Rev. Mat. Iberoam. 38 (2022), no. 2, 355–432
DOI 10.4171/RMI/1296

Classification of finite Morse index solutions to the elliptic sine-Gordon equation in the plane

Yong Liu and Juncheng Wei

Abstract. The elliptic sine-Gordon equation is a semilinear elliptic equation with a special double well potential. It has a family of explicit multiple-end solutions. We show that all finite Morse index solutions belong to this family. It will also be proved that these solutions are nondegenerate, in the sense that the corresponding linearized operators have no nontrivial bounded kernel. Finally, we prove that the Morse index of $2n$-end solutions is equal to $n(n-1)/2$.

1. Introduction and statement of the main results

This paper is concerned with the finite Morse index solutions to the *elliptic* sine-Gordon equation in the plane. Before explicitly writing down the equation and stating our results, let us briefly mention the classical sine-Gordon equation, which originated from the study of surfaces with constant negative curvature in the nineteenth century. We shall call it *hyperbolic* sine-Gordon equation throughout the paper. The hyperbolic sine-Gordon equation also appears in various physical contexts such as the Josephson junction. It has been extensively studied partly due to the facts that it is integrable and that one can use the technique of the inverse scattering transform to analyze its solutions. There exists a vast literature on this subject. We refer to the book [10] and the references therein for more information about the background and detailed discussion for the hyperbolic sine-Gordon equation.

In the laboratory coordinates, the hyperbolic sine-Gordon equation takes the form

$$\partial_z^2 u - \partial_x^2 u + \sin u = 0. \tag{1.1}$$

In this paper, the elliptic version of this equation will be investigated. More precisely, we shall consider the following elliptic sine-Gordon equation:

$$-\Delta u = \sin u \quad \text{in } \mathbb{R}^2, \ |u| < \pi, \tag{1.2}$$

where $\Delta = \partial_x^2 + \partial_y^2$.

2020 Mathematics Subject Classification: Primary 35B08; Secondary 37K35.
Keywords: Sine-Gordon equation, finite Morse index, multiple-end solution, inverse scattering.

The reason why we are interested in this equation stems from the fact that (1.2) is actually a special case of the Allen–Cahn type equations

$$\Delta u = W'(u) \quad \text{in } \mathbb{R}^N, \tag{1.3}$$

where W are double well potentials. This equation is the Euler–Lagrangian equation of the energy functional

$$J := \int \Big(\frac{1}{2} |\nabla u|^2 + W(u) \Big).$$

Choosing $W = \cos u$, we obtain equation (1.2). On the other hand, if $W(u) = \frac{1}{4}(u^2-1)^2$, then (1.3) reduces to the classical Allen–Cahn equation:

$$-\Delta u = u - u^3 \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

This is an important model in the theory of phase transitions.

A crucial property of Allen–Cahn type equations (1.3) is that they possess one-dimensional monotone increasing heteroclinic solutions, which connect two stable states in the phase transition phenomenon. In the case of (1.2), the one-dimensional heteroclinic solution is given explicitly by

$$H(x) = 4 \arctan e^x - \pi.$$

The celebrated De Giorgi conjecture concerns the classification of monotone bounded solutions of the Allen–Cahn type equation (1.3). Many works have been done towards a complete resolution of this conjecture. In particular, it is known that in dimension two and three, monotone bounded solutions must be one dimensional. We refer to [2, 13, 16–18, 23, 35, 42, 52] and the references cited there for results in this direction. A natural generalization of the De Giorgi conjecture is to classify those solutions not necessary monotone. This seems to be a difficult problem for general nonlinearities W. In this paper, we are interested in those non-monotone solutions in the plane for the special case of the elliptic sine-Gordon equation.

Without any assumption on the asymptotic behavior of the solutions at infinity, the structure of the solution set could be extremely complicated. To bypass this difficulty, let us recall the following.

Definition 1.1 (See [11, 12]). A solution u of (1.2) is called a multiple-end ($2n$-end) solution if, outside a large ball, the nodal set of u is asymptotic to $2n$ half straight lines.

These asymptotic half straight lines are called ends of the solution. One can show that actually along these lines, the multiple-end solution u behaves like the one dimensional solution H in the transverse direction. The set of $2n$-end solution will be denoted by $\mathcal{M}_{2n}$. By the curvature decay estimates of Wang–Wei [55], a solution is multiple-end if and only if it has finite Morse index.

In [12], the infinite dimensional Lyapunov–Schmidt reduction method has been used to construct a family of $2n$-end solutions for the Allen–Cahn equation (1.4). The method there can also be applied to general double well potentials, including the elliptic sine-Gordon equation (1.2). The nodal sets of these solutions consist of almost parallel curves. In particular, the angles between consecutive ends are close to 0 or π. Actually, the nodal curves are given approximately by suitable rescaled solutions of the Toda system. It is also known that locally around each $2n$-end solution, the moduli space of $2n$-end solutions has the structure of a real analytic variety. If the solution happens to be nondegenerate, then locally around it, the moduli space is indeed a $2n$-dimensional manifold [11]. For general nonlinearities, little is known for the structure of the moduli space of $2n$-end solutions, except in the $n = 2$ case. In this case, a Hamiltonian identity has been used in [26, 27] to study the symmetry properties of these solutions. It is now known [36–38] that the space of four-end solutions is diffeomorphic to the open interval $(0, 1)$, modulo translation and rotation (they give 3 free parameters in the moduli space). Based on these four-end solutions, an end-to-end construction for $2n$-end solutions has been carried out in [39]. Roughly speaking, solutions arising from this construction are near the "boundary" of the moduli space.

The classification of $\mathcal{M}_{2n}$ is still largely open for general double well nonlinearities. Important open questions include: are solutions in $\mathcal{M}_{2n}$ nondegenerate? Is $\mathcal{M}_{2n}$ connected? What is the Morse index of the solutions in $\mathcal{M}_{2n}$? In a recent paper, Mantoulidis [43] proves a lower bound $n - 1$ on the Morse index of solutions in $\mathcal{M}_{2n}$ for the Allen–Cahn equation. Here we shall give a complete answer to the above questions in the case of the elliptic sine-Gordon equation (1.2).

It is well known that the classical sine-Gordon equation (1.1) is an integrable system. Methods from the theory of integrable systems can be used to find solutions of this system. In particular, it has soliton solutions. Note that (1.2) is elliptic, while (1.1) is hyperbolic in nature. We show in this paper that the Hirota direct method of integrable systems also gives us real nonsingular solutions of (1.2). Let U_n be the functions defined by (2.15). Then U_n are solutions to (1.2). They depend on $2n$ parameters, p_j, η_j^0, $j = 1, \ldots, n$. We are interested in the spectral property of these solutions and shall prove the following.

Theorem 1.2. *Each $U_n \in \mathcal{M}_{2n}$ is L^∞-nondegenerate in the following sense: if φ is a bounded solution of the linearized equation*

$$-\Delta\varphi - \varphi \cos U_n = 0,$$

then there exist constants c_j, $j = 1, \ldots, n$, such that

$$\varphi = \sum_{j=1}^{n} (c_j \partial_{\eta_j^0} U_n).$$

We remark that the nonlinear stability of 2-soliton solutions of the classical hyperbolic sine-Gordon equation (1.1) has been proved recently by Muñoz–Palacios [44], also using the Bäcklund transformation. We refer to the references therein for more discussion on the dynamical properties of the hyperbolic sine-Gordon equation. For general background and applications of the Bäcklund transformation, we refer to [50, 51].

The Morse index of U_n is by definition the number of negative eigenvalues of the operator $-\Delta - \cos U_n$, in the space $H^1(\mathbb{R}^2)$, counted with multiplicity. It can also be defined as the maximal dimension of the space of compactly supported smooth functions where the associated quadratic form of the energy functional J is negative. Our next result is:

Theorem 1.3. *The set $\mathcal{M}_{2n}$ of $2n$-end solutions of the elliptic sine-Gordon equation (1.2) is a $2n$-dimensional connected smooth manifold. The Morse index of U_n is equal to $n(n-1)/2$. Moreover, all the finite Morse index solutions of (1.2) are of the form U_n, with suitable choice of the parameters p_j, q_j, η_j^0, $j = 1, \ldots, n$.*

We emphasize that the parameters p_j and q_j are not independent. Actually they have to satisfy $p_j^2 + q_j^2 = 1$. The classification result stated in this theorem follows from a direct application of the inverse scattering transform studied in [28]. Inverse scattering of elliptic sine-Gordon equation has also been used in [3, 4] to study solutions with periodic asymptotic behavior or vortex type singularities. Note that certain classes of vortex type solutions were analyzed through the Bäcklund transformation or the direct method in [34, 41, 45, 53], and finite energy solutions with point-like singularities have been studied in [47]. It is also worth mentioning that more recently, some classes of quite involved boundary value problems of the elliptic sine-Gordon equation have been investigated via Fokas' direct method in [19, 20, 48, 49].

Theorem 1.3 implies that in the special case $n = 2$, the four-end solutions of the equation (1.2) have Morse index one. In the family of four-end solutions, there is a special one, called saddle solution (see (2.16)), explicitly given by

$$4 \arctan\left(\frac{\cosh(y/\sqrt{2})}{\cosh(x/\sqrt{2})}\right) - \pi.$$

The nodal set of this solution consists of two orthogonally intersected straight lines. Saddle-shaped solutions of Allen–Cahn type equation $\Delta u = W'(u)$ in $\mathbb{R}^{2k}$ with $k \geq 2$ have been studied by Cabré and Terra in a series of papers. In [5–7], it is proved that in $\mathbb{R}^4$ and $\mathbb{R}^6$, the saddle-shaped solutions are unstable, while in $\mathbb{R}^{2k}$ with $k \geq 7$, they are stable. It is also conjectured in [5] that for $k \geq 4$, the saddle-shaped solution should be a global minimizer of the energy functional. However, the generalized elliptic sine-Gordon equation ($-\Delta u = \sin u$) in even dimension higher than two is believed to be non-integrable, hence no explicit formulas are available for these saddle-shaped solutions. We expect that the nondegeneracy results in this paper will be useful in the construction of solutions of the generalized elliptic sine-Gordon equation in higher dimensions.

It is worth pointing out that $W(u) = 1 + \cos u$ is essentially the only double well potential such that the corresponding equation is integrable [14]. Note that the sine nonlinearity also appears in the Peierls–Nabarro equation, whose solutions have been classified in [54]. A classification result like Theorem 1.3 for general double well potentials could be very difficult.

Finally, we mention that recently there have been some interesting works on the construction of minimal surfaces using Allen–Cahn type equations. See, for instance, [9, 21, 22, 25, 43]. Based on these links between minimal surfaces and Allen–Cahn type equations, it is expected that the classification results obtained in this paper could be used to provide another proof of the existence of infinitely many closed geodesics on any given Riemann surface. Actually this is one of our main motivations to study the elliptic sine-Gordon equation.

The paper is organized as follows. In Section 2, we write down an explicit family of $2n$-end solutions U_n for the elliptic sine-Gordon equation. We investigate the Bäcklund transformation of these solutions in Section 3. The nondegeneracy of U_n and Theorem 1.2 will be proved in Section 4. In Section 5, we classify all the $2n$-end solutions by their asymptotic behavior at infinity. Finally, in Section 6, we compute the Morse index of these solutions using a deformation argument and prove Theorem 1.3.

2. A family of multiple-end solutions of the elliptic sine-Gordon equation

In this section, for each $n \in \mathbb{N}$, we shall write down a family of explicit, *real valued*, *nonsingular* solutions of the elliptic sine-Gordon equation:

$$-\partial_x^2 u - \partial_y^2 u = \sin u \quad \text{in } \mathbb{R}^2. \tag{2.1}$$

We will see that these solutions are indeed $2n$-ended, hence of finite Morse index. It turns out that this family of solutions has $2n$ free parameters. This also means that this set of solutions is a $2n$-dimensional manifold.

Equation (2.1) has been studied by Leibbrandt in [41] using the Bäcklund transformation, with an application to the Josephson effect. However, the solutions he found are singular somewhere in the plane. Gutshabash–Lipovskiĭ [28] studied the boundary value problem of the elliptic sine-Gordon equation in the half plane using the inverse scattering transform and obtained multi-soliton solutions in the determinant form, with certain parameters. The question that for which parameters will the solutions be real and nonsingular was not considered there. The boundary problems of (2.1) in a half plane or a quarter have also been studied by the Fokas direct method, see [19, 20, 48, 49].

The construction of *explicit* multi-soliton solutions of the hyperbolic (classical) sine-Gordon equation (1.1) was carried out in [29], using the Hirota direct method. It is worth mentioning that there are also related results on certain soliton solutions in higher dimensions, we refer to [24, 30, 31, 53, 56] for further discussions in this direction. Note that the solutions of the hyperbolic sine-Gordon equation obtained in [29] contain free parameters. At this point, let us emphasize that for many integrable systems, it is usually a delicate issue to determine for which parameters the solutions are real and nonsingular. As we will see, this issue is actually closely related with our analysis of the elliptic sine-Gordon equation (2.1) in this paper.

It turns out to be more convenient to replace u by $u + \pi$ in (2.1). The equation then transforms into

$$\partial_x^2 u + \partial_y^2 u = \sin u. \tag{2.2}$$

Our first observation is the following: in the hyperbolic sine-Gordon equation (1.1), if we introduce the change of variable $z = yi$, where i represents the complex unit, then we arrive at equation (2.2). Based on this, by choosing certain complex parameters for the solutions of the hyperbolic sine-Gordon equation of [29], we then get multiple-end solutions of the elliptic sine-Gordon equation. The case of 2-soliton has been studied in [53].

To obtain solutions in closed form, we shall write the sine-Gordon equation in bilinear form. Let D be the bilinear derivative operator (see [32], page 27). For any $j, k \in \mathbb{N}$, and two functions ϕ, η, we have

$$D_x^j D_y^k \phi \cdot \eta := \left[(\partial_x - \partial_{x'})^j (\partial_y - \partial_{y'})^k \right] \left[\phi(x, y) \eta(x', y') \right] \big|_{x'=x, y'=y}.$$

For instance,

$$D_x D_y \phi \cdot \eta = \eta \partial_x \partial_y \phi - \partial_x \phi \partial_y \eta - \partial_y \phi \partial_x \eta + \phi \partial_x \partial_y \eta.$$

Throughout the paper, we use $\bar{F}$ to denote the complex conjugate of F. Let us take the bi-logarithmic transformation:

$$u = 2i \ln \frac{\bar{F}}{F}.$$

Note that the log function is multiple-valued. Here we simply take the principle branch. One can also choose other branches, which amounts to add $4k\pi, k \in \mathbb{Z}$, to the function u. We compute

$$\sin u = \frac{e^{iu} - e^{-iu}}{2i} = \frac{1}{2i}\Big(\frac{F^2}{\bar{F}^2} - \frac{\bar{F}^2}{F^2}\Big), \quad \partial_x^2 u = i\Big(\frac{D_x^2 \bar{F} \cdot \bar{F}}{\bar{F}^2} - \frac{D_x^2 F \cdot F}{F^2}\Big).$$

Then equation (2.2) can be written as

$$\big[(D_x^2 + D_y^2)F \cdot F + \tfrac{1}{2}(\bar{F}^2 - F^2)\big]\bar{F}^2 - \big[(D_x^2 + D_y^2)\bar{F} \cdot \bar{F} + \tfrac{1}{2}(F^2 - \bar{F}^2)\big]F^2 = 0.$$

We also refer to [32], page 45, for the derivation of the bilinear form in the case of the hyperbolic sine-Gordon equation. We then get the following bilinear form of equation (2.2):

$$(D_x^2 + D_y^2)F \cdot F + \frac{1}{2}(\bar{F}^2 - F^2) = \lambda F^2, \tag{2.3}$$

where λ is a real parameter. This means that if F satisfies (2.3), then u will be a solution to (2.2). On the other hand, if (2.2) is true, then we can consider the function

$$\rho(x, y) := \frac{(D_x^2 + D_y^2)F \cdot F + \frac{1}{2}(\bar{F}^2 - F^2)}{F^2}.$$

Writing ρ into real and imaginary parts, $\rho_1(x, y) + i\rho_2(x, y)$, we see that $\rho_2 = 0$. Hence necessary (at least when $F \neq 0$) there holds

$$(D_x^2 + D_y^2)F \cdot F + \frac{1}{2}(\bar{F}^2 - F^2) = \rho_1 F^2.$$

Fix an integer $n \in \mathbb{N}$. Let $p_j, q_j, j = 1, \ldots, n$, be *real* numbers satisfying $p_j^2 + q_j^2 = 1$. Define

$$\alpha(j, k) := \frac{(p_j - p_k)^2 + (q_j - q_k)^2}{(p_j + p_k)^2 + (q_j + q_k)^2}. \tag{2.4}$$

We will always assume throughout the paper that $(p_j, q_j) \neq \pm(p_l, q_l)$, for $j \neq l$. This assumption is consistent with our classification result in Section 4. Note that $\alpha(j, k) = \alpha(k, j) \geq 0$. Moreover, since $p_j^2 + q_j^2 = 1$, we have

$$p_j - iq_j = \frac{1}{p_j + iq_j}. \tag{2.5}$$

Therefore, we can rewrite α in the form

$$\begin{aligned} \alpha(j, k) &= \frac{(p_j - p_k + iq_j - iq_k)\left(\frac{1}{p_j+iq_j} - \frac{1}{p_k+iq_k}\right)}{(p_j + p_k + iq_j + iq_k)\left(\frac{1}{p_j+iq_j} + \frac{1}{p_k+iq_k}\right)} \\ &= -\frac{(p_j - p_k + iq_j - iq_k)^2}{(p_j + p_k + iq_j + iq_k)^2}. \end{aligned} \tag{2.6}$$

We then define

$$
\begin{aligned}
&a(j_1,\dots,j_m) := 1, \quad \text{if } m = 0,1,\\
&a(j_1,\dots,j_m) := \prod_{k<l\le m} \alpha(j_k,j_l), \quad \text{if } m \ge 2.
\end{aligned}
\tag{2.7}
$$

Let us introduce the notation $\eta_j = p_j x + q_j y + \eta_j^0$, $j = 1,\dots,n$, where at this moment, the η_j^0 are *real* parameters. Then we define

$$
f_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \Big(\sum_{\{n,2k\}} \left[a\left(j_1,\dots,j_{2k}\right) \exp\left(\eta_{j_1} + \dots + \eta_{j_{2k}}\right)\right]\Big), \tag{2.8}
$$

$$
g_n = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \Big(\sum_{\{n,2k+1\}} \left[a\left(j_1,\dots,j_{2k+1}\right) \exp\left(\eta_{j_1} + \dots + \eta_{j_{2k+1}}\right)\right]\Big). \tag{2.9}
$$

Here the notation $\sum_{\{n,k\}}$ means summing over all possible k different integers $j_1,\dots,j_k$ from the set of integers $\{1,\dots,n\}$. The floor function $\lfloor x \rfloor$ represents the greatest integer less than or equal to x.

In the special case $n = 3$, we have

$$
\begin{aligned}
f_3 &= \sum_{k=0}^{1} \Big(\sum_{\{3,2k\}} a(j_1,\dots,j_{2n}) \exp(\eta_{j_1} + \dots + \eta_{j_{2k}})\Big)\\
&= 1 + a(1,2)\exp(\eta_1+\eta_2) + a(1,3)\exp(\eta_1+\eta_3) + a(2,3)\exp(\eta_2+\eta_3)\\
&= 1 + \alpha(1,2)\exp(\eta_1+\eta_2) + \alpha(1,3)\exp(\eta_1+\eta_3) + \alpha(2,3)\exp(\eta_2+\eta_3),\\
g_3 &= \sum_{k=0}^{1} \Big(\sum_{\{3,2k+1\}} a(j_1,\dots,j_{2k+1}) \exp(\eta_{j_1} + \dots + \eta_{j_{2k+1}})\Big)\\
&= \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + a(1,2,3)\exp(\eta_1+\eta_2+\eta_3)\\
&= \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + \alpha(1,2)\alpha(1,3)\alpha(2,3)\exp(\eta_1+\eta_2+\eta_3).
\end{aligned}
$$

It is worth mentioning that these solutions can also be written in the determinant form ([46]). Here we choose to use the form (2.8), (2.9) because it is more convenient to check the positiveness of the function.

Theorem 2.1. *For each n, let f_n and g_n be defined by* (2.8) *and* (2.9)*. Then the function* $4\arctan(g_n/f_n)$ *is a solution to the elliptic sine-Gordon equation* (2.2).

Proof. The proof is similar to that of [29]. We sketch it for completeness.

For fixed integer n, we would like to find explicit n-soliton solutions of the bilinear equation (2.3), with the parameter λ being zero. The equation to be solved becomes

$$
(D_x^2 + D_y^2) F \cdot F + \frac{1}{2}(\bar{F}^2 - F^2) = 0. \tag{2.10}
$$

Note that the constant 1 is a solution to this equation. The key idea is to seek solutions with formal expansion in powers of ε:

$$F = 1 + \varepsilon F_1 + \varepsilon^2 F_2 + \cdots \tag{2.11}$$

We will see that for the n-soliton solutions stated in Theorem 2.1, this power series truncates into a polynomial of ε with degree n.

Inserting (2.11) into (2.10), we find that for the $O(\varepsilon)$ terms there holds

$$(D_x^2 + D_y^2) F_1 \cdot 1 + \frac{1}{2}\left(\bar{F}_1 - F_1\right) = 0. \tag{2.12}$$

For the $O(\varepsilon^2)$ terms,

$$2(D_x^2 + D_y^2) F_2 \cdot 1 + \left(D_x^2 + D_y^2\right) F_1 \cdot F_1 = -\frac{1}{2}\left(\bar{F}_1^2 - F_1^2 + 2\bar{F}_2 - 2F_2\right). \tag{2.13}$$

The $O(\varepsilon^3)$ terms are

$$(D_x^2 + D_y^2) F_3 \cdot 1 + \left(D_x^2 + D_y^2\right) F_2 \cdot F_1 - \frac{1}{2}\left(\bar{F}_2 \bar{F}_1 - F_2 F_1 + \bar{F}_3 - F_3\right). \tag{2.14}$$

The expansion can be further performed to any higher order.

Let us choose

$$F_1 := i \sum_{j=1}^{n} \exp(\eta_j).$$

Since $p_j^2 + q_j^2 = 1$, we see that (2.12) is satisfied by this choice. Moreover, a direct computation shows that

$$(D_x^2 + D_y^2) F_1 \cdot F_1 = -2 \sum_{j_1 < j_2} \left[\left((p_{j_1} - p_{j_2})^2 + (q_{j_1} - q_{j_2})^2\right) \exp(\eta_{j_1} + \eta_{j_2})\right].$$

We now define

$$F_2 := \sum_{j_1 < j_2} \left[a(j_1, j_2) \exp(\eta_{j_1} + \eta_{j_2})\right].$$

Here the index $j_2 \leq n$. Then we can compute

$$(D_x^2 + D_y^2) F_2 \cdot 1 = \sum_{j_1 < j_2} \left[a(j_1, j_2)\left((p_{j_1} + p_{j_2})^2 + (q_{j_1} + q_{j_2})^2\right) \exp(\eta_{j_1} + \eta_{j_2})\right].$$

From this, using the definition (2.4) of $a(j_1, j_2)$, we find that

$$2(D_x^2 + D_y^2) F_2 \cdot 1 + \left(D_x^2 + D_y^2\right) F_1 \cdot F_1 = 0.$$

Hence equation (2.13) also holds.

To proceed, we define

$$F_3 := i \sum_{j_1 < j_2 < j_3} \left[a(j_1, j_2, j_3) \exp(\eta_{j_1} + \eta_{j_2} + \eta_{j_3})\right].$$

We would like to show that with this choice, the ε^3 order terms (2.14) sum up to zero. Indeed, for fixed triple $j_1 < j_2 < j_3$, a direct computation tells us that in (2.14), the coefficient J before $i\exp(\eta_{j_1} + \eta_{j_2} + \eta_{j_3})$ is

$$\begin{aligned} &a(j_1, j_2)\left((p_{j_1} + p_{j_2} - p_{j_3})^2 + (q_{j_1} + q_{j_2} - q_{j_3})^2 - 1\right) \\ &\quad + a(j_2, j_3)\left((p_{j_2} + p_{j_3} - p_{j_1})^2 + (q_{j_2} + q_{j_3} - q_{j_1})^2 - 1\right) \\ &\quad + a(j_1, j_3)\left((p_{j_1} + p_{j_3} - p_{j_2})^2 + (q_{j_1} + q_{j_3} - q_{j_2})^2 - 1\right) \\ &\quad + a(j_1, j_2, j_3)\left((p_{j_1} + p_{j_2} + p_{j_3})^2 + (q_{j_1} + q_{j_2} + q_{j_3})^2 - 1\right). \end{aligned}$$

Using (2.5) and (2.6), setting $v_j := p_j + iq_j$, we find that J is equal to

$$\begin{aligned} &\frac{(v_{j_1} - v_{j_2})^2}{(v_{j_1} + v_{j_2})^2}\left(1 - (v_{j_1} + v_{j_2} - v_{j_3})\left(\frac{1}{v_{j_1}} + \frac{1}{v_{j_2}} - \frac{1}{v_{j_3}}\right)\right) \\ &+ \frac{(v_{j_2} - v_{j_3})^2}{(v_{j_2} + v_{j_3})^2}\left(1 - (v_{j_2} + v_{j_3} - v_{j_1})\left(\frac{1}{v_{j_2}} + \frac{1}{v_{j_3}} - \frac{1}{v_{j_1}}\right)\right) \\ &+ \frac{(v_{j_1} - v_{j_3})^2}{(v_{j_1} + v_{j_3})^2}\left(1 - (v_{j_1} + v_{j_3} - v_{j_2})\left(\frac{1}{v_{j_1}} + \frac{1}{v_{j_3}} - \frac{1}{v_{j_2}}\right)\right) \\ &+ \frac{(v_{j_1} - v_{j_2})^2(v_{j_2} - v_{j_3})^2(v_{j_1} - v_{j_3})^2}{(v_{j_1} + v_{j_2})^2(v_{j_2} + v_{j_3})^2(v_{j_1} + v_{j_3})^2}\left(1 - (v_{j_1} + v_{j_2} + v_{j_3})\left(\frac{1}{v_{j_1}} + \frac{1}{v_{j_2}} + \frac{1}{v_{j_3}}\right)\right). \end{aligned}$$

Multiplying it by $(v_{j_1} + v_{j_2})^2(v_{j_2} + v_{j_3})^2(v_{j_1} + v_{j_3})^2 v_{j_1} v_{j_2} v_{j_3}$, we obtain a homogeneous polynomial in $v_{j_1}, v_{j_2}, v_{j_3}$, of degree 9. Let us denote this polynomial by $L(v_{j_1}, v_{j_2}, v_{j_3})$. Observe that $(v_{j_1}^2 - v_{j_2}^2)^2$ is a factor of L. Due to symmetry, this implies that L is a polynomial of degree at least 12. Hence L has to be identically zero. Next we consider the special case that the triple (j_1, j_2, j_3) has repeated indices, for instance, $j_1 = j_2 < j_3$. Observe that L is continuous respect to $v_{j_1}, v_{j_2}, v_{j_3}$. Hence sending v_{j_2} to v_{j_1}, we see that in this special case, we also have $L = 0$. This proves that (2.14) is zero. Note that the case of repeated indices can also be directly proved in the same way as the general case, by regarding $v_{j_1}, v_{j_2}, v_{j_3}$ as abstract variables.

Now for $4 \le j \le n$, let us define

$$F_j := \exp\left((1 - (-1)^j)\frac{\pi i}{4}\right) \sum_{l_1 < \cdots < l_j \le n} \left[a(l_1, \ldots, l_j)\exp(\eta_{l_1} + \cdots + \eta_{l_j})\right].$$

In particular, this implies that for odd j, F_j is purely imaginary; while for even j, F_j is real valued. We also set $F_j = 0$ if $j > n$.

We claim that the $O(\varepsilon^k)$ terms sum up to zero in the power series expansion of ε for each $k \ge 4$. We only consider the case of k being odd. The proof is similar if k is even.

For fixed indices $j_1 \le \cdots \le j_k$, the coefficient before $i\exp(\eta_{j_1} + \eta_{j_2} + \cdots + \eta_{j_k})$ is equal to $\sum_l G_l$, where

$$G_l := \sum_{m(l)} \left[\alpha(j_{m_1}, \ldots, j_{m_l})\,\alpha(j_{m_{l+1}}, \ldots, j_{m_k})(h - 1)\right].$$

Here

$$h := (v_{j_{m_1}} + \cdots + v_{j_{m_l}} - v_{j_{m_{l+1}}} - \cdots - v_{j_{m_k}})(v_{j_{m_1}}^{-1} + \cdots + v_{j_{m_l}}^{-1} - v_{j_{m_{l+1}}}^{-1} - \cdots - v_{j_{m_k}}^{-1}),$$

the notation $\sum_{m(l)}$ means summation over indices $m_1, \ldots, m_k$ satisfying $m_j \leq k$, and

$$m_1 < \cdots < m_l; \quad m_{l+1} < \cdots < m_k.$$

Multiplying G_l by $\left(\prod_{l=1}^{k} v_{j_l}\right)\left(\prod_{a<b\leq k}(v_{j_a} + v_{j_b})^2\right)$, we get a homogeneous polynomial L with degree k^2. On the other hand, the function $(v_{j_l}^2 - v_{j_m}^2)^2$ is a factor of L. Hence the degree of L is at least $2k(k-1)$. It follows that L is identically zero. This finishes the proof of the claim.

Finally, we take $\varepsilon = 1$ and set $f_n = \operatorname{Re} F$, $g_n = \operatorname{Im} F$. Then we have

$$2i \ln \frac{\bar{F}}{F} = 4 \arctan \frac{g_n}{f_n} \cdot$$

The proof of the theorem is thereby completed. ∎

Note that f_n and g_n are both positive functions. By Theorem 2.1, the functions

$$U_n := 4 \arctan \frac{g_n}{f_n} - \pi \tag{2.15}$$

are a family of smooth solution to the elliptic sine-Gordon equation (2.1), with p_j, q_j, η_j^0 being parameters. Note that $-\pi < U_n < \pi$.

Next, we would like to analyze the asymptotic behavior of U_n at infinity. We have the following.

Lemma 2.2. *Let $c \in \mathbb{R}$ be a fixed constant and let k be a fixed index. Suppose (x_j, y_j) is a sequence of points such that $\eta_k(x_j, y_j) = c$ and $x_j^2 + y_j^2 \to +\infty$ as $j \to +\infty$. Moreover, relabeling (p_m, q_m), $m = 1, \ldots, n$, if necessary, we can assume that as $j \to +\infty$,*

$$\eta_m(x_j, y_j) \to +\infty, \quad m = 1, \ldots, k-1,$$
$$\eta_m(x_j, y_j) \to -\infty, \quad m = k+1, \ldots, n.$$

Then we have

$$\lim_{j\to+\infty} U_n(x_j, y_j) = \begin{cases} 4 \arctan\left(\exp\left(\eta_k - \beta_k\right)\right) - \pi, & \text{if } k \text{ is odd,} \\ 4 \arctan\left(\exp\left(-\eta_k - \beta_k\right)\right) - \pi, & \text{if } k \text{ is even,} \end{cases}$$

where $\beta_k = \sum_{j=1}^{k-1} \ln\left(\alpha(j,k)\right)$.

Proof. We first consider the case that k is odd. Then as $j \to +\infty$, the main order term of f_n is

$$a\,(1, \ldots, k-1) \exp(\eta_1 + \cdots + \eta_{k-1}).$$

At the same time, the main order of g_n is

$$a(1, \ldots, k) \exp\left(\eta_1 + \cdots + \eta_k\right).$$

Hence along this sequence, U_n converges to

$$4 \arctan\left(\frac{a(1, \ldots, k-1)}{a(1, \ldots, k)}\, e^c\right) - \pi = 4 \arctan(\exp(\eta_k - \beta_k)) - \pi.$$

If k is even, then as $j \to +\infty$, the main order term of f_n is

$$a(1, \ldots, k) \exp(\eta_1 + \cdots + \eta_k);$$

while the main order term of g_n will be

$$a(1,\dots,k-1)\exp(\eta_1+\cdots+\eta_{k-1}).$$

Hence in this case,

$$U_n \to 4\arctan\left(\frac{a(1,\dots,k-1)}{a(1,\dots,k)}e^{-c}\right)-\pi = 4\arctan(\exp(-\eta_k-\beta_k))-\pi. \qquad \blacksquare$$

By Lemma 2.2, away from the origin, the nodal set of the solutions U_n is asymptotic to $2n$ half straight lines, each line is parallel to one of the lines $\eta_j=0$, $j=1,\dots,n$, with the phase shift determined by the constants β_k appearing in Lemma 2.2. Hence U_n is a $2n$-end solution. Note that U_n contains $2n$ free real parameters: p_j,η_j^0, $j=1,\dots,n$. Hence this solution set is a $2n$ dimensional manifold. Note that the dimension $2n$ is consistent with the prediction given by the moduli space theory [11] of the Allen–Cahn type equation.

In the special case $n=2$, if we choose $p_1=p_2=p$ and $q_1=-q_2=q$, $\eta_1^0=\eta_2^0=\ln\frac{p}{q}$, then we get the solution

$$\varphi_{p,q}(x,y) := 4\arctan\left(\frac{p\cosh(qy)}{q\cosh(px)}\right)-\pi.$$

This corresponds to a 4-end solution of the elliptic sine-Gordon equation (1.2). Note that on the lines $px=\pm qy$, $\varphi_{p,q}=4\arctan(p/q)-\pi$. In the special case $p=q=\sqrt{2}/2$, the solution is

$$4\arctan\left(\frac{\cosh(y/\sqrt{2})}{\cosh(x/\sqrt{2})}\right)-\pi. \tag{2.16}$$

This is the classical saddle solution.

We remark that this family of 4-end solutions $\varphi_{p,q}$ has analogous in the theory of minimal surfaces. It is the so called Scherk second surface family, which contains embedded singly periodic minimal surfaces in $\mathbb{R}^3$. Explicitly, these surfaces can be described by

$$\cos^2\theta\cosh\frac{x}{\cos\theta}-\sin^2\theta\sinh\frac{y}{\sin\theta}=\cos z.$$

Here θ is a parameter. Each of these surfaces has four wings, called ends of the surfaces. Geometrically, they are obtained by desingularizing two intersected planes with intersection angle θ.

3. Bäcklund transformation of the multiple-end solutions

Lamb [40] has established a superposition formula for the Bäcklund transformation of the hyperbolic sine-Gordon equation. In particular, the formula enables us to get multi-soliton solutions in an algebraic way. However, in this formulation, for n-soliton solutions with n large, it will be quite tedious to write down the explicit expressions for the solutions. Nevertheless, it turns out that the soliton solutions in Theorem 2.1 can be obtained through the Bäcklund transformation. This will be discussed in detail in this section.

In the light-cone coordinates, the hyperbolic sine-Gordon equation has the form

$$\partial_s \partial_t u = \sin u, \quad (s,t) \in \mathbb{R}^2. \tag{3.1}$$

Let k be a real parameter. The Bäcklund transformation between two solutions u_1 and u_2 of (3.1) is given by (see for instance [51])

$$\begin{cases} \partial_s u_1 = \partial_s u_2 - 2k \sin \frac{u_1+u_2}{2}, \\ \partial_t u_1 = -\partial_t u_2 - 2k^{-1} \sin \frac{u_1-u_2}{2}. \end{cases} \tag{3.2}$$

An interesting property of this transformation is the following: if two functions u_1 and u_2 solve the system (3.2), then they satisfy (3.1) *simultaneously*.

Next we recall the bilinear form of the hyperbolic sine-Gordon equation ([32]). Let $F = f + ig$. We still write u in bi-logrithmic form:

$$u = 2i \ln \frac{\bar{F}}{F} = 4 \arctan \frac{g}{f}.$$

Here the log and arctan function are also taken to be the principle branch. Then (3.1) has the bilinear form

$$D_s D_t F \cdot F = \frac{1}{2} (F^2 - \bar{F}^2).$$

The following result can be found in [32].

Lemma 3.1. *Suppose $u_1 = 2i \ln(\bar{F}/F)$, $u_2 = 2i \ln(\bar{G}/G)$ satisfy*

$$\begin{cases} D_s G \cdot F = -\frac{k}{2} \bar{G} \bar{F}, \\ D_t G \cdot \bar{F} = -\frac{1}{2k} \bar{G} F. \end{cases} \tag{3.3}$$

Assume k is real. Then u_1 and u_2 satisfy (3.2).

Proof. We sketch the proof for completeness. We have

$$\begin{aligned} \partial_s u_1 - \partial_s u_2 &= 2i \left(\frac{\partial_s \bar{F}}{\bar{F}} - \frac{\partial_s F}{F} \right) - 2i \left(\frac{\partial_s \bar{G}}{\bar{G}} - \frac{\partial_s G}{G} \right) \\ &= 2i \, \frac{\bar{G} \partial_s \bar{F} - \bar{F} \partial_s \bar{G}}{\bar{F} \bar{G}} - 2i \, \frac{G \partial_s F - F \partial_s G}{FG}. \end{aligned} \tag{3.4}$$

On the other hand,

$$\sin \frac{u_1 + u_2}{2} = \sin \left(i \ln \frac{\bar{F} \bar{G}}{FG} \right) = \frac{1}{2i} \left(\frac{FG}{\bar{F} \bar{G}} - \frac{\bar{F} \bar{G}}{FG} \right). \tag{3.5}$$

From (3.4) and (3.5), using (3.3) and the assumption that k is real, we deduce

$$\partial_s u_1 - \partial_s u_2 = -ki \, \frac{\bar{F} \bar{G}}{FG} + ki \, \frac{FG}{\bar{F} \bar{G}} = -2k \sin \frac{u_1 + u_2}{2}.$$

Similarly, we have

$$\partial_t u_1 + \partial_t u_2 = -2k^{-1} \sin \frac{u_1 - u_2}{2}. \qquad \blacksquare$$

Fix $n \in \mathbb{N}$. Let k_j, δ_j, $j = 1, \dots, n$, be real parameters. We now set

$$\beta_j := k_j s + k_j^{-1} t + \delta_j, \quad j = 1, \dots, n.$$

At this moment, they are regarded as functions of the real variables s and t. We define

$$G_n := \sum_{\varepsilon} \Big(\exp\Big[\sum_{j=1}^{n} \Big(\frac{\varepsilon_j}{2} \Big(\beta_j + \frac{\pi i}{2} \Big) \Big) + \frac{n\pi i}{4} \Big] \prod_{j<l\le n} (k_j - \varepsilon_j \varepsilon_l k_l) \Big).$$

Here the summation $\sum_{\varepsilon}$ is taken over all possible n-tuples $(\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_j = \pm 1$, $j = 1, \dots, n$. Note that the G_n are complex-valued functions. By this definition, we have

$$\begin{aligned} G_1 &= \exp\Big(-\frac{\beta_1}{2} \Big) + i \exp\Big(\frac{\beta_1}{2} \Big), \\ G_2 &= -(k_1 - k_2) \exp\Big(\frac{1}{2}(\beta_1 + \beta_2) \Big) + (k_1 - k_2) \exp\Big(\frac{1}{2}(-\beta_1 - \beta_2) \Big) \\ &\quad + i(k_1 + k_2) \exp\Big(\frac{1}{2}(\beta_1 - \beta_2) \Big) + i(k_1 + k_2) \exp\Big(\frac{1}{2}(-\beta_1 + \beta_2) \Big). \end{aligned}$$

When $n = 0$, G_n is understood to be 1.

Lemma 3.2. *Assume that k_j, δ_j, $j = 1, \dots, n$, are* real *numbers, $k_j \neq 0$. Then G_{n-1} and G_n are connected through the following Bäcklund transformation*:

$$\begin{cases} D_s G_n \cdot G_{n-1} = -\frac{k_n}{2} \bar{G}_n \bar{G}_{n-1}, \\ D_t G_n \cdot \bar{G}_{n-1} = -\frac{1}{2k_n} \bar{G}_n G_{n-1}. \end{cases}$$

Results of this type for the KdV equation and certain superposition formulas can be found in [33]. Since we are not able to locate the precise references for a direct proof of this lemma, here we sketch the proof for the first identity. The second one follows from same arguments.

Proof. Fix the integer n and let us introduce the notation

$$\varepsilon = (\varepsilon_1, \dots, \varepsilon_n), \quad \varepsilon' = (\varepsilon'_1, \dots, \varepsilon'_{n-1}).$$

To simplify notations, we also set

$$\begin{aligned} h_1 &:= \exp\Big[\sum_{j=1}^{n} \Big(\frac{\varepsilon_j}{2} \Big(\beta_j + \frac{\pi i}{2} \Big) \Big) + \frac{n\pi i}{4} \Big] \prod_{j<l\le n} (k_j - \varepsilon_j \varepsilon_l k_l), \\ h_2 &:= \exp\Big[\sum_{j=1}^{n-1} \Big(\frac{\varepsilon'_j}{2} \Big(\beta_j + \frac{\pi i}{2} \Big) \Big) + \frac{(n-1)\pi i}{4} \Big] \prod_{j<l\le n-1} (k_j - \varepsilon'_j \varepsilon'_l k_l). \end{aligned}$$

Using $\partial_s \beta_j = k_j$, we can compute

$$\begin{aligned} 2D_s h_1 \cdot h_2 &= \Big(\sum_{j=1}^{n} (\varepsilon_j k_j) - \sum_{j=1}^{n-1} (\varepsilon'_j k_j) \Big) h_1 h_2 \\ &= \Big(\sum_{j=1}^{n} (\varepsilon_j k_j) - \sum_{j=1}^{n-1} (\varepsilon'_j k_j) \Big) \prod_{j<n} (k_j - \varepsilon_j \varepsilon_n k_n) W. \end{aligned} \tag{3.6}$$

Here,

$$W := \prod_{m<l\leq n-1} \left[\left(k_m - \varepsilon_m \varepsilon_l k_l\right) \left(k_m - \varepsilon'_m \varepsilon'_l k_l\right) \right] \exp\left(\frac{\varepsilon_n}{2} \left(\beta_n + \frac{\pi i}{2} \right) \right)$$
$$\times \exp\left(\sum_{j=1}^{n-1} \left(\frac{(\varepsilon'_j + \varepsilon_j)}{2} \left(\beta_j + \frac{\pi i}{2} \right) \right) + \frac{(2n-1)\pi i}{4} \right).$$

With all these notations, we have

$$2D_s G_n \cdot G_{n-1} = \sum_{\varepsilon,\varepsilon'} \left(\left[\sum_{j=1}^{n} (\varepsilon_j k_j) - \sum_{j=1}^{n-1} (\varepsilon'_j k_j) \right] \prod_{j<n} (k_j - \varepsilon_j \varepsilon_n k_n) W \right). \tag{3.7}$$

It turns out that this expression can be further simplified, due to cancellations between some terms. Observe that if for some index $j_0 \leq n-1$, $\varepsilon_{j_0} = \varepsilon'_{j_0}$, then the corresponding term does not contribute to the coefficient

$$\sum_{j=1}^{n} (\varepsilon_j k_j) - \sum_{j=1}^{n-1} (\varepsilon'_j k_j).$$

To compute the right-hand side of (3.7), we first consider two simple cases for the summation indices.

Case 1: In the summation, $\varepsilon_1 = -\varepsilon'_1$ and for $2 \leq j \leq n-1$, $\varepsilon_j = \varepsilon'_j$.

Fix the indices $\varepsilon_j = \varepsilon'_j$ with $j \geq 2$. Then in this case, for different $\varepsilon_1 = -\varepsilon'_1$, each term in the right-hand side of (3.7) has the common factor

$$\prod_{1<l\leq n-1} (k_1^2 - k_l^2) \prod_{1<m<l\leq n-1} (k_m - \varepsilon_m \varepsilon_l k_l)^2 \prod_{2\leq j\leq n-2} (k_j - \varepsilon_j \varepsilon_n k_n)$$
$$\times \exp\left(\frac{\varepsilon_n}{2} \left(\beta_n + \frac{\pi i}{2} \right) + \sum_{j=2}^{n-1} \left(\varepsilon_j \left(\beta_j + \frac{\pi i}{2} \right) \right) + \frac{(2n-1)\pi i}{4} \right).$$

Taking out this common factor and freezing the indices $\varepsilon_2, \ldots, \varepsilon_n$, we are led to compute

$$I_1 := \sum_{\varepsilon_1} \left[\left(\varepsilon_n k_n + 2\varepsilon_1 k_1\right) \left(k_1 - \varepsilon_1 \varepsilon_n k_n\right) \right].$$

Here the summation is over the index $\varepsilon_1 = \pm 1$, since we impose the restriction that $\varepsilon_1 = -\varepsilon'_1$. Using the fact that $\varepsilon_j^2 = 1$, we deduce

$$I_1 = \sum_{\varepsilon_1} \left(\varepsilon_n k_n k_1 - \varepsilon_1 k_n^2 + 2\varepsilon_1 k_1^2 - 2\varepsilon_n k_1 k_n \right).$$

The summation over the second term is zero, since the terms with $\varepsilon_1 = 1$ and $\varepsilon_1 = -1$ cancel each other. The same occurs for the third term. Hence we obtain $I_1 = -2\varepsilon_n k_1 k_n$. On the other hand, we compute

$$\sum_{\varepsilon_1} \left[\varepsilon_n k_n \left(k_1 - \varepsilon_1 \varepsilon_n k_n\right) \right] = 2\varepsilon_n k_n k_1.$$

It then follows that

$$I_1 = -\sum_{\varepsilon_1} [\varepsilon_n k_n (k_1 - \varepsilon_1 \varepsilon_n k_n)].$$

Using this identity, we find that, when the indices $\varepsilon_j = \varepsilon'_j$, $j \geq 2$, are fixed,

$$\begin{aligned}
&\sum_{\varepsilon_1 = -\varepsilon'_1} \Big(\Big[\sum_{j=1}^{n} (\varepsilon_j k_j) - \sum_{j=1}^{n-1} (\varepsilon'_j k_j) \Big] \prod_{j<n} (k_j - \varepsilon_j \varepsilon_n k_n) W \Big) \\
&= - \sum_{\varepsilon_1 = -\varepsilon'_1} \Big[\varepsilon_n k_n \prod_{l \leq n-1} (k_l - \varepsilon_l \varepsilon_n k_n) \prod_{m<l\leq n-1} [(k_m - \varepsilon_m \varepsilon_l k_l)\,(k_m - \varepsilon'_m \varepsilon'_l k_l)] \\
&\qquad \times \exp\Big(\frac{\varepsilon_n}{2}\Big(\beta_n + \frac{\pi i}{2}\Big) + \sum_{j=1}^{n-1} \Big(\frac{(\varepsilon_j + \varepsilon'_j)}{2} \Big(\beta_j + \frac{\pi i}{2}\Big)\Big) + \frac{(2n-1)\pi i}{4} \Big) \Big].
\end{aligned}$$

Denote the right-hand side by F_1. On the other hand, for the same fixed indices $\varepsilon_j = \varepsilon'_j$, $j \geq 2$, in $\bar{G}_n \bar{G}_{n-1}$, we have the term

$$\begin{aligned}
F_1^* := &\sum_{\varepsilon_1 = -\varepsilon'_1} \Big[\prod_{l \leq n-1} (k_l - \varepsilon_l \varepsilon_n k_n) \prod_{m<l\leq n-1} [(k_m - \varepsilon_m \varepsilon_l k_l)\,(k_m - \varepsilon'_m \varepsilon'_l k_l)] \\
&\qquad \times \exp\Big(\frac{\varepsilon_n}{2}\Big(\beta_n - \frac{\pi i}{2}\Big) + \sum_{j=1}^{n-1} \Big(\frac{(\varepsilon_j + \varepsilon'_j)}{2} \Big(\beta_j - \frac{\pi i}{2}\Big)\Big) - \frac{(2n-1)\pi i}{4} \Big) \Big].
\end{aligned}$$

Since $\varepsilon_1 = -\varepsilon'_1$ and $\varepsilon_j = \varepsilon'_j$ for $j \geq 2$, we always have

$$\varepsilon_n \exp\Big[\Big(\varepsilon_n + \sum_{j=1}^{n-1} (\varepsilon_j + \varepsilon'_j) + 2n - 1 \Big) \frac{\pi i}{2} \Big] = 1.$$

Hence

$$F_1 = -k_n F_1^*. \tag{3.8}$$

Case 2: The indices satisfy $\varepsilon_1 = -\varepsilon'_1, \varepsilon_2 = -\varepsilon'_2$, and for $3 \leq j \leq n-1$, $\varepsilon_j = \varepsilon'_j$.

In this case, for fixed indices $\varepsilon_j = \varepsilon'_j$ with $j \geq 3$, terms in (3.7) have the common factor

$$\begin{aligned}
&\prod_{2<l\leq n-1} [(k_1^2 - k_l^2)(k_2^2 - k_l^2)] \prod_{2<m<l\leq n-1} (k_m - \varepsilon_m \varepsilon_l k_l)^2 \prod_{2 \leq j \leq n-2} (k_j - \varepsilon_j \varepsilon_n k_n) \\
&\qquad \times \exp\Big(\frac{\varepsilon_n}{2}\Big(\beta_n + \frac{\pi i}{2}\Big) + \sum_{j=3}^{n-1} \Big(\varepsilon_j \Big(\beta_j + \frac{\pi i}{2}\Big)\Big) + \frac{(2n-1)\pi i}{4} \Big).
\end{aligned}$$

Taking out this common factor and freezing the indices $\varepsilon_3, \ldots, \varepsilon_n$, in view of the assumption $\varepsilon_1 = -\varepsilon'_1$ and $\varepsilon_2 = -\varepsilon'_2$, we are led to compute

$$I_2 := \sum_{\varepsilon_1, \varepsilon_2} \big[(\varepsilon_n k_n + 2\varepsilon_1 k_1 + 2\varepsilon_2 k_2)\,(k_1 - \varepsilon_1 \varepsilon_n k_n)\,(k_2 - \varepsilon_2 \varepsilon_n k_n)\,(k_1 - \varepsilon_1 \varepsilon_2 k_2)^2 \big].$$

To simplify I_2, let us first of all compute

$$I_{2,2} := \sum_{\varepsilon_1,\varepsilon_2} \left[(\varepsilon_1 k_1 + \varepsilon_2 k_2)\,(k_1 - \varepsilon_1 \varepsilon_n k_n)\,(k_2 - \varepsilon_2 \varepsilon_n k_n)\,(k_1 - \varepsilon_1 \varepsilon_2 k_2)^2 \right].$$

We can expand the bracket into individual terms. Observe that if a term has odd power of ε_1 or ε_2, then taking the summation over this term will yield zero, due to cancellation between $+1$ and -1. Hence we obtain

$$\begin{aligned}
I_{2,2} = &\sum_{\varepsilon_1,\varepsilon_2} \left[(\varepsilon_1 k_1) k_1\,(-\varepsilon_2 \varepsilon_n k_n)\,(-2\varepsilon_1 \varepsilon_2 k_1 k_2) + (\varepsilon_1 k_1)\,(-\varepsilon_1 \varepsilon_n k_n)\, k_2\,(k_1^2 + k_2^2)\right] \\
&+ \sum_{\varepsilon_1,\varepsilon_2} \left[(\varepsilon_2 k_2)\, k_1\,(-\varepsilon_2 \varepsilon_n k_n)\,(k_1^2 + k_2^2) + (\varepsilon_2 k_2)\,(-\varepsilon_1\varepsilon_n k_n)\, k_2\,(-2k_1\varepsilon_1\varepsilon_2 k_2) \right] \\
= &\sum_{\varepsilon_1,\varepsilon_2} \left[2\varepsilon_n k_1^3 k_2 k_n - \varepsilon_n k_1 k_2 k_n\,(k_1^2 + k_2^2) - \varepsilon_n k_1 k_2 k_n\,(k_1^2 + k_2^2) + 2\varepsilon_n k_1 k_2^3 k_n\right] \\
= &\ 0.
\end{aligned}$$

Therefore,

$$I_2 = \sum_{\varepsilon_1,\varepsilon_2} \left[\varepsilon_n k_n\,(k_1 - \varepsilon_1 \varepsilon_n k_n)\,(k_2 - \varepsilon_2 \varepsilon_n k_n)\,(k_1 - \varepsilon_1 \varepsilon_2 k_2)^2 \right].$$

It follows from this identity that when the indices $\varepsilon_j = \varepsilon_j'$, $j \geq 2$, are fixed, we have

$$\begin{aligned}
&\sum_{\varepsilon_1=-\varepsilon_1',\varepsilon_2=-\varepsilon_2'} \left(\Big[\sum_{j=1}^{n} (\varepsilon_j k_j) - \sum_{j=1}^{n-1} (\varepsilon_j' k_j) \Big] \prod_{j<n} (k_j - \varepsilon_j \varepsilon_n k_n) W \right) \\
&= \sum_{\varepsilon_1=-\varepsilon_1',\varepsilon_2=-\varepsilon_2'} \Bigg[\varepsilon_n k_n \prod_{l\leq n-1} (k_l - \varepsilon_l \varepsilon_n k_n) \prod_{m<l\leq n-1} \left[(k_m - \varepsilon_m \varepsilon_l k_l)\left(k_m - \varepsilon_m' \varepsilon_l' k_l\right)\right] \\
&\qquad \times \exp\Big(\frac{\varepsilon_n}{2}\Big(\beta_n + \frac{\pi i}{2}\Big) + \sum_{j=1}^{n-1} \Big(\frac{(\varepsilon_j + \varepsilon_j')}{2} \Big(\beta_j + \frac{\pi i}{2}\Big)\Big) + \frac{(2n-1)\pi i}{4} \Big) \Bigg].
\end{aligned}$$

Denote the right-hand side by F_2. On the other hand, for the same fixed indices $\varepsilon_j = \varepsilon_j'$, $j \geq 3$, in $\bar{G}_n \bar{G}_{n-1}$, we have the term

$$\begin{aligned}
F_2^* := &\sum_{\varepsilon_1=-\varepsilon_1',\varepsilon_2=-\varepsilon_2'} \Bigg[\prod_{l\leq n-1} (k_l - \varepsilon_l \varepsilon_n k_n) \prod_{m<l\leq n-1} \left[(k_m - \varepsilon_m \varepsilon_l k_l)\left(k_m - \varepsilon_m' \varepsilon_l' k_l\right)\right] \\
&\qquad \times \exp\Big(\frac{\varepsilon_n}{2}\Big(\beta_n - \frac{\pi i}{2}\Big) + \sum_{j=1}^{n-1} \Big(\frac{(\varepsilon_j + \varepsilon_j')}{2} \Big(\beta_j - \frac{\pi i}{2}\Big)\Big) - \frac{(2n-1)\pi i}{4} \Big) \Bigg].
\end{aligned}$$

Since $\varepsilon_1 = -\varepsilon_1'$, $\varepsilon_2 = -\varepsilon_2'$, and $\varepsilon_j = \varepsilon_j'$ for $j \geq 3$, we always have

$$\varepsilon_n \exp\Big[\Big(\varepsilon_n + \sum_{j=1}^{n-1} (\varepsilon_j + \varepsilon_j') + 2n - 1\Big)\frac{\pi i}{2}\Big] = -1.$$

It follows that

$$F_2 = -k_n F_2^*. \tag{3.9}$$

Having understood Case 1 and Case 2, we proceed to consider the general case. Assume without loss of generality that the indices satisfy, for some integer m_0,

$$\varepsilon_j = -\varepsilon_j', \ j = 1, \ldots, m_0, \quad \text{and} \quad \varepsilon_j = \varepsilon_j', \ j = m_0 + 1, \ldots, n-1.$$

Then we can compute (3.7) by separating these indices into pairs $(\varepsilon_1, \varepsilon_2), (\varepsilon_3, \varepsilon_4), \ldots$ Applying formula (3.9) for each pair and using (3.8) in case m_0 is odd, we finally deduce

$$2D_s G_n \cdot G_{n-1} = -k_n \bar{G}_n \bar{G}_{n-1}$$

The proof is thus completed. ∎

In view of the definition of G_n, we now define ω_n to be

$$\sum_{\varepsilon:\prod_{m=1}^n \varepsilon_m = (-1)^n} \left(\exp\left[\sum_{j=1}^n \left(\frac{\varepsilon_j}{2} \left(\beta_j + \frac{\pi i}{2} \right) \right) + \frac{n\pi i}{4} \right] \prod_{j<l\leq n} (k_j - \varepsilon_j \varepsilon_l k_l) \right),$$

where $\varepsilon_j = \pm 1$. Similarly, we define ρ_n by

$$\sum_{\varepsilon:\prod_{m=1}^n \varepsilon_m = (-1)^{n+1}} \left(\exp\left[\sum_{j=1}^n \left(\frac{\varepsilon_j}{2} \left(\beta_j + \frac{\pi i}{2} \right) \right) + \frac{(n-2)\pi i}{4} \right] \prod_{j<l\leq n} (k_j - \varepsilon_j \varepsilon_l k_l) \right).$$

Note that if k_j, δ_j are real numbers, and s, t are real variables, then

$$\omega_n = \operatorname{Re} G_n, \quad \rho_n = \operatorname{Im} G_n.$$

In particular, we have

$$\begin{aligned}
\omega_0 &= 1, \ \rho_0 = 0, \\
\omega_1 &= \exp\left(-\frac{\beta_1}{2}\right), \ \rho_1 = \exp\left(\frac{\beta_1}{2}\right), \\
\omega_2 &= -(k_1 - k_2) \exp\left(\frac{1}{2}(\beta_1 + \beta_2)\right) + (k_1 - k_2) \exp\left(\frac{1}{2}(-\beta_1 - \beta_2)\right) \\
\rho_2 &= (k_1 + k_2) \exp\left(\frac{1}{2}(\beta_1 - \beta_2)\right) + (k_1 + k_2) \exp\left(\frac{1}{2}(-\beta_1 + \beta_2)\right).
\end{aligned}$$

Applying Lemmas 3.1 and 3.2, we see that the real valued function $\tilde{u}_n := 4 \arctan(\rho_n/\omega_n)$ satisfies

$$\begin{cases} \partial_s \tilde{u}_{n-1} = \partial_s \tilde{u}_n - 2k_n \sin \frac{\tilde{u}_{n-1} + \tilde{u}_n}{2}, \\ \partial_t \tilde{u}_{n-1} = -\partial_t \tilde{u}_n - 2k_n^{-1} \sin \frac{\tilde{u}_{n-1} - \tilde{u}_n}{2}. \end{cases} \tag{3.10}$$

For later applications, we would like to generalize this system to complex valued functions (the function arctan is understood to be the principle branch). This is the content of the following.

Lemma 3.3. *Assume k_j and δ_j are complex numbers, and s and t are complex variables. Then (3.10) is still true.*

Proof. We already know that (3.10) is true for real parameters. The assertion of the lemma then follows from the fact that the functions involved are analytic with respect to those parameters and variables. ∎

We now come back to the solutions U_n of the elliptic sine-Gordon equation appeared in Theorem 2.1. We would like to show that they are indeed the Bäcklund transformation of certain $(n-1)$-soliton type solutions. This will be achieved by applying Lemma 3.3. To do this, first of all, we need to write the functions f_n and g_n in a form adapted to Lemma 3.2.

Recall that p_j, q_j are parameters in U_n. For $j = 1, \ldots, n$, let $k_j = p_j + iq_j$ and choose a complex number ι_j such that

$$e^{\iota_j} = \prod_{l<j} \frac{k_l + k_j}{k_l - k_j} \prod_{l>j} \frac{k_j + k_l}{k_j - k_l}.$$

For instance, one can simply choose ι_j to be the principle value of the log function evaluated at the right-hand side.

Since $p_j^2 + q_j^2 = 1$, we know that $k_j^{-1} = p_j - iq_j = \bar{k}_j$. Recall that $\eta_j = p_j x + q_j y + \eta_j^0$. We emphasize that here x, y are regarded as real variables. Let us now define

$$\tilde{\eta}_j := \eta_j - \iota_j. \tag{3.11}$$

We then set

$$\tilde{f}_n := \sum_{\varepsilon:\ \prod_{m=1}^{n} \varepsilon_m = (-1)^n} \left(\exp\left[\sum_{j=1}^{n} \left(\frac{\varepsilon_j}{2} \left(\tilde{\eta}_j + \frac{\pi i}{2} \right) \right) + \frac{n\pi i}{4} \right] \prod_{j<l\le n} (k_j - \varepsilon_j \varepsilon_l k_l) \right), \tag{3.12}$$

where $\varepsilon_j = \pm 1$. We also define

$$\tilde{g}_n = \sum_{\varepsilon:\ \prod_{m=1}^{n} \varepsilon_m = (-1)^{n-1}} \left(\exp\left[\sum_{j=1}^{n} \left(\frac{\varepsilon_j}{2} \left(\tilde{\eta}_j + \frac{\pi i}{2} \right) \right) + \frac{(n-2)\pi i}{4} \right] \prod_{j<l\le n} (k_j - \varepsilon_j \varepsilon_l k_l) \right). \tag{3.13}$$

Lemma 3.4. *Let f_n and g_n be defined by (2.8) and (2.9). There holds*

$$\frac{g_n}{f_n} = \frac{\tilde{g}_n}{\tilde{f}_n}.$$

Proof. Since $\eta_j = \tilde{\eta}_j + \iota_j$, f_n can be written in the form

$$\sum_{m=0}^{\lfloor n/2 \rfloor} \left(\sum_{\{n,2m\}} \left[a(i_1, \ldots, i_{2m}) \exp\left(\iota_{i_1} + \cdots + \iota_{i_{2m}}\right) \exp\left(\tilde{\eta}_{i_1} + \cdots + \tilde{\eta}_{i_{2m}}\right) \right] \right).$$

For fixed indices $(i_1, \ldots, i_{2m})$, using the definition (2.7) of a, we have

$$\begin{aligned} a\,(i_1, \ldots, i_{2m}) \exp\left(\iota_{i_1} + \cdots + \iota_{i_{2m}}\right) \\ &= (-1)^{m(2m-1)} \prod_{j<l\le 2m} \left(\frac{k_{i_j} - k_{i_l}}{k_{i_j} + k_{i_l}}\right)^2 \exp\left(\iota_{i_1} + \cdots + \iota_{i_{2m}}\right) \\ &= (-1)^{m(2m-1)} \prod_{j<l\le n} \frac{k_j - \varepsilon_j \varepsilon_l k_l}{k_j - k_l}, \end{aligned}$$

where $\varepsilon_j = 1$ if $j = i_1, \ldots, i_{2m}$; otherwise $\varepsilon_j = -1$. Note that in this case,

$$\sum_{j=1}^{n} \varepsilon_j = 4m - n.$$

Hence the sign satisfies

$$(-1)^{m(2m-1)} = \exp\left(\frac{\pi i}{4}\left(\sum_{j=1}^{n} \varepsilon_j + n\right)\right).$$

It follows that

$$\begin{aligned} & f_n \exp\left(-\frac{1}{2}(\tilde\eta_1 + \cdots + \tilde\eta_n)\right) \\ &= \frac{1}{\prod\limits_{j<l\le n} (k_j - k_l)} \sum_{\varepsilon:\, \prod\limits_{m=1}^{n} \varepsilon_m = (-1)^n} \left[\exp\left(\sum_{j=1}^{n}\left(\frac{\varepsilon_j}{2}\left(\tilde\eta_j + \frac{\pi i}{2}\right)\right) + \frac{n\pi i}{4}\right) \prod_{j<l\le n} (k_j - \varepsilon_j \varepsilon_l k_l)\right] \\ &= \frac{1}{\prod_{j<l\le n}(k_j - k_l)} \tilde f_n. \end{aligned}$$

Similarly, we have

$$g_n \exp\left(-\frac{1}{2}(\tilde\eta_1 + \cdots + \tilde\eta_n)\right) = \frac{1}{\prod_{j<l\le n}(k_j - k_l)} \tilde g_n.$$

As a consequence,

$$\frac{g_n}{f_n} = \frac{\tilde g_n}{\tilde f_n}.$$

This finishes the proof. ∎

Let $\tilde\eta_j$, $j = 1, \ldots, n-1$, be defined by (3.11). We define $\gamma = \gamma_{n-1}$ to be

$$\sum_{\varepsilon: \prod_{m=1}^{n-1} \varepsilon_m = (-1)^{n-1}} \left(\exp\left[\sum_{j=1}^{n-1}\left(\frac{\varepsilon_j}{2}\left(\tilde\eta_j + \frac{\pi i}{2}\right)\right) + \frac{(n-1)\pi i}{4}\right] \prod_{j<l\le n-1} (k_j - \varepsilon_j \varepsilon_l k_l)\right).$$

Moreover, we define $\tau = \tau_{n-1}$ by

$$\sum_{\varepsilon:\prod_{m=1}^{n-1}\varepsilon_m=(-1)^n}\left(\exp\Big[\sum_{j=1}^{n-1}\Big(\frac{\varepsilon_j}{2}\Big(\tilde{\eta}_j+\frac{\pi i}{2}\Big)\Big)+\frac{(n-3)\pi i}{4}\Big]\prod_{j<l\le n-1}(k_j-\varepsilon_j\,\varepsilon_l\,k_l)\right).$$

We emphasize that $\tilde{\eta}_j$, $j=1,\dots,n-1$, actually also depends on k_n.

Lemma 3.5. *The function τ/γ is purely imaginary.*

Proof. For each fixed j, we choose η_j' such that

$$\exp(\eta_j) := \exp(\eta_j')\prod_{l<j}\frac{k_l+k_j}{k_l-k_j}\prod_{j<l\le n-1}\frac{k_l+k_j}{k_l-k_j}.$$

Note that there are infinitely many choices for such η_j'. We may just choose one of them, for instance, the one arising from the principle branch of the log function. Consider the functions γ' and τ' defined by

$$\gamma' := \sum_{\varepsilon:\prod_{m=1}^{n-1}\varepsilon_m=(-1)^{n-1}}\left(\exp\Big(\sum_{j=1}^{n-1}\Big(\frac{\varepsilon_j}{2}\Big(\eta_j'+\frac{\pi i}{2}\Big)\Big)+\frac{(n-1)\pi i}{4}\Big)\prod_{j<l\le n-1}(k_j-\varepsilon_j\varepsilon_l k_l)\right),$$

$$\tau' := \sum_{\varepsilon:\prod_{m=1}^{n-1}\varepsilon_m=(-1)^{n}}\left(\exp\Big(\sum_{j=1}^{n-1}\Big(\frac{\varepsilon_j}{2}\Big(\eta_j'+\frac{\pi i}{2}\Big)\Big)+\frac{(n-3)\pi i}{4}\Big)\prod_{j<l\le n-1}(k_j-\varepsilon_j\varepsilon_l k_l)\right).$$

By the proof of Lemma 3.4, we have

$$f_{n-1} = \exp\Big(\frac{1}{2}(\tilde{\eta}_1+\cdots+\tilde{\eta}_{n-1})\Big)\gamma'\prod_{j<l\le n-1}\frac{1}{k_j-k_l},$$
$$g_{n-1} = \exp\Big(\frac{1}{2}(\tilde{\eta}_1+\cdots+\tilde{\eta}_{n-1})\Big)\tau'\prod_{j<l\le n-1}\frac{1}{k_j-k_l}.$$

Since $\tilde{\eta}_j=\eta_j-\iota_j$, using the definition of η_j', we find that

$$\exp(\tilde{\eta}_j)=\exp(\eta_j')\,\frac{k_j+k_n}{k_j-k_n} := \exp(\eta_j'+\eta_j'^0).$$

Then γ is equal to

$$\sum_{\varepsilon:\prod_{m=1}^{n-1}\varepsilon_m=(-1)^{n-1}}\left(\exp\Big[\sum_{j=1}^{n-1}\Big(\frac{\varepsilon_j}{2}\Big(\eta_j'+\eta_j'^0+\frac{\pi i}{2}\Big)\Big)+\frac{(n-1)\pi i}{4}\Big]\prod_{j<l\le n-1}(k_j-\varepsilon_j\varepsilon_l k_l)\right).$$

Therefore, still using the proof of Lemma 3.4 (with the phase constant η_j^0 replaced by $\eta_j^0 + \eta_j'^0$), we can also write τ/γ as

$$\frac{\sum_{m=0}^{\lfloor (n-2)/2 \rfloor} \Big(\sum_{\{n-1,2m+1\}} \Big[a(i_1,\dots,i_{2m+1}) \prod_{j=1}^{2m+1} \frac{k_{i_j}+k_n}{k_{i_j}-k_n} \exp(\eta_{i_1}+\cdots+\eta_{i_{2m+1}}) \Big] \Big)}{\sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \Big(\sum_{\{n-1,2m\}} \Big[a(i_1,\dots,i_{2m}) \prod_{j=1}^{2m} \frac{k_{i_j}+k_n}{k_{i_j}-k_n} \exp(\eta_{i_1}+\cdots+\eta_{i_{2m}}) \Big] \Big)}. \tag{3.14}$$

On the other hand, from the fact that

$$a(j,n) = -\Big(\frac{k_j - k_n}{k_j + k_n}\Big)^2,$$

we infer that $\frac{k_j+k_n}{k_j-k_n}$ is imaginary. This together with (3.14) tell us that τ/γ is imaginary. ∎

Let us set $u = 4\arctan(g_n/f_n) = 4\arctan(\tilde{g}_n/\tilde{f}_n)$ and $v = 4\arctan(\tau/\gamma)$. Here the arctan function is still understood to be the principle value. Let us define

$$\begin{cases} x = s + t, \\ y = i\,(s - t)\,. \end{cases} \tag{3.15}$$

A direct consequence of Lemma 3.3 is the following.

Lemma 3.6. *The functions u and v are connected through the following Bäcklund transformation*:

$$\begin{cases} \partial_x v = i\,\partial_y u - k_n \sin\frac{v+u}{2} - \bar{k}_n \sin\frac{v-u}{2}\,, \\ i\,\partial_y v = \partial_x u - k_n \sin\frac{v+u}{2} + \bar{k}_n \sin\frac{v-u}{2}\,. \end{cases} \tag{3.16}$$

Proof. Applying Lemma 3.3, using the fact that $k_n^{-1} = \bar{k}_n$, we see that the functions u, v satisfy

$$\begin{cases} \partial_s v = \partial_s u - 2k_n \sin\frac{v+u}{2}\,, \\ \partial_t v = -\partial_t u - 2\bar{k}_n \sin\frac{v-u}{2}\,. \end{cases}$$

Note that $\partial_s = \partial_x + i\,\partial_y, \partial_t = \partial_x - i\,\partial_y$. Hence,

$$\begin{cases} \partial_x v + i\,\partial_y v = \partial_x u + i\,\partial_y u - 2k_n \sin\frac{v+u}{2}\,, \\ \partial_x v - i\,\partial_y v = -\partial_x u + i\,\partial_y u - 2\bar{k}_n \sin\frac{v-u}{2}\,. \end{cases}$$

The system (3.16) follows immediately. ∎

We point out that since the function τ/γ is purely imaginary, $\sin(v/2)$ and $\cos(v/2)$ should be understood as

$$\sin\Big(2\arctan\frac{\tau}{\gamma}\Big) = \frac{2\gamma\tau}{\gamma^2+\tau^2}, \quad \cos\Big(2\arctan\frac{\tau}{\gamma}\Big) = \frac{\gamma^2-\tau^2}{\gamma^2+\tau^2}. \tag{3.17}$$

Moreover, $\partial_x v = 4\frac{\gamma\partial_x\tau-\tau\partial_x\gamma}{\gamma^2+\tau^2}$. Hence, $\sin(\frac{u\pm v}{2})$ and $\cos(\frac{u\pm v}{2})$ are *complex valued* functions, with possible singularities at those points where $\gamma^2+\tau^2=0$. The analysis of these singularities will be carried out in the next section.

Let n be fixed and let $\tilde{\eta}_j$ be defined as before. For $\delta = 1,\ldots,n-2$, we now define γ_δ to be

$$\sum_{\varepsilon:\prod_{m=1}^{\delta}\varepsilon_m=(-1)^\delta}\left(\exp\left[\sum_{j=1}^{\delta}\left(\frac{\varepsilon_j}{2}\left(\tilde{\eta}_j+\frac{\pi i}{2}\right)\right)+\frac{\delta\pi i}{4}\right]\prod_{j<l\le\delta}(k_j-\varepsilon_j\varepsilon_l k_l)\right).$$

Moreover, we define τ_δ by

$$\sum_{\varepsilon:\prod_{m=1}^{\delta}\varepsilon_m=(-1)^{\delta-1}}\left(\exp\left[\sum_{j=1}^{\delta}\left(\frac{\varepsilon_j}{2}\left(\tilde{\eta}_j+\frac{\pi i}{2}\right)\right)+\frac{(\delta-2)\pi i}{4}\right]\prod_{j<l\le\delta}(k_j-\varepsilon_j\varepsilon_l k_l)\right).$$

Moreover, we define $\gamma_0=1$ and $\tau_0=0$. Let $v_\delta=\arctan(\tau_\delta/\gamma_\delta)$. Arguing similarly as in Lemma 3.5, we know that for $\delta=1,\ldots$, the function $\frac{\tau_{n-2\delta}}{\gamma_{n-2\delta}}$ is real valued, while $\frac{\tau_{n-2\delta+1}}{\gamma_{n-2\delta+1}}$ is purely imaginary (except τ_0/γ_0, which is always equal to 0).

A direct generalization of Lemma 3.6 is the following.

Lemma 3.7. *For $\delta=1,\ldots,n-1$, the functions v_δ and $v_{\delta-1}$ are connected through the following Bäcklund transformation*:

$$\begin{cases} \partial_x v_{\delta-1} = i\partial_y v_\delta - k_\delta\sin\frac{v_{\delta-1}+v_\delta}{2} - \bar{k}_\delta\sin\frac{v_{\delta-1}-v_\delta}{2}, \\ i\partial_y v_{\delta-1} = \partial_x v_\delta - k_\delta\sin\frac{v_{\delta-1}+v_\delta}{2} + \bar{k}_\delta\sin\frac{v_{\delta-1}-v_\delta}{2}. \end{cases} \tag{3.18}$$

4. Linearized Bäcklund transformation and nondegeneracy of the 2*n*-end solutions

This section will be devoted to prove the nondegeneracy of the multiple-end solutions. To state our result in a more precise way, let us recall that U_n is the $2n$-end solution defined in (2.15), and η_j^0 are "phase" parameters in U_n. Let $u = U_n+\pi = 4\arctan(g_n/f_n) = 4\arctan(\tilde{g}_n/\tilde{f}_n)$. In this section, the differentiation of u with respect to these parameters will be denoted by ζ_j. That is, $\zeta_j := \partial_{\eta_j^0}u$, $j=1,\ldots,n$. Since for any η_j^0, U_n is a solution to the elliptic sine-Gordon equation, ζ_j automatically solves the linearized equation:

$$\Delta\zeta_j = \zeta_j\cos u.$$

For convenience, let us restate Theorem 1.2, which is already claimed in the first section.

Theorem 4.1. *Suppose η is bounded in $\mathbb{R}^2$ and satisfies the linearized equation*

$$\Delta\eta = \eta\cos u.$$

Then there exist constants $c_1,\ldots,c_n$ such that

$$\eta = \sum_{j=1}^{n} c_j\zeta_j.$$

Roughly speaking, this result tells us that the solution U_n is L^∞ nondegenerate. The main idea of the proof is as follows. Using the linearized Bäcklund transformation, we transform η into a kernel χ of the linearized operator at the trivial solution 0. Hence $\Delta\chi - \chi = 0$. The solutions to this equation can be classified. By analyzing the reversed Bäcklund transformation from the trivial solution 0 to u, we then conclude that η has to be of the form stated in Theorem 4.1.

Linearizing the Bäcklund transformation (3.16) at (v, u) (with perturbation of the form $(\varepsilon\phi, \varepsilon\eta)$ and ε tends to 0), we get the linearized system

$$\begin{cases} \partial_x\phi = i\partial_y\eta - k_n\cos\frac{u+v}{2}\left(\frac{\phi+\eta}{2}\right) - \bar{k}_n\cos\frac{u-v}{2}\left(\frac{\phi-\eta}{2}\right), \\ i\partial_y\phi = \partial_x\eta - k_n\cos\frac{u+v}{2}\left(\frac{\phi+\eta}{2}\right) + \bar{k}_n\cos\frac{u-v}{2}\left(\frac{\phi-\eta}{2}\right). \end{cases}$$

Intuitively, given function η, we would like to solve this system and find a solution ϕ. For this purpose, we write it in the form

$$\begin{cases} L\phi = M\eta, \\ T\phi = N\eta, \end{cases} \tag{4.1}$$

where

$$\begin{aligned} L\phi &:= \partial_x\phi + \left(k_n\cos\frac{u+v}{2} + \bar{k}_n\cos\frac{u-v}{2}\right)\frac{\phi}{2}, \\ T\phi &:= i\partial_y\phi + \left(k_n\cos\frac{u+v}{2} - \bar{k}_n\cos\frac{u-v}{2}\right)\frac{\phi}{2}, \\ M\eta &:= i\partial_y\eta - \left(k_n\cos\frac{u+v}{2} - \bar{k}_n\cos\frac{u-v}{2}\right)\frac{\eta}{2}, \\ N\eta &:= \partial_x\eta - \left(k_n\cos\frac{u+v}{2} + \bar{k}_n\cos\frac{u-v}{2}\right)\frac{\eta}{2}. \end{aligned}$$

To simplify the notation, we write $\tilde{f}_n$ as f, and $\tilde{g}_n$ as g. Using (3.17), we see that explicitly, $L\phi$ is equal to

$$\begin{aligned} &\partial_x\phi + \left(k_n\left(\frac{2(f\gamma - g\tau)^2}{(f^2+g^2)(\gamma^2+\tau^2} - 1\right) + \bar{k}_n\left(\frac{2(f\gamma + g\tau)^2}{(f^2+g^2)(\gamma^2+\tau^2} - 1\right)\right)\frac{\phi}{2} \\ &\quad := \partial_x\phi + \operatorname{Re}(\Gamma - k_n)\phi, \end{aligned}$$

where the function Γ is defined to be

$$2k_n\frac{(f\gamma - g\tau)^2}{(f^2+g^2)(\gamma^2+\tau^2)}. \tag{4.2}$$

Similarly, we have

$$T\phi = i\partial_y\phi + i\operatorname{Im}(\Gamma - k_n)\phi.$$

Note that by Lemma 3.5, τ/γ is purely imaginary. As a consequence, the function $\gamma^2 + \tau^2$ could be equal to zero somewhere in $\mathbb{R}^2$. We define this singular set to be

$$\mathcal{S} = \mathcal{S}(v) := \left\{(x, y) \in \mathbb{R}^2 : \gamma^2 + \tau^2 = 0\right\}.$$

To analyze $\mathcal{S}$, we also define

$$S_0 := \{(x, y) \in \mathcal{S} : \gamma = 0\} \quad \text{and} \quad S_* := \{(x, y) \in \mathcal{S} : \gamma \neq 0\}.$$

The closure of S_* will be denoted by $\bar{S}_*$. These sets depend on the function v, which is determined by the parameters p_j, q_j, η_j^0. Observe that $\bar{S}_*$ is also a subset of $\mathcal{S}$. Rotating the axis if necessary, we can assume $p_j \neq 0$, for all j. By the classification results to be proved in the next section, we actually can assume that $p_j < 0$ for all j. Using the identity

$$\frac{\cos\theta_1 + i\sin\theta_1 - (\cos\theta_2 + i\sin\theta_2)}{\cos\theta_1 + i\sin\theta_1 + (\cos\theta_2 + i\sin\theta_2)} = i\tan\frac{\theta_1 - \theta_2}{2},$$

we may further assume (by relabeling the indices if necessary) that

$$\frac{k_j - k_l}{k_j + k_l} i < 0, \quad \text{if } j < l.$$

This property together with an induction argument based on formula (3.14) ensure that in the Bäcklund transformation sequence $\{v_1, \ldots, v_{n-1}\}$, the functions $v_{n-2\delta}$ are real and nonsingular for $\delta = 1, 2, \ldots$

Lemma 4.2. *Let R_0 be a large constant and let B_{R_0} be the ball of radius R_0 centered at the origin. The set $\mathcal{S} \backslash B_{R_0}$ consists of $2n-2$ curves. Each curve is asymptotic to a line which is parallel to one of the lines of the form $p_j x + q_j y = 0$, $j = 1, \ldots, n-1$.*

Proof. We first recall that γ is the sum of all those terms of the form:

$$\exp\Big(\sum_{j=1}^{n-1}\Big(\frac{\varepsilon_j}{2}\Big(\tilde{\eta}_j + \frac{\pi i}{2}\Big)\Big) + \frac{(n-1)\pi i}{4}\Big)\prod_{j<l\le n-1}(k_j - \varepsilon_j\varepsilon_l k_l),$$

where $\prod_{j=1}^{n-1}\varepsilon_j = (-1)^{n-1}$. At the same time, τ is the sum of terms

$$\exp\Big(\sum_{j=1}^{n-1}\Big(\frac{\varepsilon_j}{2}\Big(\tilde{\eta}_j + \frac{\pi i}{2}\Big)\Big) + \frac{(n-3)\pi i}{4}\Big)\prod_{j<l\le n-1}(k_j - \varepsilon_j\varepsilon_l k_l),$$

where $\prod_{j=1}^{n-1}\varepsilon_j = (-1)^n$.

Let $\{(x_j, y_j)\}_{j=1}^{+\infty}$ be a sequence of points in $\mathcal{S}$ such that $x_j^2 + y_j^2 \to +\infty$. Using the fact that $|\gamma| = |\tau|$ in $\mathcal{S}$, we infer that, up to a subsequence, there exist an index j_0 and a universal constant C such that

$$|\eta_{j_0}(x_j, y_j)| \le C, \quad j = 1, \ldots$$

Otherwise, $|\tau/\gamma|$ will be tending to $+\infty$ or 0, depending on the parity of n. Then without loss of generality, we can assume that as $j \to +\infty$,

$$\eta_m \to -\infty, \quad \text{for } m = 1, \ldots, j_0 - 1,$$
$$\eta_m \to +\infty, \quad \text{for } m = j_0 + 1, \ldots, n.$$

Suppose $n - j_0 = 2k + 1$ is odd; then the main order term in τ is

$$A \exp\Big(\frac{1}{2}(-\tilde{\eta}_1 - \cdots - \tilde{\eta}_{j_0-1} + \tilde{\eta}_{j_0} + \tilde{\eta}_{j_0+1} + \cdots + \tilde{\eta}_{n-1})\Big) \prod_{j<l\le n-1} (k_j - \varepsilon'_j \varepsilon'_l k_l),$$

where $\varepsilon'_1 = \cdots = \varepsilon'_{j_0-1} = -1$, $\varepsilon'_{j_0} = \cdots = \varepsilon'_{n-1} = 1$, and

$$A = \exp\Big(\frac{\pi i}{4}\Big(\sum_{j=1}^{n-1} \varepsilon'_j + n - 3\Big)\Big) = \exp(k\pi i).$$

On the other hand, the main order term in γ is

$$B \exp\Big(\frac{1}{2}(-\tilde{\eta}_1 - \cdots - \tilde{\eta}_{j_0-1} - \tilde{\eta}_{j_0} + \tilde{\eta}_{j_0+1} + \cdots + \tilde{\eta}_{n-1})\Big) \prod_{j<l\le n-1} (k_j - \varepsilon_j \varepsilon_l k_l),$$

where $\varepsilon_1 = \cdots = \varepsilon_{j_0} = -1$, $\varepsilon_{j_0+1} = \cdots = \varepsilon_{n-1} = 1$, and

$$B = \exp\Big(\frac{\pi i}{4}\Big(\sum_{j=1}^{n-1} \varepsilon_j + n - 1\Big)\Big) = \exp(k\pi i).$$

It follows that as $j \to +\infty$,

$$\frac{\tau}{\gamma}\Big|_{(x_j,y_j)} \to \exp(\tilde{\eta}_{j_0}) \prod_{j=1}^{j_0-1} \frac{k_j + k_{j_0}}{k_j - k_{j_0}} \prod_{j=j_0+1}^{n-1} \frac{k_{j_0} - k_j}{k_{j_0} + k_j}. \tag{4.3}$$

Note that if $n - j_0$ is even, then (4.3) still holds. We know that $\tau = \pm\gamma i$. Let μ_{j_0} be the complex number defined by

$$\exp(\mu_{j_0}) = \pm i \exp(-\iota_{j_0}) \prod_{j=1}^{j_0-1} \frac{k_j + k_{j_0}}{k_j - k_{j_0}} \prod_{j=j_0+1}^{n-1} \frac{k_{j_0} - k_j}{k_{j_0} + k_j}.$$

Note that μ_{j_0} is real. Then using the fact that $\tilde{\eta}_{j_0} = \eta_{j_0} - \iota_{j_0}$ and (4.3), we find that

$$\exp(\eta_{j_0} + \mu_{j_0})|_{(x_j,y_j)} \to 1.$$

This implies that (x_j, y_j) is on the curve in $\mathcal{S}$ which is asymptotic to the line

$$p_{j_0} x + q_{j_0} y + \eta^0_j + \mu_{j_0} = 0.$$

This finishes the proof. ∎

Lemma 4.3. *As* $\min_{j=1,\ldots,n} |p_j x + q_j y| \to +\infty$*, we have*

$$\begin{aligned} \Gamma(x,y) &\to 0, && \text{if } p_n x + q_n y \to +\infty, \\ \Gamma(x,y) &\to 2k_n, && \text{if } p_n x + q_n y \to -\infty. \end{aligned}$$

Proof. Suppose $\min_{j=1,\ldots,n} |p_j x + q_j y| \to +\infty$ and $p_n x + q_n y \to +\infty$. Without loss of generality, we assume that $\eta_j(x, y) \to -\infty$ for $j = 1, \ldots, m_0$, and $\eta_j(x, y) \to +\infty$ for $j = m_0 + 1, \ldots, n$.

If $n - m_0$ is even, then the main order term (up to a coefficient) in f is

$$\exp\Big(\frac{1}{2}(-\eta_1 - \cdots - \eta_{m_0} + \eta_{m_0+1} + \cdots + \eta_n)\Big).$$

This implies that $g/f \to 0$. On the other hand, the main order term (up to a coefficient) in τ is

$$\exp\Big(\frac{1}{2}(-\eta_1 - \cdots - \eta_{m_0} + \eta_{m_0+1} + \cdots + \eta_{n-1})\Big).$$

Hence $\gamma/\tau \to 0$. It follows that for each fixed y,

$$\Gamma = 2k_n \frac{(\gamma/\tau - g/f)^2}{(1 + (g/f)^2)(1 + (\gamma/\tau)^2)} \to 0.$$

If $n - m_0$ is odd, then the main order term (up to a coefficient) in g is

$$\exp\Big(\frac{1}{2}(-\eta_1 - \cdots - \eta_{m_0} + \eta_{m_0+1} + \cdots + \eta_n)\Big).$$

Hence $f/g \to 0$. Similarly, the main order term in γ is

$$\exp\Big(\frac{1}{2}(-\eta_1 - \cdots - \eta_{m_0} + \eta_{m_0+1} + \cdots + \eta_{n-1})\Big),$$

and $\tau/\gamma \to 0$. Therefore, we still have

$$\Gamma = 2k_n \frac{(\gamma/\tau - g/f)^2}{(1 + (g/f)^2)(1 + (\gamma/\tau)^2)} \to 0.$$

Next, we suppose $\min_{j=1,\ldots,n} |p_j x + q_j y| \to +\infty$ and $p_n x + q_n y \to -\infty$. We may assume that, for some index m_0, there holds $\eta_j(x, y) \to +\infty$ for $j = 1, \ldots, m_0$, and $\eta_j(x, y) \to -\infty$ for $j = m_0 + 1, \ldots, n$.

If m_0 is even, then the main order term (up to a coefficient) in f is

$$\exp\Big(\frac{1}{2}(\eta_1 + \cdots + \eta_{m_0} - \eta_{m_0+1} - \cdots - \eta_n)\Big).$$

As a consequence, $g/f \to 0$. The main order term (up to a coefficient) in γ is

$$\exp\Big(\frac{1}{2}(\eta_1 + \cdots + \eta_{m_0} - \eta_{m_0+1} - \cdots - \eta_{n-1})\Big),$$

which implies that $\tau/\gamma \to 0$. We then deduce that

$$\Gamma = 2k_n \frac{(1 - g\tau/(f\gamma))^2}{(1 + (g/f)^2)(1 + (\tau/\gamma)^2)} \to 2k_n.$$

If m_0 is odd, then the main order term (up to a coefficient) in g is

$$\exp\Big(\frac{1}{2}(\eta_1 + \cdots + \eta_{m_0} - \eta_{m_0+1} - \cdots - \eta_n)\Big).$$

Hence $f/g \to 0$. Similarly, $\gamma/\tau \to 0$. We then deduce that

$$\Gamma = 2k_n \frac{(f\gamma/(g\tau) - 1)^2}{(1 + (f/g)^2)(1 + (\gamma/\tau)^2)} \to 2k_n.$$

This finishes the proof. ∎

For each fixed y, let us consider the homogeneous first order ODE $L\xi = 0$, that is,

$$\partial_x \xi + \mathrm{Re}(\Gamma - k_n)\xi = 0. \tag{4.4}$$

If Γ is a *smooth* function, then Lemma 4.3 tells us that the integral $\int_{-\infty}^{x} \Gamma(l, y)\, dl$ is well defined and (4.4) has a solution of the form

$$\xi(x, y) := \exp\Big(p_n x + q_n y - \int_{-\infty}^{x} \mathrm{Re}(\Gamma(l, y))\, dl\Big).$$

However, since in reality Γ has singularities, we need to define ξ in a rigorous way. To do this, it will be important to understand the function

$$\vartheta := k_n \frac{(f\gamma - g\tau)^2}{(f^2 + g^2)(\gamma\partial_s\gamma + \tau\partial_s\tau)}.$$

Let us first consider the simple case $n = 2$. We then have $\gamma = \exp\left(-\frac{1}{2}\tilde{\eta}_1\right)$, $\tau = \exp\left(\frac{1}{2}\tilde{\eta}_1\right)$,

$$f = -\exp\Big(\frac{1}{2}(\tilde{\eta}_1 + \tilde{\eta}_2)\Big)(k_1 - k_2) + \exp\Big(\frac{1}{2}(-\tilde{\eta}_1 - \tilde{\eta}_2)\Big)(k_1 - k_2),$$

$$g = \exp\Big(\frac{1}{2}(\tilde{\eta}_1 - \tilde{\eta}_2)\Big)(k_1 + k_2) + \exp\Big(\frac{1}{2}(-\tilde{\eta}_1 + \tilde{\eta}_2)\Big)(k_1 + k_2).$$

By definition, $\gamma^2 + \tau^2 = 0$ on S_*, which implies that $1 + \exp(2\tilde{\eta}_1) = 0$. If $\exp(\tilde{\eta}_1) = i$, then

$$\frac{g}{f} = \frac{\exp\left(\frac{1}{2}(\tilde{\eta}_1 - \tilde{\eta}_2)\right)(k_1 + k_2) + \exp\left(\frac{1}{2}(-\tilde{\eta}_1 + \tilde{\eta}_2)\right)(k_1 + k_2)}{-\exp\left(\frac{1}{2}(\tilde{\eta}_1 + \tilde{\eta}_2)\right)(k_1 - k_2) + \exp\left(\frac{1}{2}(-\tilde{\eta}_1 - \tilde{\eta}_2)\right)(k_1 - k_2)} = \frac{k_1 + k_2}{k_1 - k_2}\, i.$$

Moreover, recalling the relation (3.15) between (x, y) and (s, t), we get

$$\frac{\partial_s\gamma + i\partial_s\tau}{\gamma} = -k_1.$$

If follows that

$$\vartheta = -\frac{k_2}{k_1} \frac{\left(1 + \frac{k_1+k_2}{k_1-k_2}\right)^2}{1 - \left(\frac{k_1+k_2}{k_1-k_2}\right)^2} = 1 \quad \text{on } S_*.$$

One can show that if $\exp(\tilde{\eta}_1) = -i$, we still have $\vartheta = 1$ on S_*. We would like to prove that this identity is true for all n. For this purpose, we first show the following.

Lemma 4.4. *Let (x_j, y_j) be a sequence of points in S_* such that $x_j^2 + y_j^2 \to +\infty$, as $j \to +\infty$. Then*

$$\vartheta(x_j, y_j) \to 1 \quad \text{as } j \to +\infty. \tag{4.5}$$

Proof. As in the proof of Lemma 4.2, we still assume that as $j \to +\infty$,

$$\eta_m \to -\infty, \quad \text{for } m = 1, \dots, j_0 - 1,$$
$$\eta_m \to +\infty, \quad \text{for } m = j_0 + 1, \dots, n.$$

It follows that as $j \to +\infty$,

$$\frac{\tau}{\gamma}\Big|_{(x_j,y_j)} \to \exp\left(\tilde{\eta}_{j_0}\right) \prod_{j=1}^{j_0-1} \frac{k_j + k_{j_0}}{k_j - k_{j_0}} \prod_{j=j_0+1}^{n-1} \frac{k_{j_0} - k_j}{k_{j_0} + k_j}. \tag{4.6}$$

Similarly, we have

$$\frac{g}{f} \to -\exp\left(-\tilde{\eta}_{j_0}\right) \prod_{j=1}^{j_0-1} \frac{k_j - k_{j_0}}{k_j + k_{j_0}} \prod_{j=j_0+1}^{n} \frac{k_{j_0} + k_j}{k_{j_0} - k_j}. \tag{4.7}$$

Since $\gamma = \pm i\tau$ at (x_j, y_j), from (4.6) and (4.7), we get

$$\frac{g^2}{f^2} \to -\left(\frac{k_{j_0} + k_n}{k_{j_0} - k_n}\right)^2. \tag{4.8}$$

We also have

$$\frac{g\tau}{f\gamma} \to -\frac{k_{j_0} + k_n}{k_{j_0} - k_n}.$$

Hence as $j \to +\infty$, at (x_j, y_j),

$$k_n \frac{(f - g\tau/\gamma)^2}{(f^2 + g^2)} = k_n \frac{\left(1 - \frac{g\tau}{f\gamma}\right)^2}{\left(1 + \frac{g^2}{f^2}\right)} \to k_n \frac{\left(1 + \frac{k_{j_0}+k_n}{k_{j_0}-k_n}\right)^2}{1 - \left(\frac{k_{j_0}+k_n}{k_{j_0}-k_n}\right)^2} = -k_{j_0}.$$

Then, (4.5) follows from the fact that

$$\frac{\gamma\partial_s\gamma + \tau\partial_s\tau}{\gamma^2} \to -k_{j_0}.$$ ■

Lemma 4.5. $\vartheta = 1$ *on* S_*.

We point out that a simplified proof of this result will be sketched in the proof of Lemma 4.9. However, the proof given below may be also of independent interest.

Proof. On S_*, $\tau = \pm i\gamma$. We may assume without loss of generality that $\tau = \gamma i$. The case of $\tau = -\gamma i$ is similar. We then would like to prove that

$$k_n\gamma(f - gi)^2 - (f^2 + g^2)(\partial_s\gamma + i\partial_s\tau) = 0 \quad \text{on } S_*. \tag{4.9}$$

Let us consider the case of $n = 3$. The idea for the general case is same, but the notations would be heavy. We denote

$$a^*(i_1, \ldots, i_m) := a(i_1, \ldots, i_m) \prod_{j=1}^{m} \frac{k_{i_j} - k_3}{k_{i_j} + k_3}.$$

Recall that (see (3.14))

$$\gamma = 1 + a^*(1,2) \exp(\eta_1 + \eta_2) \quad \text{and} \quad \tau = a^*(1) \exp(\eta_1) + a^*(2) \exp(\eta_2).$$

On S, from $\tau = \gamma i$, we get

$$\exp(\eta_2) = \frac{i - a^*(1) \exp(\eta_1)}{a^*(2) - i a^*(1,2) \exp(\eta_1)}. \tag{4.10}$$

It follows that

$$\begin{aligned}
\gamma &= 1 + a^*(1,2) \exp(\eta_1) \frac{[i - a^*(1) \exp(\eta_1)]}{a^*(2) - i a^*(1,2) \exp(\eta_1)} \\
&= \frac{a^*(2) - i a^*(1,2) \exp(\eta_1) + a^*(1,2) \exp(\eta_1)[i - a^*(1) \exp(\eta_1)]}{a^*(2) - i a^*(1,2) \exp(\eta_1)} \\
&:= \frac{J_1}{a^*(2) - i a^*(1,2) \exp(\eta_1)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
f &= 1 + a(1,3) \exp(\eta_1 + \eta_3) + [a(1,2) \exp(\eta_1) + a(2,3) \exp(\eta_3)] \exp(\eta_2) \\
&= \frac{[1 + a(1,3) \exp(\eta_1 + \eta_3)][a^*(2) - i a^*(1,2) \exp(\eta_1)]}{a^*(2) - i a^*(1,2) \exp(\eta_1)} \\
&\quad + \frac{[a(1,2) \exp(\eta_1) + a(2,3) \exp(\eta_3)][i - a^*(1) \exp(\eta_1)]}{a^*(2) - i a^*(1,2) \exp(\eta_1)} \\
&:= \frac{J_2}{a^*(2) - i a^*(1,2) \exp(\eta_1)}. \\
g &= \exp(\eta_1) + \exp(\eta_2) + \exp(\eta_3) + a(1,2,3) \exp(\eta_1 + \eta_2 + \eta_3) \\
&= \frac{[\exp(\eta_1) + \exp(\eta_3)][a^*(2) - i a^*(1,2) \exp(\eta_1)]}{a^*(2) - i a^*(1,2) \exp(\eta_1)} \\
&\quad + \frac{[1 + a(1,2,3) \exp(\eta_1 + \eta_3)][i - a^*(1) \exp(\eta_1)]}{a^*(2) - i a^*(1,2) \exp(\eta_1)} \\
&:= \frac{J_3}{a^*(2) - i a^*(1,2) \exp(\eta_1)}.
\end{aligned}$$

We also have

$$\begin{aligned}
\partial_s \gamma + i \partial_s \tau &= [(k_1 + k_2) a^*(1,2) \exp(\eta_1) + i a^*(2) k_2] \frac{[i - a^*(1) \exp(\eta_1)]}{a^*(2) - i a^*(1,2) \exp(\eta_1)} \\
&\quad + i k_1 a^*(1) \exp(\eta_1) \frac{[a^*(2) - i a^*(1,2) \exp(\eta_1)]}{a^*(2) - i a^*(1,2) \exp(\eta_1)} \\
&= \frac{J_4}{a^*(2) - i a^*(1,2) \exp(\eta_1)}.
\end{aligned}$$

We then get

$$k_n\gamma(f-gi)^2-(f^2+g^2)(\partial_s\gamma+i\partial_s\tau)=\frac{k_nJ_1(J_2-J_3i)^2-(J_2^2+J_3^2)J_4}{[a^*(2)-ia^*(1,2)\exp(\eta_1)]^3}.$$

Let us write

$$k_nJ_1(J_2-J_3i)^2-(J_2^2+J_3^2)J_4=\sum_{j,k}A_{j,k}\exp(j\eta_1+k\eta_3).$$

We would like to show that $A_{j,k}=0$. To see this, we assume without loss of generality that along a sequence (x_j,y_j) with $|\eta_2|$ bounded, both η_1 and η_3 tend to $+\infty$, and $\eta_1>\eta_3$. Observe that the main order term is $A_{6,2}\exp(6\eta_1+2\eta_3)$. By Lemma 4.4, along this sequence, $\vartheta\to 1$. This implies that $A_{6,2}$ has to be zero, otherwise the limit of ϑ will not be equal to 1. Once we know $A_{6,2}$ is zero, the main order term becomes $A_{6,1}\exp(6\eta_1+\eta_3)$. Using again the fact that $\vartheta\to 1$ along (x_j,y_j), we deduce that $A_{6,1}$ is 0. Repeating this argument, we see that $A_{j,k}=0$ for all j,k. The identity (4.9) is then proved.

We remark that, in the case of $n=3$, one can also explicitly compute $A_{j,k}$. For instance, $A_{0,0}$ is

$$\begin{aligned}&k_3\,a^*(2)(a^*(2)-i(i))^2-(a^*(2)^2+i^2)i^2a^*(2)k_2\\&\quad=a^*(2)\Big[k_3\,\frac{(2k_2)^2}{(k_2-k_3)^2}+k_2\Big(1-\frac{(k_2+k_3)^2}{(k_2-k_3)^2}\Big)\Big]=0.\end{aligned}$$

The coefficient $A_{6,0}$ of $\exp(6\eta_1)$ is equal to

$$\begin{aligned}&-k_3\,a^*(1,2)a^*(1)\big[-a(1,2)a^*(1)-i(-ia^*(1,2))\big]^2\\&-\big((-a(1,2)a^*(1))^2+(-ia^*(1,2))^2\big)\big[(k_1+k_2)a^*(1,2)(-a^*(1))+k_1a^*(1)a^*(1,2)\big]\\&=a^*(1,2)a^*(1)(a(1,2)a^*(1))^2\big[-k_3(1+a^*(2))^2-k_2(1-(a^*(2))^2)\big]=0.\end{aligned}$$

For general n, this computation would be tedious. ∎

At this stage, we emphasize that the function ϑ is not well defined on the set $S_0(v)$. For a given function v with parameters p_j,q_j,η_j^0, it is not clear whether the corresponding set $S_0(v)$ is empty or only consists of finitely many points. In principle, it is even possible that S_0 contains a smooth curve. The following result deals with some special cases of parameters, but it will not be relevant to our later proof in this section.

Lemma 4.6. *There exist parameters p_j,q_j,η_j^0, $j=1,\ldots,n-1$, such that for the corresponding solution v, the set S_0 is empty.*

Proof. Let δ be a small positive number to be determined later on. Let us denote the lines $\eta_j=0$ by l_j. For $j=1,\ldots,n$, we choose $p_j=j/(2n)$, $q_j=\sqrt{1-p_j^2}$ and $\eta_j^0=j^2\ln\delta$. Note that for this choice, when δ is small, no three lines l_j intersect at same point. Moreover, as $\delta\to 0$, the distance between the intersection points tends to infinity. We also remark that there are many other different choices.

Let $M > 0$ be a constant independent of δ, also to be determined later on. Consider the region Ω which consists of those points (x, y) satisfying: there exists at most one η_j such that $|\eta_j(x, y)| \leq M$.

In view of (3.14), $\tau/\gamma = H_1/H_2$, where

$$H_1 := \sum_{m=0}^{\lfloor (n-2)/2 \rfloor} \left(\sum_{\{n-1,2m+1\}} \left[a(i_1, \ldots, i_{2m+1}) \prod_{j=1}^{2m+1} \frac{k_{i_j}+k_n}{k_{i_j}-k_n} \exp(\eta_{i_1} + \cdots + \eta_{i_{2m+1}}) \right] \right),$$

$$H_2 := \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \left(\sum_{\{n-1,2m\}} \left[a(i_1, \ldots, i_{2m}) \prod_{j=1}^{2m} \frac{k_{i_j}+k_n}{k_{i_j}-k_n} \exp(\eta_{i_1} + \cdots + \eta_{i_{2m}}) \right] \right).$$

Let

$$m_0 := \min_{(j_1,\ldots,j_l), l \leq n-1} \left\{ \left| a\left(j_1, \ldots, j_l\right) \prod_{b=1}^{l} \left(\frac{k_{j_b}+k_n}{k_{j_b}-k_n} \right) \right| \right\},$$

$$m_1 := \max_{(j_1,\ldots,j_l), l \leq n-1} \left\{ \left| a\left(j_1, \ldots, j_l\right) \prod_{b=1}^{l} \left(\frac{k_{j_b}+k_n}{k_{j_b}-k_n} \right) \right| \right\}.$$

We claim that if $\exp(M/4) > \frac{m_1}{m_0} 2^n$, then $\Omega \cap S_0 = \emptyset$. Indeed, suppose (x_0, y_0) is a point in Ω. Assume without loss of generality that $|\eta_1(x_0, y_0)| \leq M$. We can also assume that for some k_0,

$$\eta_j > M, \quad \text{for } j = 2, \ldots, k_0, \quad \text{and} \quad \eta_j < -M, \quad \text{for } j = k_0 + 1, \ldots, n.$$

We consider two different cases.

Case 1. k_0 is even.

If $\eta_1(x_0, y_0) > M/2$, then the main order term in H_1 is

$$\exp(\eta_1 + \cdots + \eta_{k_0}).$$

This term dominates the sum of other terms in H_1. More precisely, since $\exp(M/4) > \frac{m_1}{m_0} 2^n$, we have

$$|H_1(x_0, y_0)| \geq \exp\left(\eta_1 + \cdots + \eta_{k_0}\right) \left(1 - \frac{1}{2}\right) > 0.$$

Hence $\tau(x_0, y_0) \neq 0$. On the other hand, if $\eta_1(x_0, y_0) \leq M/2$, then the main order term in H_2 is

$$\exp(\eta_2 + \cdots + \eta_{k_0}).$$

This term dominates the sum of other terms in H_2. Hence $\gamma(x_0, y_0) \neq 0$.

Case 2. k_0 is odd.

If $\eta_1(x_0, y_0) > M/2$, then the main order term in H_2 is

$$\exp(\eta_1 + \cdots + \eta_{k_0}).$$

This term dominates the sum of other terms in H_2, hence $\gamma(x_0, y_0) \neq 0$. If $\eta_1(x_0, y_0) \leq M/2$, then the main order term in H_1 is

$$\exp(\eta_2 + \cdots + \eta_{k_0}).$$

This term dominates the sum of other terms in H_1. Hence $\tau(x_0, y_0)$ does not vanish. The claim is thus proved.

Now fix an M satisfying $\exp(M/4) > \frac{m_1}{m_0} 2^n$. Consider $(x_0, y_0) \in \mathbb{R}^2 \backslash \Omega$. If δ is sufficiently small, then by the choice of p_j, q_j, η_j^0, there exist precisely two η_j such that their absolute value at (x_0, y_0) is not larger than M. Assume they are η_1 and η_2. The function H_1 has the form

$$\frac{k_1 + k_n}{k_1 - k_n} \exp(\eta_1) + \frac{k_2 + k_n}{k_2 - k_n} \exp(\eta_2) + C_1(\delta),$$

The function H_2 has the form

$$1 - \left(\frac{k_1 - k_2}{k_1 + k_2}\right)^2 \frac{k_1 + k_n}{k_1 - k_n} \frac{k_2 + k_n}{k_2 - k_n} \exp(\eta_1 + \eta_2) + C_2(\delta).$$

Here $C_1(\delta), C_2(\delta)$ tend to zero as $\delta \to 0$. Note that $\frac{k_j + k_n}{k_j - k_n}$ is purely imaginary. Hence for δ sufficiently small, either the equation $H_1 = 0$ has no solution, or the equation $H_2 = 0$ has no solution. Hence the set S_0 is empty. Actually, in this case, by our choice of k_j, necessarily the equation $H_1 = 0$ has no solution. The proof is completed. ■

Throughout the section, we shall use $B_\epsilon(x_0, y_0)$ to denote the open ball of radius ϵ centered at (x_0, y_0). Roughly speaking, the following lemma states that the set $\bar{S}_*$ cannot contain several curves intersect at one point.

Lemma 4.7. *Suppose* $(x_0, y_0) \in \bar{S}_*$, *and* $S_0 \cap B_\epsilon(x_0, y_0) = \{(x_0, y_0)\}$ *for some* $\epsilon > 0$. *Then locally around* (x_0, y_0), $\bar{S}_*$ *is a smooth curve. More precisely, there exists* $\delta > 0$ *such that either*

$$\bar{S}_* \cap \{(x, y) : |x - x_0| < \delta, |y - y_0| < \delta\} = \{(F(y), y), y \in (y_0 - \delta, y_0 + \delta)\},$$

where F *is a smooth function, or*

$$\bar{S}_* \cap \{(x, y) : |x - x_0| < \delta, |y - y_0| < \delta\} = \{(x, F_*(x)), x \in (y_0 - \delta, y_0 + \delta)\},$$

where F_* *is a smooth function.*

Proof. Without loss of generality, we can assume that γ is real valued and τ is purely imaginary. Hence $\tau = i\tau^*$, where τ^* is real valued.

If $(x_0, y_0) \in S_*$, then $|\gamma| = |\tau| \neq 0$ and by (4.9), we have

$$\gamma \partial_s \gamma + \tau \partial_s \tau = \frac{k_n (f\gamma - g\tau)^2}{2(f^2 + g^2)}.$$

This implies that as a complex valued function, at (x_0, y_0),

$$|\partial_s(\gamma^2 + \tau^2)| = \frac{|k_n \gamma^2|}{2}.$$

This also means that

$$|\nabla(\gamma^2 - \tau^{*2})| = \frac{|k_n \gamma^2|}{2} \neq 0.$$

Note that the function $\gamma^2 - \tau^{*2}$ can be regarded as a map from $\mathbb{R}^2$ to $\mathbb{R}$. Therefore, by the implicit function theorem, the result of the lemma is true in the case that $(x_0, y_0) \in S_*$. In the rest of the proof we may assume that $(x_0, y_0) \in \bar{S}_* \backslash S_*$. In particular, $(x_0, y_0) \in S_0$.

Since γ and τ^* are real analytic functions, for δ small, the set

$$S_* \cap \{(x, y) : |x - x_0| < \delta, |y - y_0| < \delta\}$$

consists of finitely many disjoint smooth curves, $\mathrm{c}_1, \ldots, \mathrm{c}_m$. Each curve c_j is determined by a smooth map $\mathfrak{M}_j : (0, 1) \to \mathbb{R}^2$, where $\lim_{r \to 0} \mathfrak{M}_j(r) = (x_0, y_0)$. The direction of these curves at (x_0, y_0) will be denoted by $e_j := \mathfrak{M}'_j(0)$. We also write $e_j = (e_{j,1}, e_{j,2})$. To prove the lemma, it will be suffice to show that $m = 2$ and $e_1 = -ce_2$, for some $c > 0$.

We define α and β by

$$\exp(i\beta) = p_n + iq_n, \quad \exp(i\alpha) = \frac{f^2 - g^2}{f^2 + g^2} - i\,\frac{2fg}{f^2 + g^2}.$$

The function α is indeed a function of x, y. Since $f, g > 0$, we can choose α to be taking values in $(-\pi, 0)$. On S_*, if $\gamma = \tau^*$, we have

$$\vartheta = \frac{\gamma^2}{\partial_s\left(\gamma^2 - \tau^{*2}\right)} \exp\left(i(\beta + \alpha)\right).$$

If $\gamma = -\tau^*$, then

$$\vartheta = \frac{\gamma^2}{\partial_s\left(\gamma^2 - \tau^{*2}\right)} \exp\left(i(\beta - \alpha)\right).$$

To avoid confusion, we call $\alpha_j := \alpha|_{\mathrm{c}_j}$ the restriction of α to the curve c_j.

The key observation of the proof is the following: the fact that u, v are connected through the Bäcklund transformation does not depend on the choice of the coordinate system. Hence if we rotate the coordinate system by an angle θ, then the corresponding function

$$\vartheta' := \exp\left((\theta + \beta)i\right) \frac{(f\gamma - g\tau)^2}{\partial_{s'}\left(\gamma^2 - \tau^2\right)}$$

in the new coordinate system is still equal to 1. That is, if we denote the new coordinate system by (x', y'), then on S_*, if $\gamma = \tau^*$,

$$\vartheta' = \exp\left((\theta + \beta + \alpha)i\right) \frac{\gamma^2}{\partial_{s'}\left(\gamma^2 - \tau^2\right)} = 1, \tag{4.11}$$

and if $\gamma = -\tau^*$, then

$$\vartheta' = \exp\left((\theta + \beta - \alpha)i\right) \frac{\gamma^2}{\partial_{s'}\left(\gamma^2 - \tau^2\right)} = 1. \tag{4.12}$$

We split the proof into two different cases.

Case 1. $f(x_0, y_0) \neq g(x_0, y_0)$.

Since we have the freedom of choosing different coordinate systems, we may choose $\theta_1 = -\beta$. We set

$$e'_{j,1} + ie'_{j,2} = (e_{j,1} + ie_{j,2}) \exp\left(i\theta_1\right), \quad j = 1, \ldots, m.$$

We claim that there exists at most one j such that $e'_{j,2} > 0$. Assume to the contrary that $0 < e'_{j_1,2} < \cdots < e'_{j_l,2}$, where $l \geq 2$. Since $f(x_0, y_0) \neq g(x_0, y_0)$, we find that if (x, y) is close to (x_0, y_0), then $\cos\alpha(x, y) \neq 0$. On the other hand, since $S_0 \cap B_\epsilon(x_0, y_0) = \{(x_0, y_0)\}$, the function $\partial_{x'}(\gamma^2 - \tau^{*2})$ will have different signs on the curves c_{j_1} and c_{j_2}. This contradicts with the identity (4.11) and (4.12). This proves the claim. Similarly, there is at most one direction with $e'_{j,2} < 0$. We can then assume, by relabeling the indices if necessary, that $e'_{1,2} > 0, e'_{2,2} < 0$, and $e'_{j,2} = 0$, $j = 3, \ldots, m$.

Next we choose $\theta_2 = \theta_1 + \sigma$, with $|\sigma|$ being small. We denote the new coordinate system by $(x^\wedge, y^\wedge)$. Assume $\gamma = \tau^*$ on c_1 and $\gamma = -\tau^*$ on c_3. Then

$$\frac{\gamma^2}{\partial_{s^\wedge}(\gamma^2 - \tau^{*2})} \exp\left(i\,(\sigma + \alpha_1)\right) = 1, \quad \text{on } c_1, \tag{4.13}$$

$$\frac{\gamma^2}{\partial_{s^\wedge}(\gamma^2 - \tau^{*2})} \exp\left(i\,(\sigma - \alpha_3)\right) = 1, \quad \text{on } c_3. \tag{4.14}$$

Observe that α_1 and α_3 tend to $\alpha(x_0, y_0) \neq -\pi/2$, as $(x, y) \to (x_0, y_0)$. Hence $\cos(\sigma+\alpha_1)$ and $\cos(\sigma - \alpha_3)$ have the same sign when $|\sigma|$ is small. On the other hand, since the direction of c_3 is parallel to the x' coordinate axis, the function $\partial_{x^\wedge}(\gamma^2 - \tau^2)$ has different sign on c_3, for the two different choices of $\sigma = \pm\sigma_0$, where σ_0 is a fixed small positive constant. This contradicts with (4.13) and (4.14). Hence m has to be equal to 2. Note that this argument also tells us that there at most two indices of j such that $e'_{j,2} = 0$. Now we deduce that the function $\gamma\tau^*$ has same sign on c_1 and c_2 (otherwise, $m \geq 3$). We only consider the case $\gamma = \tau^*$. Then

$$\vartheta = \frac{\gamma^2}{\partial_s(\gamma^2 - \tau^{*2})} \exp\left(i\,(\beta + \alpha)\right), \quad \text{on } c_1 \text{ and } c_3.$$

If $e_1 \neq -e_2$, we can always rotate the coordinate system (x, y) into a new one $(x^\#, y^\#)$, such that $\partial_{x^\#}(\gamma^2 - \tau^{*2})$ has different sign on c_1 and c_3. This is a contradiction.

Case 2. $f(x_0, y_0) = g(x_0, y_0)$.

In this case, the proof is similar to Case 1, with minor modifications. More precisely, in Case 1, we have taken $\theta_1 = -\beta$. Now we take $\theta_1 = -\beta + \varepsilon_0$, where $\varepsilon_0 > 0$ is a small constant. Observe that for (x, y) close to (x_0, y_0), $\cos(\varepsilon_0 + \alpha_1)$ and $\cos(\varepsilon_0 - \alpha_1)$ have the same sign. The rest of the proof is same as that of Case 1. ∎

We remark that without the assumption that $S_0 \cap B_\epsilon(x_0, y_0) = \{(x_0, y_0)\}$, Lemma 4.7 is still true. This generalization will be proved in Lemma 4.9.

Lemma 4.8. *Suppose $(x_0, y_0) \in S_0$ and $|\gamma| \geq |\tau|$ in $B_\delta(x_0, y_0)$, for some $\delta > 0$. Then (x_0, y_0) is a removable singularity of Γ. That is, the limit*

$$\lim_{(x,y)\to(x_0,y_0),(x,y)\notin S} \Gamma(x, y)$$

exists.

Proof. Lemma 3.6 tells us that

$$\begin{cases} \partial_x v = i\partial_y u - k_n \sin\frac{v+u}{2} - \bar{k}_n \sin\frac{v-u}{2}, \\ i\partial_y v = \partial_x u - k_n \sin\frac{v+u}{2} + \bar{k}_n \sin\frac{v-u}{2}. \end{cases}$$

The first equation in this system can be written as

$$\begin{aligned} 4\frac{\partial_x(\tau/\gamma)}{1+(\tau/\gamma)^2} &= 4i\,\frac{\partial_y(g/f)}{1+(g/f)^2} \\ &\quad - (p_n + q_n i)\Big(\frac{2\gamma\tau}{\gamma^2+\tau^2}\frac{f^2-g^2}{f^2+g^2} - \frac{\gamma^2-\tau^2}{\gamma^2+\tau^2}\frac{2fg}{f^2+g^2}\Big) \\ &\quad - (p_n - q_n i)\Big(\frac{2\gamma\tau}{\gamma^2+\tau^2}\frac{f^2-g^2}{f^2+g^2} + \frac{\gamma^2-\tau^2}{\gamma^2+\tau^2}\frac{2fg}{f^2+g^2}\Big). \end{aligned}$$

Still setting $\tau = i\tau^*$, we get

(4.15)

$$\partial_x\Big(\frac{\tau^*}{\gamma}\Big) = \frac{\partial_y(g/f)}{1+(g/f)^2}\Big(1-\Big(\frac{\tau^*}{\gamma}\Big)^2\Big) - p_n\frac{\tau^*}{\gamma}\frac{f^2-g^2}{f^2+g^2} + \Big(1+\Big(\frac{\tau^*}{\gamma}\Big)^2\Big)\frac{q_n fg}{f^2+g^2}.$$

Similarly, the second equation of the system has the form

(4.16)

$$-\partial_y\Big(\frac{\tau^*}{\gamma}\Big) = \frac{\partial_x(g/f)}{1+(g/f)^2}\Big(1-\Big(\frac{\tau^*}{\gamma}\Big)^2\Big) + p_n\Big(1+\Big(\frac{\tau^*}{\gamma}\Big)^2\Big)\frac{fg}{f^2+g^2} + q_n\frac{\tau^*}{\gamma}\frac{f^2-g^2}{f^2+g^2}.$$

Differentiating equation (4.15) with respect to x and equation (4.16) with respect to y, we get

$$\begin{aligned} \Delta\Big(\frac{\tau^*}{\gamma}\Big) &= \frac{2\tau^*/\gamma}{1+(g/f)^2}\Big(\partial_y\Big(\frac{\tau^*}{\gamma}\Big)\partial_x\Big(\frac{g}{f}\Big) - \partial_x\Big(\frac{\tau^*}{\gamma}\Big)\partial_y\Big(\frac{g}{f}\Big)\Big) \\ &\quad + \Big(1+\Big(\frac{\tau^*}{\gamma}\Big)^2\Big)\Big(q_n\partial_y\Big(\frac{fg}{f^2+g^2}\Big) - p_n\partial_x\Big(\frac{fg}{f^2+g^2}\Big)\Big) \\ &\quad + \frac{2fg}{f^2+g^2}\frac{\tau^*}{\gamma}\Big(q_n\partial_y\Big(\frac{\tau^*}{\gamma}\Big) - p_n\partial_x\Big(\frac{fg}{f^2+g^2}\Big)\Big) \\ &\quad - \frac{\tau^*}{\gamma}\Big(p_n\partial_y\Big(\frac{f^2-g^2}{f^2+g^2}\Big) + q_n\partial_x\Big(\frac{f^2-g^2}{f^2+g^2}\Big)\Big) \\ &\quad - \frac{f^2-g^2}{f^2+g^2}\Big(p_n\partial_y\Big(\frac{\tau^*}{\gamma}\Big) + q_n\partial_x\Big(\frac{\tau^*}{\gamma}\Big)\Big). \end{aligned}$$

Inserting (4.15) and (4.16) into this equation, we find that τ^*/γ satisfies an equation of the form

(4.17)
$$\Delta\Big(\frac{\tau^*}{\gamma}\Big) = \sum_{j=0}^{3}\Big(\alpha_j(x,y)\Big(\frac{\tau^*}{\gamma}\Big)^j\Big),$$

where α_j are smooth functions determined by f, g. Since γ and τ^* are both real analytic and $|\tau^*/\gamma| \le 1$, the function τ^*/γ can be smoothly extended to the punctured ball $B_\delta(x_0, y_0)\backslash\{(x_0, y_0)\}$. Since $|\tau^*/\gamma| \le 1$, elliptic regularity and the removable singularity theorem of harmonic functions tell us that actually τ^*/γ can be regarded as a smooth function in $B_\delta(x_0, y_0)$.

Now we distinguish two cases.

Case 1. $\lim_{(x,y)\to(x_0,y_0)} \frac{\tau^*}{\gamma} = A_0 \in (-1,1)$.

In this case, we have

$$\lim_{(x,y)\to(x_0,y_0)} \Gamma(x,y) = \lim_{(x,y)\to(x_0,y_0)} \frac{k_n (f\gamma - ig\tau^*)^2}{(f^2+g^2)(\gamma^2-\tau^{*2})}$$
$$= \frac{k_n (f-igA_0)^2}{(f^2+g^2)(1-A_0^2)}\Big|_{(x,y)=(x_0,y_0)}.$$

Case 2. $\lim_{(x,y)\to(x_0,y_0)} \frac{\tau^*}{\gamma} = \pm 1$.

We first consider the case where the limit is equal 1. From (4.15), (4.16), we deduce that at the point (x_0, y_0),

$$\partial_x\left(\frac{\tau^*}{\gamma}\right) = -p_n \frac{f^2-g^2}{f^2+g^2} + \frac{2q_n fg}{f^2+g^2} := c, \tag{4.18}$$

$$\partial_y\left(\frac{\tau^*}{\gamma}\right) = -2p_n \frac{fg}{f^2+g^2} - q_n \frac{f^2-g^2}{f^2+g^2} := d. \tag{4.19}$$

Observe that $c^2+d^2=1$. Hence

$$\frac{\tau^*}{\gamma} = 1 + c(x-x_0) + d(y-y_0) + O((x-x_0)^2+(y-y_0)^2), \quad \text{as } (x,y)\to(x_0,y_0).$$

But this contradicts with the assumption that $|\gamma| \geq |\tau|$ in $B_\delta(x_0,y_0)$. Hence the limit cannot be 1. Similarly, it cannot be -1. Therefore Case 2 will not happen. ∎

In view of the proof this lemma, we now define

$$S = \left\{(x_0,y_0) \in \mathcal{S}\colon \lim_{x\to x_0} \left|\frac{\tau^*}{\gamma}(x,y_0)\right| = 1\right\}.$$

By this definition, automatically we have $S_* \subset S$.

Lemma 4.9. *Suppose $(x_0,y_0) \in S$. Then locally around (x_0,y_0), S is a smooth curve. Moreover, there exist real numbers c, d, with $c^2+d^2=1$, such that as $(x,y)\to(x_0,y_0)$,*

$$\Gamma(x,y) = \frac{c+di+O(|x-x_0|+|y-y_0|)}{c(x-x_0)+d(y-y_0)+O((x-x_0)^2+(y-y_0)^2)}.$$

Proof. If $(x_0,y_0)\in S_*$, then the result follows from the implicit function theorem and the fact that $\vartheta = 1$ on S_*.

If $(x_0,y_0) \in \bar{S}_* \backslash S_*$, then for δ small, the set $\bar{S}_* \cap B_\delta(x_0,y_0)$ separates $B_\delta(x_0,y_0)$ into several disjoint connected open components Ω_j, $j=1,\dots$ Since

$$\partial_s\left(\frac{\tau^*}{\gamma}\right) = \frac{\gamma\partial_s\tau^* - \tau^*\partial_s\gamma}{\gamma^2},$$

we find that

$$\partial_s\Big(\frac{\tau^*}{\gamma}\Big) = \begin{cases} -\frac{\partial_s(\gamma^2-\tau^{*2})}{\gamma^2}, & \text{if } \gamma = \tau^* \neq 0, \\ \frac{\partial_s(\gamma^2-\tau^{*2})}{\gamma^2}, & \text{if } \gamma = -\tau^* \neq 0. \end{cases}$$

Hence using equations (4.15) and (4.16), we deduce that for any $(x_1, y_1) \in S$, there holds

$$\vartheta(x, y_1) \to 1, \quad \text{as } x \to x_1. \tag{4.20}$$

We observe that the proof of Lemma 4.8 yields that any point $(x_1, y_1) \in S$ is not isolated in S (τ^*/γ satisfies equation (4.17) and is smooth around (x_1, y_1)). We also observe that if $(x_2, y_2) \in \Omega_1 \cap (S_0 \backslash S)$, then in a small neighborhood of (x_2, y_2), either $\gamma^2 \geq \tau^{*2}$, or $\gamma^2 \leq \tau^{*2}$. Now with (4.20) at hand, we can deal with the arcs contained in $\Omega_1 \cap S$ in a similar way as that of S^*. Hence we can apply arguments of Lemma 4.7 to infer that $\Omega_1 \cap S = \emptyset$. At this point, we emphasize that in principle, $\Omega_1 \cap S_0$ could be nonempty. Note that this argument also tells us that the set $\bar{S}_* \cap B_\delta(x_0, y_0)$ separates $B_\delta(x_0, y_0)$ precisely into two disjoint connected open components Ω_1, Ω_2, each component being diffeomorphic to a half ball.

Now we can assume without loss of generality that at some points in Ω_1, there holds $|\tau^*/\gamma| < 1$. Since $\Omega_1 \cap S = \emptyset$, we must have $|\tau^*| \leq |\gamma|$ in Ω_1. Note that the function τ^*/γ still satisfies equation (4.17). That is,

$$\Delta\Big(\frac{\tau^*}{\gamma}\Big) = \sum_{j=0}^{3}\Big(\alpha_j(x, y)\Big(\frac{\tau^*}{\gamma}\Big)^j\Big) \quad \text{in } \Omega_1.$$

Elliptic regularity and $|\tau^*/\gamma| \leq 1$ imply that τ^*/γ is smooth and that the limit $A_0 = \lim_{(x,y)\to(x_0,y_0)} \frac{\tau^*}{\gamma}$ exists. Since $B_\delta(x_0, y_0) \cap S_*$ is not empty, there holds $|A_0| = 1$. Hence it follows from same arguments as that of the previous lemma that as $(x, y) \to (x_0, y_0)$, if $A_0 = 1$, then

$$\frac{\tau^*}{\gamma} = A_0 + c(x - x_0) + d(y - y_0) + O((x - x_0)^2 + (y - y_0)^2),$$

where c and d are defined in (4.18) and (4.19). As a consequence, in a small neighborhood of (x_0, y_0),

$$\Gamma(x, y) = \frac{c + di + O(|x - x_0| + |y - y_0|)}{c(x - x_0) + d(y - y_0) + O((x - x_0)^2 + (y - y_0)^2)}.$$

A similar formula holds in the case of $A_0 = -1$.

Finally, suppose $(x_0, y_0) \in S \backslash \bar{S}_*$. By Lemma 4.8, if

$$|\gamma| \geq |\tau|, \text{ or } |\gamma| \leq |\tau|, \text{ in } B_\delta(x_0, y_0), \quad \text{for some } \delta > 0, \tag{4.21}$$

then the limit $\lim_{(x,y)\to(x_0,y_0} \Gamma(x, y) \neq \pm 1$ and $(x_0, y_0) \notin S$. On the other hand, if (4.21) does not hold, then by the previous arguments, one can show that $(x_0, y_0) \in S$, and the set $B_\delta(x_0, y_0) \cap S$ is a smooth curve. Moreover, one still has

$$\Gamma(x, y) = \frac{c + di + O(|x - x_0| + |y - y_0|)}{c(x - x_0) + d(y - y_0) + O((x - x_0)^2 + (y - y_0)^2)},$$

for some constants c, d with $c^2 + d^2 = 1$. ∎

Recall that we have defined

$$\xi(x,y)=\exp\Big(p_n x+q_n y-\int_{-\infty}^{x}\mathrm{Re}(\Gamma(l,y))\,dl\Big).$$

Let $(x_0,y_0)\in S$. By Lemma 4.9, we may assume that around this point, S is the graph of a smooth function $x=F(y)$ (the case that S is the graph of a function $y=F_*(x)$ can be handled in a similar way). Then we can define the integral in ξ in the principle value sense. Applying Lemma 4.9 and using the fact that τ^2/γ^2 is real valued, we find that in a small neighborhood Ω of (x_0,y_0),

$$\xi(x,y)=\frac{G(x,y)}{x-F(y)}, \tag{4.22}$$

where G is a function smooth in Ω.

At this moment, ξ only satisfies the first equation of (4.1). However, it "asymptotically" satisfies the second equation of (4.1), which means that $\xi^{-1}T\xi\to 0$ as $x\to-\infty$. Later on we shall prove that indeed ξ satisfies the second equation of (4.1), in certain sense. On the other hand, with the help of the function ξ, for given η, we can solve the first equation in (4.1) using the variation of parameters formula. However, to simultaneously solve the system (4.1), we need the following.

Lemma 4.10. *Let u and v be the functions defined in Lemma 3.6. Suppose that two functions ϕ and η satisfy $L\phi=M\eta$ and*

$$\Delta\eta-\eta\cos u=0.$$

Let $\Phi:=T\phi-N\eta$. Then Φ satisfies the following ODE:

$$\partial_x\Phi=-\Big(\frac{k_n}{2}\cos\frac{v+u}{2}+\frac{\bar{k}_n}{2}\cos\frac{v-u}{2}\Big)\Phi. \tag{4.23}$$

Proof. Lemma 3.6 tells us that u,v satisfy

$$\begin{cases}-\partial_x v+i\partial_y u-k_n\sin\frac{v+u}{2}-\bar{k}_n\sin\frac{v-u}{2}=0,\\ -i\partial_y v+\partial_x u-k_n\sin\frac{v+u}{2}+\bar{k}_n\sin\frac{v-u}{2}=0.\end{cases}$$

We denote the left-hand side of the first equation by A_1, and that of the second equation by A_2. Then we compute

$$\begin{aligned}i\partial_y A_1-\partial_x A_2=&-\Delta u-\frac{k_n i}{2}\big(\partial_y v+\partial_y u\big)\cos\frac{v+u}{2}-\frac{\bar{k}_n i}{2}\big(\partial_y v-\partial_y u\big)\cos\frac{v-u}{2}\\&+\frac{k_n}{2}(\partial_x v+\partial_x u)\cos\frac{v+u}{2}-\frac{\bar{k}_n}{2}(\partial_x v-\partial_x u)\sin\frac{v-u}{2}.\end{aligned}$$

In view of the identities:

$$\begin{aligned}-\partial_x v+i\partial_y u&=A_1+k_n\sin\frac{v+u}{2}+\bar{k}_n\sin\frac{v-u}{2},\\ -i\partial_y v+\partial_x u&=A_2+k_n\sin\frac{v+u}{2}-\bar{k}_n\sin\frac{v-u}{2},\end{aligned}$$

we find that $i\partial_y A_1 - \partial_x A_2$ is equal to

$$-\Delta u + \Big(A_1 + k_n \sin\frac{v+u}{2} + \bar{k}_n \sin\frac{v-u}{2}\Big)\Big(\frac{\bar{k}_n}{2}\cos\frac{v-u}{2} - \frac{k_n}{2}\cos\frac{v+u}{2}\Big)$$
$$+ \Big(A_2 + k_n \sin\frac{v+u}{2} - \bar{k}_n \sin\frac{v-u}{2}\Big)\Big(\frac{\bar{k}_n}{2}\cos\frac{v-u}{2} + \frac{k_n}{2}\cos\frac{v+u}{2}\Big).$$

Using the fact that $|k_n| = 1$, we obtain

$$\begin{aligned} i\partial_y A_1 - \partial_x A_2 = -\Delta u + \sin u + A_1\Big(\frac{\bar{k}_n}{2}\cos\frac{v-u}{2} - \frac{k_n}{2}\cos\frac{v+u}{2}\Big) \\ + A_2\Big(\frac{\bar{k}_n}{2}\cos\frac{v-u}{2} + \frac{k_n}{2}\cos\frac{v+u}{2}\Big). \end{aligned} \tag{4.24}$$

Note that the linearization of $-\Delta u + \sin u = 0$ is

$$\Delta\eta - \eta\cos u = 0.$$

Moreover, the linearization of $A_1 = 0$ is $L\phi = M\eta$; while that of the equation $A_2 = 0$ is $T\phi = N\eta$. Hence differentiating equation (4.24) in u, v, we get the desired identity (4.23). ■

With Lemma 4.10 at hand, we proceed to prove the following.

Lemma 4.11. *$T\xi = 0$ in $\mathbb{R}^2 \backslash S$.*

Proof. For each fixed $y_0 \in \mathbb{R}$, we consider the set

$$E_{y_0} := \{x : (x, y_0) \in S\}.$$

Observe that the functions γ and τ are explicitly given by suitable combination of exponential functions. Hence S is the zero set of a real analytic function. This together with Lemma 4.2 tell us that for fixed y_0, the set E_{y_0} has no accumulation points (the existence of an accumulation point would imply that E_{y_0} contains a whole straight line). Hence E_{y_0} has finitely many elements, denoted by $\xi_j(y_0)$, $j = 1, \ldots$, in increasing order.

We claim that $T\xi = 0$, if $x \in (-\infty, \xi_1(y_0))$.

To see this, let $\varepsilon > 0$ be a small constant. We choose $x_0 \in (-\infty, \xi_1(y_0))$ and let $\rho(y)$ be a function to be determined, with the initial condition $\rho(y_0) = 1$ and

$$T(\rho\xi)(x_0, y) = 0, \quad \text{for } y \in (y_0, y_0 + \varepsilon). \tag{4.25}$$

This equation can be written as

$$\rho' + (\xi^{-1}\partial_y\xi - \mathrm{Im}(\Gamma - k_n))\rho = 0. \tag{4.26}$$

This is an ODE for ρ and can be locally solved, yielding a solution for (4.25).

Since ρ only depends on y, the function $\rho\xi$ satisfies the first equation of (4.1). Hence by Lemma 4.10, the function $T(\rho\xi)$ satisfies the ODE

$$\partial_x(T(\rho\xi)) = -\Big(\frac{k_n}{2}\cos\frac{v+u}{2} + \frac{\bar{k}_n}{2}\cos\frac{v-u}{2}\Big)(T(\rho\xi)),$$

for $x \in (-\infty, x_0)$, $y \in (y_0, y_0 + \varepsilon)$. It then follows from (4.25) and the uniqueness of solutions to ODEs that

$$T(\rho\xi) = 0, \quad \text{for } x \in (-\infty, x_0), y \in (y_0, y_0 + \varepsilon). \tag{4.27}$$

In this equation, let us send x to $-\infty$. Then from (4.26) and the asymptotic behavior of ξ and Γ, we get that

$$\rho'(y) = 0, \quad \text{for } y \in (y_0, y_0 + \varepsilon).$$

This together with the initial condition $\rho(y_0) = 1$ tell us that indeed $\rho \equiv 1$. In view of (4.27),

$$T(\xi) = 0, \quad \text{for } x \in (-\infty, x_0), y \in (y_0, y_0 + \varepsilon).$$

The claim is then proved.

Next let us choose $x_1 \in (\xi_1(y_0), \xi_2(y_0))$. Let $\rho_1(y)$ be the function with initial condition $\rho_1(y_0) = 1$ and

$$T(\rho_1\xi)(x_1, y) = 0, \quad \text{for } y \in (y_1, y_1 + \varepsilon).$$

Then same arguments as before tell us that

$$T(\rho_1\xi) = 0, \quad \text{for } x \in (\xi_1(y_0), x_1), y \in (y_1, y_1 + \varepsilon). \tag{4.28}$$

We would like to show that $\rho_1' = 0$. To do this, we will send x to $\xi_1(y_0)$ in equation (4.28). We have, for $y \in (y_0, y_0 + \varepsilon)$,

$$\rho_1' + (\xi^{-1}\partial_y\xi - \operatorname{Im}(\Gamma - k_n))\rho_1 = 0, \quad \text{for } x > \xi_1(y_0). \tag{4.29}$$

On the other hand, we already know that $T(\xi) = 0$ for $x < \xi_1(y_0)$. This means

$$\xi^{-1}\partial_y\xi - \operatorname{Im}(\Gamma - k_n) = 0, \quad \text{for } x < \xi_1(y_0).$$

Denote

$$\Pi := \xi^{-1}\partial_y\xi - \operatorname{Im}(\Gamma - k_n).$$

The asymptotic behavior (4.22) of ξ near $(\xi_1(y_0), y_0)$ implies that

$$\lim_{x \to (\xi_1(y_0))^+} \Pi(x, y_0) = \lim_{x \to (\xi_1(y_0))^-} \Pi(x, y_0). \tag{4.30}$$

Combining this with (4.29), we find that $\rho_1' = 0$. Hence ρ_1 is a constant and

$$T(\xi) = 0, \quad \text{for } x \in (\xi_1(y_0), \xi_2(y_0)), y = y_0.$$

Repeating these arguments in the interval $(\xi_j(y_0), \xi_{j+1}(y_0))$, $j = 2, \ldots,$ we see that

$$T(\xi) = 0, \quad \text{for } x \neq \xi_j(y_0), y = y_0.$$

Since y_0 is arbitrary chosen, the lemma is then proved. ∎

Let η be a bounded kernel of the linearized elliptic sine-Gordon equation. That is,

$$-\Delta\eta + \eta\cos u = 0. \tag{4.31}$$

For each fixed y, the variation of parameters formula tells us that the first equation in (4.1) has a solution of the form

$$\phi(x,y) = \xi(x,y)\int_{-\infty}^{x} \xi^{-1} M\eta\, dl, \tag{4.32}$$

where the function $\xi^{-1}M\eta$ is evaluated at (l,y). Note that $\xi^{-1}M\eta$ is smooth in $\mathbb{R}^2$. This together with the assumption that $p_n < 0$ imply that the integral is well defined. However, since ξ has singularities on S, ϕ is also *singular* along S, but the singular behavior is well controlled. The following result can be regarded as a generalization of Lemma 4.11.

Lemma 4.12. *Let η be a bounded solution of* (4.31). *The function ϕ defined by* (4.32) *satisfies system* (4.1) *in $\mathbb{R}^2\backslash S$. As a consequence, ϕ is a kernel of the linearized elliptic sine-Gordon equation at v in the following sense*:

$$-\Delta\phi + \phi\cos v = 0 \quad \text{in } \mathbb{R}^2\backslash S. \tag{4.33}$$

Proof. We follow the same idea as in the proof of Lemma 4.11. We wish to show that

$$T\phi = N\eta \quad \text{in } \mathbb{R}^2\backslash S. \tag{4.34}$$

Choose $x_0 \in (-\infty, \xi_1(y_0))$ and let $\rho(y)$ be the function satisfying the initial condition $\rho(y_0) = 0$ and

$$T(\rho\xi + \phi)(x_0, y) = N\eta, \quad \text{for } y \in (y_0, y_0 + \varepsilon). \tag{4.35}$$

Then the function $\mathcal{G} := T(\rho\xi + \phi) - N\eta$ satisfies

$$\partial_x \mathcal{G} = -\Big(\frac{k_n}{2}\cos\frac{v+u}{2} + \frac{\bar{k}_n}{2}\cos\frac{v-u}{2}\Big)\mathcal{G},$$

for $x \in (-\infty, x_0)$, $y \in (y_0, y_0+\varepsilon)$. The initial condition (4.35) then implies that $\mathcal{G} = 0$ and hence

$$T(\rho\xi + \phi) = N\eta, \quad \text{for } x \in (-\infty, x_0),\, y \in (y_0, y_0 + \varepsilon).$$

Sending x to $-\infty$, using the fact that $N\eta \to 0$ as $x \to -\infty$, we find that $\rho' = 0$. Thus $\rho \equiv 0$. We deduce that

$$T\phi = N\eta, \quad \text{for } x \in (-\infty, x_0),\, y \in (y_0, y_0 + \varepsilon).$$

Next we choose $x_1 \in (\xi_1(y_0), \xi_2(y_0))$. Let $\rho_1(y)$ be the function with initial condition $\rho_1(y_0) = 0$ and

$$T(\rho_1\xi + \phi)(x_1, y) = N\eta, \quad \text{for } y \in (y_1, y_1 + \varepsilon).$$

Then same arguments as before tell us that

$$T(\rho_1\xi + \phi) = N\eta, \quad \text{for } x \in (\xi_1(y_0), x_1),\, y \in (y_1, y_1 + \varepsilon).$$

Sending x to $\xi_1(y_0)$, we have, for $y \in (y_0, y_0+\varepsilon)$,

$$\rho_1' + (\xi^{-1}\partial_y \xi - \operatorname{Im}(\Gamma - k_n))\rho_1 + \xi^{-1} T\phi = \xi^{-1} N\eta, \quad \text{for } x > \xi_1(y_0). \tag{4.36}$$

Denote $\Pi_1 := \xi^{-1}(N\eta - T\phi)$. The asymptotic behavior (4.22) of ξ near $(\xi_1(y_0), y_0)$ again implies that

$$\lim_{x\to(\xi_1(y_0))^+} \Pi_1(x, y_0) = \lim_{x\to(\xi_1(y_0))^-} \Pi_1(x, y_0).$$

This combined with (4.30) and (4.36) yields $\rho_1' = 0$. Hence $\rho_1 = 0$ and

$$T\phi = N\eta, \quad \text{for } x \in (\xi_1(y_0), \xi_2(y_0)), y = y_0.$$

Once (4.34) is proved, it then follows from the linearization of the Bäcklund transformation that ϕ satisfies (4.33). The proof is completed. ■

Now we are ready to prove Theorem 4.1 (Theorem 1.2). That is, the nondegeneracy of $2n$-end solution (it can be regarded as an n-soliton).

Proof of Theorem 4.1. Let us fix a solution $u = U_n + \pi$. Suppose η is a nontrivial bounded kernel of the corresponding linearized operator:

$$\Delta\eta = \eta \cos u.$$

By the linear decomposition lemma of [11] and the asymptotic behavior of ζ_j, there exist $c_1, \ldots, c_n$ such that the function

$$\eta^* := \eta - \sum_{j=1}^{n} c_j \zeta_j$$

decays exponentially fast to 0 as $x \to -\infty$, uniformly in y. That is, there exist constants $C, \delta > 0$ such that

$$|\eta^*(x, y)| < C \exp(-\delta|x|), \quad x < 0.$$

We point out that for each fixed y, η always decays to zero as $|x| \to \infty$. Note that at this moment, we do not know whether η^* decays to zero as $x \to +\infty$, uniformly in y. Nevertheless, we would like to prove that $\eta^* = 0$.

Applying Lemma 4.12 to the function η^*, we get a corresponding kernel ϕ of the linearized operator at the function $v = 4\arctan(\tau/\gamma)$. That is,

$$\Delta\phi = \phi \cos v.$$

Explicitly,

$$\phi(x, y) = \xi(x, y) \int_{-\infty}^{x} \xi^{-1} M\eta^* \, dl. \tag{4.37}$$

Here the function $\xi^{-1}M\eta^*$ in the integral is evaluated at (l, y). Since η^* decays exponentially fast to 0 as x tends to $-\infty$, ϕ also decays to zero as $x \to -\infty$. Note that ϕ is singular at S. However, the singular behavior of ϕ is well controlled.

Let us write τ as τ_{n-1}, and γ as γ_{n-1}. By Lemma 3.7, the function $v_{n-1} := v = 4\arctan\frac{\tau_{n-1}}{\gamma_{n-1}}$ is the Bäcklund transformation of v_{n-2}. That is, v_{n-2} and v_{n-1} satisfy

$$\begin{cases} \partial_x v_{n-2} = i\partial_y v_{n-1} - k_{n-1}\sin\frac{v_{n-2}+v_{n-1}}{2} - \bar{k}_{n-1}\sin\frac{v_{n-2}-v_{n-1}}{2}, \\ i\partial_y v_{n-2} = \partial_x v_{n-1} - k_{n-1}\sin\frac{v_{n-2}+v_{n-1}}{2} + \bar{k}_{n-1}\sin\frac{v_{n-2}-v_{n-1}}{2}. \end{cases} \tag{4.38}$$

Recall that τ_{n-2}/γ_{n-2} is a real valued function.

Let us write the function ϕ by ϕ_{n-1}. Linearizing system (4.38) and denoting

$$\Gamma_{n-1} = 2k_{n-1}\frac{(\gamma_{n-1}\gamma_{n-2} - \tau_{n-1}\tau_{n-2})^2}{(\gamma_{n-1}^2 + \tau_{n-1}^2)(\gamma_{n-2}^2 + \tau_{n-2}^2)},$$

we get the following equation to be solved for the unknown function ϕ_{n-2}:

$$\begin{cases} \partial_x\phi_{n-2} + \operatorname{Re}(\Gamma_{n-1} - k_{n-1})\phi_{n-2} = i\partial_y\phi_{n-1} - i\phi_{n-1}\operatorname{Im}(\Gamma_{n-1} - k_{n-1}), \\ i\partial_y\phi_{n-2} + i\operatorname{Im}(\Gamma_{n-1} - k_{n-1})\phi_{n-2} = \partial_x\phi_{n-1} - \phi_{n-1}\operatorname{Re}(\Gamma_{n-1} - k_{n-1}). \end{cases} \tag{4.39}$$

Since τ_{n-2}/γ_{n-2} is real valued, the function Γ_{n-1} has the same singular set S as Γ_n. Indeed, if P is a point outside S such that $\tau_{n-2}(P) = \gamma_{n-2}(P) = 0$, then by dividing the numerator and denominator of Γ_{n-1} by $\tau_{n-2}(P)$ or $\gamma_{n-2}(P)$, we see that P is actually a removable singularity. The explicit formula (4.37) of ϕ_{n-1} tells us that near a singular point $(x_0, y_0) \in S$, there exist smooth functions F, G such that $\phi_{n-1} - \frac{G(x,y)}{x-F(y)}$ is smooth. As a consequence, near (x_0, y_0), for some function $\tilde{G}$,

$$M_{n-1}\phi_{n-1} := i\big[\partial_y\phi_{n-1} - \phi_{n-1}\operatorname{Im}(\Gamma_{n-1} - k_{n-1})\big] \sim \frac{\tilde{G}(x,y)}{x - F(y)}. \tag{4.40}$$

Define

$$\xi_{n-2}(x,y) := \exp\Big(p_{n-1}x + q_{n-1}y + \int_{-\infty}^{x}\Gamma_{n-1}(l,y)\,dl\Big).$$

By Lemma 4.12, the system (4.39) has a solution

$$\phi_{n-2}(x,y) = \xi_{n-2}(x,y)\int_{-\infty}^{x}\frac{M_{n-1}\phi_{n-1}}{\xi_{n-2}}\,dl. \tag{4.41}$$

Note that $\xi_{n-2}(x,y) = O(x - F(y))$ around the singular set S. Here one need to be careful about the definition of ϕ_{n-2}. More precisely, suppose $(x_0, y_0) \in S$, then for $x > x_0$, with $x - x_0$ small, the right-hand side of (4.41) is defined to be

$$\lim_{\varepsilon\to 0^+}\Big[\xi_{n-2}(x,y)\Big(\int_{-\infty}^{x_0-\varepsilon} + \int_{x_0+\varepsilon}^{x}\Big)\frac{M_{n-1}\phi_{n-1}}{\xi_{n-2}}\,dl\Big].$$

Using (4.40), we find that ϕ_{n-2} is continuous in $\mathbb{R}^2$. We would like to show that ϕ_{n-2} is actually smooth. To see this, we use the fact that ϕ_{n-2} satisfies the linearized equation away from the singular set S. That is,

$$\Delta\phi_{n-2} = \phi_{n-2}\cos v_{n-2}. \tag{4.42}$$

Let $(x_0, y_0) \in S$. From (4.41), we see that there exists a smooth function g, such that near (x_0, y_0), the function

$$\phi_{n-2} - g(y)(x - F(y)) \ln |x - F(y)|$$

is smooth. Inserting it into (4.42), we find that the function $g \equiv 0$. As a consequence, ϕ_{n-2} is smooth.

With the function ϕ_{n-2} at hand, now let us consider the linearized Bäcklund transformation between $v_{n-3} = 4 \arctan \frac{\tau_{n-3}}{\gamma_{n-3}}$ and $v_{n-2} = 4 \arctan \frac{\tau_{n-2}}{\gamma_{n-2}}$:

$$\begin{cases} \partial_x \phi_{n-3} + \operatorname{Re}(\Gamma_{n-2} - k_{n-2})\phi_{n-3} = i\partial_y \phi_{n-2} - i\phi_{n-2} \operatorname{Im}(\Gamma_{n-2} - k_{n-2}), \\ i\partial_y \phi_{n-3} + i \operatorname{Im}(\Gamma_{n-2} - k_{n-2})\phi_{n-3} = \partial_x \phi_{n-2} - \phi_{n-2} \operatorname{Re}(\Gamma_{n-2} - k_{n-2}). \end{cases}$$

Here,

$$\Gamma_{n-2} = 2k_{n-2} \frac{(\gamma_{n-2}\gamma_{n-3} - \tau_{n-2}\tau_{n-3})^2}{(\gamma_{n-2}^2 + \tau_{n-2}^2)(\gamma_{n-3}^2 + \tau_{n-3}^2)}.$$

Note that the function τ_{n-3}/γ_{n-3} is purely imaginary. Hence it is now singular at the set

$$\mathcal{S}_{n-3} := \{(x, y) \in \mathbb{R}^2 : \gamma_{n-3}^2 + \tau_{n-3}^2 = 0\}.$$

We can also define the set $S_{0,n-3}, S_{*,n-3}, S_{n-3}$. Following the same proof as that of Lemma 4.5, one can show that on $S_{*,n-3}$, there still holds

$$\vartheta_{n-3} := k_{n-2} \frac{(\gamma_{n-2}\gamma_{n-3} - \tau_{n-2}\tau_{n-3})^2}{(\gamma_{n-2}^2 + \tau_{n-2}^2)(\gamma_{n-3}\partial_s \gamma_{n-3} + \tau_{n-3}\partial_s \tau_{n-3})} = 1.$$

Hence the same arguments as above tell us that the corresponding function ξ_{n-3} has similar asymptotic behavior near the singular set S_{n-3} as the function ξ_{n-1} near S. Using this information, we can further analyze the linearized Bäcklund transformation between v_{n-4} and v_{n-3} and get a smooth solution ϕ_{n-4} of the equation

$$\Delta \phi_{n-4} = \phi_{n-4} \cos v_{n-4}.$$

Repeating the above procedure, we may consider the Bäcklund transformation between $v_j = 4 \arctan \frac{\tau_j}{\gamma_j}$ and $v_{j-1} = 4 \arctan \frac{\tau_{j-1}}{\gamma_{j-1}}$, $j = n-4, \dots, 1$. Linearizing these Bäcklund transformations and solving them similarly as in Lemma 4.12 (one also need to be careful about the point singularities in these systems), we finally get a solution ϕ_0 of the equation

$$\Delta \phi_0 - \cos(v_0)\phi_0 = 0.$$

Observe that whether or not τ_1/γ_1 is real valued, the function $v_0 = 4 \arctan \tau_0/\gamma_0$ is always equal to 0. Hence from the previous argument, one can actually show that ϕ_0 is smooth.

We claim that ϕ_0 is bounded in $\mathbb{R}^2$. To see this, let us first estimate ϕ_{n-1}, which is defined by (4.37). In view of this definition, we need to analyze the function ξ. Observe that by Lemma 4.3, the function Γ tends to the limit 0 or $2k_n$ away from the ends. Moreover, since we have assumed that $p_n < 0$, this limit is 0 in the region $\Xi_- := \{(x, y) : p_n x + q_n y > 0\}$; while in $\Xi_+ := \{(x, y) : p_n x + q_n y < 0\}$, the limit is $2k_n$.

Let us define

$$\Theta_- := \{(x, y) \in \Xi_- : \mathrm{dist}((x, y), S) > 1\}.$$

Recall that by Lemma 4.2, outside a large ball, the set S consists of finitely many curves asymptotic to rays, with each ray being parallel to one of the ends. In Θ_-, using the exponential decay of Γ away from the ends, we have

$$\exp(-p_n x - q_n y)\xi = \exp\Big(-\int_{-\infty}^{x} \mathrm{Re}(\Gamma(l, y))\, dl\Big) \le C. \tag{4.43}$$

Therefore, in Θ_-, we can estimate

$$\phi_{n-1} = \xi \int_{-\infty}^{x} \xi^{-1} M \eta^* \, dl \le C.$$

This estimate can be refined. Indeed, since $\eta^* \to 0$ as $x \to -\infty$, uniformly in y, we have, in Ξ_-,

$$\phi_{n-1} \to 0, \quad \text{as } x \to -\infty, \text{ uniformly in } y. \tag{4.44}$$

Similarly, we define

$$\Theta_+ := \{(x, y) \in \Xi_+ : \mathrm{dist}((x, y), S) > 1\}.$$

In Θ_+, since Γ converges to $2k_n$ away from the ends, we have

$$\int_{-\infty}^{x} \mathrm{Re}(\Gamma(l, y))\, dl = 2p_n\Big(x + \frac{q_n}{p_n} y\Big) + O(1).$$

Therefore, in Θ_+, there holds

$$\exp(p_n x + q_n y)\,\xi = \exp\Big(2p_n x + 2q_n y - \int_{-\infty}^{x} \mathrm{Re}\,(\Gamma(l, y))\, dl\Big) \le C. \tag{4.45}$$

To estimate ϕ_{n-1} in Θ_+, we define

$$\mathcal{B}(y) := \int_{-\infty}^{+\infty} \xi^{-1} M \eta^* \, dl.$$

Note that this is well defined, because ξ is exponential growing as $x \to \pm\infty$. We have $\phi_{n-1} \to \xi(x, y)\mathcal{B}(y)$, as $x \to +\infty$. Inserting this into the equation

$$\partial_y \phi_{n-1} + \mathrm{Im}(\Gamma - k_n)\phi_{n-1} = -iN\eta^*,$$

and using the fact that ξ also solves the equation

$$\partial_y \xi + \mathrm{Im}(\Gamma - k_n)\xi = 0,$$

we infer that $\frac{d}{dy}\mathcal{B} = 0$ and hence $\mathcal{B}$ is a constant. Using the estimates (4.43) and (4.45) of ξ, and the fact that η^* converges to 0 as $|y| \to +\infty$ for all $x < 0$, we find that, if $q_n > 0$, then $\mathcal{B}(y) \to 0$ as $y \to -\infty$, and if $q_n < 0$, then $\mathcal{B}(y) \to 0$ as $y \to +\infty$. As a consequence, $\mathcal{B} = 0$. Then in Θ_+, we can write

$$\phi_{n-1} = \xi \int_{-\infty}^{x} \xi^{-1} M \eta^* \, dl = \xi \int_{+\infty}^{x} \xi^{-1} M \eta^* \, dl.$$

This together with the estimate (4.45) of ξ imply that $\phi_{n-1} \le C$. Note that in the region $\{(x, y) : \mathrm{dist}((x, y), S) \le 1\}$, the asymptotic behavior of ϕ_{n-1} is determined by that of ξ, and we can estimate

$$|\phi_{n-1}| \le \Big|\frac{C}{x - F(y)}\Big|,$$

provided that S is locally determined by $x = F(y)$; and $|\phi_{n-1}| \le |\frac{C}{y - F_*(x)}|$, if S is locally determined by $y = F_*(y)$. With this information at hand, we can proceed to estimate ϕ_{n-2} using similar arguments as for ϕ_{n-1}. Recall that ϕ_{n-2} is smooth. One then can show that actually $|\phi_{n-2}| \le C$ in $\mathbb{R}^2$. Repeating this arguments, we finally deduce that ϕ_0 is also bounded.

Having proved that ϕ_0 is bounded, we can use the Liouville theorem to conclude that $\phi_0 = 0$.

Up to now, we have defined ϕ_j, $j = 1, \ldots, n-1$, and proved that ϕ_0 is zero. We would like to show that $\phi_1 \equiv 0$. To see this, we analyze the *reverse* linearized Bäcklund transformation from v_0 to v_1:

$$\begin{cases} \partial_x \phi_0 + \mathrm{Re}(\Gamma_1 - k_1)\phi_0 = i\partial_y \phi_1 - i\phi_1 \,\mathrm{Im}(\Gamma_1 - k_1), \\ i\partial_y \phi_0 + i\,\mathrm{Im}(\Gamma_1 - k_1)\phi_0 = \partial_x \phi_1 - \phi_1 \,\mathrm{Re}(\Gamma_1 - k_1). \end{cases}$$

Since $\phi_0 = 0$, we see that necessarily, $\phi_1 = c\xi^*$, for some constant c, where

$$\xi^* := \exp\Big(- p_1 x - q_1 y + \int_{-\infty}^{x} \Gamma_1(l, y)\, dl \Big).$$

Note that $\xi^* = \xi_1^{-1}$. By the asymptotic behavior of Γ_1, ξ^* does not decay to zero along the line $p_1 x + q_1 y = 0$. But on the other hand, an estimate of the form (4.44) also holds for the function ϕ_1 in the region

$$\{(x, y) : p_1 x + q_1 y > 0\},$$

Hence necessarily there holds $c = 0$ and $\phi_1 = 0$. We remark that the function ξ^* arises from differentiating the function v_1 with the phase parameter η_1^0. That is, $\xi^* = c_0 \partial_{\eta_1^0} v_1$, where c_0 is a constant. Repeating the above arguments, we see that $\phi_{n-1} = 0$, and $\eta^* = 0$. Hence by the definition of η^*, we obtain $\eta = \sum_{j=1}^n c_j \zeta_j$. This finishes the proof. ■

5. Inverse scattering transform and the classification of multiple-end solutions

We consider the elliptic sine-Gordon equation in the form

$$\Delta u = \sin u, \quad 0 < u < 2\pi. \tag{5.1}$$

Under the correspondence $\phi + \pi \leftrightarrow u$, multiple-end solutions of the equation $-\Delta\phi = \sin\phi$ correspond to those solutions of (5.1) whose π level sets are asymptotic to finitely many half straight lines at infinity. Along these rays, the solutions u resemble the one dimensional heteroclinic solution $4\arctan e^x$ in the transverse direction. In this section, we would like to classify these solutions using the inverse scattering transform of the elliptic sine-Gordon equation, developed in [28]. For inverse scattering of the classical hyperbolic sine-Gordon equation, we refer to [1, 8, 15].

The main result of this section is the following.

Proposition 5.1. *Suppose ϕ is a $2n$-end solution of the equation $-\Delta\phi = \sin\phi$. Then there exist parameters p_j, q_j, η_j^0, $j = 1, \ldots, n$, such that $\phi = U_n$, where U_n is defined in (2.15).*

Let us denote $\phi + \pi$ by u and let use I to denote the 2×2 identity matrix. Let λ be a complex spectral parameter, and let σ_j be the Pauli spin matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that $\sigma_j^2 = I$, $\sigma_3\sigma_1 = i\sigma_2 = -\sigma_1\sigma_3$, $\sigma_3\sigma_2 = -i\sigma_1 = -\sigma_2\sigma_3$, and $\sigma_2\sigma_1 = -i\sigma_3 = -\sigma_1\sigma_2$. Equation (5.1) has a Lax pair

$$\Phi_x = A\Phi, \tag{5.2}$$

$$\Phi_y = B\Phi. \tag{5.3}$$

Here Φ is vector valued or 2×2 matrix valued, depending on the contexts. Moreover, the matrices A and B are defined by

$$\begin{aligned} A &:= \frac{i}{4}\left[\left(\lambda - \frac{\cos u}{\lambda}\right)\sigma_3 - (u_x - iu_y)\sigma_2 - \frac{\sin u}{\lambda}\sigma_1\right], \\ B &:= \frac{1}{4}\left[-\left(\lambda + \frac{\cos u}{\lambda}\right)\sigma_3 + (u_x - iu_y)\sigma_2 - \frac{\sin u}{\lambda}\sigma_1\right]. \end{aligned}$$

Indeed, the compatibility of (5.2) and (5.3) yields

$$A_y + AB = B_x + BA.$$

A direct computation shows that this is equivalent to equation (5.1).

Define $K(\lambda) := \lambda - 1/\lambda$. For each fixed $y \in \mathbb{R}$, as $x \to \pm\infty$, due to the exponential decay of u to 0 or 2π, we see that

$$A \to \frac{Ki}{4}\sigma_3.$$

We would like to investigate the existence of matrix valued solutions $\Phi_\pm$ of (5.2) such that $\Phi_\pm(x,y) \to \exp\left(\frac{Ki}{4}\sigma_3 x\right)$, as $x \to \pm\infty$, using Picard iteration under certain assumptions on λ. It turns out that different columns of $\Phi_\pm$ have different analytic properties (with respect to λ). This is the content of the following result.

Lemma 5.2. *Assume* $\operatorname{Im}\lambda \geq 0$ *and* $\lambda \neq 0$. *There exists a solution $\Phi_{+,1}$ to the equation $\partial_x\Phi_{+,1} = A\Phi_{+,1}$, satisfying* $\Phi_{+,1}\exp(-Kix/4) - (1,0)^T \to 0$, *as $x \to +\infty$. There also exists a solution $\Phi_{-,2}$ to the equation $\partial_x\Phi_{-,2} = A\Phi_{-,2}$, satisfying* $\Phi_{-,2}\exp(Kix/4) - (0,1)^T \to 0$, *as $x \to -\infty$. Moreover, $\Phi_{+,1}$ and $\Phi_{-,2}$ are analytic with respect to λ in the region* $\{\lambda : \operatorname{Im}\lambda > 0, \lambda \neq 0\}$.

Proof. Let us define

$$A^*(u,\lambda) := A(u,\lambda) - \frac{Ki\sigma_3}{4}\cdot \tag{5.4}$$

We write

$$A^* = \begin{pmatrix} A^*_{11} & A^*_{12} \\ A^*_{21} & A^*_{22} \end{pmatrix}.$$

Note that each entry of A^* tends to 0 as $|x| \to +\infty$. Let us introduce the column vector

$$\varphi_{+,1} = \Phi_{+,1} \exp\Big(-\frac{Kix}{4}\Big) = (\varphi_{+,11}, \varphi_{+,21})^T.$$

For each fixed (y,λ), we consider the integral equation

$$\begin{cases} \varphi_{+,11}(x,y,\lambda) = 1 + \int_{+\infty}^x \left[A^*_{11}\varphi_{+,11} + A^*_{12}\varphi_{+,21}\right](s,y,\lambda)\,ds, \\ \varphi_{+,21}(x,y,\lambda) = \int_{+\infty}^x \exp\left(\frac{Ki}{2}(s-x)\right)\left[A^*_{21}\varphi_{+,11} + A^*_{22}\varphi_{+,21}\right](s,y,\lambda)\,ds. \end{cases} \tag{5.5}$$

If $\varphi_{+,1}$ satisfies (5.5), then $\partial_x \Phi_{+,1} = A\Phi_{+,1}$.

Now suppose $\operatorname{Im}\lambda \geq 0$ and impose the boundary condition $\varphi_{+,1}(x,y,\lambda) \to (1,0)^T$, as $x \to +\infty$. Under this boundary condition, the system (5.5) has a unique solution. This can be proved by Picard iteration, starting from the constant vector $(1,0)^T$. More precisely, we define the sequence $\big(\varphi^{(n)}_{+,11}, \varphi^{(n)}_{+,21}\big)$ in the following way. Let $\big(\varphi^{(0)}_{+,11}, \varphi^{(0)}_{+,21}\big) := (1,0)$ and

$$\begin{cases} \varphi^{(n)}_{+,11}(x,y,\lambda) = 1 + \int_{+\infty}^x \left[A^*_{11}\varphi^{(n-1)}_{+,11} + A^*_{12}\varphi^{(n-1)}_{+,21}\right](s,y,\lambda)\,ds, \\ \varphi^{(n)}_{+,21}(x,y,\lambda) = \int_{+\infty}^x \exp\left(\frac{Ki}{2}(s-x)\right)\left[A^*_{21}\varphi^{(n-1)}_{+,11} + A^*_{22}\varphi^{(n-1)}_{+,21}\right](s,y,\lambda)\,ds. \end{cases}$$

If $\operatorname{Im}\lambda \geq 0$ and $\lambda \neq 0$, then

$$\operatorname{Re}\Big(\frac{Ki}{2}\Big) = -\frac{1}{2}\Big(1 + \frac{1}{|\lambda|}\Big)\operatorname{Im}\lambda \leq 0. \tag{5.6}$$

This condition ensures that the integral

$$\int_{+\infty}^x \exp\Big(\frac{Ki}{2}(s-x)\Big)\big[A^*_{21}\varphi^{(n-1)}_{+,11} + A^*_{22}\varphi^{(n-1)}_{+,21}\big](s,y,\lambda)\,ds$$

is well defined. Note that the integrand depending analytically on λ.

To simplify the notation, let us suppress the y and λ dependences of these functions. We have the following estimates:

$$|\varphi^{(1)}_{+,11}(x)| \leq 1 + \int_x^{+\infty} |A^*_{11}(s)|\,ds, \quad |\varphi^{(1)}_{+,21}(x)| \leq \int_x^{+\infty} |A^*_{21}(s)|\,ds.$$

Let us define

$$Q(x) := \int_x^{+\infty} \big(|A^*_{11}(s)| + |A^*_{12}(s)| + |A^*_{21}(s)| + |A^*_{22}(s)|\big)\,ds. \tag{5.7}$$

Then

$$|\varphi^{(1)}_{+,11}(x)| \le 1 + Q(x) \quad \text{and} \quad |\varphi^{(1)}_{+,21}(x)| \le Q(x).$$

Inserting these estimates into the integral equation defining $\varphi^{(2)}_{+,j1}$ and integrating by parts, we obtain

$$|\varphi^{(2)}_{+,11}(x)| \le 1 + Q(x) + \frac{1}{2}Q^2(x) \quad \text{and} \quad |\varphi^{(2)}_{+,21}(x)| \le Q(x) + \frac{1}{2}Q^2(x).$$

Using an induction argument, we get

$$|\varphi^{(n)}_{+,11}(x)| \le \sum_{j=0}^{n} \frac{Q^j(x)}{j!} \quad \text{and} \quad |\varphi^{(n)}_{+,21}(x)| \le \sum_{j=1}^{n} \frac{Q^j(x)}{j!}. \tag{5.8}$$

It follows that $(\varphi^{(n)}_{+,11}, \varphi^{(n)}_{+,21})$ converges to a solution $(\varphi_{+,11}, \varphi_{+,21})$, which is analytic in λ in the region $\{\lambda : \operatorname{Im}\lambda > 0, \lambda \neq 0\}$. By (5.8), we also have

$$|\varphi_{+,11}(x)| \le \exp(Q(x)) \quad \text{and} \quad |\varphi_{+,12}(x)| \le \exp(Q(x)) - 1. \tag{5.9}$$

Observe that since the integral in $\varphi^{(n)}_{+,1}$ is from $+\infty$ to x, we have $(\varphi_{+,11}, \varphi_{+,21}) \to (1, 0)$, as $x \to +\infty$. We also have $\partial_x \Phi_{+,1} = A\Phi_{+,1}$. We emphasize that if the lower limit $+\infty$ in the integrand defining $\varphi_{+,21}$ is replaced by other numbers, then $\varphi_{+,21}$ will not have the desired asymptotic behavior.

Same arguments as above yield a solution $(\varphi_{-,12}, \varphi_{-,22})$ satisfying $\varphi_{-,2}(x, y, \lambda) \to (0, 1)^T$, as $x \to -\infty$, and the integral equation

$$\begin{cases} \varphi_{-,12}(x, y, \lambda) = \int_{-\infty}^{x} \exp\left(-\frac{Ki}{2}(s - x)\right) \left[A^*_{11}\varphi_{-,12} + A^*_{21}\varphi_{-,22}\right](s, y, \lambda)\, ds, \\ \varphi_{-,22}(x, y, \lambda) = 1 + \int_{-\infty}^{x} \left[A^*_{21}\varphi_{-,12} + A^*_{22}\varphi_{-,22}\right](s, y, \lambda)\, ds. \end{cases}$$

This solution is also analytic in $\{\lambda : \operatorname{Im}\lambda > 0, \lambda \neq 0\}$. This finishes the proof. ∎

For each fixed $y \in \mathbb{R}$, Φ_+ and Φ_- are solutions of the same ODE system. Hence they are related by

$$\Phi_+(x, y, \lambda) = \Phi_-(x, y, \lambda) \begin{bmatrix} a(\lambda, y) & b(\lambda, y) \\ b^*(\lambda, y) & a^*(\lambda, y) \end{bmatrix}, \tag{5.10}$$

for some functions a, b, a^*, b^*, which are independent of x. We emphasize that the function a defined here is not the same as that defined in Section 2.

Lemma 5.3. *For each $\lambda \in \mathbb{C}\backslash\{0\}$ with $\operatorname{Im}\lambda \ge 0$, there holds*

$$\Phi_{+,1}(x, y, \lambda) = i\sigma_2\, \Phi_{+,2}(x, y, -\lambda).$$

Similarly, for each $\lambda \in \mathbb{C}\backslash\{0\}$ with $\operatorname{Im}\lambda \le 0$, there holds

$$\Phi_{-,1}(x, y, \lambda) = i\sigma_2\, \Phi_{-,2}(x, y, -\lambda).$$

Proof. Let us write $\Phi_\pm$ into columns: $\Phi_\pm = [\Phi_{\pm,1}, \Phi_{\pm,2}]$, where

$$\Phi_{\pm,j} = \begin{bmatrix} \Phi_{\pm,1j} \\ \Phi_{\pm,2j} \end{bmatrix}, \quad j = 1, 2.$$

For $j = 1, 2$, we define

$$\Theta_{\pm,j} := \begin{bmatrix} \Phi_{+,2j} \\ -\Phi_{+,1j} \end{bmatrix} = i\sigma_2 \Phi_{\pm,j}.$$

By the symmetry of A, we know that $\Theta_{+,1}$ satisfies

$$\partial_x \Theta_{+,1}(x, y, \lambda) = A(u, -\lambda)\, \Theta_{+,1}(x, y, \lambda).$$

It follows from the asymptotic behavior of $\Phi_{\pm,j}$ at infinity and the uniqueness of solutions to the ODE that

$$\Theta_{+,1}(x, y, \lambda) = -\Phi_{+,2}(x, y, -\lambda). \tag{5.11}$$

Similarly, $\Theta_{-,1}(x, y, \lambda) = -\Phi_{-,2}(x, y, -\lambda)$. ∎

Lemma 5.4. *Suppose $\lambda \in \mathbb{R}\backslash\{0\}$. We have that $a^*(\lambda, y) = a(-\lambda, y)$ and $b^*(\lambda, y) = -b(-\lambda, y)$. As a consequence,*

$$\Phi_+(x, y, \lambda) = \Phi_-(x, y, \lambda) \begin{bmatrix} a(\lambda, y) & b(\lambda, y) \\ -b(-\lambda, y) & a(-\lambda, y) \end{bmatrix}.$$

Proof. By definition, Φ_+ and Φ_- are related by

$$\begin{cases} \Phi_{+,1} = a\, \Phi_{-,1} + b^*\, \Phi_{-,2}, \\ \Phi_{+,2} = b\, \Phi_{-,1} + a^*\, \Phi_{-,2}. \end{cases} \tag{5.12}$$

From the second equation of (5.12), we get

$$\Theta_{+,2} = b\, \Theta_{-,1} + a^*\, \Theta_{-,2}.$$

Using this and Lemma 5.3, we obtain

$$\Phi_{+,1}(x, y, -\lambda) = -b(\lambda, y)\, \Phi_{-,2}(x, y, -\lambda) + a^*(\lambda, y)\, \Phi_{-,1}(x, y, -\lambda). \tag{5.13}$$

On the other hand, by the first equation of (5.12),

$$\Phi_{+,1}(x, y, -\lambda) = a(-\lambda, y)\, \Phi_{-,1}(x, y, -\lambda) + b^*(-\lambda, y)\, \Phi_{-,2}(x, y, -\lambda). \tag{5.14}$$

Comparing (5.13) with (5.14), we finally deduce

$$a^*(\lambda, y) = a(-\lambda, y), b^*(\lambda, y) = -b(-\lambda, y).$$ ∎

The functions $a(\lambda, y)$ and $b(\lambda, y)$ are a priori depending on y and the spectral parameter λ. Nevertheless, we have the following.

Lemma 5.5. *Suppose u is a solution to (5.1). Assume $\lambda \in \mathbb{R}\backslash\{0\}$. Then $a(\lambda, y) = a(\lambda, 0)$, and*

$$b(\lambda, y) = b(\lambda, 0) \exp\Big(-\frac{1}{2}(\lambda + \lambda^{-1})\, y\Big). \tag{5.15}$$

Proof. Recall that Φ_+ satisfies (5.2), but it does not satisfy (5.3). However, the function $\Phi^* := \Phi_+ \exp(-\frac{1}{4}(\lambda + \frac{1}{\lambda})\sigma_3 y)$ satisfies the equation

$$\partial_y \Phi^* = B\Phi^*.$$

Inserting (5.10) into this equation, we get

$$\begin{aligned}
&\partial_y \Phi_- \begin{bmatrix} a(\lambda, y) & b(\lambda, y) \\ -b(-\lambda, y) & a(-\lambda, y) \end{bmatrix} \exp\Big(-\frac{1}{4}\Big(\lambda + \frac{1}{\lambda}\Big)\sigma_3 y\Big) \\
&+ \Phi_- \begin{bmatrix} \partial_y a(\lambda, y) & \partial_y b(\lambda, y) \\ -\partial_y b(-\lambda, y) & \partial_y a(-\lambda, y) \end{bmatrix} \exp\Big(-\frac{1}{4}\Big(\lambda + \frac{1}{\lambda}\Big)\sigma_3 y\Big) \\
&+ \Phi_- \begin{bmatrix} a(\lambda, y) & b(\lambda, y) \\ -b(-\lambda, y) & a(-\lambda, y) \end{bmatrix} \partial_y\Big[\exp\Big(-\frac{1}{4}\Big(\lambda + \frac{1}{\lambda}\Big)\sigma_3 y\Big)\Big] \\
&= B\Phi_- \begin{bmatrix} a(\lambda, y) & b(\lambda, y) \\ -b(-\lambda, y) & a(-\lambda, y) \end{bmatrix} \exp\Big(-\frac{1}{4}\Big(\lambda + \frac{1}{\lambda}\Big)\sigma_3 y\Big).
\end{aligned}$$

Sending x to $-\infty$ and using the fact that Φ_- tends exponentially fast to $\exp(\frac{Ki}{4}\sigma_3 x)$, we obtain

$$\begin{aligned}
&\begin{bmatrix} \partial_y a(\lambda, y) & \partial_y b(\lambda, y) \\ -\partial_y b(-\lambda, y) & \partial_y a(-\lambda, y) \end{bmatrix} + \begin{bmatrix} a(\lambda, y) & b(\lambda, y) \\ -b(-\lambda, y) & a(-\lambda, y) \end{bmatrix}\Big(-\frac{1}{4}\Big(\lambda + \frac{1}{\lambda}\Big)\sigma_3\Big) \\
&\quad = -\frac{1}{4}\Big(\lambda + \frac{1}{\lambda}\Big)\sigma_3 \begin{bmatrix} a(\lambda, y) & b(\lambda, y) \\ -b(-\lambda, y) & a(-\lambda, y) \end{bmatrix}.
\end{aligned}$$

It follows that

$$\partial_y a = 0, \quad \partial_y b = -\frac{1}{2}\Big(\lambda + \frac{1}{\lambda}\Big) b.$$

The assertion of the lemma follows immediately from these two equations. ∎

Without loss of generality, we may assume that ϕ is rotated so that no end is parallel to the x-axis. Since ϕ is a multiple-end solution of (1.2), there exists a choice of parameters p_j, q_j, η_j^0, with $p_j > 0$, $j = 1, \ldots, n$, such that the zero level set of the corresponding solution U_n has the same asymptotic lines as that of ϕ, as $y \to +\infty$. We denote the a part of the scattering data of $U_n + \pi$ by $\hat{a}(\lambda, y)$.

Lemma 5.6. *Assume $\lambda \in \mathbb{R}\backslash\{0\}$. We have $a(\lambda, y) = \hat{a}(\lambda, y)$ and $b(\lambda, y) = 0$.*

Proof. By (5.12),

$$\Phi_{+,1}(x, y, \lambda) = a(\lambda, y)\, \Phi_{-,1}(x, y, \lambda) - b(-\lambda, y)\, \Phi_{-,2}(x, y, \lambda). \tag{5.16}$$

We rewrite $\Phi_+ = \exp\big(\frac{Ki\sigma_3}{4} x\big)\, \Phi_+^*$. Then Φ_+^* satisfies

$$\partial_x \Phi_+^* = \exp\Big(-\frac{Ki\sigma_3 x}{4}\Big) A^* \exp\Big(\frac{Ki\sigma_3 x}{4}\Big)\, \Phi_+^*. \tag{5.17}$$

Consider the norm $\|M\| := \left(\sum_{j,k} |m_{jk}|^2\right)^{1/2}$, where m_{jk} are entries of a matrix M. We have, by (5.17), for some constant C_0,

$$\partial_x \|\Phi_+^*\| \leq C_0 \|A^*\| \|\Phi_+^*\|. \tag{5.18}$$

Applying the refined asymptotics theorem (Theorem 2.1 of [11]), we deduce that A^* decays exponentially fast to 0 away from each end. It then follows from (5.18) and the Gronwall inequality that $\|\Phi_+^*\| \leq C$ in $\mathbb{R}^2$, for a universal constant C. Hence $\|\Phi_+\| \leq C$. Similarly, $\|\Phi_-\| \leq C$. Then in view of the relation (5.16), by sending x to $-\infty$, we see that for each fixed λ, $|b(\lambda, y)|$ is uniformly bounded with respect to y. This together with (5.15) implies $b(\lambda, y) \equiv 0$.

We use $\hat{A}$ to denote the matrix obtained from replacing u by U_n in A. Let $\hat{\Phi}_\pm$ be the matrix valued solutions of the equation $\partial_x \hat{\Phi}_\pm = \hat{A}\hat{\Phi}_\pm$, with the same asymptotic behavior as that of $\Phi_\pm$. To compare $\hat{\Phi}_\pm$ with $\Phi_\pm$, we write

$$\partial_x \Phi_+ = \hat{A}\Phi_+ + (A - \hat{A})\,\Phi_+.$$

By the variation of parameters formula, we have

$$\Phi_+ = \hat{\Phi}_+\Big(I + \int_{+\infty}^{x} (\hat{\Phi}_+)^{-1}(A - \hat{A})\,\Phi_+\,ds\Big). \tag{5.19}$$

By the choice of U_n, there exists $\delta > 0$ such that

$$|\phi - U_n| \leq C \exp\big(-\delta\sqrt{x^2+y^2}\big), \quad \text{for } y > 0. \tag{5.20}$$

Similar estimates hold for the derivatives of $\phi - U_n$. Hence from (5.19), we deduce

$$\|\Phi_+ - \hat{\Phi}_+\| \leq C \exp\big(-\delta\sqrt{x^2+y^2}\big), \quad \text{for } y > 0.$$

Arguing in the same manner,

$$\|\Phi_- - \hat{\Phi}_-\| \leq C \exp\big(-\delta\sqrt{x^2+y^2}\big), \quad \text{for } y > 0.$$

Now, in view of the relations

$$\Phi_{+,1}(x,y,\lambda) = a(\lambda,y)\,\Phi_{-,1}(x,y,\lambda), \quad \hat{\Phi}_{+,1}(x,y,\lambda) = \hat{a}(\lambda,y)\,\hat{\Phi}_{-,1}(x,y,\lambda),$$

we conclude that for fixed λ,

$$\lim_{y\to+\infty} (a(\lambda,y) - \hat{a}(\lambda,y)) = 0.$$

This together with Lemma 5.5 implies that for any $y \in \mathbb{R}$, $a(\lambda, y) = \hat{a}(\lambda, y)$. ∎

Observe that

$$\operatorname{Im} K = \Big(1 + \frac{1}{|\lambda|^2}\Big) \operatorname{Im}\lambda.$$

By Lemma 5.2, we now know that the functions $\Phi_{+,1}$ and $\Phi_{-,2}$ are analytic in the upper half λ-plane $\mathbb{R}^{2,+}$; while $\Phi_{+,2}$ and $\Phi_{-,1}$ are analytic in the lower half λ-plane.

We use $W(\Phi_{+,1},\Phi_{-,2})$ to denote the Wronskian determinant of $\Phi_{+,1}$ and $\Phi_{-,2}$. That is, $W(\Phi_{+,1},\Phi_{-,2}) = |\Phi_{+,1},\Phi_{-,2}|$. Note that for $\lambda \in \mathbb{R}\backslash\{0\}$, we have $\Phi_{+,1} = a(\lambda)\Phi_{-,1} - b(-\lambda)\Phi_{-,2}$, hence we obtain

$$W(\Phi_{+,1},\Phi_{-,2}) = W(a(\lambda)\Phi_{-,1},\Phi_{-,2}) - W(b(-\lambda)\Phi_{-,2},\Phi_{-,2}) = aW(\Phi_{-,1},\Phi_{-,2}).$$

Using the asymptotic behavior of $\Phi_{-,1},\Phi_{-,2}$ as $x\to-\infty$, we have $W(\Phi_{-,1},\Phi_{-,2}) = 1$. This then implies that for $\lambda\in\mathbb{R}\backslash\{0\}$,

$$a(\lambda,y) = W(\Phi_{+,1},\Phi_{-,2}). \tag{5.21}$$

Hence a can be analytically extended into $\mathbb{R}^{2,+}$ using (5.21). By the asymptotic behavior of $\Phi_{+,1},\Phi_{-,2}$ as $\lambda\to 0$, a will be continuous up to the boundary of $\mathbb{R}^{2,+}$. We also remark that if λ is in the lower half plane, then the behavior of $\Phi_{+,1}$ is much more delicate, because in general, solutions with the desired asymptotic behavior at $+\infty$ may not be unique.

We have the following generalization of Lemma 5.6.

Lemma 5.7. *Assume* $\operatorname{Im}\lambda\geq 0$ *and* $\lambda\neq 0$. *Let a be defined by* (5.21). *Then* $a(\lambda,y) = \hat{a}(\lambda,y)$.

Proof. Recall that by Lemma 5.2, the function $\varphi_{+,1} = \Phi_{+,1}\exp(-\frac{Kix}{4})$ satisfies the integral equations

$$\begin{cases} \varphi_{+,11}(x,y,\lambda) = 1 + \int_{+\infty}^{x}\left[A_{11}^*\varphi_{+,11} + A_{21}^*\varphi_{+,21}\right](s,y,\lambda)\,ds, \\ \varphi_{+,21}(x,y,\lambda) = \int_{+\infty}^{x}\exp\left(\frac{Ki}{2}(s-x)\right)\left[A_{21}^*\varphi_{+,11} + A_{22}^*\varphi_{+,21}\right](s,y,\lambda)\,ds. \end{cases}$$

This solution is analytic in the upper half λ-plane. Similarly, for the corresponding functions $\hat{\varphi}_{+,1}$ associated with the potential U_n, we have

$$\begin{cases} \hat{\varphi}_{+,11}(x,y,\lambda) = 1 + \int_{+\infty}^{x}\left[\hat{A}_{11}^*\varphi_{+,11} + \hat{A}_{12}^*\varphi_{+,21}\right](s,y,\lambda)\,ds, \\ \hat{\varphi}_{+,21}(x,y,\lambda) = \int_{+\infty}^{x}\exp\left(\frac{Ki}{2}(s-x)\right)\left[\hat{A}_{21}^*\varphi_{+,11} + \hat{A}_{22}^*\varphi_{+,21}\right](s,y,\lambda)\,ds. \end{cases}$$

If we set $\rho_j := \varphi_{+,j1}(x,y,\lambda) - \hat{\varphi}_{+,j1}(x,y,\lambda)$, $j = 1,2$, then

$$\begin{cases} \rho_1 = \int_{+\infty}^{x}\left[\hat{A}_{11}^*\rho_1 + \hat{A}_{21}^*\rho_2 + \mathbf{f}_1\right](s,y,\lambda)\,ds, \\ \rho_2 = \int_{+\infty}^{x}\exp\left(\frac{Ki}{2}(s-x)\right)\left[\hat{A}_{21}^*\rho_1 + \hat{A}_{22}^*\rho_2 + \mathbf{f}_2\right]ds, \end{cases} \tag{5.22}$$

where

$$\mathbf{f}_1 := (A_{11}^* - \hat{A}_{11}^*)\,\varphi_{+,11} + (A_{12}^* - \hat{A}_{12}^*)\,\varphi_{+,21},$$
$$\mathbf{f}_2 := (A_{21}^* - \hat{A}_{21}^*)\,\varphi_{+,11} + (A_{22}^* - \hat{A}_{22}^*)\,\varphi_{+,21}.$$

Due to the estimate (5.9), $\varphi_{+,1},\hat{\varphi}_{+,1}$ are uniformly bounded for (x,y) in the whole plane. Similarly, using the decay estimate (5.20), we infer from (5.22) and the Picard iteration of (ρ_1,ρ_2) that

$$|\rho_1(x)| \leq \int_x^{+\infty}\left(|\mathbf{f}_1(s)| + |\mathbf{f}_2(s)|\right)ds\,\exp(Q(x)),$$
$$|\rho_2(x)| \leq \int_x^{+\infty}\left(|\mathbf{f}_1(s)| + |\mathbf{f}_2(s)|\right)ds\,\exp(Q(x)).$$

Here $Q(x)$ is defined by (5.7). It follows that

$$\lim_{y\to+\infty}\left[\varphi_{+,1}(0,y,\lambda)-\hat{\varphi}_{+,1}(0,y,\lambda)\right]=0.$$

Similarly, letting $\varphi_{-,2}=\Phi_{-,2}\exp\left(\frac{Kix}{4}\right)$, we have

$$\lim_{y\to+\infty}\left[\varphi_{-,2}(0,y,\lambda)-\hat{\varphi}_{-,2}(0,y,\lambda)\right]=0.$$

Using the definition of a, we then deduce

$$\lim_{y\to+\infty}\left[a(\lambda,y)-\hat{a}(\lambda,y)\right]=0.$$

On the other hand, we can still prove that $\partial_y a(y,\lambda)=0$. Hence $a(\lambda,y)=\hat{a}(\lambda,y)$. This finishes the proof. ■

Let λ_j, $j=1,\ldots,m$, be the zeros of a in $\mathbb{R}^{2,+}$. At these points, by the definition of a, there holds $W(\Phi_{+,1},\Phi_{-,2})=0$. Hence the vectors $\Phi_{+,1}$ and $\Phi_{-,2}$ are co-linear to each other. Let us define c_j by the formula

$$\Phi_{+,1}(x,y,\lambda_j)=c_j(y)\,\Phi_{-,2}(x,y,\lambda_j).$$

Then $c_j'=-\frac{1}{2}(\lambda_j+1/\lambda_j)c_j$ and therefore $c_j(y)=c_j(0)\exp(-\frac{1}{2}(\lambda_j+1/\lambda_j)y)$. It is worth pointing out that unlike b, the function c_j is in general not uniformly bounded with respect to y. Let us use $\hat{c}_j(y)$ to denote the corresponding function of U_n. It is a natural question that whether one can prove $c_j(y)=\hat{c}_j(y)$, following a similar idea as that of Lemma 5.6. It turns out that, to do this, one need to directly analyze the *precise* asymptotic behavior of $\Phi_{+,1}$ as $y\to\infty$. While in principle this can be done, we choose to bypass this difficulty and verify it *a posteriori*, after we prove that $\phi=U_n$.

Now we have all the necessary *scattering data* at hand, which are a,b,λ_j,c_j.

Lemma 5.8. *Suppose all the zeros of a in the upper half λ-plane are simple. Then $u=U_n+\pi$.*

Before proceeding to the proof, we emphasize that the result of this lemma is proved under the *additional assumption* that all the zeros of a in the upper half λ-plane are simple. However, we will show in Lemma 5.9 that for the standard solution $U_n+\pi$, the corresponding scattering data $\hat{a}$ only has simple zeros, which in turn implies that a only has simple zeros. The proof of Lemma 5.9 does not depend on the result of Lemma 5.8; however, the *construction* of *explicit* Jost functions in Lemma 5.9 is *inspired* by the formula (5.25) of the proof of Lemma 5.8.

Proof of Lemma 5.8. We would like to carry out a simplified version of the inverse scattering procedure to construct the potential u from the scattering data, following [28]. Part of the arguments here are more or less standard. Since it is not easy to locate the precise references, we sketch the proof below for completeness.

For fixed $y\in\mathbb{R}$, by (5.16), we have, for $\lambda\in\mathbb{R}$,

$$\Phi_{-,1}(x,y,\lambda)=\frac{\Phi_{+,1}(x,y,\lambda)}{a(\lambda,y)}. \tag{5.23}$$

Consider the operator

$$(\mathcal{P} f)(\xi) := \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(\lambda)}{\lambda - \xi}\, d\lambda.$$

Let us rewrite equation (5.23) as

(5.24)

$$\Phi_{-,1}(x, y, \lambda) \exp\Big(-\frac{K(\lambda) i}{4} x\Big) - (1,0)^T = \frac{\Phi_{+,1}(x, y, \lambda)}{a(\lambda, y)} \exp\Big(-\frac{K(\lambda) i}{4} x\Big) - (1,0)^T.$$

The left-hand side is analytic in the *lower* half λ plane, while the right-hand side is meromorphic in the *upper* half plane with simple poles λ_j, $j = 1, \dots, m$. Here $\operatorname{Im} \lambda_j > 0$. Note that the function $\exp\big(-\frac{K(\lambda)i}{4} x\big)$ has two essential singularities: $\lambda = \infty$ and $\lambda = 0$. However, one can show that

$$\Phi_{-,1}(x, y, \lambda) \exp\Big(-\frac{K(\lambda) i}{4} x\Big) - (1,0)^T \to 0 \quad \text{as } \lambda \to \infty.$$

Moreover, $\Phi_{-,1}(x, y, \lambda) \exp\big(-\frac{K(\lambda) i}{4} x\big)$ can be continued to the origin. We refer to [15], page 396, for related discussion on this issue for the hyperbolic sine-Gordon equation. For each fixed $\xi \in \mathbb{C}$ with $\operatorname{Im} \xi < 0$, applying the operator $\mathcal{P}$ to both sides of equation (5.24), using the residue theorem and the fact that $\Phi_{+,1}(\lambda_j) = c_j \Phi_{-,2}(\lambda_j)$, we obtain

$$\begin{aligned} \text{(5.25)} \qquad & \Phi_{-,1}(x, y, \xi) \exp\Big(-\frac{K(\xi) i}{4} x\Big) - (1,0)^T \\ &= \sum_{j=1}^{m} \Big[\frac{\tilde{c}_j}{\xi - \lambda_j} \exp\Big(-\frac{K(\lambda_j) i}{4} x\Big) \Phi_{-,2}(x, y, \lambda_j)\Big], \end{aligned}$$

where

$$\text{(5.26)} \qquad \tilde{c}_j(y) := \frac{c_j(y)}{\partial_\lambda a(\lambda_j, y)}.$$

On the other hand, by Lemma 5.3, $\Phi_{-,2}(x, y, -\xi) = -i\sigma_2 \Phi_{-,1}(x, y, \xi)$. Hence taking $\xi = -\lambda_l$ in (5.25), we get

$$\begin{aligned} & i\sigma_2 \Phi_{-,2}(x, y, \lambda_l) \exp\Big(\frac{K(\lambda_l) i}{4} x\Big) - (1,0)^T \\ &= -\sum_{j=1}^{m} \Big[\frac{\tilde{c}_j}{\lambda_l + \lambda_j} \exp\Big(-\frac{K(\lambda_j) i}{4} x\Big) \Phi_{-,2}(x, y, \lambda_j)\Big]. \end{aligned}$$

This is a system of m equations for the functions $\Phi_{-,2}(x, y, \lambda_j)$, $j = 1, \dots, m$. Let M be the matrix with entries

$$\mathbf{m}_{lj} := \frac{\tilde{c}_j(y)}{\lambda_l + \lambda_j} \exp\Big(-\frac{K(\lambda_j)}{2} ix\Big).$$

Let $\eta := (\eta_1, \dots, \eta_{2m})^T$, where

$$\eta_l = \begin{cases} \exp\big(\frac{K(\lambda_l) ix}{4}\big) \Phi_{-,22}(x, y, \lambda_l), & \text{if } l = 1, \dots, m, \\ \exp\big(\frac{K(\lambda_{l-m}) ix}{4}\big) \Phi_{-,12}(x, y, \lambda_{l-m}), & \text{if } l = m+1, \dots, 2m. \end{cases}$$

Then we get

$$\begin{pmatrix} I & M \\ -M & I \end{pmatrix} \eta = \mathbf{e}_1, \tag{5.27}$$

where I is the $m \times m$ identity matrix and $\mathbf{e}_1 = (1, \ldots, 1, 0, \ldots, 0)^T$. Observe that

$$\begin{pmatrix} I & M \\ -M & I \end{pmatrix} = \begin{pmatrix} I & 0 \\ iI & I \end{pmatrix} \begin{pmatrix} I + iM & M \\ 0 & I - iM \end{pmatrix} \begin{pmatrix} I & 0 \\ -iI & I \end{pmatrix}.$$

Defining

$$\eta_+^* = \begin{pmatrix} I & 0 \\ -iI & I \end{pmatrix} \eta, \quad e_+^* = \begin{pmatrix} I & 0 \\ -iI & I \end{pmatrix} \mathbf{e}_1, \quad Z_+ = \begin{pmatrix} I + iM & M \\ 0 & I - iM \end{pmatrix},$$

we can transform equation (5.27) into $Z_+ \eta_+^* = e_+^*$. It follows that for $j = 1, \ldots, m$,

$$\eta_j = \frac{\det H_{+,j}}{\det Z_+}, \tag{5.28}$$

where the matrix $H_{+,j}$ is obtained from replacing the j-th column of Z_+ by the vector e_+^*. Similarly, we have

$$\eta_j = \frac{\det H_{-,j}}{\det Z_-}, \quad j = 1, \ldots, m,$$

where

$$e_-^* = \begin{pmatrix} I & 0 \\ iI & I \end{pmatrix} \mathbf{e}_1, \quad Z_- = \begin{pmatrix} I - iM & M \\ 0 & I + iM \end{pmatrix},$$

and $H_{-,j}$ is obtained from replacing the j-th column of Z_- by e_-^*.

Inserting (5.25) into the vector equation $\partial_x \Phi_{-,1} = A \Phi_{-,1}$, expanding both sides in terms of ξ (for ξ large), and comparing the $O(1)$ term in the *second* component, we get

$$u_x - i u_y = 2i \sum_{j=1}^{m} \left[\tilde{c}_j(y) \exp\left(-\frac{iK(\lambda_j)}{4} x \right) \Phi_{-,22}(x, y, \lambda_j) \right].$$

Hence by (5.28),

$$u_x - i u_y = 2i \sum_{j=1}^{m} \left[\tilde{c}_j(y) \exp\left(-\frac{iK(\lambda_j)}{2} x \right) \frac{\det H_{+,j}}{\det Z_+} \right].$$

We would like to simplify this expression. To do this, let us set

$$\nu_j := \tilde{c}_j(y) \exp\left(-\frac{iK(\lambda_j)}{2} x \right).$$

Note that in terms of ν_j, the entries of M are of the form $\nu_j/(\lambda_l + \lambda_j)$. We use $\tilde{Z}_+$ to represent the matrix obtained from Z_+ by multiplying the l and $l+m$-th *rows* of Z_+

by ν_l, $l = 1, \dots, m$. For each fixed $j = 1, \dots, m$, applying the same operation to the matrix $H_{+,j}$, we get the corresponding matrix $\tilde{H}_{+,j}$. Then

$$u_x - i u_y = 2i \sum_{j=1}^{m} \frac{\nu_j \det \tilde{H}_{+,j}}{\det \tilde{Z}_+}. \tag{5.29}$$

Similarly, we also have

$$u_x - i u_y = 2i \sum_{j=1}^{m} \frac{\nu_j \det \tilde{H}_{-,j}}{\det \tilde{Z}_-}. \tag{5.30}$$

Observe that $(\partial_x - i\partial_y)(\nu_l \nu_j) = -i(\lambda_l + \lambda_j)\nu_l \nu_j$. We define the matrix $\tilde{M}$ whose entries are $(\lambda_l + \lambda_j)^{-1}\nu_l \nu_j$. Let $\tilde{I}$ be the diagonal matrix whose entries on the diagonal is ν_j, $j = 1, \dots, m$. For fixed j, observe that in $\det \tilde{H}_{+,j} + \det \tilde{H}_{-,j}$, terms involving the last m components of the j-th column of $\det \tilde{H}_{+,j}$ and $\det \tilde{H}_{-,j}$ cancel. Hence we have

$$\begin{aligned}\sum_{j=1}^{m} \left(\nu_j \det \tilde{H}_{+,j} + \nu_j \det \tilde{H}_{-,j}\right) = {}& 2\det\left(\tilde{I} + i\tilde{M}\right)\left(\partial_x - i\partial_y\right)\det\left(\tilde{I} - i\tilde{M}\right) \\ & - 2\det\left(\tilde{I} - i\tilde{M}\right)\left(\partial_x - i\partial_y\right)\det\left(\tilde{I} + i\tilde{M}\right).\end{aligned}$$

In view of the fact that $\det \tilde{Z}_\pm = \det(\tilde{I} + i\tilde{M})\det(\tilde{I} - i\tilde{M})$, we infer

$$u_x - i u_y = 2i\left(\partial_x - i\partial_y\right)\ln\frac{\det(\tilde{I} - i\tilde{M})}{\det(\tilde{I} + i\tilde{M})} = 2i\left(\partial_x - i\partial_y\right)\ln\frac{\det(iI + M)}{\det(-iI + M)}.$$

Next we show that u can be written in the Hirota form appeared in Section 2. Indeed, if we define $\nu_j^* = \lambda_j^{-1}\nu_j$, then the entries of M become $(\lambda_l + \lambda_j)^{-1}\lambda_j \nu_j^*$ and there holds

$$\det(iI + M) = \sum_{j=1}^{m}\Big(\sum_{l_1 < \cdots < l_j}\left[i^{m-j} b(l_1, \dots, l_j)\nu_{l_1}^* \cdots \nu_{l_j}^*\right]\Big), \tag{5.31}$$

where

$$b(l_1, \dots, l_j) = \prod_{1 \le \alpha < \beta \le j}\left(\frac{\lambda_{l_\alpha} - \lambda_{l_\beta}}{\lambda_{l_\alpha} + \lambda_{l_\beta}}\right)^2.$$

This precisely means that u has the Hirota form given in Section 2. The identity (5.31) can be proved by considering the coefficients of the polynomial

$$g(r) := \det|irI + M|.$$

For instance, since the determinant of the matrix $\left(\frac{2\lambda_j}{\lambda_n + \lambda_j}\right)_{n,j}$ is equal to

$$\prod_{1 \le \alpha < \beta \le m}\left(\frac{\lambda_\alpha - \lambda_\beta}{\lambda_\alpha + \lambda_\beta}\right)^2,$$

then $g(0)$ can be explicitly computed and is equal to

$$\det M = b(1, \ldots, m)\, \nu_1^* \cdots \nu_m^*;$$

while the coefficient of ir is the sum of all the $(m-1)$-th order principle minors M:

$$\sum_{l_1 < \cdots < l_{m-1}} \left[b(l_1, \ldots, l_{m-1})\, \nu_{l_1}^* \cdots \nu_{l_{m-1}}^* \right].$$

Now we would like to compare u with $U_n + \pi$. Recall that in the expression of $U_n + \pi = 4 \arctan(\tilde{g}_n / \tilde{f}_n)$, there are parameters p_j, q_j, η_j^0, $j = 1, \ldots, m$, and p_j are chosen to be positive. On the other hand, in ν_j^*, the coefficient before x is $-\frac{K(\lambda_j)}{2} i$, which is equal to

$$\frac{\operatorname{Im} \lambda_j}{2} \left(1 + \frac{1}{(\operatorname{Re} \lambda_j)^2 + (\operatorname{Im} \lambda_j)^2} \right) - i\, \frac{\operatorname{Re} \lambda_j}{2} \left(1 - \frac{1}{(\operatorname{Re} \lambda_j)^2 + (\operatorname{Im} \lambda_j)^2} \right).$$

The coefficient before y is $-\frac{1}{2}(\lambda_j + \lambda_j^{-1})$, which is equal to

$$-\frac{\operatorname{Re} \lambda_j}{2} \left(1 + \frac{1}{(\operatorname{Re} \lambda_j)^2 + (\operatorname{Im} \lambda_j)^2} \right) - i\, \frac{\operatorname{Im} \lambda_j}{2} \left(1 - \frac{1}{(\operatorname{Re} \lambda_j)^2 + (\operatorname{Im} \lambda_j)^2} \right).$$

Since u is real valued and has the same asymptotic behavior as $U_n + \pi$ as $y \to +\infty$, it then follows, from the Hirota form of u, that $(\operatorname{Re} \lambda_j)^2 + (\operatorname{Im} \lambda_j)^2 = 1$, $m = n$. Moreover,

$$\operatorname{Im} \lambda_j = p_j,\ \operatorname{Re} \lambda_j = -q_j, \quad \text{for } j = 1, \ldots, m, \tag{5.32}$$

and $u = U_n + \pi$, $c_j(0) = \hat{c}_j(0)$.

We would like to point out that for $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda \geq 0$,

$$a(\lambda) = \hat{a}(\lambda) = \prod_{j=1}^{m} \frac{\lambda - \lambda_j}{\lambda + \lambda_j}. \tag{5.33}$$

Indeed, for $\lambda \in \mathbb{R} \backslash \{0\}$, from (5.23) and $\det \Phi_\pm = 1$, and $b = 0$, we get $a(\lambda) a(-\lambda) = 1$. Let us define

$$\beta(\lambda) = a(\lambda) \prod_{j=1}^{m} \frac{\lambda + \lambda_j}{\lambda - \lambda_j}.$$

The function β is analytic in the upper half λ-plane $\mathbb{R}^{2,+}$. By (5.23) and (5.25), using the asymptotic behavior of $\Phi_{+,1}$(as $x \to +\infty$), we find that for some constants d_j,

$$\frac{1}{a(\lambda)} = 1 + \sum_{j=1}^{m} \frac{d_j}{\lambda - \lambda_j}, \quad \text{if } \lambda \in \mathbb{R}.$$

Now in view of $a(\lambda) a(-\lambda) = 1$, we deduce that

$$a(\lambda) = \prod_{j=1}^{m} \frac{\lambda - \lambda_j}{\lambda + \lambda_j}, \quad \text{if } \lambda \in \mathbb{R}.$$

That is, $\beta(\lambda) = 1$ for $\lambda \in \mathbb{R}$. Hence by the Liouville theorem, $\beta(\lambda) = 1$ in $\mathbb{R}^{2,+}$. We then get (5.33). The proof is completed. ∎

Next, we proceed to compute the scattering data of the "standard" solution $U_n + \pi$. We first point out that the scattering data $\hat{a}, \hat{b}, \lambda_j, \hat{c}_j$ of $U_n + \pi$ is well defined through functions $\hat{\Phi}_\pm$, which are solutions of ODEs in the Lax pair. We have the following.

Lemma 5.9. *Let p_j, q_j be the parameters appearing in the solution $U_n + \pi$ and let λ_j be defined through (5.32). Then the scattering data $\hat{a}$ of $U_n + \pi$ is given by*

$$\hat{a}(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \lambda_j}{\lambda + \lambda_j}, \quad \text{for } \lambda \in \mathbb{R}^{2,+}.$$

Proof. Before proceeding to the details, which requires tedious computation, let us sketch the main idea of the proof. The proof has two main steps. In the first step, we compute the scattering data of the simplest two-end solution $U_1 + \pi$ by finding the *explicit* form of the corresponding $\hat{\Phi}_\pm$ (the so called Jost function). In the second step, for $n > 1$, we analyze the behavior of $\hat{\Phi}_\pm$ for $y \to \infty$, using the asymptotic behavior of $U_n + \pi$. The reason we can do this is that $\hat{a}$ is independent of y. Now our key observation is that as y tends to ∞, $U_n + \pi$ asymptotically splits into n heteroclinic solutions ($U_1 + \pi$ with suitable parameters), passing each one of these heteroclinic solutions along the x direction, we gain a factor $\frac{\lambda-\lambda_j}{\lambda+\lambda_j}$ in $\hat{a}$ (for $\lambda \in \mathbb{R}\backslash\{0\}$), because $\hat{a}$ is the "ratio" between $\hat{\Phi}_{+,1}$ and $\hat{\Phi}_{-,1}$.

Step 1. Compute $\hat{a}$ for $U_1 + \pi$.

We shall *define* $\hat{\Phi}_{-,1}$ directly. The definition given below is inspired by (5.25). More precisely, define

$$\begin{aligned}\hat{\Phi}_{-,1}(x, y, \lambda) &= \exp\Big(\frac{K(\lambda)i}{4}x\Big)(1,0)^T \\ &\quad + \exp\Big(\frac{K(\lambda)i}{4}x\Big)\sum_{j=1}^{n}\Big[\frac{\tilde{c}_j(y)}{\lambda - \lambda_j}\exp\Big(-\frac{K(\lambda_j)i}{4}x\Big)\hat{\Phi}_{-,2}(x, y, \lambda_j)\Big].\end{aligned} \tag{5.34}$$

Here,

$$\begin{aligned}\tilde{c}_j(y) &:= \hat{c}_j(y)\Big[\partial_\lambda\Big(\prod_{l=1}^{n}\frac{\lambda-\lambda_l}{\lambda+\lambda_l}\Big)\Big|_{\lambda=\lambda_j}\Big]^{-1} \\ &= \frac{\hat{c}_j(0)\exp(-\frac{1}{2}(\lambda_j + 1/\lambda_j)y)}{2\lambda_j}\prod_{l\neq j}\frac{\lambda_j+\lambda_l}{\lambda_j-\lambda_l},\end{aligned} \tag{5.35}$$

the $\hat{c}_j(0)$ are parameters, and $\hat{\Phi}_{-,2}(x, y, \lambda_j) = (\hat{\Phi}_{-,12}(x, y, \lambda_j), \hat{\Phi}_{-,22}(x, y, \lambda_j))^T$ is given by

$$\begin{aligned}\hat{\Phi}_{-,12}(x, y, \lambda_j) &= \exp\Big(-\frac{K(\lambda_j)ix}{4}\Big)\frac{\det H_{+,j} + i\det H_{+,j+n}}{\det Z_+}, \\ \hat{\Phi}_{-,22}(x, y, \lambda_j) &= \exp\Big(-\frac{K(\lambda_j)ix}{4}\Big)\Big(\frac{\det H_{+,j}}{\det Z_+}\Big).\end{aligned}$$

With the definition of $\tilde{c}_j$ given by (5.35), $\mathbf{m}_{lj}$ is defined by

$$\mathbf{m}_{lj} := \frac{\tilde{c}_j(y)}{\lambda_l + \lambda_j} \exp\Big(-\frac{K(\lambda_j)}{2} i x\Big).$$

We emphasize that in this lemma, $\tilde{c}_j(y)$ is not defined through (5.26), but through (5.35). Hence the definition of $\tilde{c}_j(y)$ here does *not* require any assumption of simpleness of the zeros of $\hat{a}$.

Intuitively, the function $\hat{\Phi}_{-,1}$ should satisfy

$$\partial_x \hat{\Phi}_{-,1} = \hat{A}\hat{\Phi}_{-,1}. \tag{5.36}$$

However, a direct proof of this fact for general n seems to be quite tedious. Nevertheless, in what follows, we will see that in the case of $n = 1$, we can verify (5.36) by direct computation. Indeed, in this case, we have

$$U_1 + \pi = 2i \ln \frac{i + \mathbf{m}_{11}}{-i + \mathbf{m}_{11}}\cdot$$

We also have

$$\sin U_1 = \frac{1}{2i}\Big[\Big(\frac{i + \mathbf{m}_{11}}{-i + \mathbf{m}_{11}}\Big)^2 - \Big(\frac{-i + \mathbf{m}_{11}}{i + \mathbf{m}_{11}}\Big)^2\Big], \tag{5.37}$$

$$\cos U_1 = -\frac{1}{2}\Big[\Big(\frac{i + \mathbf{m}_{11}}{-i + \mathbf{m}_{11}}\Big)^2 + \Big(\frac{-i + \mathbf{m}_{11}}{i + \mathbf{m}_{11}}\Big)^2\Big]. \tag{5.38}$$

Moreover,

$$\begin{aligned}\hat{\Phi}_{-,1}(x, y, \lambda) = \exp\Big(\frac{K(\lambda) i}{4} x\Big)(1, 0)^T \\ + \exp\Big(\frac{K(\lambda) i}{4} x\Big)\Big[\frac{\hat{c}_1(y)}{2(\lambda - \lambda_1)\lambda_1} \exp\Big(-\frac{K(\lambda_1) i}{4} x\Big)\hat{\Phi}_{-,2}(x, y, \lambda_1)\Big],\end{aligned}$$

where $\hat{\Phi}_{-,2}(x, y, \lambda_1) = (\hat{\Phi}_{-,12}(x, y, \lambda_1), \hat{\Phi}_{-,22}(x, y, \lambda_1))^T$,

$$\hat{\Phi}_{-,12}(x, y, \lambda_1) = \exp\Big(-\frac{K(\lambda_1) i x}{4}\Big)\frac{\mathbf{m}_{11}}{1 + \mathbf{m}_{11}^2},$$

$$\hat{\Phi}_{-,22}(x, y, \lambda_1) = \exp\Big(-\frac{K(\lambda_1) i x}{4}\Big)\frac{1}{1 + \mathbf{m}_{11}^2}\cdot$$

Recall that

$$\hat{A}\hat{\Phi}_{-,1} = \frac{i}{4}\Big[\Big(\lambda + \frac{\cos U_1}{\lambda}\Big)\sigma_3 - \big[(\partial_x - i\partial_y) U_1\big]\sigma_2 + \frac{\sin U_1}{\lambda}\sigma_1\Big]\hat{\Phi}_{-,1}.$$

The first component J_1 of the vector $\hat{A}\hat{\Phi}_{-,1}$ is

$$\begin{aligned}&\frac{i}{4}\Big(\lambda + \frac{\cos U_1}{\lambda}\Big)\exp\Big(\frac{K(\lambda) i}{4} x\Big)\Big[1 + \frac{\hat{c}_1(y)}{2(\lambda - \lambda_1)\lambda_1}\exp\Big(-\frac{K(\lambda_1) i}{2} x\Big)\frac{\mathbf{m}_{11}}{1 + \mathbf{m}_{11}^2}\Big] \\ &+ \frac{1}{4}\big[(\partial_x - i\partial_y) U_1\big]\exp\Big(\frac{K(\lambda) i}{4} x\Big)\Big[\frac{\hat{c}_1(y)}{2(\lambda - \lambda_1)\lambda_1}\exp\Big(-\frac{K(\lambda_1) i}{2} x\Big)\frac{1}{1 + \mathbf{m}_{11}^2}\Big] \\ &+ \frac{i}{4}\frac{\sin U_1}{\lambda}\exp\Big(\frac{K(\lambda) i}{4} x\Big)\Big[\frac{\hat{c}_1(y)}{2(\lambda - \lambda_1)\lambda_1}\exp\Big(-\frac{K(\lambda_1) i x}{2}\Big)\frac{1}{1 + \mathbf{m}_{11}^2}\Big].\end{aligned}$$

Recall that the function $\mathbf{m}_{11}$ is defined by

$$\mathbf{m}_{11} = \frac{\hat{c}_1(y)}{4\lambda_1^2} \exp\Big(-\frac{K(\lambda_1)i}{2} x\Big).$$

Using this, we find that $J_1 \exp\big(-\frac{K(\lambda)i}{4} x\big)$ is equal to

$$\frac{i}{4}\Big(\lambda + \frac{\cos U_1}{\lambda}\Big)\Big(1 + \frac{2\lambda_1}{\lambda-\lambda_1}\frac{\mathbf{m}_{11}^2}{1+\mathbf{m}_{11}^2}\Big) + \frac{1}{4}\big[(\partial_x - i\partial_y)\,U_1\big]\Big(\frac{2\lambda_1}{\lambda-\lambda_1}\frac{\mathbf{m}_{11}}{1+\mathbf{m}_{11}^2}\Big) + \frac{i}{4}\frac{\sin U_1}{\lambda}\Big(\frac{2\lambda_1}{\lambda-\lambda_1}\frac{\mathbf{m}_{11}}{1+\mathbf{m}_{11}^2}\Big).$$

On the other hand, the first component J_1^* of $\partial_x \hat{\Phi}_{-,1}$ has the form

$$\frac{K(\lambda)i}{4}\exp\Big(\frac{K(\lambda)i}{4}x\Big)\Big(1 + \frac{2\lambda_1}{\lambda-\lambda_1}\frac{\mathbf{m}_{11}^2}{1+\mathbf{m}_{11}^2}\Big) + \exp\Big(\frac{K(\lambda)i}{4}x\Big)\frac{2\lambda_1}{\lambda-\lambda_1}\frac{-K(\lambda_1)i\mathbf{m}_{11}^2}{(1+\mathbf{m}_{11}^2)^2}.$$

Now we can compute

$$4(J_1 - J_1^*)\exp\Big(-\frac{K(\lambda)i}{4}x\Big) = i\,\frac{1+\cos U_1}{\lambda}\Big(1 + \frac{2\lambda_1}{\lambda-\lambda_1}\frac{\mathbf{m}_{11}^2}{1+\mathbf{m}_{11}^2}\Big) + i\,\frac{\sin U_1}{\lambda}\frac{2\lambda_1}{\lambda-\lambda_1}\frac{\mathbf{m}_{11}}{1+\mathbf{m}_{11}^2} - \frac{8i}{\lambda-\lambda_1}\frac{\mathbf{m}_{11}^2}{(1+\mathbf{m}_{11}^2)^2}.$$

Inserting (5.37), (5.38) into the right-hand side, we see that it is identically zero. Therefore, the first component of $\partial_x\hat{\Phi}_{-,1} - \hat{A}\hat{\Phi}_{-,1}$ vanishes. Similarly, its second component is 0. We then obtain $\partial_x\hat{\Phi}_{-,1} = \hat{A}\hat{\Phi}_{-,1}$. We also observe that $\hat{\Phi}_{-1}$ has the required asymptotic behavior:

$$\hat{\Phi}_{-1}\exp(-Kix/4) \to (1,0), \quad \text{as } x \to -\infty.$$

With the explicit form of the function $\hat{\Phi}_{-,1}$ at hand, using the relation $\hat{\Phi}_{+,1} = a\hat{\Phi}_{-,1}$ for $\lambda \in \mathbb{R}\backslash\{0\}$, we directly compute that

$$\hat{a}(\lambda)^{-1} = 1 + \frac{d_1}{\lambda-\lambda_1}, \quad \text{if } \lambda \in \mathbb{R}\backslash\{0\}, \tag{5.39}$$

for some constant d_1 (actually one can calculate directly that $d_1 = 2\lambda_1$). In view of $\hat{a}(\lambda)\hat{a}(-\lambda) = 1$ for $\lambda \in \mathbb{R}\backslash\{0\}$, we deduce from (5.39) that

$$\hat{a}(\lambda) = \frac{\lambda-\lambda_1}{\lambda+\lambda_1}, \quad \text{if } \lambda \in \mathbb{R}\backslash\{0\}. \tag{5.40}$$

We should point out that at this moment we still do not know whether λ_1 is a zero of $\hat{a}$. Hence we cannot use the argument of the last paragraph in the proof of Lemma 5.8 to conclude that $\hat{a}(\lambda) = \frac{\lambda-\lambda_1}{\lambda+\lambda_1}$ in $\mathbb{R}^{2,+}$. To bypass this difficulty, we would like to show that $\hat{a}$

cannot have repeated zeros in $\mathbb{R}^{2,+}$. Indeed, suppose to the contrary that λ_j^* is a zero of $\hat{a}$ in the upper half λ plane with multiplicity $\kappa > 1$. Then using the residue theorem as that of (5.25), we find that in $\hat{\Phi}_{-,1}(x, y, \xi)$, there are terms like

$$\frac{\Phi_{+,1}(x, y, \lambda_j^*) \exp\left(-\frac{K(\lambda_j^*)i}{4} x\right)}{(\xi - \lambda_j^*)^\kappa}.$$

This together with the relation $\hat{\Phi}_{+,1} = a\hat{\Phi}_{-,1}$ implies that $\hat{a}^{-1}$ will not have the form $\frac{\lambda+\lambda_1}{\lambda-\lambda_1}$ on $\mathbb{R}$, which is a contradiction. Hence all the zeros of $\hat{a}$ has to be simple and then by Lemma 5.8, the scattering data of $U_1 + \pi$ is given by

$$a(\lambda) = \hat{a}(\lambda) = \frac{\lambda - \lambda_1}{\lambda + \lambda_1}, \quad \text{for } \lambda \in \mathbb{R}^{2,+}.$$

Step 2. Compute $\hat{a}$ for $U_n + \pi, n > 1$.

Let us first compute the scattering data $\hat{a}$ of the four-end solution $U_2 + \pi$. To carry out the analysis in full detail, we need to introduce some additional notation. $U_2 + \pi$ has two ends in the upper half x-y plane, which are two half straight lines denoted by L_1, L_2. Along each end, as $y \to +\infty$, it converges to the one dimensional solution $U_1 + \pi$ with suitable parameters, $p_j, q_j, \eta_{j,0}$. Let us denote the one dimensional solution around L_1 by $U_{1,\alpha} + \pi$, and the one around L_2 by $U_{1,\beta} + \pi$. We also assume without loss of generality that L_1 is at the left of L_2 in the upper half plane.

For $U_{1,\alpha} + \pi$ and $U_{1,\beta} + \pi$, we have corresponding Jost functions $\hat{\Phi}_{-,1,\alpha}$ and $\hat{\Phi}_{-,1,\beta}$, defined in the first step. Hence

$$\partial_x \hat{\Phi}_{-,1,\alpha} = \hat{A}_\alpha \hat{\Phi}_{-,1,\alpha}, \partial_x \hat{\Phi}_{-,1,\beta} = \hat{A}_\beta \hat{\Phi}_{-,1,\beta}.$$

Moreover, $\hat{\Phi}_{-,1,\alpha} \exp(-Kix/4) \to (1,0)^T$, and $\hat{\Phi}_{-,1,\beta} \exp(-Kix/4) \to (1,0)^T$, as $x \to -\infty$. We emphasize that $\hat{\Phi}_{-1,\alpha}$ and $\hat{\Phi}_{-1,\beta}$ also depend on the y variable.

The Jost function of $U_2 + \pi$ will still be denoted by $\hat{\Phi}_{-,1}$, but at this moment we do not have explicit formula for it (although it is expected to be of the form (5.34), we did not prove that, because the computation is tedious). We also have

$$\partial_x \hat{\Phi}_{-,1} = \hat{A} \hat{\Phi}_{-,1}$$

and $\hat{\Phi}_{-,1} \exp(-Kix/4) \to (1,0)^T$, as $x \to -\infty$. Recall that for $\lambda \in \mathbb{R}\backslash\{0\}$, $\hat{a}(\lambda)$ is defined by the relation

$$\hat{\Phi}_{+,1} = a \hat{\Phi}_{-,1}, \tag{5.41}$$

where $\hat{\Phi}_{+,1}$ is the Jost function with $\hat{\Phi}_{+,1} \exp(-Kix/4) \to (1,0)^T$, as $x \to +\infty$. Hence computing $\hat{a}$ amounts to analyzing the asymptotic behavior of $\hat{\Phi}_{-,1}$ as $x \to +\infty$.

In the following, we consider the relevant functions in the upper half plane. The half straight lines L_1 and L_2 form an angle. Let us denote its angular bisector as L^*. Since $U_2 + \pi$ tends to $U_{1,\alpha} + \pi$ along the end L_1 exponentially fast, the proof of Lemma 5.6 tells us that for some positive constant δ_1,

$$|\hat{\Phi}_{-,1} - \hat{\Phi}_{-,1,\alpha}| \le C \exp\left(-\delta_1 \sqrt{x^2 + y^2}\right), \quad \text{if } y > 0 \text{ and } (x, y) \text{ is at the left of } L^*. \tag{5.42}$$

We remark that although Lemma 5.6 deals with matrix valued solutions, the argument also can be applied to vector valued solutions with straightforward changes. On the other hand, by the explicit formula of $\hat{\Phi}_{-1,\alpha}$ (or using the fact that the scattering data $\hat{a}$ of $U_{1,\alpha}$ is $(\lambda-\lambda_1)/(\lambda+\lambda_1)$), we have, if (x,y) lies in the right of L_1, then

$$\left|\hat{\Phi}_{-1,\alpha}(x,y)\exp(-Kix/4)-\frac{\lambda+\lambda_1}{\lambda-\lambda_1}(1,0)^T\right|\leq C\exp(-\delta_2 d(x,y)). \tag{5.43}$$

where $\delta_2>0$ is a small positive constant and $d(x,y)$ is the distance of (x,y) to L_1. Combining (5.42) and (5.43), we find that on the line L^*,

$$\left|\hat{\Phi}_{-,1}(x,y)\exp(-Kix/4)-\frac{\lambda+\lambda_1}{\lambda-\lambda_1}(1,0)^T\right|\leq C\exp(-\delta\, d(x,y)), \tag{5.44}$$

for some small positive constant δ.

Next let us consider the function $\hat{\Phi}^*_{-,1,\beta}$, defined by

$$\hat{\Phi}^*_{-,1,\beta}:=\frac{\lambda+\lambda_1}{\lambda-\lambda_1}\hat{\Phi}_{-,1,\beta}.$$

Note that $\hat{\Phi}^*_{-,1,\beta}$ still satisfies the equation $\partial_x\hat{\Phi}^*_{-,1,\beta}=\hat{A}_\beta\hat{\Phi}^*_{-,1,\beta}$. We have

$$\left|\hat{\Phi}^*_{-,1,\beta}\exp(-Kix/4)-\frac{\lambda+\lambda_1}{\lambda-\lambda_1}(1,0)^T\right|\leq C\exp\big(-\tilde{\delta}\tilde{d}(x,y)\big),\quad \text{on } L^*,$$

for some positive constant $\tilde{\delta}$, where $\tilde{d}(x,y)$ denotes the distance of (x,y) to L_2. Hence by (5.44), reducing δ if necessary, we get, for $(x,y)\in L^*$ in the upper half plane,

$$\left|\hat{\Phi}_{-,1}(x,y)-\hat{\Phi}^*_{-,1,\beta}\right|\leq C\exp(-\delta y).$$

Again by the proof of Lemma 5.6, we find that for (x,y) at the left of L^*,

$$\left|\hat{\Phi}_{-,1}(x,y)-\hat{\Phi}^*_{-,1,\beta}\right|\leq C\exp(-\delta y)+C\exp\big(-\delta\sqrt{x^2+y^2}\big). \tag{5.45}$$

Here we emphasize that in the right-hand side of the above inequality we have the term $C\exp(-\delta y)$. The reason is that, on the line L^*, $\hat{\Phi}_{-,1}(x,y)$ and $\hat{\Phi}^*_{-,1,\beta}$ are not identical. Nevertheless, we also know that a solution η of the equation $\partial_x\eta=\hat{A}\eta$ with initial condition $\eta=\hat{\Phi}_{-,1}(x,y)-\hat{\Phi}^*_{-,1,\beta}$ at L^* is bounded by $C\exp(-\delta y)$ at the right of L^*. This fact again follows from the proof of Lemma 5.6, which uses the assumption $\lambda\in\mathbb{R}$ in an essential way.

Now by the asymptotic behavior of $\hat{\Phi}_{-,1,\beta}$ as $x\to+\infty$, (5.45) implies that

$$\lim_{x\to+\infty}\left|\hat{\Phi}_{-,1}(x,y)\exp(-Kix/4)-\frac{\lambda+\lambda_1}{\lambda-\lambda_1}\frac{\lambda+\lambda_2}{\lambda-\lambda_2}(1,0)^T\right|\leq C\exp(-\delta y).$$

Sending y to $+\infty$ and using (5.41), we deduce

$$\hat{a}(\lambda)=\frac{\lambda-\lambda_1}{\lambda+\lambda_1}\frac{\lambda-\lambda_2}{\lambda+\lambda_2},\quad \text{for } \lambda\in\mathbb{R}\backslash\{0\}.$$

For general $U_n + \pi, n \geq 2$, we can repeat the above arguments as we pass across each end along the x direction, and conclude that

$$\hat{a}(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \lambda_j}{\lambda + \lambda_j}, \quad \text{for } \lambda \in \mathbb{R} \backslash \{0\}.$$

Then we can use the arguments in the last paragraph of Step 1 to conclude that all the zeros of $\hat{a}$ are simple and

$$\hat{a}(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \lambda_j}{\lambda + \lambda_j}, \quad \text{for } \lambda \in \mathbb{R}^{2,+}.$$

This finishes the proof. ∎

With these preparations, we are now ready to prove the main result of this section.

Proof of Proposition 5.1. Recall that a is the scattering data of our original solution u. Lemma 5.7 tells us that u and $U_n + \pi$ have the same a part of the scattering data. Hence

$$a(\lambda) = \prod_{j=1}^{n} \frac{\lambda - \lambda_j}{\lambda + \lambda_j}, \quad \text{for } \lambda \in \mathbb{R}^{2,+}.$$

In particular, all the zeros of a in the upper half λ-plane are simple. We then apply Lemma 5.8 to conclude that $u = U_n + \pi$. The proof is completed. ∎

6. Morse index of the multiple-end solutions

In this section, we shall compute the Morse index of the multiple-end solutions U_n of the elliptic sine-Gordon equation $-\Delta u = \sin u$ through a deformation argument. By definition, the Morse index of U_n is the total number of negative eigenvalues of the operator $\eta \to -\Delta \eta - \eta \cos U_n$ defined on $L^2(\mathbb{R}^2)$. The main result of this section is the following.

Proposition 6.1. *The Morse index of the* $2n$*-end solutions to the elliptic sine-Gordon equation is equal to* $n(n-1)/2$.

We shall split the proof of this result into several lemmas. Before proceeding, let us first of all briefly recall the so called end-to-end construction of multiple-end solutions of the Allen–Cahn type equation, developed in [39]. Roughly speaking, for each $n \geq 2$, we can glue $n(n-1)/2$ number of four-end solutions together by matching their ends and obtain a solution with $2n$ ends.

To explain the construction more precisely, we choose n straight lines $L_1, \ldots, L_n$ such that these lines intersect at $n(n-1)/2$ distinct points. The intersection point of L_i with L_j will be denoted by $\omega_{i,j}$. We assume the minimal distance between those points $\omega_{i,j}$ is equal to 2.

For each k large, the end-to-end construction in [39] tells us that we can "desingularize" the configuration of n rescaled lines $kL_1, \ldots, kL_n$. Actually, we can put four-end solutions $g_{i,j}$ near each rescaled intersection point $k\omega_{i,j}$ at a distance of $O(1)$ order in a suitable way and match their ends to form an approximate solution $\tilde{u}_k$. The center of $g_{i,j}$ will be denoted by $z_{i,j} = z_{i,j}(u_k)$. Around each $z_{i,j}$, $\tilde{u}_k$ is equal to $g_{i,j}$. By slightly adjusting their ends, we can perturb the approximate solution $\tilde{u}_k$ into a true solution u_k of the Allen–Cahn type equation.

Throughout this section, we shall use $B_r(p)$ to denote be the ball of radius r centered at the point p. Let c_0 be a fixed large constant. The following estimate is a direct byproduct of the end-to-end construction: there exists $\delta > 0$ such that

$$|u_k - \tilde{u}_k| \le C\exp(-\delta k), \quad \text{in } B_{c_0 k}(0). \tag{6.1}$$

This essentially follows from the fact that the error $\Delta\tilde{u}_k + \sin\tilde{u}_k$ of the approximate solution $\tilde{u}_k$ is of the order $O(e^{-\delta k})$.

Lemma 6.2. *Let u_k be a solution obtained from the end-to-end construction discussed above. The Morse index of u_k is at least $n(n-1)/2$ for k large.*

Proof. For each pair of indices $(i,j), i,j = 1,\ldots,n, i<j$, we use $\eta_{i,j}$ with $\|\eta_{i,j}\|_{L^\infty} = 1$ to denote a choice of the negative eigenfunction of the operator $-\Delta - \cos g_{i,j}$, corresponding to the (unique) negative eigenvalue $\sigma_{i,j}$. That is,

$$-\Delta\eta_{i,j} - \eta_{i,j}\cos g_{i,j} = \sigma_{i,j}\eta_{i,j}.$$

The total number of such functions is $n(n-1)/2$.

Let $\rho_{i,j}$ be a cutoff function localized near $z_{i,j}$, such that

$$\rho_{i,j} = \begin{cases} 1, & \text{in } B_{\sqrt{k}}(z_{i,j}), \\ 0, & \text{in } \mathbb{R}^2 \setminus B_{2\sqrt{k}}(z_{i,j}). \end{cases}$$

We can also assume that $\rho_{i,j}$ and its first derivatives are uniformly bounded with respect to k. Let $\eta^*_{i,j} := \rho_{i,j}\eta_{i,j}$. Since the mutual distance between those point $z_{i,j}$ are of the order $O(k)$, we see that the $\eta^*_{i,j}$ have disjoint supports. Using the fact that $\eta_{i,j}$ decays exponentially fast to zero away from $z_{i,j}$, we can show that for k large,

$$\begin{aligned} &\int_{\mathbb{R}^2} \left(|\nabla\eta^*_{i,j}|^2 - (\eta^*_{i,j})^2\cos u_k\right) \\ &\quad = \int_{\mathbb{R}^2} \left(\left(|\nabla\eta_{i,j}|^2 - \eta^2_{i,j}\cos u_k\right)\rho^2_{i,j} + 2\rho_{i,j}\eta_{i,j}\nabla\rho_{i,j}\nabla\eta_{i,j} + \eta^2_{i,j}|\nabla\rho_{i,j}|^2\right) < 0. \end{aligned}$$

Hence the Morse index of u_k is *at least* $n(n-1)/2$. ■

Before proceeding, we need to introduce some notations. Let $\mathcal{N}(u_k)$ be the nodal set of u_k and let $\mathbf{d}(p, \mathcal{N}(u_k))$ be the distance of a point p to the set $\mathcal{N}(u_k)$. Let r_0 be a large constant. We set

$$\Omega = \Omega_{r_0} := \bigcup_{i,j,i<j} B_{r_0}(z_{i,j}(u_k)).$$

We use H to denote the one dimensional heteroclinic solution. Explicitly,

$$H(s) = 4\arctan(e^s) - \pi.$$

Throughout the section, we use C to denote a universal constant. One of the main ingredients in the proof of Proposition 6.1 is the following.

Lemma 6.3. *Let* $-\lambda_k^2$ *(with* $\lambda_k > 0$*) be a negative eigenvalue of the operator* $-\Delta - \cos u_k$*. Then there exists a constant* $\vartheta < 0$ *independent of* k*, such that* $-\lambda_k^2 < \vartheta$ *for all* k*.*

Proof. Let ϕ_k be the corresponding eigenfunction of the eigenvalue $-\lambda_k^2$, normalized such that $\|\phi_k\|_{L^\infty} = 1$.

First of all, we would like to prove that if r_0 is a fixed constant chosen to be large enough, then

$$\|\phi_k\|_{L^\infty(\Omega_{r_0})} \geq \alpha,$$

where α is some positive constant independent of k.

By definition, ϕ_k satisfies

$$-\Delta\phi_k - \phi_k \cos u_k = -\lambda_k^2 \phi_k. \tag{6.2}$$

As $\mathbf{d}(p, \mathcal{N}(u_k)) \to +\infty$, there holds $|u_k| \to \pi$ and $\cos u_k \to -1$. It follows that when $\mathbf{d}(p, \mathcal{N}(u_k))$ is sufficiently large, $-\cos u_k + \lambda_k^2 \geq 1/2$. Hence by constructing suitable barrier functions of exponential type, we find that for some positive constant $\delta > 0$,

$$|\phi_k(p)| \leq C \exp\left(-\delta\, \mathbf{d}(p, \mathcal{N}(u_k))\right), \quad \text{for } p \in \mathbb{R}^2. \tag{6.3}$$

Let us estimate ϕ_k in the region $\mathbb{R}^2 \setminus \Omega$. To be more specific, we focus on the region around the nodal line l^*, which connects two, say $g_{1,2}$ and $g_{1,3}$, adjacent four-end solutions. Without loss of generality, using (6.1), we may assume that this nodal line is given by the graph of the function $y = w(x)$, and reducing δ if necessary,

$$|w(x)| \leq C \exp\left(-\delta \min\{|x - t_1|, |x - t_2|\}\right), \quad x \in [t_1, t_2],$$

with $(t_1, w(t_1)) \in \partial B_{r_0}(z_{1,2}(u_k))$, $(t_2, w(t_2)) \in \partial B_{r_0}(z_{1,3}(u_k))$. Note the $|t_1 - t_2|$ is of the order $O(k)$, and t_1, t_2 actually also depend on k.

Let us define the function

$$h(x) := \int_{-\infty}^{+\infty} \phi_k(x, y) H'(y)\, dy.$$

Since ϕ_k satisfies (6.2), for $x \in [t_1, t_2]$, h satisfies

$$-h''(x) = -\lambda_k^2 h(x) + \underbrace{O\left(\exp\left(-\delta \min\{|x - x_1|, |x - x_2|\}\right)\right)}_{\mu(x)}.$$

The variation of parameters formula then tells us that for some constants a, b,

$$\begin{aligned} h(x) = {} & a \exp(\lambda_k x) + b \exp(-\lambda_k x) + \frac{1}{2\lambda_k} \exp(\lambda_k x) \int_{t_1}^{x} \exp(-\lambda_k s)\, \mu(s)\, ds \\ & - \frac{1}{2\lambda_k} \exp(-\lambda_k x) \int_{t_1}^{x} \exp(\lambda_k s)\, \mu(s)\, ds. \end{aligned} \tag{6.4}$$

Let us define

$$f(s) = \int_{(t_1+t_2)/2}^{s} \mu(s)\, ds.$$

By the estimate of μ, we have

$$|f(s)| \leq C \exp\left(-\delta \min\{|x-t_1|, |x-t_2|\}\right).$$

Integrating by parts leads to

$$\begin{aligned} I &:= \frac{1}{2\lambda_k} \exp(\lambda_k x) \int_{t_1}^{x} \exp(-\lambda_k s)\, \mu(s)\, ds - \frac{1}{2\lambda_k} \exp(-\lambda_k x) \int_{t_1}^{x} \exp(\lambda_k s)\, \mu(s)\, ds \\ &= \frac{1}{2} \exp(\lambda_k x) \int_{t_1}^{x} f(s) \exp(-\lambda_k s)\, ds + \frac{1}{2} \exp(-\lambda_k x) \int_{t_1}^{x} f(s) \exp(\lambda_k s)\, ds. \end{aligned}$$

Then I can be estimated by

$$|I| \leq C \exp\left(-\delta \min\{|x-t_1|, |x-t_2|\}\right).$$

Let $I_0(x) := a \exp(\lambda_k x) + b \exp(-\lambda_k x)$. By the maximum principle, we have

$$|I_0(x)| \leq \max\{|I_0(t_1)|, |I_0(t_2)|\}, \quad \text{for } x \in [t_1, t_2].$$

Therefore,

$$|h(x)| \leq C\left(|h(t_1)| + |h(t_2)| + \exp\left(-\delta \min\{|x-t_1|, |x-t_2|\}\right)\right).$$

In particular, this implies that

$$|h(x)| \leq C \|\phi_k\|_{L^\infty(\Omega)} + C \exp\left(-\delta \min\{|x-t_1|, |x-t_2|\}\right), \quad x \in [t_1, t_2]. \tag{6.5}$$

On the other hand, we define

$$\upsilon^* := \phi_k(x, y) - h(x) H'(y).$$

Let $\sigma_0 > 0$ be a fixed small constant and let $y_0(x) := \sigma_0 \min\{|x-t_1|, |x-t_2|\} + 10$. Consider the region

$$E := \{(x, y) : x \in (t_1, t_2),\ y \in (-y_0(x), y_0(x))\}.$$

Let ρ be a cutoff function such that $\rho = 0$ in $\mathbb{R}^2 \backslash E$, and $\rho = 1$ in

$$\{(x, y) : x \in (t_1 + 1, t_2 - 1),\ y \in (-y_0(x) + 1, y_0(x) - 1)\}.$$

Define $\upsilon = \rho \upsilon^*$. Observe that although υ is not necessary orthogonal to H', we still have

$$\begin{aligned} \int_{\mathbb{R}^2} \upsilon(x, y) H'(y) dy &= \int [\phi_k(x, y) - h(x) H'(y)] \rho\, dy \\ &= O\left(\exp(-\delta \min\{|x-t_1|, |x-t_2|\})\right). \end{aligned}$$

By the decay estimate (6.3) of ϕ_k, we have

$$-\Delta \upsilon - \upsilon \cos H(y) = O\left(\exp(-\delta \min\{|x-t_1|, |x-t_2|\})\right).$$

Applying the estimates established in Lemma 3.5 of [12], reducing δ if necessary, we get

$$|\upsilon| \leq C \exp\left(-\delta \min\{|x-t_1|, |x-t_2|\}\right). \tag{6.6}$$

Estimates (6.3), (6.5) and (6.6) tell us that (enlarging the constant r_0 if necessary)

$$\|\phi_k\|_{L^\infty(\Omega)} \geq \alpha,$$

where α is some positive constant independent of k.

To prove the lemma, we assume to the contrary that for a sequence $k_n \to +\infty$, the eigenvalues $\lambda_{k_n}(u_{k_n})$ were tending to 0. We still denote k_n by k and $\lambda_{k_n}(u_{k_n})$ by $\lambda_k(u_k)$.

Suppose for some constant $\alpha > 0$, a pair of indices $(\bar{\imath}, \bar{\jmath})$ satisfies

$$\|\phi_k\|_{L^\infty(B_{r_0}(z_{\bar{\imath},\bar{\jmath}}))} \geq \alpha > 0 \quad \text{for all } k.$$

Then the function $\phi_k(z - z_{\bar{\imath},\bar{\jmath}})$ converges to a nontrivial bounded kernel $\beta_{\bar{\imath},\bar{\jmath}}$ of the operator $-\Delta - \cos \tilde{g}_{\bar{\imath},\bar{\jmath}}$, where $\tilde{g}_{\bar{\imath},\bar{\jmath}}$ is the four-end solution centered at the origin obtained from suitable translation of $g_{\bar{\imath},\bar{\jmath}}$. We would like to analyze the asymptotic behavior of ϕ_k around $z_{\bar{\imath},\bar{\jmath}}$ in a more precise way.

To simplify the notation, we assume $z_{\bar{\imath},\bar{\jmath}} = 0$. After a possible rotation, the four-end solution $\tilde{g}_{\bar{\imath},\bar{\jmath}}$ has the form

$$4 \arctan \frac{p \cosh(qy)}{q \cosh(px)} - \pi,$$

where p, q are positive constants with $p^2 + q^2 = 1$. Then by the L^∞-nondegeneracy of four-end solutions, $\beta_{i,j} = \tau_1 \partial_x \tilde{g}_{\bar{\imath},\bar{\jmath}} + \tau_2 \partial_y \tilde{g}_{\bar{\imath},\bar{\jmath}}$, for some constants τ_1, τ_2. The nodal curve of $\tilde{g}_{\bar{\imath},\bar{\jmath}}$ in the first quadrant is asymptotic to the line

$$l_1 : qy - px = \ln \frac{q}{p} \cdot$$

The ends in the second, third and fourth quadrants are asymptotic to l_2, l_3 and l_4 respectively, where

$$l_2 : qy + px = \ln \frac{q}{p}, \quad l_3 : -qy - px = \ln \frac{q}{p} \quad \text{and} \quad l_4 : -qy + px = \ln \frac{q}{p} \cdot$$

Without loss of generality, we assume $p < q$. The case of $p \geq q$ is similar. The line l_1 intersects with the y-axis at the point $(0, \frac{1}{q} \ln \frac{q}{p})$. This point will be denoted by P_+. The intersection point of the line l_3 with the y axis will be denoted by $P_- := (0, -\frac{1}{q} \ln \frac{q}{p})$. We also introduce the coordinate system (x_1, y_1) adapted to the end in the first quadrant, where the x_1 axis is on l_1, and the y_1 axis is orthogonal to l_1. Hence the angle between the x and x_1 axes is equal to $\arctan \frac{p}{q}$, which is also equal to the angle between the y and y_1 axes. The origin of the (x_1, y_1) coordinate system will be the point P_+. Similarly, for $j = 2, 3, 4$, we have the coordinate system (x_j, y_j) corresponding to the end in the j-th quadrant, where the x_j axis is on l_j. The origin of (x_2, y_2)-system is P_+, while the origin of the (x_3, y_3) and (x_4, y_4) systems is P_-.

By the linear decomposition lemma (Lemma 4.2 of [11]), or using the explicit formula of the four-end solutions, there exists constant $\delta > 0$, such that

$$|\partial_x \tilde{g}_{\bar{\imath},\bar{\jmath}} + qH'(y_1)| + |\partial_y \tilde{g}_{\bar{\imath},\bar{\jmath}} - pH'(y_1)| \leq C \exp(-\delta x_1), \quad \text{in the first quadrant.}$$

Hence in this region,

$$\beta_{\bar{\imath},\bar{j}} = (-\tau_1 q + \tau_2 p)\, H'(y_1) + O(\exp(-\delta x_1)). \tag{6.7}$$

Similar asymptotic behaviors hold in other quadrants. Let us list them below for later purpose:

$$\begin{aligned}
\beta_{\bar{\imath},\bar{j}} &= (\tau_1 q + \tau_2 p)\, H'(y_2) + O(\exp(-\delta x_2)), && \text{in the second quadrant,}\\
\beta_{\bar{\imath},\bar{j}} &= (\tau_1 q - \tau_2 p)\, H'(y_3) + O(\exp(-\delta x_3)), && \text{in the third quadrant,}\\
\beta_{\bar{\imath},\bar{j}} &= (-\tau_1 q - \tau_2 p)\, H'(y_4) + O(\exp(-\delta x_4)), && \text{in the fourth quadrant.}
\end{aligned}$$

We also set $a_1 := -\tau_1 q + \tau_2 p$, $a_2 := \tau_1 q + \tau_2 p$, $a_3 := \tau_1 q - \tau_2 p$, $a_4 := -\tau_1 q - \tau_2 p$.

By the end-to-end construction (see the construction of the kernel ξ_k at the end of the proof of this lemma), there exists a solution γ_k solving

$$-\Delta\gamma_k - \gamma_k \cos u_k = 0, \tag{6.8}$$

such that for some constant $\delta > 0$, $|\gamma_k - \beta_{\bar{\imath},\bar{j}}| \le C\exp(-\delta k)$ in $B_k(z_{\bar{\imath},\bar{j}})$. The bound $\exp(-\delta k)$ essentially follows from the estimate (6.1). Recall that

$$-\Delta\phi_k - \phi_k \cos u_k = -\lambda_k^2 \phi_k. \tag{6.9}$$

If we denote the outward normal derivative with respect to the boundary of the ball $B_k := B_k(z_{\bar{\imath},\bar{j}})$ by ∂_ν, then from (6.8) and (6.9), we deduce

$$\lambda_k^2 = \frac{\int_{\partial B_k} (\gamma_k \partial_\nu \phi_k - \phi_k \partial_\nu \gamma_k)}{\int_{B_k} (\phi_k \gamma_k)}. \tag{6.10}$$

For $j = 1, \ldots, 4$, in the j-th quadrant, by (6.4) and (6.6),

$$\phi_k = \left[b_{k,j} \exp(-\lambda_k x_j) + m_{k,j} \exp(\lambda_k x_j)\right] H'(y_j) + \zeta_j(x, y), \tag{6.11}$$

where $b_{k,j}, m_{k,j}$ are constants depending on k and

$$|\zeta_j| \le C\exp(-\delta x_j), \quad \text{in the } j\text{-th quadrant.}$$

We emphasize that in the decomposition of the form (6.11), the constants $b_{k,j}, m_{k,j}$ may not be uniquely determined and may not be uniformly bounded with respect to k. However, we know that as $k \to +\infty$, around $z_{\bar{\imath},\bar{j}}$, $\phi_k \to \beta_{\bar{\imath},\bar{j}}$ and $\lambda_k \to 0$. This implies that as $k \to +\infty$,

$$b_{k,j} + m_{k,j} \to a_j, \quad \text{for } j = 1, \ldots, 4.$$

Recall that the minimal distance between points $k\omega_{i,j}$ is equal to $2k$. Using the asymptotic behavior of $\beta_{\bar{\imath},\bar{j}}$ and (6.11), we have

$$\begin{aligned}
&\int_{\partial B_k} (\gamma_k \partial_\nu \phi_k - \phi_k \partial_\nu \gamma_k)\\
&= \sum_{j=1}^{4} (a_j \lambda_k [-b_{k,j} \exp(-\lambda_k k) + m_{k,j} \exp(\lambda_k k)]) + O(\exp(-\delta k)).
\end{aligned} \tag{6.12}$$

On the other hand, still by (6.7) and (6.11), we have

(6.13)

$$\int_{B_k} (\phi_k \gamma_k) = \lambda_k^{-1} \sum_{j=1}^{4} \left(a_j [-b_{k,j} (\exp(-\lambda_k k) - 1) + m_{k,j} (\exp(\lambda_k k) - 1)] \right) + O(1).$$

To simplify the notation, let us set

$$M := \sum_{j=1}^{4} \left(a_j [-b_{k,j} \exp(-\lambda_k k) + m_{k,j} \exp(\lambda_k k)] \right), \quad \text{and}$$

$$N := \sum_{j=1}^{4} \left(a_j (b_{k,j} - m_{k,j}) \right). \tag{6.14}$$

Using these notations and (6.12), (6.13), we see from the identity (6.10) that

$$\lambda_k^2 = \frac{\lambda_k M + O(\exp(-\delta k))}{\lambda_k^{-1} M + \lambda_k^{-1} N + O(1)}.$$

This implies that

$$N = \lambda_k^{-1} O(\exp(-\delta k)) + o(1). \tag{6.15}$$

Claim: $\lambda_k k \to 0$ as $k \to +\infty$.

To prove this claim, we assume to the contrary that the claim is not true. Then we can find a subsequence, still denoted by λ_k, such that $\lambda_k \geq c_1 k^{-1}$ for some fixed positive constant c_1. Then by (6.15),

$$N \to 0, \quad \text{as } k \to +\infty. \tag{6.16}$$

Note that for each pair of indices (i_0, j_0), we can associate to it the corresponding quantity N, which satisfies (6.16). To make things more rigorous, let us introduce some notation.

For any pair of indices (i, j), we have the rescaled lines kL_i, kL_j introduced at the beginning of this section. They intersect at the point $k\omega_{i,j}$. We also designate a direction for each of these lines. We know that around the point $z_{i,j}$, we have put the four-end solution $g_{i,j}$ as a building block for the approximate solution for u_k. As $k \to +\infty$, $\phi_k(z - z_{i,j})$ tends to a kernel $\beta_{i,j}$ of the operator $-\Delta - \cos \tilde{g}_{i,j}$. Previous analysis tell us that along the four ends of $\tilde{g}_{i,j}$, we can associate the data $a_j, b_{k,j}, m_{k,j}$. To distinguish between different intersection points, we write those "a" part of the data as $a^*_{i,+,j}$ and $a^*_{i,-,j}$. More precisely, $a^*_{i,+,j}$ will be the "a" along the end of $\tilde{g}_{i,j}$ corresponding to the positive direction of kL_i, while $a^*_{i,-,j}$ will be the "a" along the end of $\tilde{g}_{i,j}$ corresponding to the negative direction of kL_i. Similarly, we have $b^*_{i,+,j}, m^*_{i,+,j}$ and $b^*_{i,-,j}, m^*_{i,-,j}$, which actually depend on k. We also point out that some of $a^*_{i,\pm,j}$ could be zero.

For each fixed $j = 1, \ldots, n$, we associate the following quantities to the line kL_j:

$$P_j := \sum_{i \neq j} \left[a^*_{j,+,i} (b^*_{j,+,i} - m^*_{j,+,i}) + a^*_{j,-,i} (b^*_{j,-,i} - m^*_{j,-,i}) \right],$$

$$Q_j := \sum_{i \neq j} \left[a^*_{i,+,j} (b^*_{i,+,j} - m^*_{i,+,j}) + a^*_{i,-,j} (b^*_{i,-,j} - m^*_{i,-,j}) \right].$$

Summing up the identities (6.16) for all pairs of indices (i, j), we find that as $k \to +\infty$,

$$\sum_{l=1}^{n} Q_l \to 0.$$

There are two possible cases.

Case 1. There exist a constant $\sigma > 0$, and an index j_0, both independent of k, such that $Q_{j_0} > \sigma$ for all k large.

In this case, summing up the identities (6.16) for all pairs of indices of the form (i, j_0), we find that

$$P_{j_0} \leq -\frac{\sigma}{2}, \quad \text{for } k \text{ large.} \tag{6.17}$$

We can relabel the indices such that $j_0 = n$, and the intersection points $\omega_{1,n}, \ldots, \omega_{n-1,n}$ are in the order consistent with the positive kL_n direction. Fix an index i and write the line segment connecting $k\omega_{i,n}$ with $k\omega_{i+1,n}$ as L^*. For the four-end solution $g_{i,n}$, the coordinate system adapted to its end corresponding to L^* will be written as $(\mathbf{x}_i, \mathbf{y}_i)$. For the four-end solution $g_{i+1,n}$, the coordinate system adapted to its end corresponding to L^* will be written as $(\mathbf{x}_{i+1}, \mathbf{y}_{i+1})$. As we have analyzed above, around L^*, the main order (the part parallel to H') of ϕ_k in the $(\mathbf{x}_i, \mathbf{y}_i)$-coordinate has the form

$$\left[b^*_{n,+,i} \exp\left(-\lambda_k \mathbf{x}_i\right) + m^*_{n,+,i} \exp\left(\lambda_k \mathbf{x}_i\right)\right] H'(\mathbf{y}_i);$$

while the main order of ϕ_k in $(\mathbf{x}_{i+1}, \mathbf{y}_{i+1})$-coordinate has the form

$$\left[b^*_{n,-,i+1} \exp\left(-\lambda_k \mathbf{x}_{i+1}\right) + m^*_{n,-,i+1} \exp\left(\lambda_k \mathbf{x}_{i+1}\right)\right] H'\left(\mathbf{y}_{i+1}\right).$$

Choose any point on L^* and let d_i be the sum of its $\mathbf{x}_i$ and $\mathbf{x}_{i+1}$ coordinates. Note that $d_i = O(k)$. Then we have the following relation:

$$b^*_{n,+,i} = m^*_{n,-,i+1} \exp\left(d_i \lambda_k\right). \tag{6.18}$$

Similarly,

$$b^*_{n,-,i+1} = m^*_{n,+,i} \exp\left(d_i \lambda_k\right).$$

It follows that

$$b^{*2}_{n,+,i} - m^{*2}_{n,+,i} + b^{*2}_{n,-,i+1} - m^{*2}_{n,-,i+1} = \left(m^{*2}_{n,+,i} + m^{*2}_{n,-,i+1}\right)\left(\exp(2d_i\lambda_k) - 1\right). \tag{6.19}$$

In view of the fact that $b^*_{n,+,i} + m^*_{n,+,i} = a^*_{n,+,i} + o(1)$, we obtain

$$\begin{aligned} a^*_{n,+,i}\left(b^*_{n,+,i} - m^*_{n,+,i}\right) &= b^{*2}_{n,+,i} - m^{*2}_{n,+,i} + o\left(\left|b^*_{n,+,i} - m^*_{n,+,i}\right|\right), \\ &= b^{*2}_{n,+,i} - m^{*2}_{n,+,i} + o(1)\left(1 + \left|m^*_{n,+,i}\right|\right), \end{aligned}$$

and

$$a^*_{n,-,i}\left(b^*_{n,-,i} - m^*_{n,-,i}\right) = b^{*2}_{n,-,i} - m^{*2}_{n,-,i} + o(1)\left(1 + \left|m^*_{n,-,i}\right|\right).$$

Here we remark that under the assumption that $\lambda_k \geq c_1 k^{-1}$, we can actually show that the $m^*_{i,\pm,j}$ are uniformly bounded with respect to k. But the proof below does not need this.

Now by (6.19), for the line kL_n, we have

$$\begin{aligned} P_n &= \sum_{i \neq n} \left[a^*_{n,+,i} \left(b^*_{n,+,i} - m^*_{n,+,i} \right) + a^*_{n,-,i} \left(b^*_{n,-,i} - m^*_{n,-,i} \right) \right] \\ &= \sum_{i=1}^{n-2} \left[\left(m^{*2}_{n,+,i} + m^{*2}_{n,-,i+1} \right) \left(\exp\left(2 d_i \lambda_k \right) - 1 \right) \right] \\ &\quad + \left(b^{*2}_{n,-,1} - m^{*2}_{n,-,1} \right) + \left(b^{*2}_{n,+,n-1} - m^{*2}_{n,+,n-1} \right) \\ &\quad + o(1) \sum_{i=1}^{n-1} \left(1 + \left| m^*_{n,+,i} \right| + \left| m^*_{n,-,i} \right| \right). \end{aligned} \tag{6.20}$$

Due to the fact that ϕ_k decays to zero at infinity, there holds $m^*_{n,-,1} = m^*_{n,+,n-1} = 0$. It follows that

$$b^{*2}_{n,-,1} - m^{*2}_{n,-,1} + b^{*2}_{n,+,n-1} - m^{*2}_{n,+,n-1} = a^{*2}_{n,-,1} + a^{*2}_{n,+,n-1} + o(1).$$

Therefore using the assumption that $\lambda_k \geq c_1 k^{-1}$, we get

$$\exp(2 d_i \lambda_k) - 1 \geq \exp(2 c_1 d_i k^{-1}) - 1 \geq c_2 > 0$$

for some fixed constant c_2 and thus $\liminf_{k \to +\infty} P_n \geq 0$. This contradicts with (6.17) and hence Case 1 cannot happen.

Case 2. For any index l, $Q_l \to 0$ as $k \to +\infty$.

In this case, we should have

$$\lim_{k \to +\infty} P_l = 0, \quad \text{for any fixed index } l. \tag{6.21}$$

On the other hand, we still have identities similar to the form (6.20), for any line kL_j. In view of the assumption that $\|\phi_k\|_{L^\infty} = 1$, we know that for at least one pair of indices (i_0, j_0), the constant $a_{j_0,+,i_0}$ is nonzero. Without loss of generality, we assume $j_0 = n$.

If $a^*_{n,-,1}$ is nonzero, by (6.20), we have $\liminf_{k \to +\infty} P_n > 0$, which contradicts with (6.21). If $a^*_{n,-,1} = 0$, then we consider $m^*_{n,+,1}$. There are two possible subcases.

Subcase 1. Up to a subsequence, $|m^*_{n,+,1}| \geq \alpha_0 > 0$, where α_0 is a constant independent of k.

In this subcase, still by (6.20), we have $\liminf_{k \to +\infty} P_n > 0$, which again contradicts with (6.21).

Subcase 2. $m^*_{n,+,1} \to 0$ as $k \to +\infty$.

In this subcase, using the fact that $a^*_{n,+,1} = a^*_{n,-,1} = 0$, we have $b^*_{n,+,1} \to 0$ as $k \to +\infty$. Hence $m^*_{n,-,2}$ also tends to 0, by (6.18). Now instead of $a^*_{n,-,1}$, we can consider $a^*_{n,-,2}$. If $a^*_{n,-,2}$ is nonzero, then we again get a contradiction by using (6.20).

This procedure can be repeated until we arrive at $a_{n,-,i_0}$ and get a contradiction. Hence Case 2 cannot happen. The Claim is then proved.

Let c be a fixed large constant. With the information on λ_k at hand, next we would like to prove: there exists a function ξ_k satisfying $\|\xi_k\|_{L^\infty} < +\infty$,

$$\|\xi_k - \phi_k\|_{L^\infty(B_{ck})} = o(1),$$

and

$$-\Delta\xi_k - \xi_k \cos u_k = 0. \tag{6.22}$$

The proof of this fact will be based on the end-to-end construction. Let us explain it in the sequel. More details about the end-to-end construction can be found in Section 3 of [39].

We recall that around each $z_{i,j}$, the sequence of functions $\phi_k(z - z_{i,j})$ converges to $\beta_{i,j}$, where $\beta_{i,j}$ is bounded and

$$-\Delta\beta_{i,j} - \beta_{i,j} \cos \tilde{g}_{i,j} = 0.$$

Up to a rotation of the coordinate system, we can choose positive direction $e_j = (e_{j,1}, e_{j,2})$ for each line kL_j such that $e_{j,1} > 0$. We also assume $|e_j| = 1$ and $e_{j,2} < e_{j+1,2}$ for all j. For each fixed index j, the line kL_j intersects with other $n-1$ lines. The one with the rightmost intersection point with kL_j will be denoted by kL_{ι_j}. The ends of u_k in the right half plane are asymptotic to the lines kL_j, $j = 1, \ldots, n$. For the functions $\beta_{\iota_j,j}$, recall that we have introduced the constants $a_{j,+,\iota_j}$.

Let $\varepsilon > 0$ be a small parameter. Let $kL_{j,\varepsilon}$ be the line obtained by parallel translation of kL_j in the direction orthogonal to e_j with a distance equal to $\varepsilon|a_{j,+,\iota_j}|$. If $a_{j,+,\iota_j}$ is positive, then $kL_{j,\varepsilon}$ is above kL_j, and if $a_{j,+,\iota_j}$ is negative, then $kL_{j,\varepsilon}$ will be below kL_j. By the end-to-end construction, there exists a solution $u_{k,\varepsilon}$ to the equation $-\Delta u_{k,\varepsilon} = \sin u_{k,\varepsilon}$, whose ends in the right half plane are asymptotic to the lines $kL_{j,\varepsilon}$, $j = 1, \ldots, n$. This construction relies on the fact that we can consecutively adjust the centers of the four-end solutions according to the new set of lines $kL_{j,\varepsilon}$, from right to left. Let us define

$$\xi_k := \lim_{\varepsilon \to 0} \frac{u_{k,\varepsilon} - u_k}{\varepsilon}.$$

Then ξ_k is the desired function. To see this, we first observe that by the construction, ξ_k satisfies (6.22) and has the same asymptotic behavior as $\beta_{\iota_j,j}$ along the end kL_j in the positive kL_j direction. Note that for any bounded kernel of the four-end solution, its asymptotic behavior (the part parallel to H') at infinity is determined by its asymptotic behavior along two of its ends. The estimate $\lambda_k = o(k^{-1})$ tells us that away from the centers of $g_{i,j}$, the projection of ϕ_k onto H' is not too far from a constant, indeed, its error is of order $o(1)$. We then deduce that $\|\phi_k - \xi_k\|_{L^\infty(B_{ck})} = o(1)$. It remains to prove that ξ_k is bounded. To show this, let us recall that u_k is equal to U_n with suitable parameters p_j, q_j, η_j^0. We then consider the solutions $U_{n,\varepsilon}$ with the same p_j, q_j as U_n, and with $\eta_{j,\varepsilon}^0$ being close to η_j^0, chosen in such a way that the ends of $U_{n,\varepsilon}$ in the right half plane is asymptotic to $kL_{j,\varepsilon}$. Then we define the function

$$\xi_k^* := \lim_{\varepsilon \to 0} \frac{U_{n,\varepsilon} - U_n}{\varepsilon}.$$

Since the ends of $U_{n,\varepsilon}$ in the left half plane are also parallel to kL_j, $j = 1, \ldots, n$, we see that $\|\xi_k^*\|_{L^\infty} < +\infty$. Now we consider the function $\Phi := \xi_k - \xi_k^*$. Then $\Phi(x, y) \to 0$, along the ends of u_k in the right half plane. Then by the proof of nondegeneracy in Section 4, $\Phi = 0$. The fact that $\Phi = 0$ can also be proved in the following way. We know that the dimension of the kernel of the operator $-\Delta - \cos u_k$ in the space of functions with at most linearly growing rate along each end is equal to $2n$, which follows from the nondegeneracy of U_n. On the other hand, differentiating U_n with respect to q_j yields a kernel linearly growing along kL_j, both in the positive and negative directions. These provides us with n-linearly independent unbounded kernels. We also observe that differentiating U_n with respect to η_j^0 yields a bounded kernel which does not decay to zero along kL_j, both in the positive and negative directions. Hence Φ has to be zero. We then conclude that ξ_k is bounded and hence is the desired function. Note that there is a delicate issue here. Namely we are not choosing ξ_k to be ξ_k^* directly, because at the beginning we do not have very precise asymptotic behavior of ξ_k^* and we cannot immediately infer that $\|\xi_k^* - \phi_k\|_{L^\infty} = o(1)$. This is why we use the end-to-end construction to get better asymptotic behavior of ξ_k.

Along each end, ϕ_k decays to zero. Let $(\mathbf{x}, \mathbf{y})$ be the coordinates adapted to this end. Then ϕ_k has the form

$$\phi_k = b_k \exp(-\lambda_k \mathbf{x})\, H'(\mathbf{y}) + O\left(\mathbf{d}(z, \cup z_{i,j})\right).$$

Along this same end,

$$\xi_k = a_k H'(\mathbf{y}) + O\left(\mathbf{d}(z, \cup z_{i,j})\right)$$

Moreover, using the properties of ξ_k, we have $b_k - a_k \to 0$. We also know that there exists at least one end such that the corresponding $|a_k|$ is bounded away from 0 uniformly with respect to k. We then compute that $\int_{\mathbb{R}^2}(\xi_k \phi_k) > 0$, which implies $\lambda_k = 0$. We remark that one can also use similar arguments as that of the proof of the claim above to conclude directly that $\lambda_k = 0$ (here one uses the fact that along each end, the m^* part of the function ϕ_k vanishes). In any case, this contradicts with $-\lambda_k^2 < 0$. Hence the lemma is proved. ∎

Lemma 6.4. *The Morse index of u_k is at most $n(n-1)/2$ for k large.*

Proof. Suppose to the contrary that there were $n(n-1)/2+1$ negative eigenvalues (counted with multiplicity), with corresponding eigenfunctions $\phi_{k,j}$, $j = 1, \ldots, n(n-1)/2+1$, normalized such that $\|\phi_{k,j}\|_{L^2(\mathbb{R}^2)} = 1$, and $\int_{\mathbb{R}^2}(\phi_{k,i}\phi_{k,j}) = 0$ for $i \neq j$.

For each index l and for each pair of indices (i_0, j_0), as $k \to +\infty$, the sequence $\varphi_k(\cdot) := \phi_{k,l}(\cdot - z_{i_0,j_0})$ converges, up to a subsequence, to a function φ_∞ satisfying

$$-\Delta\varphi_\infty - \varphi_\infty \cos \tilde{g}_{i_0,j_0} = \sigma_{i_0,j_0}\varphi_\infty,$$

where σ_{i_0,j_0} is the unique negative eigenvalue of the operator $-\Delta - \cos \tilde{g}_{i_0,j_0}$. Note that φ_∞ could be the trivial zero function. However, for at least one pair of indices, it will be nontrivial.

Let $\eta_{i,j}^*$ be the function introduced in Lemma 6.2. Let $\mathbf{d}(p, \cup z_{i,j})$ be the distance of a point p to the set of all points $z_{i,j}$, $i, j = 1, \ldots, n$, $i \neq j$. For each fixed index l, up to a subsequence, we can assume that for some constants $\alpha_{i,j,l}$, $i, j = 1, \ldots, n$, $i \neq j$, independent of k, and some $\delta > 0$,

$$\phi_{k,l}(z) = \sum_{i,j,i\neq j}(\alpha_{i,j,l}\,\eta_{i,j}^*) + \varpi_l(z)\exp\left(-\delta\,\mathbf{d}(z, \cup z_{i,j})\right),$$

where $\|\varpi_l\|_{L^\infty(\mathbb{R}^2)} \to 0$ as $k \to +\infty$.

Observe that there exist constants $c_s, s = 1, \ldots, n(n-1)/2 + 1$, at least one of them being nonzero, such that for each fixed pair of indices (i, j),

$$\sum_{s=1}^{n(n-1)/2+1} \alpha_{i,j,s}\, c_s = 0.$$

Hence

$$\sum_{s=1}^{n(n-1)/2+1} c_s\, \phi_{k,s} = \sum_{s=1}^{n(n-1)/2+1} \big(c_s\, \varpi_s(z) \exp(-\delta\, \mathbf{d}(z, \cup z_{i,j}))\big).$$

Since $\phi_{k,i}$ and $\phi_{k,j}$ are L^2-orthogonal to each other for $i \neq j$, the L^2 norm of the left-hand side is equal to $\big(\sum_{s=1}^{n(n-1)/2+1} c_s^2\big)^{1/2} > 0$; while the L^2 norm of the right-hand side tends to 0 as $k \to +\infty$. This is a contradiction. Hence the Morse index of u_k cannot be greater than $n(n-1)/2$ for k large.

We remark that from technical point of view, there is an alternative way to prove this lemma: first, one can perturb the function $\eta^*_{i,j}$ into a true eigenfunction $\hat{\eta}_{i,j}$ using the implicit function theorem; then one can show that any eigenfunction corresponding to a negative eigenvalue cannot be orthogonal to all these eigenfunctions $\hat{\eta}_{i,j}$. ∎

Proof of Proposition 6.1. We have proved that the Morse index of u_k equals $n(n-1)/2$ if k is large. Now observe that any $2n$-end solution U_n can be deformed to a solution of the above form, through a family of $2n$-end solutions. As we proved in Section 4, all the solutions in this family are L^∞-nondegenerate. Due to the continuous dependence of the eigenfunction upon this deformation, the Morse indices of all these solutions have to be same. This implies that the Morse index of any $2n$-end solutions is equal to $n(n-1)/2$. ∎

Proof of Theorem 1.3. Proposition 5.1 tells us that any $2n$-end solution belongs to the family U_n. All solutions in this family are L^∞-nondegenerate and this family has $2n$ free parameters. Hence the set $\mathcal{M}_{2n}$ of the $2n$-end solutions is a $2n$ dimensional manifold. Proposition 6.1 tells us that their Morse index is equal to $n(n-1)/2$. This finishes the proof of Theorem 1.3. ∎

Acknowledgements. Part of this work was finished while the first author was visiting the University of British Columbia in 2017. He thanks the institution for the financial support.

Funding. The first author is partially supported by the Fundamental Research Funds for the Central Universities (WK3470000014) and NSFC under grant 11971026. The research of J. Wei is partially supported by NSERC of Canada.

References

[1] Ablowitz, M. J., Kaup, D. J., Newell, A. C. and Segur, H.: Method for solving the sine-Gordon equation. *Phys. Rev. Lett.* **30** (1973), 1262–1264.

[2] Ambrosio, L. and Cabré, X.: Entire solutions of semilinear elliptic equations in $\mathbb{R}^3$ and a conjecture of De Giorgi. *J. Amer. Math. Soc.* **13** (2000), no. 4, 725–739.

[3] Borisov, A. B. and Kiseliev, V. V.: Inverse problem for an elliptic sine-Gordon equation with an asymptotic behaviour of the cnoidal-wave type. *Inverse problem* **5** (1989), no. 6, 959–982.

[4] Borisov, A. B., Tankeyev, A. P., Shagalov, A. G. and Bezmaternih, G. V.: Multivortex-like solutions of the sine-Gordon equation. *Phys. Lett. A* **111** (1985), no. 1-2, 15–18.

[5] Cabré, X.: Uniqueness and stability of saddle-shaped solutions to the Allen–Cahn equation. *J. Math. Pures Appl. (9)* **98** (2012), no. 3, 239–256.

[6] Cabré, X. and Terra, J.: Saddle-shaped solutions of bistable diffusion equations in all of $\mathbb{R}^{2m}$. *J. Eur. Math. Soc. (JEMS)* **11** (2009), no. 4, 819–843.

[7] Cabré, X. and Terra, J.: Qualitative properties of saddle-shaped solutions to bistable diffusion equations. *Comm. Partial Differential Equations* **35** (2010), no. 11, 1923–1957.

[8] Cheng, P.-J., Venakides, S. and Zhou, X.: Long-time asymptotics for the pure radiation solution of the sine-Gordon equation. *Comm. Partial Differential Equations* **24** (1999), no. 7-8, 1195–1262.

[9] Chodosh, O. and Mantoulidis, C.: Minimal surfaces and the Allen–Cahn equation on 3-manifolds: index, multiplicity, and curvature estimates. *Ann. of Math. (2)* **191** (2020), no. 1, 213–328.

[10] Cuevas-Maraver, J., Kevrekidis, P. G. and Williams, F. (eds.): *The sine-Gordon model and its applications. From pendula and Josephson junctions to gravity and high-energy physics*. Nonlinear Systems and Complexity 10, Springer, Cham, 2014.

[11] Del Pino, M., Kowalczyk, M. and Pacard, F.: Moduli space theory for the Allen–Cahn equation in the plane. *Trans. Amer. Math. Soc.* **365** (2013), no. 2, 721–766.

[12] Del Pino, M., Kowalczyk, M., Pacard, F. and Wei, J.: Multiple-end solutions to the Allen–Cahn equation in $\mathbb{R}^2$. *J. Funct. Anal.* **258** (2010), no. 2, 458–503.

[13] Del Pino, M., Kowalczyk, M. and Wei, J.: On De Giorgi's conjecture in dimension $N \geq 9$. *Ann. of Math. (2)* **174** (2011), no. 3, 1485–1569.

[14] Dodd, R. K. and Bulough, R. K.: Bäcklund transformations for the sine-Gordon equations. *Proc. Roy. Soc. London Ser. A.* **351** (1976), no. 1667, 499–523.

[15] Faddeev, L. D. and Takhtajan, L. A.: *Hamiltonian methods in the theory of solitons*. Classics in Mathematics, Springer, Berlin, 2007.

[16] Farina, A., Mari, L. and Valdinoci, E.: Splitting theorems, symmetry results and overdetermined problems for Riemannian manifolds. *Comm. Partial Differential Equations* **38** (2013), no. 10, 1818–1862.

[17] Farina, A. and Valdinoci, E.: 1D symmetry for solutions of semilinear and quasilinear elliptic equations. *Trans. Amer. Math. Soc.* **363** (2011), no. 2, 579–609.

[18] Farina, A. and Valdinoci, E.: 1D symmetry for semilinear PDEs from the limit interface of the solution. *Comm. Partial Differential Equations* **41** (2016), no. 4, 665–682.

[19] Fokas, A. S., Lenells, J. and Pelloni, B.: Boundary value problems for the elliptic sine-Gordon equation in a semi-strip. *J. Nonlinear Sci.* **23** (2013), no. 2, 241–282.

[20] Fokas, A. S. and Pelloni, B.: The Dirichlet-to-Neumann map for the elliptic sine-Gordon equation. *Nonlinearity* **25** (2012), no. 4, 1011–1031.

[21] Gaspar, P.: The second inner variation of energy and the Morse index of limit interfaces. *J. Geom. Anal.* **30** (2020), no. 1, 69–85.

[22] Gaspar, P. and Guaraco, M. A.: The Allen–Cahn equation on closed manifolds. *Calc. Var. Partial Differential Equations* **57** (2018), no. 4, paper no. 101, 42 pp.

[23] Ghoussoub, N. and Gui, C.: On a conjecture of De Giorgi and some related problems. *Math. Ann.* **311** (1998), no. 3, 481–491.

[24] Gibbon, J. D. and Zambotti, G.: The interaction of n-dimensional soliton wave fronts. *Nuovo Cimento B (11)* **28** (1975), no. 1, 1–17.

[25] Guaraco, M. A. M.: Min-max for phase transitions and the existence of embedded minimal hypersurfaces. *J. Differential Geom.* **108** (2018), no. 1, 91–133.

[26] Gui, C.: Hamiltonian identities for elliptic partial differential equations. *J. Funct. Anal.* **254** (2008), no. 4, 904–933.

[27] Gui, C.: Symmetry of some entire solutions to the Allen–Cahn equation in two dimensions. *J. Differential Equations* **252** (2012), no. 11, 5853–5874.

[28] Gutshabash, E. S. and Lipovskiĭ, V. D.: A boundary value problem for a two-dimensional elliptic sine-Gordon equation and its application to the theory of the stationary Josephson effect. *J. Math. Sci.* **68** (1994), no. 2, 197–201.

[29] Hirota, R.: Exact solution of the sine-Gordon equation for multiple collisions of solitons. *J. Phys. Soc. Japan* **33** (1972), no. 5, 1459–1463.

[30] Hirota, R.: Exact three-soliton solution of the two-dimensional sine-Gordon equation. *J. Phys. Soc. Japan* **35** (1973), no. 5, 1566.

[31] Hirota, R.: A new form of Bäcklund transformations and its relation to the inverse scattering problem. *Progr. Theoret. Phys.* **52** (1974), no. 5, 1498–1512.

[32] Hirota, R.: *The direct method in soliton theory*. Cambridge Tracts in Mathematics 155, Cambridge University Press, Cambridge, 2004.

[33] Hirota, R. and Satsuma, J.: A simple structure of superposition formula of the Bäcklund transformation. *J. Phys. Soc. Japan* **45** (1978), no. 5, 1741–1750.

[34] Hudak, O.: On vortex configurations in two-dimensional sine-Gordon systems with applications to phase transitions of the Kosterlitz–Thouless type and to Josephson junctions. *Phys. Lett. A* **89** (1982), no. 5, 245–248.

[35] Jerison, D. and Monneau, R.: Towards a counter-example to a conjecture of De Giorgi in high dimensions. *Ann. Mat. Pura Appl. (4)* **183** (2004), no. 4, 439–467.

[36] Kowalczyk, M. and Liu, Y.: Nondegeneracy of the saddle solution of the Allen–Cahn equation. *Proc. Amer. Math. Soc.* **139** (2011), no. 12, 4319–4329.

[37] Kowalczyk, M., Liu, Y. and Pacard, F.: The space of 4-ended solutions to the Allen–Cahn equation in the plane. *Ann. Inst. H. Poincaré Anal. Non Lineaire* **29** (2012), no. 5, 761–781.

[38] Kowalczyk, M., Liu, Y. and Pacard, F.: The classification of four-end solutions to the Allen–Cahn equation on the plane. *Anal. PDE* **6** (2013), no. 7, 1675–1718.

[39] Kowalczyk, M., Liu, Y., Pacard, F. and Wei, J.: End-to-end construction for the Allen–Cahn equation in the plane. *Calc. Var. Partial Differential Equations* **52** (2015), no. 1-2, 281–302.

[40] Lamb, G. L. Jr.: Analytical description of ultrashort optical pulse propagation in a resonant medium. *Rev. Modern Phys.* **43** (1971), no. 2, 99–124.

[41] Leibbrandt, G.: Exact solutions of the elliptic sine equation in two space dimensions with application to the Josephson effect. *Physical Review B* **15** (1977), no. 7, 3353–3361.

[42] Liu, Y., Wang, K. and Wei, J.: Global minimizers of the Allen–Cahn equation in dimension $n \geq 8$. *J. Math Pures Appl. (9)* **108** (2017), no. 6, 818–840.

[43] Mantoulidis, C.: A note on the Morse index of $2k$-ended phase transitions in $\mathbb{R}^2$. Preprint 2017, arXiv: 1705.07580.

[44] Muñoz, C. and Palacios, J. M.: Nonlinear stability of 2-solitons of the Sine-Gordon equation in the energy space. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **36** (2019), no. 4, 977–1034.

[45] Nakamura, A.: Relation between certain quasivortex solutions and solitons of the sine-Gordon equation and other nonlinear equations. *J. Phys. Soc. Japan* **52** (1983), no. 6, 1918–1920.

[46] Nimmo, J. J. C. and Freeman, N. C.: The use of Bäcklund transformations in obtaining N-soliton solutions in Wronskian form. *J. Phys. A.: Math Gen.* **17** (1984), 1415.

[47] Novokshenov, V. Y. and Shagalov, A. G.: Bound states of the elliptic sine-Gordon equation. *Phys. D* **106** (1997), no. 1-2, 81–94.

[48] Pelloni, B.: Spectral analysis of the elliptic sine-Gordon equation in the quarter plane. (Russian). *Teoret. Mat. Fiz.* **160** (2009), no. 1, 189–201. Translation in *Theoret. and Math. Phys.* **160** (2009), no. 1, 1031–1041.

[49] Pelloni, B. and Pinotsis, D. A.: The elliptic sine-Gordon equation in a half plane. *Nonlinearity* **23** (2010), no. 1, 77–88.

[50] Rogers, C. and Schief, W. K.: *Bäcklund and Darboux transformations. Geometry and modern applications in soliton theory*. Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2002.

[51] Rogers, R. and Shadwick, W. F.: *Bäcklund transformations and their applications*. Mathematics in Science and Engineering 161, Academic Press, New York-London, 1982.

[52] Savin, O.: Regularity of flat level sets in phase transitions. *Ann. of Math. (2)* **169** (2009), no. 1, 41–78.

[53] Takeno, S.: Multi-(resonant-soliton)-soliton solutions and vortex-like solutions to two- and three-dimensional sine-Gordon equations. *Progr. Theoret. Phys.* **68** (1982), no. 3, 992–995.

[54] Toland, J. F.: The Peierls–Nabarro and Benjamin–Ono equations. *J. Funct. Anal.* **145** (1997), no. 1, 136–150.

[55] Wang, K. and Wei, J.: Finite Morse index implies finite ends. *Comm. Pure Appl. Math.* **72** (2019), no. 5, 1044–1119.

[56] Wazwaz, A.-M.: N-soliton solutions for the sine-Gordon equation of different dimensions. *J. Appl. Math. Inform.* **30** (2012), no. 5-6, 925–934.

Received November 13, 2019; revised January 16, 2021. Published online August 3, 2021.

Yong Liu
Department of Mathematics, University of Science and Technology of China, 230026 Hefei, China; yliumath@ustc.edu.cn

Juncheng Wei
Department of Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z2, Canada; jcwei@math.ubc.ca