such that $(\overline{\delta}, \overline{r}) = \overline{\delta(r)} = 0$ in $R/\mathfrak{m}$ for every $\overline{r} \in R/\mathfrak{m}^{\langle n\rangle}$. Then, $\delta(r) \in \mathfrak{m}$ for every $r \in R$, and so, $\delta \in \mathcal{J}_{R|\mathbb{K}}$. We conclude that $\overline{\delta} = 0$. Similarly, let $r \in R/\mathfrak{m}^{\langle n\rangle}$ be such that$(\overline{\delta}, \overline{r}) = \overline{\delta(r)} = 0$ in $R/\mathfrak{m}$ for every $\overline{\delta} \in D^{n-1}_{R|\mathbb{K}}/\left(\mathcal{J}_{R|\mathbb{K}} \cap D^{n-1}_{R|\mathbb{K}}\right)$. Then, $\delta(r) \in \mathfrak{m}$ for every $\delta \in D^{n-1}_{R|\mathbb{K}}$, and so, $r \in \mathfrak{m}^{\langle n\rangle}$. We conclude that $(\ ,\ )$ is nondegenerate. ∎

We now introduce concepts in prime characteristic that play a role in the proof of our main result (Theorem 3.10).

Definition 3.5. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local $\mathbb{K}$-algebra with $\mathbb{K}$ as a coefficient field. Suppose that $\mathbb{K}$ has prime characteristic p. Suppose that R is a domain.

- The ring of p^e-roots of R is defined by

$$R^{1/p^e} = \{f^{1/p^e} \mid f \in R\} \subseteq \overline{\operatorname{frac}(R)}.$$

- We say that R is F-finite if R^{1/p^e} is finitely generated as an R-module.
- We say that R is F-pure if the inclusion $R \hookrightarrow R^{1/p^e}$ splits.
- We say that R is strongly F-regular if for every $c \in R \setminus \{0\}$ there exists $e \in \mathbb{Z}^+$ such that the inclusion $Rc^{1/p^e} \hookrightarrow R^{1/p^e}$ splits.

We say that a variety X satisfies one of these properties if it is satisfied for every local ring $\mathcal{O}_{X,x}$ for every closed point $x \in X$.

Definition 3.6. Let $(R, \mathfrak{m}, \mathbb{K})$ be a local $\mathbb{K}$-algebra with $\mathbb{K}$ as a coefficient field. Suppose that $\mathbb{K}$ has prime characteristic p, and that R is a domain.

- We say that an additive map $\phi\colon R \to R$ is p^{-e}-linear if $\phi(r^{p^e} f) = r\phi(f)$.
- The set of all p^{-e}-linear maps is denoted by $\mathcal{C}^e_R$.
- The set of Cartier operators is defined by $\mathcal{C}_R = \bigcup_{e\in\mathbb{N}} \mathcal{C}^e_R$.

Remark 3.7. There is a bijective correspondence between

$$\Psi : \mathcal{C}_R^e \to \operatorname{Hom}_R(R^{1/p^e}, R)$$

given by $\Psi(\phi)(r^{1/p^e}) = \phi(r)$ for $\phi \in \mathcal{C}_R^e$.

The following characterization of differential operators in prime characteristic plays a crucial role to relate them with Cartier operators. We stress that the following results holds because we are assuming that $\mathbb{K}$ is an algebraically closed field, and so, a perfect field.

Theorem 3.8 ([31]). *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local $\mathbb{K}$-domain with $\mathbb{K}$ as a coefficient field. Then,*

$$D_{R|\mathbb{K}} = \bigcup_{e \in \mathbb{N}} \operatorname{Hom}_{R^{p^e}}(R, R).$$

Remark 3.9. There is a bijective correspondence between

$$\Psi : \operatorname{Hom}_{R^{p^e}}(R, R) \to \operatorname{Hom}_R(R^{1/p^e}, R^{1/p^e})$$

given by $\Psi(\phi)(r^{1/p^e}) = (\phi(r))^{1/p^e}$ for $\phi \in \operatorname{Hom}_{R^{p^e}}(R, R)$.

Now we are ready to prove that the Nash blow-up does properly modify normal varieties.

Theorem 3.10. *Let X be a normal variety over $\mathbb{K}$ of dimension d. Suppose that $\mathbb{K}$ has prime characteristic p. If $\operatorname{Nash}_1(X) \cong X$, then X is a non-singular variety.*

Proof. Let x be a point in X. Let $R = \mathcal{O}_{X,x}$ and $\mathfrak{m}$ be its maximal ideal. By Lemma 2.4, we have that the module of principal parts $\mathcal{P}^1_{R|\mathbb{K}}$ has a free summand of rank $d+1$. As a consequence, $\dim_{\mathbb{K}}(\mathfrak{m}/\mathfrak{m}^{\langle 2 \rangle}) = d$. There exist elements $x_1, \ldots, x_d \in \mathfrak{m}$ and derivations $\partial_1, \ldots, \partial_d$ such that $\partial_i(x_j)$ is a unit if and only if $i = j$ by Lemma 3.4. Let $A = (a_{i,j})$ be the $d \times d$-matrix whose (i, j)-entry is $\partial_i(x_j)$. We note that A is an invertible matrix, as it is invertible modulo $\mathfrak{m}$. Let $C = (c_{i,j})$ be the inverse of A. Let $\delta_t = \sum_{i=1}^d c_{t,i}\partial_i$. Then, $\delta_t(x_j) = \sum_{i=1}^d c_{t,i}\partial_i(x_j) = \sum_{i=1}^d c_{t,i}a_{i,j}$. Then, $\delta_t(x_j) = 1$ if $t = j$ and zero otherwise.

Let $\mathscr{A} = \{\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d \mid \alpha_i < p \;\; \forall i\}$. Let $\frac{1}{\alpha!}\delta^\alpha = \frac{1}{\alpha_1! \cdots \alpha_d!}\delta_1^{\alpha_1} \cdots \delta_d^{\alpha_d}$ for $\alpha \in \mathscr{A}$. We point out that $\frac{1}{\alpha_i!}$ in $\mathbb{K}$ is well defined, because $\alpha_i < p$ for every i. Since δ_t is a derivation for every t, we have that $\frac{1}{\alpha!}\delta^\alpha(x^\alpha) = 1$. In addition, $\frac{1}{\alpha!}\delta^\alpha(x^\beta) \in \mathfrak{m}$ for for every $\alpha, \beta \in \mathscr{A}$ such that $\alpha \neq \beta$.

Let $\tilde{A} = (\tilde{a}_{\alpha,\beta})$ be the $p^d \times p^d$-matrix indexed by $\mathscr{A} \times \mathscr{A}$, whose (α, β)-entry is $\frac{1}{\alpha!}\delta^\alpha(x^\beta)$. For this, we need to order $\mathscr{A}$, but the choice of order does not play a role in the rest of the proof. We note that $\tilde{A}$ is an invertible matrix. Let $\tilde{C} = (\tilde{c}_{\alpha,\beta})$ be the inverse of $\tilde{A}$. Let $\phi_\gamma = \sum_\alpha \tilde{c}_{\gamma,\alpha}\frac{1}{\alpha!}\delta^\alpha$. Then, $\phi_\gamma(x^\beta) = \sum_\alpha \tilde{c}_{\gamma,\alpha}\frac{1}{\alpha!}\delta^\alpha(x^\beta) = \sum_\alpha \tilde{c}_{\gamma,\alpha}\widetilde{a}_{\alpha,\beta}$. Then, $\phi_\gamma(x^\beta) = 1$ if $\gamma = \beta$ and zero otherwise.

Since $\mathbb{K}$ has prime characteristic, we have $\operatorname{Der}_{R|\mathbb{K}} \subseteq \operatorname{Hom}_{R^p}(R, R)$. Moreover, $\frac{1}{\alpha!}\delta^\alpha \in \operatorname{Hom}_{R^p}(R, R)$ for every $\alpha \in \mathscr{A}$. As a consequence, $\phi_\alpha \in \operatorname{Hom}_{R^p}(R, R)$ for every $\alpha \in \mathscr{A}$. Let $\varphi_\alpha \in \operatorname{Hom}_R(R^{1/p}, R^{1/p})$ defined by $\varphi_\alpha(f^{1/p}) = (\phi_\alpha(f))^{1/p}$. Then, $\varphi_\alpha(x^{\beta/p}) = 1$ if $\alpha = \beta$ and zero otherwise.

We set $\psi\colon \oplus_{\alpha\in\mathscr{A}} Re_\alpha \to R^{1/p}$ defined by $e_\alpha \mapsto x^{\alpha/p}$. Let $Q \subseteq R$ be a prime ideal of R of height 1. Let ψ_Q be the map induced by ψ by the localization at Q. Since R is normal, we have that R_Q is a regular ring. Then, $R_Q^{1/p}$ is a free R_Q-module. Let $\sigma_Q\colon R_Q^{1/p} \to R_Q$ be a splitting of the inclusion $R_Q \hookrightarrow R_Q^{1/p}$. We consider the map $\rho_Q\colon R_Q^{1/p} \to \bigoplus_{\alpha\in\mathscr{A}} R_Q e_\alpha$ defined by

$$\rho_Q\Big(\frac{f^{1/p}}{s}\Big) = \bigoplus_{\alpha\in\mathscr{A}} \sigma_Q\Big(\frac{\varphi_\alpha(f^{1/p})}{s}\Big)e_\alpha.$$

Then, we have that ρ_Q is surjective because $\rho_Q(x^{\alpha/p}) = e_\alpha$. Since $R_Q^{1/p}$ is a free R_Q-module of rank p^d, we conclude that ρ_Q is an isomorphism. Furthermore, $\rho_Q \circ \psi_Q$ is the identity on $\oplus_{\alpha\in\mathscr{A}} R_Q e_\alpha$. As a consequence, ψ_Q is an isomorphism. Since R is normal, both $\oplus_{\alpha\in\mathscr{A}} Re_\alpha$ and $R^{1/p}$ are torsion-free and (S_2). Then ψ is an isomorphism [28], Tag 0AV9. Hence, $R^{1/p}$ is a free R-module, and so, R is regular by Kunz's theorem [19]. ■

We now focus on F-pure varieties, that is, varieties whose local ring $\mathcal{O}_{X,x}$ is F-pure for every closed point $x \in X$. For this, we need to recall two criterions. One for D-simplicity, and another for strong F-regularity.

Proposition 3.11 ([4], Corollary 3.16). *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local $\mathbb{K}$-algebra with $\mathbb{K}$ as a coefficient field. Then, R is simple as a $D_{R|\mathbb{K}}$-module if and only if its differential core is zero.*

Lemma 3.12. *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local $\mathbb{K}$-algebra with $\mathbb{K}$ as a coefficient field. Then, the differential core of R is a prime ideal.*

Proof. Let $\mathfrak{m}^{[\![p^e]\!]} = \{f \in R \mid \phi(f) \in \mathfrak{m}, \forall\phi \in \operatorname{Hom}_{R^{p^e}}(R, R)\}$ and let $I_e = \{f \in R \mid \phi(f) \in \mathfrak{m}, \forall\phi \in \mathcal{C}_R^e\}$. Let $\mathfrak{q} = \bigcap_{e\in\mathbb{Z}^+} I_e$. We recall that $\mathfrak{q}$ is the splitting prime of R ([1], Theorem 1.1), and so, a prime ideal. We recall that I_e is sometimes defined using the injective hull of $R/\mathfrak{m}$. This definition is equivalent to the one presented in this proof ([1], Theorem 3.3) because $1 \mapsto f^{1/p^e}$ does not split over R if and only if $f \in I_e$ (see also Remark 4.4 in [23]). Since $\mathbb{K}$ is perfect, $D_{R|\mathbb{K}} = \bigcup_{e\in\mathbb{N}} \operatorname{Hom}_{R^{p^e}}(R, R)$ by Theorem 3.8. Then,

$$\mathfrak{m}^{\langle\mu(p^e-1)\rangle} \subseteq \mathfrak{m}^{[\![p^e]\!]} \subseteq \mathfrak{m}^{\langle p^e-1\rangle},$$

where $\mu = \dim_{\mathbb{K}} \mathfrak{m}/\mathfrak{m}^2$ (see Proposition 5.14 in [4]). Since R is F-pure, we have that $\mathfrak{m}^{[\![p^e]\!]} = I_e$ (see Proposition 5.10 in [4]). As a consequence,

$$\mathfrak{q} = \bigcap_{e\in\mathbb{Z}^+} I_e = \bigcap_{e\in\mathbb{Z}^+} \mathfrak{m}^{[\![p^e]\!]} = \bigcap_{n\in\mathbb{Z}^+} \mathfrak{m}^{\langle n\rangle} = \mathfrak{p}_{\text{diff}}(R).$$

Hence, the differential core of R is a prime ideal. ■

Theorem 3.13 ([25], Theorem 2.2). *Let $(R, \mathfrak{m}, \mathbb{K})$ be a local $\mathbb{K}$-algebra with $\mathbb{K}$ as a coefficient field. Let R be an F-pure F-finite ring. Then, R is $D_{R|\mathbb{K}}$-simple if and only if R is strongly F-regular.*

We now present another of our main results. Even though we are not able to show that $\operatorname{Nash}_n(X) \cong X$ implies smoothness, this condition implies strong F-regularity. In particular, in this case X is Cohen–Macaulay and normal.

Theorem 3.14. *Let X be an F-pure variety. If $\operatorname{Nash}_n(X) \cong X$ for some $n \geq 1$, then X is a strongly F-regular variety.*

Proof. Let x be a closed point in X, and $d = \dim(X)$. Let $R = \mathcal{O}_{X,x}$ and $\mathfrak{m}$ be its maximal ideal. By Lemma 2.4, we have that the module of principal parts $\mathcal{P}^n_{R|\mathbb{K}}$ has a free summand of rank $\binom{n+d}{d}$. Then, by Proposition 3.3, $\dim_{\mathbb{K}}(R/\mathfrak{m}^{\langle n\rangle}) = \binom{n+d}{d}$. Let $\mathfrak{p}$ denote the differential core of R. Since R is an F-pure ring, $\mathfrak{p}$ is a prime ideal by Lemma 3.12. Hence, $R/\mathfrak{p}$ is a domain.

Let $\overline{R} = R/\mathfrak{p}$ and $\overline{\mathfrak{m}} = \mathfrak{m}\overline{R}$. We note that $\mathfrak{p}$ is a $D_{R|\mathbb{K}}$-ideal ([4], Proposition 3.15), so it is stable under the action of any differential operator in $D_{R|\mathbb{K}}$. Then, we have a natural map of filtered rings $D_{R|\mathbb{K}} \to D_{\overline{R}|\mathbb{K}}$. Then, $\overline{\mathfrak{m}}^{\langle n\rangle} \subseteq \mathfrak{m}^{\langle n\rangle}\overline{R}$. Since $\mathfrak{p} = \bigcap_{t\in\mathbb{N}} \mathfrak{m}^{\langle t\rangle}$, we have that

$$\binom{n+d}{d} = \dim_{\mathbb{K}}(R/\mathfrak{m}^{\langle n\rangle}) = \dim_{\mathbb{K}}(R/\mathfrak{m}^{\langle n\rangle} + \mathfrak{p}) \leq \dim_{\mathbb{K}}(\overline{R}/\overline{\mathfrak{m}}^{\langle n\rangle}) \leq \binom{n+c}{c},$$

where $c = \dim R/\mathfrak{p} \leq d$ by Proposition 3.3. Then, $c = d$. We note that $\mathfrak{p}$ contains every minimal prime. We conclude that $\mathfrak{p} = 0$; otherwise, $\mathfrak{p}$ contains a parameter and $c < d$. Hence, R is strongly F-regular by Proposition 3.11 and Theorem 3.13. ∎

We now present an analogous to Nobile's theorem for F-pure rings. It is worth mentioning that F-pure rings might not be normal.

Corollary 3.15. *Let X be an F-pure variety. If $\operatorname{Nash}_1(X) \cong X$, then X is a non-singular variety.*

Proof. By Theorem 3.14. X is a strongly F-regular variety. Then, X is a normal variety. Hence, X is nonsingular by Theorem 3.10. ∎

4. Higher-order versions of Nobile's theorem

In this section we study a higher version of Nobile's theorem for quotient varieties. We also revisit a known result for hypersurfaces ([11], Theorem 4.13) concerning the analog of Nobile's theorem for higher Nash blow-ups in prime characteristic.

4.1. Quotient varieties

Let G be a linearly reductive algebraic group acting algebraically on $\operatorname{Spec}(R)$, where R is a polynomial rings over $\mathbb{K}$. The algebraic quotient $X /\!\!/ G$ is defined by identifying two points of X whenever their orbit closures have non-empty intersection. This is an affine algebraic variety whose coordinate ring is R^G. If all the orbits are closed, then $X /\!\!/ G$ is the usual orbit space and it is called a quotient variety. If $|G|$ has a multiplicative inverse in $\mathbb{K}$, this situation happens. In this subsection, we present a higher version of Nobile's theorem in this case.

Theorem 4.1. *Let G be a finite non-trivial group such that $|G|$ has a multiplicative inverse in $\mathbb{K}$. Let G act linearly on a polynomial ring $R = \mathbb{K}[x_1, \ldots, x_d]$. Suppose that $G \setminus \{e\}$ contains no elements that fix a hyperplane in the space of one-forms $[R]_1$. Let $X = \operatorname{Spec}(R^G)$. Then,* $\operatorname{Nash}_n(X) \ncong X$.

Proof. The ramification locus of a finite group action corresponds to the union of fixed spaces of elements of G. Consequently, the assumption that no element fixes a hyperplane ensures that the extension is unramified in codimension one. The inclusion is order-differentially extensible ([4], Proposition 6.4). Let $\mathfrak{m}$ be the maximal homogeneous ideal of R, and $\eta = \mathfrak{m} \cap R^G$. We note that $\dim R^G_{\mathfrak{m}} = \dim R^G = d$. Under these conditions, we have that $\eta^{\langle n \rangle} = \mathfrak{m}^n \cap R^G$ ([4], Proposition 6.14). Then,

$$\dim_{\mathbb{K}}(\eta_\eta^{\langle j-1 \rangle}/\eta_\eta^{\langle j \rangle}) \leq \dim_{\mathbb{K}}(\mathfrak{m}_{\mathfrak{m}}^{j-1}/\mathfrak{m}_{\mathfrak{m}}^{j}).$$

By our assumptions on G, we have that $R \neq R^G$. Then,

$$\dim_{\mathbb{K}}(\eta_\eta^{\langle 1 \rangle}/\eta_\eta^{\langle 2 \rangle}) < \dim_{\mathbb{K}}(\mathfrak{m}_{\mathfrak{m}}/\mathfrak{m}_{\mathfrak{m}}^{2}) = d.$$

We conclude that

$$\dim_{\mathbb{K}}(R/\eta_\eta^{\langle n \rangle}) = \sum_{j=1}^{n} \dim_{\mathbb{K}}(\eta_\eta^{\langle j-1 \rangle}/\eta_\eta^{\langle j \rangle}) < \sum_{j=1}^{n} \dim_{\mathbb{K}}(\mathfrak{m}_{\mathfrak{m}}^{j-1}/\mathfrak{m}_{\mathfrak{m}}^{j}) = \binom{n+d}{d}.$$

Then, the free rank of $\mathcal{P}_{R^G_\eta|\mathbb{K}}$ is strictly smaller than $\binom{n+d}{d}$ by Proposition 3.3. As a consequence, $\operatorname{Nash}_n(X) \ncong X$ by Lemma 2.4. ∎

4.2. Hypersurfaces

Now we study the case of hypersurfaces. We note that the proof we present is characteristic free.

Theorem 4.2. *Let X be a normal hypersurface. If* $\operatorname{Nash}_n(X) \cong X$, *then X is a non-singular variety.*

Proof. Let $x \in X$ and $R = \mathcal{O}_{X,x}$. By Lemma 2.4, $\mathcal{P}_R^n \cong R^{\binom{n+d}{d}} \oplus T$, where T is the torsion submodule. On the other hand, R normal implies that $\mathcal{P}_R^n$ is torsion-free [3], Theorem 4.3. Therefore $\mathcal{P}_R^n \cong R^{\binom{n+d}{d}}$. Then R is a regular ring (Theorem 3.1 in [3]; see also Theorem 10.2 in [4] for a more general statement). We conclude that X is non-singular. ∎

Acknowledgements. We thank Josep Àlvarez Montaner and Jack Jeffries for helpful comments. We thank the anonymous referee for suggestions that improved this manuscript.

Funding. The first-named author was partially supported by CONACyT Grant 287622. The second-named author was partially supported by CONACyT Grant 284598 and Cátedras Marcos Moshinsky.

References

[1] Aberbach, I. M. and Enescu, F.: The structure of F-pure rings. *Math. Z.* **250** (2005), no. 4, 791–806.

[2] Atanasov, A., Lopez, C., Perry, A., Proudfoot, N. and Thaddeus, M.: Resolving toric varieties with Nash blowups. *Exp. Math.* **20** (2001), no. 3, 288–303.

[3] Barajas, P. and Duarte, D.: On the module of differentials of order n of hypersurfaces. *J. Pure Appl. Algebra* **224** (2020), no. 2, 536–550.

[4] Brenner, H., Jeffries, J. and Núñez Betancourt, L.: Quantifying singularities with differential operators. *Adv. Math.* **358** (2019), 106843, 89 pp.

[5] Chávez-Martínez, E., Duarte, D. and Giles Flores, A.: A higher-order tangent map and a conjecture on the higher nash blowup of curves. *Math. Z.* **297** (2021), no. 3-4, 1767–1791.

[6] Dao, H., De Stefani, A., Grifo, E. Huneke, C. and Núñez Betancourt, L.: Symbolic powers of ideals. In *Singularities and foliations. geometry, topology and applications*, 387–432. Springer Proc. Math. Stat. 222, Springer, Cham, 2018.

[7] De Alba, H. and Duarte, D.: On the k-torsion of the module of differentials of order n of hypersurfaces. *J. Pure Appl. Algebra* **225** (2021), no. 8, 106646, 7 pp.

[8] De Fernex, T. and Tu, Y.-C.: Towards a link theoretic characterization of smoothness. *Math. Res. Lett.* **24** (2017), no. 5, 1239–1262.

[9] Duarte, D.: Higher Nash blowup on normal toric varieties. *J. Algebra* **418** (2014), 110–128.

[10] Duarte, D.: Nash modification on toric surfaces. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **108** (2014), no. 1, 153–171.

[11] Duarte, D.: Computational aspects of the higher Nash blowup of hypersurfaces. *J. Algebra* **477** (2017), 211–230.

[12] González Pérez, P. D. and Teissier, B.: Toric geometry and the Semple–Nash modification. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM* **108** (2014), no. 1, 1–48.

[13] González-Sprinberg, G.: Transformé de Nash et éventail de dimension 2. *C. R. Acad. Sci. Paris Sér. A-B* **284** (1977), no. 1, A69–A71.

[14] González-Sprinberg, G.: Résolution de Nash des points doubles rationnels. *Ann. Inst. Fourier (Grenoble)* **32** (1982), no. 2, x, 111–178.

[15] Grigoriev, D. and Milman, P. D.: Nash resolution for binomial varieties as Euclidean division. A priori termination bound, polynomial complexity in essential dimension 2. *Adv. Math.* **231** (2012), no. 6, 3389–3428.

[16] Grothendieck, A.: *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV*. Inst. Hautes Études Sci. Publ. Math. 32, 1967.

[17] Grothendieck, A. and Dieudonné, J. A.: *Eléments de géométrie algébrique. I*. Grundlehren der Mathematischen Wissenschaften 166, Springer-Verlag, Berlin, 1971.

[18] Hironaka, H.: On Nash blowing-up. In *Arithmetic and geometry, Vol. II*, 103–111. Progr. Math. 36, Birkhäuser, Boston, MA, 1983.

[19] Kunz, E.: Characterizations of regular local rings of characteristic p. *Amer. J. Math.* **91** (1969), 772–784.

[20] Nobile, A.: Some properties of the Nash blowing-up. *Pacific J. Math.* **60** (1975), no. 1, 297–305.

[21] Oneto, A. and Zatini, E.: Remarks on Nash blowing-up. (Commutative algebra and algebraic geometry, II, Turin, 1990). (Italian). *Rend. Sem. Mat. Univ. Politec. Torino* **49** (1991), no. 1, 71–82. Libreria Editrice Universitaria Levrotto & Bella, Turin, 1993.

[22] Rebassoo, V.: *Desingularization properties of the Nash blowing-up process*. PhD Thesis, University of Washington, 1977.

[23] Schwede, K.: Centers of F-purity. *Math. Z.* **265** (2010), no. 3, 687–714.

[24] Semple, J. G.: Some investigations in the geometry of curve and surface elements. *Proc. London Math. Soc. (3)* **4** (1954), 24–49.

[25] Smith, K. E.:. The D-module structure of F-split rings. *Math. Res. Lett.* **2** (1995), no. 4, 377–386.

[26] Spivakovsky, M.: Sandwiched singularities and desingularization of surfaces by normalized Nash transformations. *Ann. of Math. (2)* **131** (1990), no. 3, 411–491.

[27] Teissier, B.: The hunting of invariants in the geometry of discriminants. In *Real and complex singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, 565–678. Sijthoff and Noordhoff, Alphen aan den Rijn, 1977.

[28] The Stacks project. The Stacks project, Tag 0AV9, 2015.

[29] Toh-Yama, R.: Higher Nash blowups of the A_3-singularity. *Comm. Algebra* **47** (2019), no. 11, 4541–4564.

[30] Yasuda, T.: Higher Nash blowups. *Compos. Math.* **143** (2007), no. 6, 1493–1510.

[31] Yekutieli, A.: *An explicit construction of the Grothendieck residue complex*. (With an appendix by Pramathanath Sastry.) Astérisque 208, 1992.

Received March 28, 2020; revised July 7, 2020. Published online June 22, 2021.

Daniel Duarte
CONACyT-Universidad Autónoma de Zacatecas, Paseo a la Bufa entronque Av. Solidaridad s/n, 98060, Zacatecas, México;
aduarte@uaz.edu.mx

Luis Núñez-Betancourt
Centro de Investigación en Matemáticas, Jalisco s/n, Col. Valenciana, 36023, Guanajuato, Gto., México;
luisnub@cimat.mx