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# On a capacity strong type inequality and related capacity estimates

Keng Hao Ooi and Nguyen Cong Phuc

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**Abstract.** We establish a Maz'ya type capacity inequality which resolves a special case of a conjecture by David R. Adams. As a consequence, we obtain several equivalent norms for Choquet integrals associated to Bessel or Riesz capacities. This enables us to obtain bounds for the Hardy–Littlewood maximal function in a sub-linear setting.

## 1. Introduction

Let  $\alpha$  be a real number and let  $s > 1$ . We define the space of Bessel potentials  $H^{\alpha,s} = H^{\alpha,s}(\mathbb{R}^n)$ ,  $n \geq 1$ , as the completion of  $C_c^\infty(\mathbb{R}^n)$  with respect to the norm

$$\|u\|_{H^{\alpha,s}} = \|\mathcal{F}^{-1}[(1 + |\xi|^2)^{\alpha/2} \mathcal{F}(u)]\|_{L^s(\mathbb{R}^n)},$$

where  $\mathcal{F}$  is the Fourier transform in  $\mathbb{R}^n$ . In the case  $\alpha > 0$ , it follows (see, e.g., [8]) that a function  $u$  belongs to  $H^{\alpha,s}$  if and only if

$$u = G_\alpha * f$$

for some  $f \in L^s(\mathbb{R}^n)$ , and moreover  $\|u\|_{H^{\alpha,s}} = \|f\|_{L^s(\mathbb{R}^n)}$ . Here,  $G_\alpha$  is the Bessel kernel of order  $\alpha$  defined by  $G_\alpha(x) := \mathcal{F}^{-1}[(1 + |\xi|^2)^{-\alpha/2}](x)$ .

Recall that the Bessel capacity associated to the Bessel potential space  $H^{\alpha,s}$  is defined for any set  $E \subset \mathbb{R}^n$  by

$$\text{Cap}_{\alpha,s}(E) := \inf \{ \|f\|_{L^s(\mathbb{R}^n)}^s : f \geq 0, G_\alpha * f \geq 1 \text{ on } E \}.$$

A function  $f: \mathbb{R}^n \rightarrow [-\infty, +\infty]$  is said to be defined quasieverywhere (q.e.) if it is defined at every point of  $\mathbb{R}^n$  except for only a set of zero capacity  $\text{Cap}_{\alpha,s}$ . The notion of Choquet integral associated to Bessel capacities will be important in this work. For a q.e. defined function  $w: \mathbb{R}^n \rightarrow [0, \infty]$ , the Choquet integrals of  $w$  is defined by

$$\int_{\mathbb{R}^n} w \, d\text{Cap}_{\alpha,s} := \int_0^\infty \text{Cap}_{\alpha,s}(\{x \in \mathbb{R}^n : w(x) > t\}) \, dt.$$

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One of the fundamental results of potential theory is the following Maz'ya's capacity inequality, originally obtained by Maz'ya, and subsequently extended by Adams, Dahlberg, and Hansson:

$$\int_{\mathbb{R}^n} (G_\alpha * f)^s d\text{Cap}_{\alpha,s} \leq A \int_{\mathbb{R}^n} f^s dx,$$

which holds for any nonnegative Lebesgue measurable function  $f$ . See, e.g., [3], [9] and [7], and in particular, see Section 2.3.1 and the historical comments in Section 2.3.13 of [7]. This kind of capacity inequalities and their many applications are discussed in Chapters 2, 3 and 11 of [7].

In [2], Adams conjectured (in the context of Riesz capacities and Riesz potentials) that another capacity strong type inequality

$$(1.1) \quad \int_{\mathbb{R}^n} (G_\alpha * f) d\text{Cap}_{\alpha,s} \leq A \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx$$

holds for any nonnegative Lebesgue measurable function  $f$  (see equation (3.11) in [2]). (The integral  $\int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx$  is understood as  $\infty$  whenever  $f = \infty$  on a set of positive Lebesgue measure. In the case  $f \equiv 0$ , it is understood as 0). Moreover, he essentially showed for the corresponding Riesz capacities and potentials that this is true provided  $\alpha$  is an integer in  $(0, n)$  (see page 23 in [2]). However, we observed that his argument does not appear to work for Bessel capacities and Bessel potentials as in (1.1) even with integers  $\alpha \in (0, n)$ .

One of the main purposes of this note is to verify (1.1) for any real  $\alpha > 0$ .

**Theorem 1.1.** *Let  $\alpha > 0$  and  $s > 1$  be such that  $\alpha s \leq n$ . There exists a constant  $A > 0$  such that (1.1) holds for any nonnegative Lebesgue measurable function  $f$ .*

Our proof of (1.1) is also applicable to the setting of Riesz capacities and potentials, and thereby extends the above mentioned results of [2] to all real  $\alpha \in (0, n)$ .

Our approach to (1.1) is based mainly in our recent work [11] in which predual spaces to a Sobolev multiplier type space were considered. For  $\alpha > 0$ ,  $s > 1$ , and  $p > 1$ , let  $M_p^{\alpha,s} = M_p^{\alpha,s}(\mathbb{R}^n)$  be the Banach space of functions  $f \in L_{\text{loc}}^p(\mathbb{R}^n)$  such that the trace inequality

$$(1.2) \quad \left( \int_{\mathbb{R}^n} (G_\alpha * h)^s |f|^p dx \right)^{1/p} \leq A \|h\|_{L^s(\mathbb{R}^n)}^{s/p}$$

holds for all nonnegative  $h \in L^s(\mathbb{R}^n)$ . A norm of a function  $f \in M_p^{\alpha,s}$  can be defined as

$$(1.3) \quad \|f\|_{M_p^{\alpha,s}} := \sup_K \left( \frac{\int_K |f(x)|^p dx}{\text{Cap}_{\alpha,s}(K)} \right)^{1/p},$$

where the supremum is taken over all compact sets  $K \subset \mathbb{R}^n$  with non-zero capacity. Note that the right-hand side of (1.3) is known to be equivalent to the least possible constant  $A$  in (1.2) (see [3, 9]).

In [11], we showed that a predual of  $M_p^{\alpha,s}$  is its Köthe dual space  $(M_p^{\alpha,s})'$  defined by

$$(M_p^{\alpha,s})' = \left\{ \text{measurable functions } f : \sup \int |fg| dx < +\infty \right\},$$

where the supremum is taken over all functions  $g$  in the unit ball of  $M_p^{\alpha,s}$ . The norm of  $f \in (M_p^{\alpha,s})'$  is defined as the above supremum. Thus we have

$$[(M_p^{\alpha,s})']^* = M_p^{\alpha,s},$$

with equality of norms. Various characterizations of  $(M_p^{\alpha,s})'$  can be found in [11]. For our purpose here the case  $p = s' = s/(s-1)$  is of special interest. In particular, as mentioned in Remark 2.10 in [11], it follows from [6, 10] that the space  $M_{s'}^{\alpha,s}$  is an intrinsic space associated to the nonlinear integral equation

$$u = G_\alpha * (u^{s'}) + f \quad \text{a.e.}$$

Another important observation in [11] is the following equivalence:

$$(1.4) \quad \int_{\mathbb{R}^n} |u| d\text{Cap}_{\alpha,s} \simeq \gamma_{\alpha,s}(u),$$

which holds for all q.e. defined functions  $u$  in  $\mathbb{R}^n$ . Here the functional  $\gamma_{\alpha,s}(\cdot)$  is defined for each q.e. defined function  $u$  by

$$\gamma_{\alpha,s}(u) := \inf \left\{ \int f^s dx : 0 \leq f \in L^s(\mathbb{R}^n) \text{ and } G_\alpha * f \geq |u|^{1/s} \text{ q.e.} \right\}.$$

Note that  $\gamma_{\alpha,s}(tu) = |t|\gamma_{\alpha,s}(u)$  for all  $t \in \mathbb{R}$  and moreover  $\gamma_{\alpha,s}(u_1 + u_2) \leq \gamma_{\alpha,s}(u_1) + \gamma_{\alpha,s}(u_2)$  (see [11]). On the other hand, the Choquet integral  $\int_{\mathbb{R}^n} |\cdot| d\text{Cap}_{\alpha,s}$  is known to be subadditive only for  $s = 2$  and  $0 < \alpha \leq 1$ . In particular, the set of all q.e. defined functions  $u$  in  $\mathbb{R}^n$  such that  $\int_{\mathbb{R}^n} |u| d\text{Cap}_{\alpha,s} < +\infty$  is a normable space. An argument as in the proof of Proposition 2.3 in [11] can be used to show that this space is complete.

As a consequence of (1.4) and the proof of Theorem 1.1, in this paper we obtain two other characterizations for the Choquet integral. For a q.e. defined function  $u$  in  $\mathbb{R}^n$ , we denote by  $\lambda_{\alpha,s}(u)$  and  $\beta_{\alpha,s}$ ,  $\alpha > 0$ ,  $s > 1$ , the following quantities:

$$\lambda_{\alpha,s}(u) := \inf \left\{ \|f\|_{(M_{s'}^{\alpha,s})'} : 0 \leq f \in (M_{s'}^{\alpha,s})' \text{ and } G_\alpha * f \geq |u| \text{ q.e.} \right\}$$

and

$$\beta_{\alpha,s}(u) := \inf \left\{ \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx : f \geq 0, G_\alpha * f \geq |u| \text{ q.e.} \right\}.$$

**Theorem 1.2.** *Let  $\alpha > 0$  and  $s > 1$  be such that  $\alpha s \leq n$ . For any q.e. defined function  $u$  in  $\mathbb{R}^n$  it holds that*

$$(1.5) \quad \int_{\mathbb{R}^n} |u| d\text{Cap}_{\alpha,s} \simeq \lambda_{\alpha,s}(u) \simeq \beta_{\alpha,s}(u).$$

*In particular, we have*

$$\text{Cap}_{\alpha,s}(E) \simeq \lambda_{\alpha,s}(\chi_E) \simeq \beta_{\alpha,s}(\chi_E)$$

*for any set  $E \subset \mathbb{R}^n$ .*

To discuss a consequence of Theorem 1.2, we now recall that the (center) local Hardy–Littlewood maximal function is defined for each  $f \in L^1_{\text{loc}}(\mathbb{R}^n)$  by

$$\mathbf{M}^{\text{loc}} f(x) = \sup_{0 < r \leq 1} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)| dy.$$

for every  $x \in \mathbb{R}^n$ .

**Theorem 1.3.** *Let  $\alpha > 0$  and  $s > 1$  be such that  $\alpha s \leq n$ . For any  $q > (n - \alpha)/n$  and any measurable and q.e. defined function  $f$ , we have*

$$\int_{\mathbb{R}^n} (\mathbf{M}^{\text{loc}} f)^q d\text{Cap}_{\alpha,s} \leq A(n, \alpha, s, q) \int_{\mathbb{R}^n} |f|^q d\text{Cap}_{\alpha,s}.$$

An interesting aspect of Theorem 1.3 is that the power  $q$  is allowed to be strictly less than 1. Moreover, here we do not assume any continuity assumption on  $f$ . See [1], Theorem 7.5 in [4], and [12] for some related results.

Finally, we remark that Theorems 1.1, 1.2, and 1.3 also hold in the homogeneous setting provided  $\alpha \in (0, n)$ ,  $s > 1$ , and Bessel potentials and capacities are replaced by the corresponding Riesz potentials and capacities. Moreover, in the homogeneous setting the local Hardy–Littlewood maximal function  $\mathbf{M}^{\text{loc}}$  can be replaced by the larger standard Hardy–Littlewood maximal function.

Recall that the Riesz kernel  $I_\alpha$ ,  $\alpha \in (0, n)$ , is defined as the inverse Fourier transform of  $|\xi|^{-\alpha}$  (in the distributional sense), and explicitly we have  $I_\alpha(x) = \gamma(n, \alpha)|x|^{\alpha-n}$ , where  $\gamma(n, \alpha) = \Gamma(\frac{n-\alpha}{2})/[\pi^{n/2} 2^\alpha \Gamma(\alpha/2)]$ . The Riesz potential of a nonnegative measure  $\mu$  is defined by the convolution  $I_\alpha * \mu$ . For  $\alpha \in (0, n)$  and  $s > 1$ , the Riesz capacity  $\text{cap}_{\alpha,s}$  is defined for each set  $E \subset \mathbb{R}^n$  by

$$\text{cap}_{\alpha,s}(E) := \inf \left\{ \|f\|_{L^s(\mathbb{R}^n)}^s : f \geq 0, I_\alpha * f \geq 1 \text{ on } E \right\}.$$

This capacity is the capacity associated to the homogeneous Sobolev space  $\dot{H}^{\alpha,s}$  (see Section 9 in [11]).

**Notation.** The characteristic function of a set  $E \subset \mathbb{R}^n$  is denoted by  $\chi_E$ . For two quantities  $A$  and  $B$ , we write  $A \simeq B$  to mean that there exist positive constants  $c_1$  and  $c_2$  such that  $c_1 A \leq B \leq c_2 A$ .

## 2. Proof of Theorem 1.1

*Proof of Theorem 1.1.* Let  $L^1(C)$  denote the space of quasicontinuous function  $f$  in  $\mathbb{R}^n$  such that

$$\|f\|_{L^1(C)} := \int_{\mathbb{R}^n} |f| d\text{Cap}_{\alpha,s} < +\infty.$$

Recall a function  $f$  is said to be quasicontinuous (with respect to  $\text{Cap}_{\alpha,s}$ ) if for any  $\epsilon > 0$  there exists an open set  $O$  such that  $\text{Cap}_{\alpha,s}(O) < \epsilon$  and  $f$  is continuous in  $O^c := \mathbb{R}^n \setminus O$ . It is known that the dual of  $L^1(C)$  can be identify with the space  $\mathfrak{M}^{\alpha,s} = \mathfrak{M}^{\alpha,s}(\mathbb{R}^n)$  which

consists of locally finite signed measures  $\mu$  in  $\mathbb{R}^n$  such that the norm  $\|\mu\|_{\mathfrak{M}^{\alpha,s}} < +\infty$  (see Theorem 2.4 in [11]). Here we define

$$\|\mu\|_{\mathfrak{M}^{\alpha,s}} := \sup_K \frac{|\mu|(K)}{\text{Cap}_{\alpha,s}(K)},$$

where the supremum is taken over all compact sets  $K \subset \mathbb{R}^n$  such that  $\text{Cap}_{\alpha,s}(K) \neq 0$ .

In view of (1.4),  $L^1(C)$  is normable and thus it follows from the Hahn–Banach theorem that for any  $u \in L^1(C)$  we have

$$(2.1) \quad \|u\|_{L^1(C)} \simeq \sup \left\{ \left| \int u d\mu \right| : \|\mu\|_{\mathfrak{M}^{\alpha,s}} \leq 1 \right\}.$$

Let  $f$  be a nonnegative measurable and bounded function with compact support. Applying (2.1) with  $u = G_\alpha * f$ , we have

$$\begin{aligned} \int_{\mathbb{R}^n} G_\alpha * f d\text{Cap}_{\alpha,s} &\leq A \sup_{\|\mu\|_{\mathfrak{M}^{\alpha,s}} \leq 1} \int G_\alpha * f d|\mu| = A \sup_{\|\mu\|_{\mathfrak{M}^{\alpha,s}} \leq 1} \int (G_\alpha * |\mu|) f dx \\ &\leq A \|f\|_{(M_{s'}^{\alpha,s})'} \sup_{\|\mu\|_{\mathfrak{M}^{\alpha,s}} \leq 1} \|G_\alpha * |\mu|\|_{M_{s'}^{\alpha,s}} \leq A \|f\|_{(M_{s'}^{\alpha,s})'}, \end{aligned}$$

where the last inequality follows from Theorem 1.2 in [10]. By density (see Remark 3.3 in [11]), we see that the inequality

$$(2.2) \quad \int_{\mathbb{R}^n} G_\alpha * f d\text{Cap}_{\alpha,s} \leq A \|f\|_{(M_{s'}^{\alpha,s})'}$$

holds for any nonnegative function  $f \in (M_{s'}^{\alpha,s})'$ .

In proving (1.1) we may assume that  $\int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx < +\infty$  and hence  $f$  is finite a.e. by our convention. In this case we must have that  $f \in (M_{s'}^{\alpha,s})'$ . Indeed, for any  $g \in M_{s'}^{\alpha,s}$  such that  $\|g\|_{M_{s'}^{\alpha,s}} \leq 1$ , by Remark 2.10 in [11] and [6], there exists a nonnegative function  $u \in L_{\text{loc}}^{s'}(\mathbb{R}^n)$  such that

$$u = G_\alpha * (u^{s'}) + \frac{|g|}{M} \quad \text{a.e.}$$

for a constant  $M > 0$  independent of  $g$  and  $u$ . Thus, as in [5] (see also [6]), we have

$$\begin{aligned} (2.3) \quad \int_{\mathbb{R}^n} f|g| dx &= M \int_{\mathbb{R}^n} f(u - G_\alpha * (u^{s'})) dx = M \int_{\mathbb{R}^n} (fu - u^{s'} G_\alpha * f) dx \\ &= M \int_{\mathbb{R}^n} G_\alpha * f \left( u \frac{f}{G_\alpha * f} - u^{s'} \right) dx \\ &\leq Ms^{-s} (s-1)^{s-1} \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx, \end{aligned}$$

where we used the Young inequality  $ab - a^{s'}/s' \leq b^s/s$ ,  $a, b \geq 0$ , in the last inequality. Thus taking the supremum over  $g \in M_{s'}^{\alpha,s}$  such that  $\|g\|_{M_{s'}^{\alpha,s}} \leq 1$  in (2.3), we find

$$(2.4) \quad \|f\|_{(M_{s'}^{\alpha,s})'} \leq A \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx < +\infty.$$

Finally, combining (2.2) with (2.4) we obtain (1.1) as desired.  $\blacksquare$

**Remark 2.1.** We remark that (1.1) and (2.2) are indeed equivalent. On one hand, the proof above shows that (2.2) implies (1.1). On the other hand, (1.1) implies that

$$\int_{\mathbb{R}^n} G_\alpha * f \, d\text{Cap}_{\alpha,s} \leq A \|f\|_{KV}$$

for any nonnegative measurable function  $f$ . Here we define

$$\|f\|_{KV} := \inf \left\{ \int_{\mathbb{R}^n} h^s (G_\alpha * h)^{1-s} \, dx : h \geq |f| \text{ a.e.} \right\}.$$

( $\|f\|_{KV}$  is understood as  $\infty$  if there is no measurable function  $h$  such that  $h \geq |f|$  a.e. and  $\int_{\mathbb{R}^n} h^s (G_\alpha * h)^{1-s} \, dx < +\infty$ .) As we observe in Remark 2.10 in [11], the two-sided bound  $\|f\|_{(M_{s'}^{\alpha,s})} \simeq \|f\|_{KV}$  follows from [6, 10]. Thus (1.1) implies (2.2).

### 3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we first prove the following ‘‘integration by parts’’ lemma.

**Lemma 3.1.** *Let  $\alpha > 0$  and  $s > 1$  be such that  $\alpha s \leq n$ . Suppose that  $\mu$  is a nonnegative measure such that the diameter of  $\text{supp}(\mu)$  is less than 1. Then there is a constant  $A = A(n, \alpha, s) > 0$  such that, for  $f = (G_\alpha * \mu)^{s'-1}$ , we have*

$$(G_\alpha * f)^s \leq A G_\alpha * [f(G_\alpha * f)^{s-1}]$$

pointwise everywhere in  $\mathbb{R}^n$ .

**Remark 3.2.** For Riesz potentials, this lemma has been established for all  $f \geq 0$  in [14] (see also [6, 13]). In our setting, which deals with Bessel potentials, it is necessary to require  $\mu$  to have compact support.

*Proof of Lemma 3.1.* Without loss of generality, we may assume that  $\text{supp}(\mu) \subset B_{1/2}(0)$ . With  $f = (G_\alpha * \mu)^{s'-1}$ , we write  $f = f_1 + f_2$ , where

$$f_1 = f \chi_{B_3(0)} \quad \text{and} \quad f_2 = f \chi_{B_3(0)^c} \quad (\text{with } B_3(0)^c = \mathbb{R}^n \setminus B_3(0)).$$

Then

$$(3.1) \quad (G_\alpha * f)^s \leq A[(G_\alpha * f_1)^s + (G_\alpha * f_2)^s].$$

We shall use the following pointwise two-sided estimates for  $G_\alpha$  (see, e.g., Section 1.2.4 in [3]):

$$(3.2) \quad G_\alpha(x) \simeq |x|^{\alpha-n}, \quad \forall |x| \leq 15, (0 < \alpha < n).$$

and

$$(3.3) \quad G_\alpha(x) \simeq G_\alpha(x+y), \quad \forall |x| \geq 3, |y| \leq 1, (\alpha > 0).$$

We mention that (3.3) follows from the asymptotic behavior  $G_\alpha$  near infinity that can be found, e.g., in equation (1.2.24) in [3].

We now write

$$[G_\alpha * f_1(x)]^s = \int_{|y| \leq 3} G_\alpha(x-y) f(y) \left[ \int_{|z| \leq 3} G_\alpha(x-z) f(z) dz \right]^{s-1} dy.$$

Thus if  $|x| \geq 10$ , then  $|x-z| \geq 7 \geq |y-z|$ , which yields that

$$G_\alpha(x-z) \leq G_\alpha(y-z).$$

Therefore, we get

$$[G_\alpha * f_1(x)]^s \leq G_\alpha * [f(G_\alpha * f)^{s-1}](x)$$

in the case  $|x| \geq 10$ .

On the other hand, if  $|x| < 10$ , then for  $|y| \leq 3$  by (3.2) we have

$$G_\alpha(x-y) \simeq |x-y|^{\alpha-n}.$$

Thus applying Lemma 2.1 in [13] we obtain

$$[G_\alpha * f_1(x)]^s \leq A G_\alpha * [f_1(G_\alpha * f_1)^{s-1}](x) \leq A G_\alpha * [f(G_\alpha * f)^{s-1}](x)$$

in the case  $|x| < 10$ .

Combining these two estimates we get that

$$(3.4) \quad [G_\alpha * f_1(x)]^s \leq A G_\alpha * [f(G_\alpha * f)^{s-1}](x), \quad \forall x \in \mathbb{R}^n.$$

To estimate  $[G_\alpha * f_2(x)]^s$  we first observe the following bound:

$$(3.5) \quad f_2(x) \leq A G_\alpha * f(x), \quad \forall x \in \mathbb{R}^n.$$

Inequality (3.5) is trivial when  $|x| < 3$ . On the other hand, for  $|x| \geq 3$ , we have by (3.3),

$$(f_2(x))^{s-1} = \int_{|y| < 1/2} G_\alpha(x-y) d\mu(y) \leq A \int_{|y| < 1/2} G_\alpha(x) d\mu(y) = A \|\mu\| G_\alpha(x).$$

Note that for  $|y-x| < 1/2$  and  $|x| \geq 3$ , by (3.3) we have

$$f(y)^{s-1} = \int_{|z| < 1/2} G_\alpha(y-z) d\mu(z) \geq c_0 G_\alpha(x) \|\mu\|,$$

and so, for  $|x| \geq 3$ ,

$$\begin{aligned} G_\alpha * f(x) &\geq \int_{|y-x| < 1/2} G_\alpha(x-y) f(y) dy \\ &\geq \int_{|y-x| < 1/2} G_\alpha(x-y) (c_0 G_\alpha(x) \|\mu\|)^{s-1} dy \\ &\geq c (\|\mu\| G_\alpha(x))^{s-1} \geq c_1 f_2(x). \end{aligned}$$

Thus (3.5) is verified. Now by Hölder's inequality and (3.5) we have

$$(3.6) \quad [G_\alpha * f_2]^s \leq A G_\alpha * (f_2^s) \leq A_1 G_\alpha * [f(G_\alpha * f)^{s-1}].$$

At this point, combining (3.1), (3.4), and (3.6), we obtain the lemma. ■

We are now ready to prove Theorem 1.5.

*Proof of Theorem 1.5.* Let  $u$  be a q.e. defined function in  $\mathbb{R}^n$ . Suppose that  $f$  is a non-negative measurable function such that  $G_\alpha * f \geq |u|$  quasi-everywhere. Then by (2.2) and (2.4) it follows that

$$\int_{\mathbb{R}^n} |u| d\text{Cap}_{\alpha,s} \leq \int_{\mathbb{R}^n} G_\alpha * f d\text{Cap}_{\alpha,s} \leq A_1 \|f\|_{(M_s^{\alpha,s})'} \leq A_2 \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx.$$

Now taking the infimum over such  $f$  we arrive at

$$\int_{\mathbb{R}^n} |u| d\text{Cap}_{\alpha,s} \lesssim \lambda_{\alpha,s}(u) \lesssim \beta_{\alpha,s}(u).$$

Thus to complete the proof, it is left to show that

$$(3.7) \quad \beta_{\alpha,s}(u) \lesssim \int_{\mathbb{R}^n} |u| d\text{Cap}_{\alpha,s}.$$

To this end, we first show (3.7) for  $u = \chi_E$ , where  $E$  is any set such that  $\text{Cap}_{\alpha,s}(E) > 0$  and the diameter of  $E$  is less than 1. By Theorems 2.5.6 and 2.6.3 in [3] one can find a nonnegative measure  $\mu = \mu^E$  with  $\text{supp}(\mu) \subset \bar{E}$  (called capacity measure for  $E$ ) such that the function  $V^E = G_\alpha * ((G_\alpha * \mu)^{s'-1})$  satisfies the following properties:

$$\mu^E(\bar{E}) = \text{Cap}_{\alpha,s}(E) = \int_{\mathbb{R}^n} V^E d\mu^E = \int_{\mathbb{R}^n} (G_\alpha * \mu^E)^{s'} dx,$$

and

$$V^E \geq 1 \quad \text{quasi-everywhere on } E.$$

Let  $f = (G_\alpha * \mu)^{s'-1}$ . By Lemma 3.1, we have

$$\chi_E \leq (V^E)^s = (G_\alpha * f)^s \leq A G_\alpha * [f(G_\alpha * f)^{s-1}] \quad \text{q.e.}$$

Thus,

$$\begin{aligned} \beta_{\alpha,s}(\chi_E) &\leq A \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{(s-1)s} \{G_\alpha * [f(G_\alpha * f)^{s-1}]\}^{1-s} dx \\ &\leq A \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{(s-1)s} (G_\alpha * f)^{(1-s)s} dx \\ &= A \int_{\mathbb{R}^n} f^s dx = A \int_{\mathbb{R}^n} (G_\alpha * \mu)^{s'} dx = A \text{Cap}_{\alpha,s}(E), \end{aligned}$$

as desired.

We now let  $\{\mathcal{B}^j\}_{j \geq 0}$  be a covering of  $\mathbb{R}^n$  by open balls with unit diameter. This covering is chosen in such a way that it has a finite multiplicity depending only on  $n$ . We shall use the following quasi-additivity of  $\text{Cap}_{\alpha,s}$ :

$$(3.8) \quad \sum_{j \geq 0} \text{Cap}_{\alpha,s}(E \cap \mathcal{B}^j) \leq M \text{Cap}_{\alpha,s}(E)$$

for any set  $E \subset \mathbb{R}^n$ . For compact sets  $E$ , a proof of (3.8) can be found in Proposition 3.1.5 in [9]. The same proof also works for any set  $E$  provided one uses Corollary 2.6.8 in [3].



In proving (3.7), we may assume  $\int_{\mathbb{R}^n} |u| d\text{Cap}_{\alpha,s} < +\infty$ . Let  $E_k = \{2^{k-1} < |u| \leq 2^k\}$ , and let  $E_{j,k} = E_k \cap \mathcal{B}^j$  for  $k \in \mathbb{Z}$  and  $j \geq 0$ . We have

$$(3.9) \quad \beta_{\alpha,s}(u) = \beta_{\alpha,s}\left(\sum_{k \in \mathbb{Z}} |u| \chi_{E_k}\right) \leq \beta_{\alpha,s}\left(\sum_{k \in \mathbb{Z}} \sum_{j \geq 0} |u| \chi_{E_{j,k}}\right).$$

For  $k \in \mathbb{Z}$  and  $j \geq 0$ , let

$$f_{j,k} = (G_\alpha * \mu^{E_{j,k}})^{s'-1} \quad \text{and} \quad F_{j,k} = f_{j,k}(G_\alpha * f_{j,k})^{s-1}.$$

By the above argument, we have

$$G_\alpha * (2^k F_{j,k}) \geq c |u| \chi_{E_{j,k}} \quad \text{q.e.}$$

and

$$\int_{\mathbb{R}^n} (2^k F_{j,k})^s (G_\alpha * (2^k F_{j,k}))^{1-s} dx \leq A 2^k \text{Cap}_{\alpha,s}(E_{j,k}).$$

By (2.4), this gives

$$(3.10) \quad \|2^k F_{j,k}\|_{(M_s^{\alpha,s})'} \leq A 2^k \text{Cap}_{\alpha,s}(E_{j,k}).$$

Set  $F = \sup_{j,k} 2^k F_{j,k}$ . Then we have  $(G_\alpha * F)^{1-s} \leq (G_\alpha * (2^k F_{j,k}))^{1-s}$  for any  $k \in \mathbb{Z}$  and  $j \geq 0$ . Moreover,

$$G_\alpha * F \geq c \sum_{k \in \mathbb{Z}} |u| \chi_{E_k} \geq c_1 \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} |u| \chi_{E_{j,k}} \quad \text{q.e.}$$

due to the finite multiplicity of  $\{\mathcal{B}^j\}_{j \geq 0}$ . Also, it follows from (3.8) and (3.10) that

$$\begin{aligned} \|F\|_{(M_s^{\alpha,s})'} &\leq C_1 \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} 2^k \text{Cap}_{\alpha,s}(E_{j,k}) \leq C_2 \sum_{k \in \mathbb{Z}} 2^k \text{Cap}_{\alpha,s}(E_k) \\ &\leq A \int_{\mathbb{R}^n} |u| d\text{Cap}_{\alpha,s} < +\infty. \end{aligned}$$

In particular,  $F$  is finite a.e. and thus there is a set  $N$  such that  $|N| = 0$  and

$$\mathbb{R}^n = \bigcup_{k \in \mathbb{Z}, j \geq 0} \{0 < F \leq 2^{k+1} F_{j,k}\} \cup \{F = 0\} \cup N.$$

Thus we find

$$\begin{aligned} \beta_{\alpha,s}\left(\sum_{k \in \mathbb{Z}} \sum_{j \geq 0} |u| \chi_{E_{j,k}}\right) &\leq A \int_{\mathbb{R}^n} F^s (G_\alpha * F)^{1-s} dx \\ &\leq A \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} \int_{\{0 < F \leq 2^{k+1} F_{j,k}\}} F^s (G_\alpha * F)^{1-s} dx \\ &\leq A \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} \int_{\mathbb{R}^n} (2^k F_{j,k})^s (G_\alpha * (2^k F_{j,k}))^{1-s} dx \\ &\leq A \sum_{k \in \mathbb{Z}} \sum_{j \geq 0} 2^k \text{Cap}_{\alpha,s}(E_{j,k}) \leq C \int_{\mathbb{R}^n} |u| d\text{Cap}_{\alpha,s}. \end{aligned}$$

Inequality (3.7) now follows from (3.9) and the last bound. This completes the proof of the theorem.  $\blacksquare$

**Remark 3.3.** For Riesz potentials  $I_\alpha * f$  and Riesz capacities  $\text{cap}_{\alpha,s}$ , where  $\alpha \in (0, n)$  and  $s > 1$ , the corresponding bound (3.7) can be obtained using (1.4) and the pointwise bound

$$(3.11) \quad (I_\alpha * f)^s \leq AI_\alpha * [f(I_\alpha * f)^{s-1}],$$

which holds for all nonnegative measurable functions  $f$  (see [13, 14]). Indeed, for any  $f \geq 0$  such that  $I_\alpha * f \geq |u|^{1/s}$  q.e., by (3.11) we have  $AI_\alpha * [f(I_\alpha * f)^{s-1}] \geq |u|$  q.e., and thus again by (3.11),

$$\beta_{\alpha,s}(u) \leq A \int_{\mathbb{R}^n} f^s (I_\alpha * f)^{(s-1)s} I_\alpha * [f(I_\alpha * f)^{s-1}]^{1-s} dx \leq A \int_{\mathbb{R}^n} f^s dx.$$

Minimizing over such  $f$  and recalling (1.4), we get the corresponding bound (3.7) as desired.

## 4. Proof of Theorem 1.3

*Proof of Theorem 1.3.* By Theorem 1.2, we have

$$\int_{\mathbb{R}^n} |f|^q d\text{Cap}_{\alpha,s} \simeq \inf \left\{ \int_{\mathbb{R}^n} h^s (G_\alpha * h)^{1-s} dx : h \geq 0, (G_\alpha * h)^{1/q} \geq |f| \text{ q.e.} \right\}.$$

On the other hand, for any  $h \geq 0$  and  $(G_\alpha * h)^{1/q} \geq |f|$  q.e., by Theorem 3.1 in [11] we have

$$\mathbf{M}^{\text{loc}} f \leq \mathbf{M}^{\text{loc}} [(G_\alpha * h)^{1/q}] \leq A(G_\alpha * h)^{1/q}$$

pointwise everywhere, provided  $q > (n - \alpha)/n$ . Thus

$$\begin{aligned} \int_{\mathbb{R}^n} |f|^q d\text{Cap}_{\alpha,s} &\geq c \inf \left\{ \int_{\mathbb{R}^n} g^s (G_\alpha * g)^{1-s} dx : g \geq 0, (G_\alpha * g)^{1/q} \geq \mathbf{M}^{\text{loc}} f \text{ q.e.} \right\} \\ &\simeq \int_{\mathbb{R}^n} (\mathbf{M}^{\text{loc}} f)^q d\text{Cap}_{\alpha,s}. \end{aligned}$$

This completes the proof of the theorem. ■

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**Keng Hao Ooi**

Department of Mathematics, Louisiana State University, 303 Lockett Hall, Baton Rouge, LA 70803, USA;  
[kooi1@lsu.edu](mailto:kooi1@lsu.edu)

**Nguyen Cong Phuc**

Department of Mathematics, Louisiana State University, 303 Lockett Hall, Baton Rouge, LA 70803, USA;  
[pcnguyen@math.lsu.edu](mailto:pcnguyen@math.lsu.edu)