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Carleson measure estimates and ε-approximation for bounded harmonic functions, without Ahlfors regularity assumptions

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Abstract. Let Ω be a domain in $\mathbb{R}^{d+1}$, where $d \geq 1$. It is known that if Ω satisfies a corkscrew condition and $\partial\Omega$ is d-Ahlfors regular, then the following are equivalent:

(a) a square function Carleson measure estimate holds for bounded harmonic functions on Ω;

(b) an ε-approximation property holds for all such functions and all $0 < \varepsilon < 1$;

(c) $\partial\Omega$ is uniformly rectifiable.

Here we explore (a) and (b) when $\partial\Omega$ is not required to be Ahlfors regular. We first observe that (a) and (b) hold for any domain Ω for which there exists a domain $\widetilde{\Omega} \subset \Omega$ such that $\partial\widetilde{\Omega}$ is uniformly rectifiable and $\partial\Omega \subset \partial\widetilde{\Omega}$. We then assume Ω satisfies a corkscrew condition and $\partial\Omega$ satisfies a capacity density condition. Under these assumptions, we prove conversely that if (a) or (b) holds for Ω then such a domain $\widetilde{\Omega} \supset \Omega$ exists. And we give two further characterizations of domains where (a) or (b) holds. The first is that harmonic measure for Ω satisfies a Carleson packing condition with respect to diameters similar to a condition comparing harmonic measures to $\mathcal{H}^d$ already known to be equivalent to uniform rectifiability. The second characterization is reminiscent of the Carleson measure description of H^∞ interpolating sequences in the unit disc.

1. Introduction

Let $\Omega \subset \mathbb{R}^{d+1}$ be an open set. For simplicity we always assume Ω is a domain, i.e., connected, although the interested reader can easily extend all our results to the case of disconnected open sets. We say bounded harmonic functions on Ω satisfy a *Carleson measure estimate* if there is a constant $C > 0$ such that

$$\frac{1}{r^d} \int_{B(x,r)\cap\Omega} |\nabla u(y)|^2 \operatorname{dist}(y, \partial\Omega)\, dy \leq C \|u\|^2_{L^\infty(\Omega)} \tag{1.1}$$

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whenever $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\Omega)$, and u is a bounded harmonic function on Ω. It is a famous result of C. Fefferman [14] that (1.1) holds for the upper half-space $\mathbb{R}^{d+1}_+$, where it characterizes Poisson integrals of BMO functions.

If u is a bounded harmonic function on Ω and if $0 < \varepsilon < 1$, we say that u is *ε-approximable* if there exist $g \in W^{1,1}_{\mathrm{loc}}(\Omega)$ and $C > 0$ such that

$$\|u - g\|_{L^\infty(\Omega)} < \varepsilon \tag{1.2}$$

and, for all $x \in \partial\Omega$ and all $r > 0$,

$$\frac{1}{r^d} \int_{B(x,r)\cap\Omega} |\nabla g(y)|\, dy \le C. \tag{1.3}$$

It is clear by normal families that (1.2) and (1.3) then hold for every bounded harmonic function on Ω with constant $C = C_\varepsilon$ depending only on ε and Ω. It is also clear that after local mollifications, (1.2) and (1.3) will hold with $g \in C^\infty(\Omega)$; see [15], page 347, or the argument concluding Section 2 below. The notion of ε-approximation was introduced by Varopoulos in [34] and [35] in his work on corona problems and H^1-BMO duality. Chapter VIII of [15] gave a proof for all $\varepsilon > 0$ on the upper half plane, and Dahlberg [8] extended the proof to Lipschitz domains using his work connecting square functions to maximal functions. Later, Kenig, Koch, Pipher and Toro [30] applied ε-approximation to more general elliptic boundary value problems and proved that on any Lipschitz domain elliptic harmonic measure is A_∞ equivalent to boundary surface measure. Further connections between ε-approximation, Carleson measure estimates, square functions, maximal functions, and A_∞ conditions for elliptic measures have been obtained on Lipschitz domains by several authors, including [13,19,28,29,33], and then on domains with Ahlfors regular boundaries by [3,20–23], and most recently by [2,4,5,18,24,25].

The papers [23] and [17] connect ε-approximation and Carleson measures to rectifiability in domains with Ahlfors regular boundaries. To explain them we give three definitions. The open set $\Omega \subset \mathbb{R}^n$ satisfies a *corkscrew condition* if there exists a constant $\alpha \in (0, 1/2)$ such that whenever $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\Omega)$, there exists a ball $B(p, \alpha r)$ so that

$$B(p, \alpha r) \subset \Omega \cap B(x, r). \tag{1.4}$$

If Ω is a connected open set with the corkscrew condition, we say Ω is a *corkscrew domain*. For $n > d \ge 1$, a set $E \subset \mathbb{R}^n$ is called *d-Ahlfors regular* (or simply Ahlfors regular if d is clear from the context) if there exists a constant $c > 0$ such that for all $x \in E$ and $0 < r < \operatorname{diam}(E)$,

$$c^{-1} r^d \le \mathcal{H}^d(B(x, r) \cap E) \le c r^d \tag{1.5}$$

where $\mathcal{H}^d$ denotes the d-dimensional Hausdorff measure. When $1 \le d < n$ is an integer, the set $E \subset \mathbb{R}^n$ is *uniformly d-rectifiable* if it is d-Ahlfors regular and there exist constants c and $M > 0$ such that for all $x \in E$ and all $0 < r \le \operatorname{diam}(E)$ there is a Lipschitz mapping g from the ball $B(0, r) \subset \mathbb{R}^d$ to $\mathbb{R}^n$ such that $\operatorname{Lip}(g) \le M$ and

$$\mathcal{H}^d(E \cap B(x, r) \cap g(B_d(0, r))) \ge c r^d. \tag{1.6}$$

Uniform rectifiability is a quantitative version of rectifiability. It was introduced in the pioneering works [11] and [12] of David and Semmes, who proved that for any $\Omega \subset \mathbb{R}^n$ the $(n-1)$-uniform rectifiability of $\partial\Omega$ is a geometric condition under which all singular integrals with sufficiently smooth odd kernels are bounded in $L^2(\partial\Omega)$. Later [31] and [32] proved conversely that the L^2 boundedness of the Cauchy integral or the Riesz transforms on an Ahlfors regular boundary $\partial\Omega$ implies $\partial\Omega$ is $(n-1)$ uniformly rectifiable. The papers [23] and [17] prove that if $\Omega \subset \mathbb{R}^{d+1}$, $d \geq 1$, is a corkscrew domain and $\partial\Omega$ is d-Ahlfors regular, then the following are equivalent:

(a) All bounded harmonic functions on Ω satisfy the Carleson measure estimate (1.1).

(b) Every bounded harmonic function on Ω is ε-approximable for all $0 < \varepsilon < 1$.

(c) $\partial\Omega$ is uniformly d-rectifiable.

In fact, [23] proved (c) implies (a) and (b) and [17] proved the converse statements.

Here our goal is to understand the conditions (a) and (b) when $\partial\Omega$ is not necessarily Ahlfors regular. To state our results we need two more definitions. We will usually assume Ω satisfies a *capacity density condition*: there is $\beta > 0$ such that for all $x \in \partial\Omega$ and $r \leq \operatorname{diam}(\Omega)$,

$$\operatorname{Cap}(B(x,r) \setminus \Omega) \geq \begin{cases} \beta r & \text{if } d+1 = 2, \\ \beta r^{d-1} & \text{if } d+1 \geq 3, \end{cases} \tag{1.7}$$

where Cap is the Newtonian capacity when $d+1 \geq 3$ and the logarithmic capacity when $d+1 = 2$. If Ω satisfies (1.7), every point of $\partial\Omega$ is regular for the Dirichlet problem, so that for each $p \in \Omega$ there exists a unique Borel probability $\omega_p = \omega(p, .\,, \Omega)$ on $\partial\Omega$ such that

$$u(p) = \int_{\partial\Omega} u(x)\, d\omega(p, x, \Omega) \tag{1.8}$$

if u is continuous on $\overline{\Omega}$ and harmonic on Ω. Moreover, if $u(x)$ is continuous on $\partial\Omega$, (1.8) defines a function harmonic on Ω which continuously extends u from $\partial\Omega$ to $\overline{\Omega}$. Since Ω is connected, it follows from Harnack's inequality that for all $p, q \in \Omega$ there is a constant $C_{p,q} = C_{p,q}(\Omega)$ such that $\omega_p \leq C_{p,q}\,\omega_q$. The measure ω_p is called the *harmonic measure for* p.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^{d+1}$, $d \geq 1$, be a domain.*

A) *If there exists a domain $\widetilde{\Omega}$ such that*

$$\widetilde{\Omega} \subset \Omega \quad \textit{and} \quad \partial\Omega \subset \partial\widetilde{\Omega}, \tag{1.9}$$

and $\partial\widetilde{\Omega}$ is uniformly rectifiable, then (a) *and* (b) *hold for Ω.*

B) *Conversely, if Ω satisfies* (1.4), (1.7) *and either* (a) *or* (b), *then there exists a domain $\widetilde{\Omega}$, with $\partial\widetilde{\Omega}$ uniformly rectifiable, such that* (1.9) *holds.*

The proof of Part A of Theorem 1.1 is an easy application via Whitney cubes of the theorem of [23] and does not require (1.4) or (1.7) to hold on Ω. It will be given in Section 2. The proof of the converse Part B involves a variation on a corona decomposition in [17]. It occupies most of this paper.

Theorem 1.2. *If* Ω *is a domain satisfying* (1.4) *and* (1.7)*, there is* $\varepsilon_0 > 0$*, depending only on the constants in* (1.4) *and* (1.7)*, such that*:

A) *If* (a) *or* (b) *holds for* Ω*, then for every* $0 < \varepsilon < \varepsilon_0$ *there is* $C(\varepsilon)$ *such that if* $p_j \in \Omega \cap B(x, R)$, $x \in \partial\Omega$, *and* $E_j \subset \partial\Omega$ *satisfy*

$$\omega(p_j, E_j, \Omega) \geq 1 - \varepsilon, \quad \text{and} \tag{1.10}$$

$$E_j \cap E_k = \emptyset \quad \text{if } k \neq j, \tag{1.11}$$

then

$$\sum \operatorname{dist}(p_j, \partial\Omega)^d \leq C(\varepsilon) R^d. \tag{1.12}$$

B) *Conversely, if for some* $0 < \varepsilon < \varepsilon_0$, (1.10) *and* (1.11) *imply* (1.12) *whenever such* $\{p_j\}$ *and* $\{E_j\}$ *exist, then* (a) *and* (b) *hold for* Ω.

The proof of Part A of Theorem 1.2 is in Section 4. It uses a construction from the beginning of [17] and some elementary properties of harmonic measure. The proof of the converse Part B is deeper. It runs parallel to the proof of Part B of Theorem 1.1.

To illustrate Theorem 1.1 and Theorem 1.2, we consider Cantor sets. Let $0 < \lambda < 1/2$ and in $\mathbb{R}^2$ set $K_\lambda = \bigcap_{n \geq 0} K_{\lambda,n}$, where $K_{\lambda,0} = [0,1] \times]0,1]$, $K_{\lambda,n+1} \subset K_{\lambda,n}$, and $K_{\lambda,n+1}$ is the union of 4^{n+1} pairwise disjoint closed squares of side λ^{n+1}, each containing one corner of a component square of $K_{\lambda,n}$. Then (1.4) and (1.7) hold for $\Omega_\lambda = \mathbb{R}^2 \setminus K_\lambda$. Theorem 1.1 implies (a) or (b) holds for Ω_λ if and only if $\lambda < 1/4$, but this can be seen without the harder proof of Theorem 1.1. If $\lambda \geq 1/4$, $\mathscr{H}^1$ and harmonic measure for $\mathbb{C} \setminus K_\lambda$ are mutually singular ([6], [16]) and then the easier half of the proof of Theorem 1.2 in Section 4 shows (a) and (b) fail. The case $\lambda < 1/4$ is easier yet because then, if u is harmonic on Ω_λ,

$$\int_{B(x,R)\setminus K_\lambda} |\nabla u| \, dy \leq \|u\|_{L^\infty(\Omega)} \int_{B(x,R)\setminus K_\lambda} \frac{dy}{\operatorname{dist}(y, K_\lambda)} \leq CR \, \|u\|_{L^\infty(\Omega)}.$$

When $\lambda < 1/4$, the domain $\widetilde{\Omega}_\lambda$ can be obtained by removing from Ω_λ a continuum of diameter $c\lambda^n$ near the center of each $K_{\lambda,n}$, and the converse proof of Theorem 1.1 amounts to constructing similar continua in the general case. There it is helpful to recall that for $\lambda < 1/4$ the harmonic measures for Ω_λ and $\widetilde{\Omega}_\lambda$ are mutually singular.

The Part B converses of Theorem 1.1 and Theorem 1.2 are both corollaries of Theorem 1.4, which asserts that under (1.4) and (1.7), (a) and (b) are both equivalent to the existence of a particular corona decomposition on $\partial\Omega$ made by comparing harmonic measures to diameters. To state Theorem 1.4 we must first explain its setting, which will be discussed more fully in Section 6. The corona decomposition in Theorem 1.4 is similar to the decomposition in [17], which in the Ahlfors regular case is proved in Proposition 3.1 and Proposition 5.1 of [17] to be equivalent to the uniform rectifiability of $\partial\Omega$ and thus also equivalent to (a) or (b). However, the decomposition in [17] used a family of subsets of $\partial\Omega$, often called Christ–David cubes, which were originally defined only when $\partial\Omega$ is Ahlfors regular. To make our decomposition satisfy its needed "small boundary condition" (1.18), we first define in Proposition 1.3 a new family of "cubes" in $\partial\Omega$. These new cubes are built by repeating the original construction of David [9] assuming Ω satisfies the condition of Theorem 1.2 but not assuming $\partial\Omega$ is Ahlfors regular, and the main difference between the corona decomposition in Theorem 1.4 and that in [17] is this definition of cubes. We note that [27] and [26] have made similar cube constructions in the general case of doubling metric spaces.

Proposition 1.3. *Assume Ω is a bounded corkscrew domain satisfying (1.7) and the conclusion of Theorem 1.2 that (1.10) and (1.11) imply (1.12). Then there exist a positive integer N and a family*

$$\mathcal{S} = \bigcup_{j\geq 0} \mathcal{S}_j$$

of Borel subsets of $\partial\Omega$ which has properties (1.13), (1.14), (1.15), (1.16), (1.17) and the "small boundary property" (1.18):

$$\operatorname{diam} S \sim 2^{-Nj} \quad \text{if } S \in \mathcal{S}_j; \tag{1.13}$$

$$\partial\Omega = \textstyle\bigcup_{\mathcal{S}_j} S \quad \text{for all } j; \tag{1.14}$$

$$S \cap S' = \emptyset \quad \text{if } S, S' \in \mathcal{S}_j \text{ and } S' \neq S; \tag{1.15}$$

$$\text{if for } j < k,\ S_j \in \mathcal{S}_j \text{ and } S_k \in \mathcal{S}_k, \quad \text{then } S_k \subset S_j \text{ or } S_k \cap S_j = \emptyset. \tag{1.16}$$

There exists a constant $c_0 > 0$ such that for all $S \in \mathcal{S}$ there exists $x_S \in S$ with

$$B(x_S, c_0\ell(S)) \cap \partial\Omega \subset S. \tag{1.17}$$

For $0 < \tau < 1$ and $S_j \in \mathcal{S}_j$, define

$$\Delta_\tau(S_j) = \big\{y \in S_j : \operatorname{dist}(y, \partial\Omega \setminus S_j) < \tau 2^{-Nj}\big\} \cup \big\{y \in \partial\Omega \setminus S_j : \operatorname{dist}(y, S_j) < \tau 2^{-Nj}\big\},$$

let

$$\mathcal{G}(\tau 2^{-Nj}) = \Big\{K = \bigcap_{1\leq i\leq d+1} \{k_i \tau 2^{-Nj} \leq x_i \leq (k_i + 1)\tau 2^{-Nj}\}, k_i \in \mathbb{Z}\Big\}$$

denote the set of closed dyadic cubes in $\mathbb{R}^{d+1}$ of side 2^{-Nj}, scaled down by τ, and define

$$N_\tau(S_j) = \#\big\{K \in \mathcal{G}(\tau 2^{-Nj}) : K \cap \Delta_\tau(S_j) \neq \emptyset\big\}.$$

Then there exists a constant C_{sb} so that

$$N_\tau(S_j) \leq C_{sb}\, \tau^{(1/C_{sb})-d} \tag{1.18}$$

for all τ and all $S_j \in \mathcal{S}_j$.

Assuming Proposition 1.3, we make the following construction: by (1.17), (1.13), and (1.4), to each $S \in \mathcal{S}$ there corresponds a "corkscrew ball" $B(p, \alpha c_0 \ell(S)) \subset \Omega$ with $\operatorname{dist}(p, S) \leq c_0\ell(S)$. Moreover, by (1.7) and Lemmas 3.1 and 3.2 from Section 3 below, for any $0 < \varepsilon < 1/2$ there exist constants

$$2^{-N-1}c_0 < c_3 < 4c_3 < c_0 \tag{1.19}$$

depending on ε, the constants in (1.4) and (1.7) and the constants c_1, c_2 from Section 3 but not on N, such that for every $S \in \mathcal{S}$ there exists a ball $B_S = B(p_S, c_3\ell(S))$ satisfying

$$B_S = B(p_S, c_3\ell(S)) \subset 4B_S = B(p_S, 4c_3\ell(S)) \subset \Omega \cap B(x_S, \tfrac{c_0}{2}\ell(S)) \tag{1.20}$$

and

$$\inf_{p \in 2B_S} \big\{\omega\big(p, S \cap B(x_S, c_0\ell(S)), \Omega \cap B(x_S, c_0\ell(S))\big)\big\} \geq 1 - \varepsilon. \tag{1.21}$$

We can also take N so large that if $S \cap S' = \emptyset$,

$$B_S \cap B_{S'} = \emptyset, \tag{1.22}$$

and if $\ell(S') > \ell(S)$,

$$2B_{S'} \cap B(x_S, c_0\ell(S)) = \emptyset. \tag{1.23}$$

If $\ell(S) \neq \ell(S')$ and N is sufficiently large, (1.22) follows from (1.13), (1.16) and (1.19). If $\ell(S) = \ell(S')$, (1.15) and (3.4) imply (1.22) since $\varepsilon < 1/2$. If $\ell(S') > \ell(S)$, then (1.23) holds by (1.19).

For $S \in \mathcal{S}$ and $\lambda > 1$, define $\lambda S = \{x : \text{dist}(x, S) \leq (\lambda - 1)\ell(S)\}$. Let $0 < \delta \lesssim 1$ and $A \gtrsim 1$ be fixed constants. For $S_0 \in \mathcal{S}$ and $S \in \mathcal{S}$ with $S \subset S_0$, we say $S \in \mathsf{HD}(S_0)$ (for "high density") if S is a maximal cube for which

$$\inf_{p \in B_{S_0}} \omega(p, 2S) \geq A\Big(\frac{\ell(S)}{\ell(S_0)}\Big)^d, \tag{1.24}$$

and we say $S \in \mathsf{LD}(S_0)$ (for "low density") if S is maximal for

$$\sup_{p \in B_{S_0}} \omega(p, S) \leq \delta\Big(\frac{\ell(S)}{\ell(S_0)}\Big)^d. \tag{1.25}$$

By (3.2) and Harnack's inequality,

$$\sup_{p \in B_{S_0}} \omega(p, S) \leq c_5 \inf_{q \in B_{S_0}} \omega(q, 2S) \tag{1.26}$$

for some constant c_5, and we can assume $A > c_5\delta$ so that $\mathsf{HD}(S_0) \cap \mathsf{LD}(S_0) = \emptyset$.

For each $S_0 \in \mathcal{S}$, let

$$G_1(S_0) = \big\{S \in \mathsf{LD}(S_0) \cup \mathsf{HD}(S_0) : S \text{ is maximal}\big\}. \tag{1.27}$$

We call $G_1(S_0)$ the *first generation of descendants* of S_0, and we define later generations inductively:

$$G_k(S_0) = \bigcup_{S \in G_{k-1}(S_0)} G_1(S). \tag{1.28}$$

Proposition 1.3 will be proved in Section 5 after Part A of Theorem 1.2 has been proved in Section 4. Therefore it is not inconsistent to assume the conclusions of Proposition 1.3 when assuming (a) or (b) in Part A of Theorem 1.4.

Theorem 1.4. *If Ω is a domain satisfying* (1.4) *and* (1.7)*, there is $\varepsilon_1 > 0$, depending only on the constants in* (1.4) *and* (1.7)*, such that*:

A) *Assume* (a) *or* (b) *holds for Ω and let $\mathcal{S}$ be a family of subsets of $\partial\Omega$ satisfying Proposition* 1.3. *Then there exists $A_0 > 0$ such that whenever $0<\varepsilon<\varepsilon_1$, $0<\delta<\varepsilon/3$, and $A>\max(A_0, c_5\delta)$, there exists a constant $C=C(\varepsilon,\delta,d,A)$ such that for any $S_0\in\mathcal{S}$,*

$$\sum_{k=1}^{\infty}\sum_{G_k(S_0)} \ell(S)^d \le C\,\ell(S_0)^d. \tag{1.29}$$

B) *Conversely, assume there exists a family $\mathcal{S}$ of subsets of $\partial\Omega$ satisfying Proposition* 1.3 *and* (1.20)–(1.22)*, assume* (1.24)–(1.27) *hold for some ε, δ and A with $0<\varepsilon<\varepsilon_1$, $0<\delta<\varepsilon/3$, and $A > c_5\delta$, and further assume*

(i) *$\mathcal{S}$ satisfies* (1.29)*, and*

(ii) *there exists $C>0$ such that if B is a ball, $\{S_j\}\subset\mathcal{S}$, $\bigcup S_j \subset B$ and $S_j\cap S_k = \emptyset$ for $j\neq k$, then $\sum \ell(S_j)^d \le C\,\mathrm{diam}(B)^d$.*

Then (a) *and* (b) *hold for Ω.*

Part A of Theorem 1.1 is proved in Section 2, without assuming (1.4) or (1.7). In Section 3 we give three lemmas from [1] and [17] which lead to the proof in Section 4 of Part A of Theorem 1.2. The proofs of Theorem 1.4, Theorem 1.1 Part B, and Theorem 1.2 Part B are convoluted. In Section 5 the conclusion of Theorem 1.2 Part A is used to define the cube family $\mathcal{S}$ and prove Proposition 1.3. In Section 6 properties of $\mathcal{S}$ and the construction from [17] yield a proof of Part A of Theorem 1.4 (and thereby extend Proposition 3.1 of [17] to domains satisfying (1.4) and (1.7)). Then, in Section 7, Part A of Theorem 1.4 and an iterated balayage argument are used to construct a subdomain $\widetilde{\Omega}\subset\Omega$ such that $\partial\Omega\subset\partial\widetilde{\Omega}$ and $\partial\widetilde{\Omega}$ is Ahlfors regular, and in Section 8 a similar balayage argument shows the crucial generation sum (1.29) for Ω controls the corresponding sum for $\widetilde{\Omega}$. Proposition 5.1 of [17] and Lemma 6.2 then imply $\partial\widetilde{\Omega}$ is uniformly rectifiable, and therefore prove Part B of Theorem 1.1. Finally, the proof of Theorem 1.4 Part B follows from Theorem 1.1 Part A and the proof of Theorem 1.1 Part B, and a word-for-word repeat of that argument yields the proof of Theorem 1.2 Part B. An outline of the logic is:

$$\begin{gathered}
\widetilde{\Omega}\ \text{exists}\\
\Downarrow\\
\text{(a) and (b)}\\
\Downarrow\\
\text{(a) or (b)}\\
\Downarrow\\
\text{Theorem 1.2 Part A}\\
\Downarrow\\
\text{Proposition 1.3 and Theorem 1.4 Part A}\\
\Downarrow\\
\widetilde{\Omega}\ \text{exists.}
\end{gathered}$$

A reading of the proofs will show that ε-approximation of all harmonic functions with $\sup_\Omega |u| \le 1$ for some fixed small ε is equivalent to the other conclusions of all three theorems.

The argument in this paper entails many constants. Constants C or C_j are large and may vary from use to use, but the constants $c_0, c_1, \dots$ are small and sometimes interdependent. They are written so that c_j can depend on c_k only if $k < j$.

2. Proof of Theorem 1.1 Part A

We recall the Whitney decomposition of Ω into cubes $\Omega = \bigcup_{\mathcal{W}} Q$. Each $Q \in \mathcal{W} = \mathcal{W}(\Omega)$ is a closed dyadic cube,

$$Q = \bigcap_{1 \le j \le d+1} \{k_j 2^{-n} \le x_j \le (k_j + 1) 2^{-n}\}, \tag{2.1}$$

with n and k_j integers. If $Q_1, Q_2 \in \mathcal{W}$, then

$$Q_1 \subset Q_2, \quad Q_2 \subset Q_1, \quad \text{or} \quad Q_1^\circ \cap Q_2^\circ = \emptyset, \tag{2.2}$$

where Q° denotes the interior of Q. There are constants $1 < c_6 < c_7 < 3$ such that for all $Q \in \mathcal{W}$,

$$c_6 Q \cap \partial\Omega = \emptyset \quad \text{but} \quad c_7 Q \cap \partial\Omega \neq \emptyset, \tag{2.3}$$

where $\ell(Q)$ is the sidelength of Q and cQ is the concentric closed cube having sidelength $c\ell(Q)$.

Assume Ω and $\widetilde{\Omega}$ satisfy condition (1.9) from Theorem 1.1, let u be an harmonic function on Ω with $\sup_\Omega |u| \le 1$, and let $Q \in \mathcal{W}(\Omega)$. We fix a constant $1 < c_8 < c_6$ and consider two cases.

Case I: $c_8 Q \cap \partial\widetilde{\Omega} = \emptyset$.

In this case there is $C_1 = C_1(d, c_7, c_8)$ such that $\operatorname{dist}(y, \partial\Omega) \le C_1 \operatorname{dist}(y, \partial\widetilde{\Omega})$ for all $y \in Q$, so that

$$\int_Q |\nabla u(y)|^2 \operatorname{dist}(y, \partial\Omega)\, dy \le C_1 \int_Q |\nabla u(y)|^2 \operatorname{dist}(y, \partial\widetilde{\Omega})\, dy. \tag{2.4}$$

Case II: $c_8 Q \cap \partial\widetilde{\Omega} \neq \emptyset$.

In this case Harnack's inequality gives $\sup_Q |\nabla u(y)| \le C_2/\ell(Q)$, for $C_2 = C_2(d, c_7)$, so that

$$\int_Q |\nabla u(y)|^2 \operatorname{dist}(y, \partial\Omega)\, dy \le C_2^2 (1 + c_8)^{(d+1)/2} \ell(Q)^d = C_3\, \ell(Q)^d. \tag{2.5}$$

Now consider a ball $B = B(x, r)$, with $x \in \partial\Omega, r < \operatorname{diam}\Omega$, and let

$$\mathcal{W}_B = \{Q \in \mathcal{W}(\Omega) : Q \cap B \neq \emptyset\},$$

and for $J = \mathrm{I}$ or II, let $\mathcal{W}_{B,J}$ be the set of Case J cubes in $\mathcal{W}_B$. Also note that by (2.3),

$$\bigcup_{\mathcal{W}_B} c_6 Q \subset B(x, C_4 r) \tag{2.6}$$

for a constant C_4 depending only c_6 and c_7. Then we have

$$\int_B |\nabla u(y)|^2 \operatorname{dist}(y, \partial\Omega)\, dy \leq \sum_{\mathcal{W}_B} \int_Q |\nabla u(y)|^2 \operatorname{dist}(y, \partial\Omega)\, dy = \sum_{\mathcal{W}_{B,\mathrm{I}}} + \sum_{\mathcal{W}_{B,\mathrm{II}}}.$$

To estimate $\sum_{\mathcal{W}_{B,\mathrm{I}}}$ we use (2.4), (2.6), the uniform rectifiability of $\partial\widetilde{\Omega}$, and the theorem of [23] to get

$$\sum_{\mathcal{W}_{B,\mathrm{I}}} \leq C_1 \int_{B(x,C_4 r)} |\nabla u(y)|^2 \operatorname{dist}(y, \partial\widetilde{\Omega})\, dy \leq C(C_4 r)^d. \tag{2.7}$$

For estimating $\sum_{\mathcal{W}_{B,\mathrm{II}}}$, the only available inequality is

$$\sum_{\mathcal{W}_{B,\mathrm{II}}} \leq C_3 \sum_{\mathcal{W}_{B,\mathrm{II}}} \ell(Q)^d$$

from (2.5). But in Case II,

$$\ell(Q)^d \leq C_5\, \mathcal{H}^d(c_6 Q \cap \partial\widetilde{\Omega}) \tag{2.8}$$

because $\partial\widetilde{\Omega}$ is Ahlfors regular and by (2.2) and (2.3) no point lies in more than $N = N(c_6, c_7, d)$ cubes $c_6 Q$, $Q \in \mathcal{W}$. Therefore (2.5), (2.6), and the Ahlfors regularity of $\partial\widetilde{\Omega}$ imply

$$\sum_{\mathcal{W}_{B,\mathrm{II}}} \leq C_5 \sum_{\mathcal{W}_{B,\mathrm{II}}} \ell(Q)^d \leq C_5 N \mathcal{H}^d(B(x, C_4 r) \cap \partial\Omega) \leq C_5 N (C_4 r)^d. \tag{2.9}$$

Thus by (2.7), (2.5) and (2.9), (a) holds for all bounded harmonic u.

To prove (b), let u be an harmonic function on Ω, let $\varepsilon > 0$ and consider the Case I and Case II cubes in $\mathcal{W}(\Omega)$. Write

$$U = \bigcup_{\text{Case I}} Q, \quad V = \bigcup_{\text{Case II}} Q,$$

and

$$\Gamma = \Omega \cap \partial V = \Omega \cap \partial U.$$

By [23], there exists $g \in W^{1,1}(\widetilde{\Omega})$ satisfying (1.2) and (1.3) for u on $\widetilde{\Omega}$. Define $G = g\chi_U + u\chi_{V \cup \Gamma}$. Then $\|u - G\|_{L^\infty(\Omega)} < \varepsilon$. Testing G against $\nabla\varphi$, $\varphi \in C^\infty(\Omega)$, with Green's theorem shows that as distributions on Ω,

$$\nabla G = \chi_U \nabla g + \chi_V \nabla u + \nu,$$

where ν is an $\mathbb{R}^{d+1}$-valued measure that accounts for the jump between g and u across Γ and has total variation $|\nu| \leq \varepsilon \chi_\Gamma \mathcal{H}^d$. Let $x \in \partial\Omega$ and $r > 0$. Then by the Case I and Case II argument in the proof of (a),

$$\int_{B(x,r)\cap(U\cup V)} |\nabla G|\, dy \leq C r^d,$$

and because $\partial\widetilde{\Omega}$ is Ahlfors regular, (2.8) implies

$$|\nu|(B(x,r)\cap\Omega) \leq C\varepsilon r^d.$$

Hence (1.3) holds for the vector measure ∇G.

To replace G by a $W^{1,1}_{\mathrm{loc}}$ function, let $\eta > 0$ be small, write

$$\psi_\eta(y) = \eta^{-(d+1)} \psi\Big(\frac{y}{\eta}\Big),$$

where $\psi \in C^\infty(\mathbb{R}^{d+1})$ is a non-negative radial function, compactly supported in $B(0,1)$, with $\int_{\mathbb{R}^{d+1}} \psi\, dy = 1$, let χ_j, $j \geq 1$, be a C_0^∞ partition of unity on Ω such that χ_j has support $\{2^{-j-1} < \mathrm{dist}(y,\partial\Omega) < 2^{-j+1}\}$, and define

$$\tilde{G}(y) = \sum_j \chi_j(y)\, G * \psi_{2^{-j}\eta}(y).$$

Then $\tilde{G} \in C^\infty(\Omega) \subset W^{1,1}_{\mathrm{loc}}(\Omega)$ and (1.2) and (1.3) hold for $\tilde{G}$ and u.

3. Three lemmas

Recall we assume (1.7), so that the harmonic measure $\omega(p,E) = \omega(p,E,\Omega)$ exists for $p \in \Omega$ and Borel $E \subset \partial\Omega$. The first lemma is Lemma 3 from [1].

Lemma 3.1. *Ω satisfies (1.7) with constant β if and only if there exists $\eta = \eta(\beta) < 1$ such that for all $x \in \partial\Omega$ and all $r > 0$,*

$$\sup_{B(x,r)\cap\Omega} \omega(p, \partial B(x,2r)\cap\Omega, \Omega\cap B(x,2r)) \leq \eta. \tag{3.1}$$

The second lemma is a well-known consequence of Lemma 3.1 and induction.

Lemma 3.2. *Assume Ω satisfies (1.4) and (1.7) and let $0 < \varepsilon < 1/2$. There are constants c_1 and c_2 depending only on ε and the constants α and β in (1.4) and (1.7), such that whenever $x \in \partial\Omega$ and $r < \mathrm{diam}\,\Omega$, there exists a ball $B = B(p, c_1 r)$ such that*

$$4B = B(p, 4c_1 r) \subset \Omega \cap B(x,r), \tag{3.2}$$

$$\mathrm{dist}(p,\partial\Omega) < c_2 r, \tag{3.3}$$

and

$$\inf_{q\in 2B} \omega(q, \partial\Omega\cap B(x,r), \Omega\cap B(x,r)) > 1-\varepsilon. \tag{3.4}$$

Proof. By the maximum principle and induction, (3.1) implies that for all $s > 0$,

$$\sup_{B(x,s)\cap\Omega} \omega(p, \partial B(x, 2^N s) \setminus \Omega, \Omega \cap B(x, 2^N s)) < \eta^N. \tag{3.5}$$

For $\varepsilon > 0$ take N with $\eta^N < \varepsilon$ and set $C_1 = 1 + 2^N$. For any $p \in \Omega$ take $x \in \partial\Omega$ such that $|x - p| = \operatorname{dist}(p, \partial\Omega)$. Applying (3.5) with $s = |x - p|$ and the maximum principle, we get

$$\omega(p, \partial\Omega \cap B(p, C_1 s), \Omega) > 1 - \varepsilon. \tag{3.6}$$

By (1.4), $\Omega \cap B(x, \frac{r}{1+C_1})$ contains a ball $B = B(p, \frac{\alpha r}{1+C_1})$. Therefore (3.2) holds with

$$c_1 = \frac{\alpha}{4(1 + C_1)}$$

and (3.3) holds with

$$c_2 = \frac{1}{1 + C_1}.$$

If $q \in 2B = B(p, \frac{\alpha r}{2(1+C_1)})$, then by (3.2) $\operatorname{dist}(q, \partial\Omega) \le |q - x| \le \frac{r}{1+C_1}$. Therefore $B(q, C_1 \operatorname{dist}(q, \partial\Omega)) \subset B(x, r)$, so that (3.6) implies (3.4). ∎

The next lemma is similar to Lemma 3.3 of [17].

Lemma 3.3. *Assume Ω satisfies (1.4) and (1.7). Then there exist $\varepsilon_0 > 0$ and constants c_9 and c_{10} depending only on d and the constants α and β of (1.4) and (1.7) such that if $0 < \varepsilon < \varepsilon_0$ and*

(i) *$S \subset \partial\Omega$ is a Borel set, $x \in S$, $0 < r < \operatorname{diam}(\Omega)$, and $B(x, r) \cap \partial\Omega \subset S$,*

(ii) *the ball $B_S = B(p_S, c_1 r)$ satisfies (3.2), (3.3) and (3.4) from Lemma 3.2,*

(iii) *$E_S \subset S \cap B(x, r)$ is a compact set such that*

$$\inf_{2B_S} \omega(q, E_S, \Omega) \ge 1 - \varepsilon, \tag{3.7}$$

then there exists a non-negative harmonic function u_S on Ω and a Borel function f_S such that

$$0 \le f_S \le \chi_{E_S}$$

and for all $p \in \Omega$,

$$u_S(p) = \int_{E_S} f_S(y)\, d\omega(p, y, \Omega), \tag{3.8}$$

$$\inf_{B_S} u_S(p) \ge c_9, \tag{3.9}$$

and there exists a unit vector $\vec{e}_S \in \mathbb{R}^{d+1}$ such that

$$\inf_{B_S} |\nabla u_S(p) \cdot \vec{e}_S| \ge \frac{c_{10}}{c_1 r}. \tag{3.10}$$

The right side of (3.10) is so written to display the radius $c_1 r$ of B_S.

Proof. Take $q_S \in S \cap \partial\Omega$ with $|q_S - p_S| < 2\,\mathrm{dist}(p_S, \partial\Omega)$. By (3.2) and (3.3) we have

$$4c_1 r < |p_S - q_S| < 2c_2 r. \tag{3.11}$$

Case I. $d \geq 2$. By (1.7) and the definition of capacity there exists a positive measure μ_S supported on $\overline{B}(q_S, c_1 r) \setminus \Omega$ with $\int d\mu_S > \beta(c_1 r)^{d-1}$ such that the potential

$$U_S(p) = \int |p-y|^{1-d} d\mu_S(y)$$

is harmonic on $\mathbb{R}^{d+1} \setminus \mathrm{supp}\,\mu_S \supset \Omega$, and satisfies

$$0 < U_S(p) \leq 1 \tag{3.12}$$

for all $p \in \mathbb{R}^{d+1}$. By Egoroff's theorem, there is a compact set $F_S \subset \overline{B}(q_0, c_1 r) \setminus \Omega$ such that $\mu_S(F_S) \geq \beta(c_1 r)^{d-1}$ and

$$\int_{B(p,\eta)} |p-y|^{1-d}\, d\mu_S(y) \to 0 \quad (\eta \to 0)$$

uniformly on F_S. Redefine U_S to be

$$U_S(p) = \int_{F_S} |p-y|^{1-d}\, d\mu_S(y). \tag{3.13}$$

Then U_S is continuous on $\mathbb{R}^{d+1}$, harmonic on $\mathbb{R}^{d+1} \setminus F_S \supset \Omega$, and satisfies (3.12).

By (3.11) and (3.13),

$$\inf_{2B_S} U_S(p) \geq \beta\Big(\frac{c_1 r}{|p_S - q_S| + 3c_1 r}\Big)^{d-1} = \beta\, 7^{1-d} = c_9'. \tag{3.14}$$

Let $\vec{e}_S = \frac{\overrightarrow{(q_S - p_S)}}{|q_S - p_S|}$. Then by (3.11) we have

$$\inf\Big\{\vec{e}_S \cdot \frac{(q-p)}{|q-p|}\, q \in F_S, p \in B_S\Big\} = \frac{c_2}{c_1} = \frac{4}{\alpha}\cdot \tag{3.15}$$

Hence by (3.11), (3.13), (3.15) and the formula

$$\nabla U_S(p) = (1-d)\int_{F_S} \frac{(p-y)}{|p-y|^{d+1}}\, d\mu_S(y), \tag{3.16}$$

we have, on B_S,

$$|\nabla U_S(p) \cdot \vec{e}_S| \geq \frac{4}{\alpha}\, \frac{(d-1)\beta c_1 r^{d-1}}{(2c_1 r + 2c_2 r)^d} = \frac{c_{10}'}{c_1 r}, \tag{3.17}$$

in which

$$c_{10}' = \frac{d-1}{2c_1{}^{d-2}}\, \frac{\beta}{\alpha}\, \Big(\frac{\alpha}{4+\alpha}\Big)^d$$

depends only on d, α and β. Since U_S is continuous on $\overline{\Omega}$,

$$U_S(p) = \int_{\partial\Omega} g_S(y)\, d\omega(p, y, \Omega)$$

with continuous $g_S = U_S|\partial\Omega$. Set $f_S = \chi_{E_S} g_S$ and define u_S by (3.8). Finally, take

$$\varepsilon_0 < \min\Big(\frac{c_9'}{2}, \frac{c_{10}'}{3}\Big),$$

assume $0 < \varepsilon < \varepsilon_0$, and assume also that Lemma 3.2 holds for c_1, c_2, α and ε. Then since $|f_S| \le 1$, (3.7) yields $\sup_{2B_S} |U_S - u_S| \le \varepsilon$. Hence (3.14) implies (3.9) for $c_9 = c_9'/2$, and by (3.7) and Harnack's inequality, $\sup_{B_S} |\nabla(U_S - u_S)| \le \frac{2\varepsilon}{c_1 r}$. so that (3.7) implies (3.10) for $c_{10} = c_{10}'/3$.

Case II: $d = 1$. Decreasing c_1 and c_2 if necessary, we have, again by Egoroff's theorem, compact sets $F_S^{\pm} \subset \overline{B}(x,r) \setminus \Omega$ such that $\mathrm{Cap}(F_S^{\pm}) \ge \beta c_1 r/2 \equiv e^{-\gamma}$ and probability measures $\mu_{\pm}$ supported on $F_S^{\pm}$ so that the logarithmic potentials

$$U_{\pm}(p) = \int_{F_S^{\pm}} \log \frac{1}{|p-y|}\, d\mu_{\pm}(y)$$

are continuous on $\mathbb{R}^2$ and harmonic on $\mathbb{R}^2 \setminus F_S^{\pm}$ and satisfy $U_{\pm} < \gamma$ on $\mathbb{R}^2 \setminus F_S^{\pm}$ and for small η, $\gamma - \eta \le U_{\pm} \le \gamma$ on $F_S^{\pm}$. Because capacity is bounded by diameter, we can, by choices of c_1 and c_2, position $F_S^{\pm}$ so that

$$F_S^+ \subset B(p_S, 2c_2 r)$$

but

$$F_S^- \subset \mathbb{R}^2 \setminus B(p_S, 4c_2 r).$$

Then on $\mathbb{R}^2 \setminus (F_Q^+ \cup F_S^-)$ the function $U^+ - U^-$ is harmonic and bounded, because the logarithmic singularities at ∞ cancel, and by the choices of $F_S^{\pm}$,

$$\sup_{F^+ \cup F^-} |U^+ - U^-| \le \gamma - \log\Big(\frac{1}{2c_r 2}\Big) = \log\Big(\frac{4c_2}{\beta c_1}\Big),$$
$$\inf_{2B_S} (U^+ - U^-) \ge \log\Big(\frac{1}{2c_2 r - 2c_1 r}\Big) - \log\Big(\frac{1}{4c_2 r + 2c_1 r}\Big) = \log\Big(\frac{2c_2 + c_1}{c_2 - c_1}\Big),$$

and for some unit vector $\vec{e}_S$,

$$\inf_{B_S} \big|\nabla(U^+ - U^-) \cdot \vec{e}_S\big| \ge \frac{c_{10}''}{r}.$$

Then (3.8), (3.9) and (3.10) hold for

$$f_S = \Big(2\log\Big(\frac{4c_2}{c_1}\Big)\Big)^{-1} \Big(\log\Big(\frac{4c_2}{c_1}\Big) + U^+ - U^-\Big)\, \chi_{E_S}. \qquad \blacksquare$$

4. Proof of Theorem 1.2 Part A

We follow the proof of Lemma 3.7 of [17]. Replacing ε by $\varepsilon/4$ and R by CR, $C > 1$, and setting $r_j = \operatorname{dist}(p_j, \partial\Omega)$ and $B_j = B(p_j, r_j)$, we can by Lemma 3.2 and Harnack's inequality assume $E_j \subset B(x, R)$, $4B_j = B(p_j, 4r_j) \subset \Omega \cap B(x, R)$ and

$$\inf_{2B_j} \omega(p, E_j, \Omega) > 1 - \frac{\varepsilon}{2}. \tag{4.1}$$

Then the conclusion of Theorem 1.2 Part A is immediate from:

Lemma 4.1. *Assume* (1.4), (1.7) *and either* (a) *or* (b) *holds for* Ω. *Then if* $0 < \varepsilon < \varepsilon_0$, *there is* $C(\varepsilon)$ *such that if for* $j = 1, 2, \dots$ *there exist balls* $B_j = B(p_j, r_j) \subset \Omega \cap B(x, R)$, $x \in \partial\Omega$, *and sets* $E_j \subset \partial\Omega$ *with* (4.1) *and*

$$E_j \cap E_k = \emptyset, \quad j \neq k, \tag{4.2}$$

then

$$\sum r_j^d \leq C(\varepsilon) R^d. \tag{4.3}$$

Proof. By Lemma 3.3 there exists a Borel function $0 \leq f_j \leq \chi_{E_j}$ such that the harmonic function

$$u_j(p) = \int_{E_j} f_j(y)\, d\omega(p, y, \Omega) \tag{4.4}$$

satisfies

$$\inf_{2B_j} u_j(p) \geq c_{12}, \tag{4.5}$$

and there exists a unit vector $\vec{e}_j \in \mathbb{R}^{d+1}$ such that

$$\inf_{B_j} \left|\nabla u_j(p) \cdot \vec{e}_j\right| \geq \frac{c_{12}}{r_j}. \tag{4.6}$$

Set $u = \sum u_j$. Then by (4.1) we have $\sup_{2B_j} |u - u_j| \leq \varepsilon/2$, so that by Harnack's inequality, $\sup_{B_j} |\nabla(u - u_j)| \leq 2\varepsilon/r_j$. Therefore

$$|\nabla u| > c_{11} - 3\varepsilon/r_j$$

on B_j and

$$\int_{B_j \cap \Omega} |\nabla u(x)|^2 \operatorname{dist}(x, \partial\Omega)\, dx \geq c_{12}\, r_j^d. \tag{4.7}$$

Assuming (a) holds on Ω with constant C and summing, we obtain

$$\sum_j (\operatorname{dist}(p_j, \partial\Omega))^d \leq \frac{1}{c_{12}} \int_{B \cap \partial\Omega} |\nabla u(x)|^2 \operatorname{dist}(x, \partial\Omega)\, dx \leq CR^d,$$

which yields (4.3) when (a) holds.

Now assume (b) holds for Ω and $\varepsilon < c_{11}/3$. If $g \in W^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies (1.2) for u and $\varepsilon < c_{11}/3$, then, using (4.6) and (4.7) for u_j, we obtain

$$\int_{B_j} |\nabla g(x)|\, dx \geq c_{13}\, r_j^d .$$

Thus from (a) or (b) we conclude that (4.3) holds. ■

We note two corollaries of Lemma 4.1.

Corollary 4.2. *Let $\Omega \subset \mathbb{R}^{d+1}$ be a corkscrew domain for which* (1.7) *holds. If* (a) *or* (b) *holds for Ω, then there is a constant $C > 0$ such that for all $x \in \partial\Omega$ and all $r > 0$,*

$$\mathcal{H}^d(B(x,r) \cap \partial\Omega) \leq C r^d . \tag{4.8}$$

Proof. Cover any compact $K \subset B(x,R) \cap \partial\Omega$ by a minimal set $\mathcal{F}$ of N_n distinct closed dyadic cubes of side 2^{-n}. Partition $\mathcal{F}$ into 3^{d+1} disjoint families $\mathcal{F}'$ so that $\mathrm{dist}(Q_1, Q_2) \geq 2^{-n}$ if $Q_1 \neq Q_2 \in \mathcal{F}'$, and fix any such family $\mathcal{F}'$. By (1.4) and (1.7) and Lemma 3.2 there exists c_{14} so that for every $Q_j \in \mathcal{F}'$ there exists a ball $B_j = B(p_j, c_{14}2^{-n}) \subset \Omega \cap \frac{5}{4}Q_j$ with $\inf_{B_j} \omega(p, Q_j \cap \partial\Omega, \Omega) > 1 - \varepsilon$, where ε fixed and small. Then by Lemma 4.1,

$$(c_{14}\, 2^{-n})^d \, \#\mathcal{F}' \leq C(\varepsilon)\, r^d ,$$

which yields

$$\mathcal{H}^d(K) \leq 3^{d+1} c_{14}^{-d}\, C(\varepsilon)\, r^d . \qquad ■$$

Merged with the results of [23] and [17], Corollary 4.2 yields:

Corollary 4.3. *If $\Omega \subset \mathbb{R}^{d+1}$ is a corkscrew domain for which there exists a constant $c > 0$ such that for all $x \in \partial\Omega$ and all $0 < R < \mathrm{diam}(\partial\Omega)$,*

$$\mathcal{H}^d(B(x,r) \cap \partial\Omega) \geq c r^d , \tag{4.9}$$

then (a) *or* (b) *holds for Ω if and only if $\partial\Omega$ is uniformly rectifiable.*

5. Modified Christ–David cubes

To prove Proposition 1.3, we follow the construction in [9] very closely, although the arguments from [7], [10], [26] or [27] would also work. To start we use (a) or (b) to get a grip on the small boundary condition (1.18).

Lemma 5.1. *Let $0 < \eta < 1$ and let N be a positive integer. Assume Ω is a bounded corkscrew domain with* (1.7) *and assume the conclusion of Theorem 1.2 Part A holds for Ω. Then for any $x \in \partial\Omega$ and any $j \in \mathbb{N}$, there exists an open ball $B_j(x) = B_j(x,r)$ having center x and radius*

$$r \in (2^{-Nj}, (1+\eta)\, 2^{-Nj})$$

such that if

$$\begin{aligned}\Delta_j(x) &= B_j(x) \cap \partial\Omega,\\ E_j(x) &= \{y \in \Delta_j(x) : \operatorname{dist}(y, \partial\Omega \setminus \Delta_j(x)) < \eta^2 2^{-Nj}\}\\ &\qquad \cup \{y \in \partial\Omega \setminus \Delta_j(x) : \operatorname{dist}(y, \Delta_j(x)) < \eta^2 2^{-Nj}\}\end{aligned}$$

and $m_j(x)$ is the minimum number of closed balls $\overline{B(p, \eta^2 2^{-Nj})}$ needed to cover $E_j(x)$, then

$$m_j(x) \le C_d \eta^{1-2d}, \tag{5.1}$$

in which the constant C_d depends only on d and the constant in (1.12).

Proof. Partition the closed ring $\Sigma = \overline{B(x, (1+\eta)2^{-Nj})} \setminus B(x, 2^{-Nj})$ into a family $\mathcal{R}$ of at most $1 + [1/\eta]$ closed rings having width $\eta^2 2^{-Nj}$ and center x. Fix $2^{-n} \sim \eta^2 2^{-Nj}$, let $\mathcal{E}$ be the set of closed dyadic cubes Q of side 2^{-n} such that $Q \cap \partial\Omega \cap \Sigma \neq \emptyset$ and let $M = \#\mathcal{E}$. Choose a maximal subset $\mathcal{E}_0 \subset \mathcal{E}$ of pairwise disjoint closed cubes. Then $\mathcal{E}_0$ has cardinality $\#\mathcal{E}_0 \ge c_{14} 3^{-d-1} M$ and the enlarged cubes $\frac{5}{4}Q$, $Q \in \mathcal{E}_0$, are pairwise disjoint. For each $Q \in \mathcal{E}_0$ there exist by (1.7) a compact set $E_Q \subset \frac{5}{4}Q \cap \partial\Omega$ and a ball $B(p_Q, \alpha\eta^2 2^{-j}) \subset \frac{5}{4}Q \cap \Omega$ satisfying the conclusions of Lemma 3.2 and Lemma 3.3. Now we can follow the proof of Corollary 4.2 to conclude that $\#\mathcal{E}_0(\eta^2 2^{-Nj})^d \le C 2^{-Njd}$. Hence $M \le C\eta^{-2d}$ and there exists a pair of adjacent closed subrings in $\mathcal{R}$ whose union meets at most $c_{15} C \eta^{1-2d}$ dyadic cubes from $\mathcal{E}$. That implies (5.1). ∎

Proof of Proposition 1.3. For $j \ge 0$, let V_j be a maximal subset of $\partial\Omega$ such that when $x, x' \in V_j$, $|x - x'| \ge 2^{-jN}$, and for $x \in V_j$ let $B_j(x)$ be the ball given by Lemma 5.1, and set $\Delta_j(x) = \partial\Omega \cap B_j(x)$. Put a total order, written $x < y$, on the finite set V_j and define

$$\Delta_j^*(x) = \Delta_j(x) \setminus \bigcup_{y<x} \Delta_j(y).$$

Then for each j, (1.10), (1.11), and (1.12) hold for the family $\{\Delta_j^*(x)\}$ and because the balls $B(x, (1-\eta)2^{-Nj})$, $x \in V_j$, are disjoint we have

$$B(x, (1-\eta)\, 2^{-Nj}) \subset \Delta_j^*(x) \tag{5.2}$$

for every $x \in V_j$. Because $\partial\Omega \subset \mathbb{R}^{d+1}$, there is constant M_d independent of j such that

$$\#\{y \in V_j : y < x \text{ and } B_j(y) \cap B_j(x) \neq \emptyset\} \le M_d. \tag{5.3}$$

Therefore by (5.1) the minimum number m_j^* of closed balls $\overline{B(p, \eta^2 2^{-Nj})}$ needed to cover

$$\begin{aligned}E_j^*(x) &= \{y \in \Delta_j^*(x) : \operatorname{dist}(y, \partial\Omega) < \eta^2 2^{-Nj}\}\\ &\qquad \cup \{y \in \partial\Omega \setminus \Delta_j^*(x) : \operatorname{dist}(y, \Delta_j^*(x)) < \eta^2 2^{-Nj}\}\end{aligned}$$

has the upper bound

$$m_j^*(x) \le C_d M_d \eta^{1-2d}. \tag{5.4}$$

Because the families $\{\Delta_j^*\}_{j\geq 0}$ may not satisfy the nesting condition (1.13) or the small boundary condition (1.15), we further refine each set Δ_j^*, still following [9]. If $x \in V_j$, $j \geq 1$, there exists by (1.11) and (1.12) a unique $\varphi(x) \in V_{j-1}$ such that $x \in \Delta_{j-1}^*(\varphi(x))$. For any j and $x \in V_j$, define $D_{j,0}(x) = \Delta_j^*(x)$, and for $n \in \mathbb{N}$,

$$D_{j,n}(x) = \bigcup \{\Delta_{j+n}^*(y) : \varphi^n(y) = x\}$$

Then for any j and n,

$$\bigcup \{D_{j,n}(x) : x \in V_j\} = \partial\Omega, \tag{5.5}$$

and by induction,

$$D_{j,n}(x) = \bigcup \{D_{j,n-k}(y) : \varphi^k(y) = x\}. \tag{5.6}$$

for $0 \leq k \leq n$.

Write $\mathrm{dist}_{\mathcal{H}}(A, B)$ for the Hausdorff distance between subsets A, B of $\mathbb{R}^{d+1}$. Since $\mathrm{diam}(\Delta_j^*) \leq (1+\eta)2^{-Nj}$, we have

$$\mathrm{dist}_{\mathcal{H}}(D_{j,1}(x), \Delta_j^*(x)) \leq (1+\eta)\, 2^{-N(j+1)},$$

so that by (5.6) and induction,

$$\mathrm{dist}_{\mathcal{H}}(D_{j,n}(x), D_{j,n+1}(x)) \leq (1+\eta)\, 2^{-N(j+n)}. \tag{5.7}$$

Hence for each j and $x \in V_j$, the sequence of $\{\overline{D_{j,n}(x)}\}$ of compact sets converges in Hausdorff metric to a compact set $R_j(x)$. It is clear from (5.5) that for any fixed j,

$$\bigcup \{R_j(x) : x \in V_j\} = \partial\Omega \tag{5.8}$$

because if $y \in \partial\Omega$ then $y \in D_{j,n}(x^{(n)})$ for some $x^{(n)} \in V(j)$ and because $V(j)$ is finite there is $x \in V(j)$ with $y \in \overline{D_{j,n}(x)}$ for infinitely many n.

Since we took closures, (1.12) may not hold for the sets $\{R_j(x)\}$, and like in [9] we must alter them one final time. By induction we can choose the ordering on the finite set V_j, $j \geq 1$, so that $x < y$ if $\varphi(x) < \varphi(y)$. Then define, for all j and $x \in V(j)$,

$$S_j(x) = R_j(x) \setminus \bigcup_{V(j) \ni y < x} R_j(y). \tag{5.9}$$

Then it is clear from (5.8) that (1.12) and (1.13) hold for the family $\mathcal{S} = \bigcup_j \{S_j\}$, and since by (5.7),

$$\mathrm{diam}(S_j(x)) \leq \mathrm{diam}(R_j(x)) \leq \sum_{k=j}^{\infty} 2(1+\eta)\, 2^{-Nk} \leq 4(1+\eta)\, 2^{-Nj}. \tag{5.10}$$

To obtain the lower bound in (1.10) and also (1.13), (1.14) and (1.15), we need 2^{-N} to be small compared to η. Assume

$$2^{-N} \sim \eta^2 < \frac{1}{9}. \tag{5.11}$$

Then by (5.2) and (5.7) we have for $x \in V_j$,

$$\operatorname{dist}(x, \partial\Omega \setminus D_{j,n}) \ge (1-\eta)\, 2^{-Nj} - \sum_{k>j} 2(1+\eta)\, 2^{-Nk}$$
$$\ge 2^{-Nj}\Big(1 - \eta - 2(1+\eta)\,\frac{2^{-N}}{1-2^{-N}}\Big) \ge \frac{2^{-Nj}}{3}.$$

This implies (1.14), and with (5.10) it also implies (1.10).

To show (1.13), suppose $u \in \Delta_j(x) \cap \Delta_{j+1}(y)$. Then by (5.7), $u = \lim x_n$, where $x_n \in V_n$, $x_{n+1} \in \Delta_n^*(x_n)$ and $x_j = x$, and $u = \lim y_n$, where $y_n \in V_n$, $y_{n+1} \in \Delta_n^*(y_n)$ and $y_{j+1} = y$. Hence $u \in \bigcap_{n \ge j} R_n(x_n) \cap \bigcap_{n \ge j+1} R_n(y_n)$ so that by the definition (5.9), $y_n = x_n$ for all $n \ge j+1$ and $S_{j+1}(y) \subset S_j(x)$.

To verify the small boundary condition (1.18) we can by (5.2) assume $\tau = 2^{-Nk}$, $k \ge 1$. Let $x \in V_j$ and write $S = S_j(x)$. Then by (5.7) and (5.10), $N_\tau(S)$ is comparable to

$$\#\{y \in V_{j+k} : S^*_{j+k}(y) \cap \Delta_\tau(S) \ne \emptyset\},$$

and by (5.4) and (5.11) this number is bounded by $(C_d M_d \eta^{1-2d})^k \sim (C_d M_d)^k \tau^{1/2}$, which, for $C > 2$ and τ small, is bounded by $C\tau^{1/C-d}$. ■

6. A corona decomposition and the proof of Theorem 1.4 Part A

Assume $\Omega \subset \mathbb{R}^{d+1}$, $d \ge 1$, is a domain satisfying (1.4), (1.7), and either (a) or (b), and let $\mathcal{S}$ be a family of subsets of $\partial\Omega$ satisfying the conclusions of Proposition 1.3. We shall prove there exist constants ε_1, A_0 and C such that (1.24) holds with constant C whenever $0 < \delta < \varepsilon/3 < \varepsilon_1/3$ and $A > A_0$, $S_0 \in \mathcal{S}$, and $G_k(S_0)$ are its generations defined for δ and A. Recall that by Proposition 1.3 the family $\mathcal{S}$ has the properties (1.17), (1.18), and (1.19).

Lemma 6.1. *Let $S \in \mathcal{S}$ and let $\{S_j\} \subset \mathcal{S}$ be a family of cubes $S_j \subset S$ satisfying $S_j \cap S_k = \emptyset$ when $j \ne k$. If $S_j \in \mathsf{HD}(S)$ for all j, then*

$$\sum \ell(S_j)^d \le \frac{C_1}{A}\, \ell(S)^d, \tag{6.1}$$

while if $S_j \in \mathsf{LD}(S)$ for all j, then

$$\sup_{B_S} \sum_{S_j} \omega(p, S_j) \le C_2\,\delta, \tag{6.2}$$

where C_1 and C_2 depend only on d, δ and the constant in (1.12).

Proof. Assertion (6.2) follows from (1.20), (1.21), (1.22), (1.25) and Lemma 4.1, with constant C_2 depending only on δ and the constants in Proposition 1.3 and (1.12).

Since the definition of HD entails $\omega(p_S, 2S_j, \Omega)$ and not $\omega(p_S, S_j, \Omega)$, the proof of (6.1) requires more work. Note that if $2S_k \cap 2S_j \ne \emptyset$ and $\ell(S_k) \le \ell(S_j)$ then, by (1.10),

$$S_k \subset B(x_{S_j}, C\ell(S_j)),$$

in which the constant C depends only on the upper bound in (1.10) and thus only on α, β and d. Hence by Theorem 1.2 Part A,

$$\sum\{\ell(S_k)^d : 2S_k \cap 2S_j \neq \emptyset,\ \ \ell(S_k) \leq \ell(S_j)\} \leq C_1 \ell(S_j)^d,$$

and by a Vitali argument there exists $\{S'_j\} \subset \{S_j\}$ with $2S'_j \cap 2S'_k = \emptyset$ and

$$\sum \ell(S_j)^d \leq C_1 \sum \ell(S'_j)^d \leq \frac{C_1}{A} \sum \omega(p_S, 2S'_j, \Omega)\, \ell(S)^d \leq \frac{C_1}{A}\, \ell(S)^d. \qquad \blacksquare$$

Turning to the proof of Theorem 1.4 Part A, we now assume $A > 2C_1$. To prove (1.29), we separate high and low density cubes. For $S \in \mathcal{S}$, let $GH_1(S)$ be the family of high density cubes $S' \in G_1(S)$ and by induction

$$GH_k(S) = \bigcup_{S' \in GH_{k-1}(S)} GH_1(S'). \tag{6.3}$$

Thus if $S_k \in GH_k(S)$, then

$$S_k \subset S_{k-1} \subset \cdots \subset S_1 \subset S_0 = S, \tag{6.4}$$

in which for $j > 0$,

$$S_{j+1} \in \mathsf{HD}(S_j),$$

so that all ancestors of S_k except possibly S_0 are HD subcubes of their predecessors. Write

$$GH(S) = \bigcup_{k \geq 1} GH_k(S).$$

Then by (6.1),

$$\sum_{GH(S)} \ell(S')^d = \sum_{k=1}^{\infty} \sum_{GH_k(S)} \ell(S')^d \leq \frac{C_1}{A - C_1}\, \ell(S)^d. \tag{6.5}$$

Similarly, let $GL_1(S)$ be the family of low density cubes $S_j \in G_1(S)$ and by induction,

$$GL_k(S) = \bigcup_{S' \in GL_{k-1}(S)} GL_1(S'). \tag{6.6}$$

Thus if $S_k \in GL_k(S)$, then

$$S_k \subset S_{k-1} \subset \cdots \subset S_1 \subset S_0 = S \tag{6.7}$$

and $S_{j+1} \in \mathsf{LD}(S_j)$ for $j > 0$. Write

$$GL(S) = \bigcup_{k \geq 1} GL_k(S).$$

Lemma 6.2. *Assume ε in (1.2) is small and $\delta \le \varepsilon$. Then there exists a constant C_2 such that for any $S_0 \in \mathcal{S}$,*

$$\sum_{GL(S_0)} \ell(S)^d = \sum_{k=1}^{\infty} \sum_{GL_k(S_0)} \ell(S)^d \le C_2\, \ell(S_0)^d. \tag{6.8}$$

Proof. The proof is like the proof of (6.1). For any $S \in GL(S)$ define

$$E_S = S \setminus \bigcup_{S' \in GL_1(S)} S'.$$

Then $E_{S_1} \cap E_{S_2} = \emptyset$ for $S_1 \ne S_2$ and that $\inf_{B_S} \omega(p, E_S, \Omega) > 1 - \varepsilon$, so that Lemma 4.1 and (1.13) imply (6.8). ∎

Now the proof of (1.29) follows by interlacing (6.5) and (6.8). Write

$$L_1(S) = \sum_{GL(S)} \ell(S')^d, \quad H_1(S) = \sum_{GH(S)} \ell(S')^d,$$

and by induction,

$$L_{k+1}(S) = \sum_{GH(S)} L_k(S'), \quad H_{k+1}(S) = \sum_{GL(S)} H_k(S').$$

Then

$$\sum_{k=1}^{\infty} \sum_{G_k(S_0)} \ell(S)^d = \sum_{k=1}^{\infty} (L_k(S_0) + H_k(S_0))$$

and by (6.5) and (6.8),

$$L_{k+1}(S) \le C_2 H_k(S) \quad \text{and} \quad H_{k+1}(S) \le \frac{C_1 L_k(S)}{A - C_1},$$

so that writing $L_0(S) = H_0(S) = 1$ and taking $A - 1 > C_1 + C_1 C_2$ yields

$$\sum_{k=1}^{\infty} \sum_{G_k(S_0)} \ell(S)^d \le \frac{AC_2 + C_1 + C_1 C_2}{A - C_1 - C_1 C_2}.$$

That proves (1.29) and Theorem 1.4 Part A.

7. A domain $\widetilde{\Omega}$

Assume Ω is a corkscrew domain satisfying (1.7) and $\mathcal{S}$ is a family of subsets of $\partial\Omega$ having properties (1.13)–(1.18) of Proposition 1.3, and their consequences (1.20), (1.21) and (1.22). Fix constants $\varepsilon, \delta, N, A$ and C with $0 < \delta < \varepsilon/3$ and A so large that (1.27) holds for any $S_0 \in \mathcal{S}$ when the generations $G_k(S_0)$ are define by (1.22) and (1.25). Also assume $\mathcal{S}$ satisfies the conclusion of Lemma 4.1 or, equivalently, hypothesis (ii) of Theorem 1.4 Part B. Under those assumptions we construct a domain $\widetilde{\Omega} \subset \Omega$ with $\partial\Omega \subset \partial\widetilde{\Omega}$ and a d-Ahlfors regular measure σ supported on $\partial\widetilde{\Omega}$ and boundedly mutually absolutely continuous with $\chi_{\partial\widetilde{\Omega}}\mathcal{H}^d$.

For any $S \in \mathcal{S}$, let

$$\Gamma_S \subset 2B_S \setminus B_S$$

be a finite union of separated closed spherical caps such that

$$\mathcal{H}^d(\Gamma_S) = c_{16}\,\ell(S)^d. \tag{7.1}$$

Since B_S has diameter $2c_1\ell(S)$ we can (and do) require Γ_S to be uniformly rectifiable with constants depending only on $c_0, \dots, c_{16}$ but not on S. Note that (taking c_{16} carefully) we have

$$\omega(p_S, \Gamma_S, \Omega^*) \sim 1/2 \tag{7.2}$$

for any domain Ω^* such that

$$(\Omega \setminus \Gamma_S) \cap B(x_S, c_0\ell(S)) \subset \Omega^* \subset \Omega,$$

and by (3.4),

$$\omega(p_S, S \cup \Gamma_S, \Omega^*) > 1 - \varepsilon \tag{7.3}$$

for all such Ω^*. Define $\Omega_0 = \Omega$ and assume $\operatorname{diam}(\partial\Omega) \sim 1$ so that $\partial\Omega = S_0 \in \mathcal{S}$. Fix $\lambda > 1$ so that

$$\lambda - 1 < \operatorname{dist}(S, 4B_S) \tag{7.4}$$

and define

$$\widetilde{\mathsf{HD}}(S_0) = \Big\{S_1 \in \mathcal{S}, S_1 \subset S_0 : \omega(p_{S_0}, \lambda S, \Omega_0) \geq A\Big(\tfrac{\ell(S_1)}{\ell(S_0)}\Big)^d,\ S_1 \text{ maximal}\Big\},$$

$$\widetilde{\mathsf{LD}}(S_0) = \Big\{S_1 \in \mathcal{S}, S_1 \subset S_0 : \omega(p_{S_0}, S_1, \Omega_n) \leq \delta\Big(\tfrac{\ell(S_1)}{\ell(S_0)}\Big)^d,\ S_1 \text{ maximal}\Big\},$$

$$\widetilde{G}_1 = \widetilde{G}_1(S_0) = \big\{S' \in \widetilde{\mathsf{HD}}(S_0) \cup \widetilde{\mathsf{LD}}(S_0),\ S' \text{ maximal}\big\},$$

$$K_1 = S_0 \setminus \bigcup_{\widetilde{G}_1(S_0)} S,$$

$$\mathsf{Tree}(S_0) = \big\{S \in \mathcal{S} : S \not\subset S' \text{ for all } S' \in \widetilde{G}_1(S_0)\big\},$$

$$\Omega_1 = \Omega \setminus \bigcup_{\widetilde{G}_1(S_0)} \Gamma_S,$$

$$\mu_1(\cdot) = \ell(S_0)^d \chi_{K_1}\omega(p_{S_0}, \cdot, \Omega_0), \quad \nu_1 = \textstyle\sum_{\widetilde{G}_1(S_0)} \chi_{\Gamma_S}\mathcal{H}^d,$$

and

$$\sigma_1 = \mu_1 + \nu_1.$$

Then σ_1 is a finite measure on $\partial\Omega_1$.

For $S \in \mathcal{S}$ define

$$S^1 = S \cup \bigcup \{\Gamma_{S'} : S' \in \widetilde{G}_1, S' \subset S\}$$

and declare $\ell(S^1) = \ell(S)$.

Lemma 7.1. *There are constants c_{17} and c_{18} such that if $S \in \mathsf{Tree}(S_0)$,*

$$c_{17}\,\ell(S)^d \leq \sigma_1(S^1) \leq c_{18}\,\ell(S)^d. \tag{7.5}$$

Proof. For the upper bound we have

$$\mu_1(S^1) \leq A\frac{\ell(S)^d}{\ell(S_0)^d},$$

since $S \in \mathsf{Tree}(S_0)$, and

$$\nu_1(S^1) \leq C_1\,\ell(S)^d$$

by Lemma 4.1.

For the lower bound note that

$$\sigma_1(S^1) = \ell(S_0)^d \omega(p_{S_0}, S, \Omega_0) - \ell(S_0)^d \sum_{\widetilde{G}_1(S_0)\ni S'\subset S} \omega(p_{S_0}, S', \Omega_0) + \sum_{\widetilde{G}_1(S_0)\ni S'\subset S} \mathcal{H}^d(\Gamma_{S'}),$$

in which

$$\ell(S_0)^d\,\omega(p_{S_0}, S, \Omega_0) \geq \delta\,\ell(S)^d,$$

while by the definition of $G_1(S_0)$,

$$\ell(S_0)^d \sum_{\widetilde{G}_1(S_0)\ni S'\subset S} \omega(p_{S_0}, S', \Omega) \leq C_1\,2^{2Nd}A \sum_{\widetilde{G}_1(S_0)\ni S'\subset S} \ell(S')^d.$$

Thus if

$$\sum_{\widetilde{G}_1(S_0)\ni S'\subset S} \ell(S')^d \leq \frac{\delta}{C_1 2^{2Nd+1}A}\,\ell(S)^d, \tag{7.6}$$

the lower bound holds with $c_{17} = \delta/2$. On the other hand, if (7.6) fails, then $\mu_1(S^1) \geq 0$ and

$$\nu_1(S^1) \geq \frac{c_{16}}{C_1\,2^{2Nd+1}A}\,\delta. \qquad \blacksquare$$

Now continue by induction. For $n \geq 1$ assume we have defined $\widetilde{G}_n = \widetilde{G}_n(S_0)$, Ω_n, and S^n for all $S \in \mathcal{S}$. Then for each $S \in \widetilde{G}_n(S_0)$ define

$$\widetilde{\mathsf{HD}}(S) = \left\{S_1 \in \mathcal{S}, S_1 \subset S : \omega(p_S, \lambda(S_1^n), \Omega_n) \geq A\Big(\frac{\ell(S_1)}{\ell(S)}\Big)^d,\ S_1 \text{ maximal}\right\},$$

$$\widetilde{\mathsf{LD}}(S) = \left\{S_1 \in \mathcal{S}, S_1 \subset S : \omega(p_S, (S_1^n), \Omega_n) \leq \delta\Big(\frac{\ell(S_1)}{\ell(S)}\Big)^d,\ S_1 \text{ maximal}\right\},$$

$$
\begin{aligned}
\widetilde{G}_1(S) &= \big\{S' \in \widetilde{\mathsf{HD}}(S) \cup \widetilde{\mathsf{LD}}(S),\ S' \text{ maximal}\big\},\\
\mathsf{Tree}(S) &= \big\{S' \in \mathcal{S} : S' \subset S,\ S' \not\subset S_1 \text{ for all } S_1 \in \widetilde{G}_1(S)\big\},\\
\widetilde{G}_{n+1}(S_0) &= \bigcup_{G_n(S_0)} \widetilde{G}_1(S),\\
K_{n+1} &= \bigcup_{\widetilde{G}_n(S_0)} \Big(S \setminus \bigcup_{\widetilde{G}_1(S)} S_1\Big),\\
\Omega_{n+1} &= \Omega_n \setminus \bigcup_{\widetilde{G}_{n+1}(S_0)} \Gamma_S,\\
\mu_{n+1}(\cdot) &= \sum_{S\in\widetilde{G}_n} \ell(S)^d \chi_{S\cap K_{n+1}} \omega(p_S, \cdot, \Omega_n),\\
\nu_{n+1} &= \sum_{\widetilde{G}_{n+1}(S_0)} \chi_{\Gamma_S} \mathcal{H}^d,
\end{aligned}
$$

and define

$$\sigma_{n+1} = \mu_{n+1} + \nu_{n+1}.$$

Then σ_{n+1} is a finite measure on $\partial\Omega_{n+1}$.

For $S \in \mathcal{S}$ define

$$S^{n+1} = S^n \cup \bigcup \{\Gamma_{S'} : S' \in \widetilde{G}_{n+1},\ S' \subset S\}$$

and declare $\ell(S^{n+1}) = \ell(S)$. Note that by the proof of Lemma 7.1,

$$c_{19}\, \ell(S)^d \le \sigma_{n+1}(S^{n+1}) \le c_{20}\, \ell(S)^d \tag{7.7}$$

for all $S \in \mathsf{Tree}(S')$, $S' \in \widetilde{G}_n(S_0)$.

Define $\widetilde{\Omega} = \cap\Omega_n$, which, as we will see, is a connected open set, and

$$\mu = \sum_{n\ge1} \mu_n, \quad \nu = \sum_{n\ge1} \nu_n, \quad \sigma = \mu + \nu,$$

and, for $S \in \mathcal{S}$,

$$S^\infty = \bigcup S^n.$$

Lemma 7.2. *Let $S \in \widetilde{G}_n$. Then*

$$\sum_{\widetilde{\mathsf{HD}}(S)} \Big(\frac{\ell(S_1)}{\ell(S)}\Big)^d \le \frac{C_1}{A} \tag{7.8}$$

and

$$\sum_{\widetilde{\mathsf{LD}}(S)} \omega(p_S, S_1, \Omega) \le C\delta + \varepsilon, \tag{7.9}$$

where

$$\inf_{T\in\mathcal{S}} \inf_{p\in\Gamma_T} \omega(p, T, \Omega) \ge 1 - \varepsilon.$$

Proof. The proof of (7.8) is the same as the proofs of (6.1) and (6.8) because by Part A of Theorem 1.2 (or hypothesis (ii) of Part B of Theorem 1.4), the Vitali argument from that proof can still be applied.

To prove (7.9), let $S \in \tilde{G}_n$ and for $1 \le k \le (n-1)$, let $T_k(S)$ be that unique $T \in \tilde{G}_k$ such that $S \subset T_k$. Let $S_1 \in \tilde{\mathsf{LD}}(S)$. Then $S_1 \subset \partial\Omega \subset \partial\Omega_n$ and

$$\omega(p_S, S_1, \Omega) = \omega(p_S, S_1, \Omega_n) + \sum_{k=1}^{n} \sum_{T \in \tilde{G}_k \setminus \{S_1\}} \int_{\Gamma_T} \omega(p, S_1, \Omega)\, d\omega(p_S, p, \Omega_n).$$

By definition and Theorem 1.2 Part A,

$$\sum_{S_1 \in \tilde{\mathsf{LD}}(S)} \omega(p_S, S_1, \Omega_n) \le \delta \sum_{\mathsf{LD}(S)} \left(\frac{\ell(S_1)}{\ell(S)}\right)^d \le C\delta,$$

while

$$\begin{aligned}
&\sum_{S_1 \in \tilde{\mathsf{LD}}(S)} \sum_{k=1}^{n} \sum_{T \in \tilde{G}_k \setminus \{S_1\}} \int_{\Gamma_T} \omega(p, S_1, \Omega)\, d\omega(p_S, dp, \Omega_n) \\
&\qquad = \int_{\Gamma_S} \sum_{S_1 \in \tilde{\mathsf{LD}}(S)} \omega(p, S_1, \Omega)\, d\omega(p_S, dp, \Omega_n) \\
&\qquad\quad + \sum_{k=1}^{n} \sum_{T \in \tilde{G}_k, T \cap S = \emptyset} \int_{\Gamma_T} \sum_{S_1 \in \tilde{\mathsf{LD}}(S)} \omega(p, S_1, \Omega)\, d\omega(p_S, dp, \Omega_n) \\
&\qquad\quad + \sum_{k=1}^{n-1} \int_{\Gamma_{T_k}} \sum_{\tilde{\mathsf{LD}}(S)} \omega(p, S_1, \Omega)\, d\omega(p_S, dp, \Omega_n) \\
&\qquad = \mathrm{I} + \mathrm{II} + \mathrm{III}.
\end{aligned}$$

By (7.2) and Harnack's inequality, we have

$$\mathrm{I} \lesssim \frac{2}{3} \sum_{\mathsf{LD}(S)} \omega(p_S, S_1, \Omega),$$

and we can move term I to the left side of (7.9).

For II, note that

$$(S \cup \Gamma_S) \cap \bigcup_{1 \le k \le n} \bigcup_{\{T \in \tilde{G}_k, T \cap S = \emptyset\}} \Gamma_T = \emptyset$$

so that by (7.3) we have $\mathrm{II} \le \varepsilon$.

For III, recall that $\mathrm{dist}(p_{T_k}, S) \ge c_2 2^{N(n-k)} \ell(S)$. Therefore

$$B(x_S, c_0 \ell(S)) \cap \bigcup_{1 \le k \le n-1} \Gamma_{T_k} = \emptyset,$$

so that by (1.23), $\mathrm{III} < C\varepsilon$.

That established (7.9) and Lemma 7.2. ∎

If $C\delta + \varepsilon$ is small, Lemma 7.2 and the proof of Lemma 6.2 yield

$$\sum_{k=1}^{\infty} \sum_{\tilde{G}_k} \Big(\frac{\ell(S_1)}{\ell(S)} \Big)^d \le C_3 \tag{7.10}$$

for any $S \in \mathcal{S}$.

By (7.1) and (7.10), $\widetilde{\Omega} = \bigcup \widetilde{\Omega}_n$ is a connected open set and

$$\partial\widetilde{\Omega} = \partial\Omega \cup \bigcup_{n=1}^{\infty} \bigcup_{S \in \tilde{G}_n} \Gamma_S.$$

By (7.7), σ is a finite measure on $\partial\widetilde{\Omega}$ such that for all $S \in \mathcal{S}$,

$$c_{21}\, \ell(S)^d \le \sigma(S^\infty) \le c_{22}\, \ell(S)^d,$$

and by Lemma 7.1 and the definition of ν_{n+1},

$$\sigma(E) = \mathcal{H}^d(E)$$

for all Borel $E \subset \bigcup \Gamma_S$. In view of properties (1.13) and (1.17) of $\mathcal{S}$, these imply that σ is a d-Ahlfors regular measure with closed support $\partial\widetilde{\Omega}$ and hence that $\partial\widetilde{\Omega}$ is d-Ahlfors regular. Moreover, the family

$$\mathcal{S}^\infty = \bigcup_{S \in \mathcal{S}} S^\infty \cup \bigcup_{S \in \cup_n \tilde{G}_n} \mathcal{F}_S,$$

where $\mathcal{F}_S$ is the dyadic decomposition of Γ_S in spherical coordinates, is a family of Christ–David cubes for $\partial\widetilde{\Omega}$, and by construction $\widetilde{\Omega}$ satisfies the corkscrew condition (1.4).

8. Proof of Theorem 1.1 Part B

To prove Theorem 1.1 Part B, we assume Ω is a corkscrew domain satisfying (1.4) and either (a) or (b) and we let $\widetilde{\Omega}$ be the domain constructed from Ω in Section 7. Recall that $\partial\widetilde{\Omega}$ is d-Ahlfors regular. We will prove $\partial\widetilde{\Omega}$ is uniformly rectifiable by repeating the proof of Lemma 6.2 and applying Proposition 5.1 of [17]. Define $G_0^*(S_0^\infty) = \{S_0^\infty\}$ and by induction, for $S^\infty \in G_n^*$ define

$$\begin{aligned}
\mathsf{HD}(S^\infty) &= \Big\{ S_1^\infty \in \mathcal{S}^\infty : S_1^\infty \subset S^\infty, \omega(p_S, \lambda S_1^\infty, \widetilde{\Omega}) \ge A\Big(\frac{\ell(S_1)}{\ell(S)}\Big)^d, S_1^\infty \text{ maximal} \Big\},\\
\mathsf{LD}(S^\infty) &= \Big\{ S_1^\infty \in \mathcal{S}^\infty : S_1^\infty \subset S^\infty, \omega(p_S, S_1^\infty, \widetilde{\Omega}) \le \delta\Big(\frac{\ell(S_1)}{\ell(S)}\Big)^d, S_1^\infty \text{ maximal} \Big\},\\
G_1^*(S^\infty) &= \big\{ S_1^\infty \in \mathsf{HD}(S^\infty) \cup \mathsf{LD}(S^\infty), S_1^\infty \text{ maximal} \big\},\\
\mathsf{Tree}(S^\infty) &= \big\{ S_1^\infty \in \mathcal{S} : S_1^\infty \subset S^\infty,\ \ S_1^\infty \not\subset S_2^\infty \text{ for all } S_2^\infty \in G_1^*(S^\infty) \big\},
\end{aligned}$$

and

$$G_{n+1}^* = \bigcup_{S^\infty \in G_n^*} G_1^*(S^\infty).$$

Lemma 8.1. *Let $S^\infty \in G_n^*$. Then*

$$\sum_{S_1^\infty \in \mathsf{HD}(S^\infty)} \left(\frac{\ell(S_1)}{\ell(S)}\right)^d \leq \frac{C_1}{A} \tag{8.1}$$

and

$$\sum_{S_1^\infty \in \mathsf{LD}(S^\infty)} \omega(p_S, S_1, \Omega) \leq C\delta + \varepsilon, \tag{8.2}$$

where

$$\inf_{T\in\mathcal{S}} \inf_{p\in\Gamma_T} \omega(p, T, \Omega) \geq 1-\varepsilon.$$

Proof. The proof of (8.1) is the same as the proof of (6.8). To prove (8.2), we follow the proof of (6.2) and (7.9). Let $S_1^\infty \in \mathsf{LD}(S^\infty)$. Then

$$\omega(p_S, S_1, \Omega) \leq \omega(p_S, S_1^\infty, \widetilde{\Omega}) + \sum_{k\geq 1} \sum_{G_k^*\setminus\{S_1\}} \int_{\Gamma_T} \omega(p, S_1, \Omega)\, d\omega(p_S, p, \widetilde{\Omega}).$$

By definition and Theorem 1.2 Part A,

$$\sum_{S_1^\infty \in \mathsf{LD}(S^\infty)} \omega(p_S, S_1^\infty, \widetilde{\Omega}) \leq \delta \sum_{\mathsf{LD}(S^\infty)} \left(\frac{\ell(S_1^\infty)}{\ell(S^\infty)}\right)^d \leq C\delta, \tag{8.3}$$

and

$$\begin{aligned}
&\sum_{S_1^\infty \in \mathsf{LD}(S^\infty)} \sum_{k=1}^{\infty} \sum_{T\in G_k^*\setminus\{S_1\}} \int_{\Gamma_T} \omega(p, S_1, \Omega)\, d\omega(p_S, dp, \widetilde{\Omega}) \\
&\quad = \int_{\Gamma_S} \sum_{S_1^\infty \in \mathsf{LD}(S^\infty)} \omega(p, S_1, \Omega)\, d\omega(p_S, dp, \widetilde{\Omega}) \\
&\qquad + \sum_{k=1}^{\infty} \sum_{T\in G_k^*, T\cap S=\emptyset} \int_{\Gamma_T} \sum_{S_1^\infty \in \mathsf{LD}(S^\infty)} \omega(p, S_1, \Omega)\, d\omega(p_S, p, \widetilde{\Omega}) \\
&\qquad + \sum_{k=1}^{n-1} \int_{\Gamma_{T_k}} \sum_{S_1^\infty \in \mathsf{LD}(S^\infty)} \omega(p, S_1, \Omega)\, d\omega(p_S, p, \widetilde{\Omega}) \\
&\qquad + \sum_{S_1\in G_1^*(S)} \sum_{T\in \bigcup_k G_k^*(S_1)} \int_{\Gamma_T} \omega(p, S_1, \Omega)\, d\omega(p_S, p, \widetilde{\Omega}) \\
&\quad = \mathrm{I}' + \mathrm{II}' + \mathrm{III}' + \mathrm{IV}'.
\end{aligned}$$

Here I', II' and III' can be handled the same way as I, II, and III were, while $\mathrm{IV}' \leq C\varepsilon$ by (8.3). ∎

Thus if $C\delta + \varepsilon$ is small, Lemma 7.2 and Lemma 6.2 yield

$$\sum_{k=1}^{\infty} \sum_{G_k^*} \Big(\frac{\ell(S_1)}{\ell(S)}\Big)^d \leq C_3 \tag{8.4}$$

for any $S^\infty \in \mathcal{S}^\infty$ and any $S_1^\infty \in \mathsf{Tree}(S^\infty)$,

$$\delta\Big(\frac{\ell(S_1)}{\ell(S)}\Big)^d \leq \omega(p_S, \lambda S_1^\infty, \widetilde{\Omega}) \leq A\Big(\frac{\ell(S_1)}{\ell(S)}\Big)^d. \tag{8.5}$$

By (8.5) and Proposition 5.1 of [17], this proves $\partial\widetilde{\Omega}$ is uniformly rectifiable, and that establishes Part B of Theorem 1.1.

9. Proof of Theorem 1.4 Part B and Theorem 1.2 Part B

To prove Part B of Theorem 1.4 note that under its hypotheses the arguments in Section 7 and Section 8 show that the constructed domain $\widetilde{\Omega}$ has uniformly rectifiable boundary. Therefore by Part A of Theorem 1.1, (a) and (b) hold for Ω.

To prove Part B of Theorem 1.2 note that its hypotheses imply Proposition 1.3 and hence condition (ii) of Part B of Theorem 1.4. Then the argument in Section 6 yields (1.29), so that Part B of Theorem 1.4 implies Part B of Theorem 1.2.

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