Rev. Mat. Iberoam. 38 (2022), no. 2, 589–599
DOI 10.4171/RMI/1285

On a capacitary strong type inequality and related capacitary estimates

Keng Hao Ooi and Nguyen Cong Phuc

Abstract. We establish a Maz'ya type capacitary inequality which resolves a special case of a conjecture by David R. Adams. As a consequence, we obtain several equivalent norms for Choquet integrals associated to Bessel or Riesz capacities. This enables us to obtain bounds for the Hardy–Littlewood maximal function in a sublinear setting.

1. Introduction

Let α be a real number and let $s > 1$. We define the space of Bessel potentials $H^{\alpha,s} = H^{\alpha,s}(\mathbb{R}^n)$, $n \geq 1$, as the completion of $C_c^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|u\|_{H^{\alpha,s}} = \|\mathcal{F}^{-1}[(1+|\xi|^2)^{\alpha/2}\mathcal{F}(u)]\|_{L^s(\mathbb{R}^n)},$$

where $\mathcal{F}$ is the Fourier transform in $\mathbb{R}^n$. In the case $\alpha > 0$, it follows (see, e.g., [8]) that a function u belongs to $H^{\alpha,s}$ if and only if

$$u = G_\alpha * f$$

for some $f \in L^s(\mathbb{R}^n)$, and moreover $\|u\|_{H^{\alpha,s}} = \|f\|_{L^s(\mathbb{R}^n)}$. Here, G_α is the Bessel kernel of order α defined by $G_\alpha(x) := \mathcal{F}^{-1}[(1+|\xi|^2)^{-\alpha/2}](x)$.

Recall that the Bessel capacity associated to the Bessel potential space $H^{\alpha,s}$ is defined for any set $E \subset \mathbb{R}^n$ by

$$\mathrm{Cap}_{\alpha,s}(E) := \inf\big\{\|f\|^s_{L^s(\mathbb{R}^n)} : f \geq 0, G_\alpha * f \geq 1 \text{ on } E\big\}.$$

A function $f\colon \mathbb{R}^n \to [-\infty, +\infty]$ is said to be defined quasieverywhere (q.e.) if it is defined at every point of $\mathbb{R}^n$ except for only a set of zero capacity $\mathrm{Cap}_{\alpha,s}$. The notion of Choquet integral associated to Bessel capacities will be important in this work. For a q.e. defined function $w\colon \mathbb{R}^n \to [0,\infty]$, the Choquet integrals of w is defined by

$$\int_{\mathbb{R}^n} w \, d\mathrm{Cap}_{\alpha,s} := \int_0^\infty \mathrm{Cap}_{\alpha,s}(\{x \in \mathbb{R}^n : w(x) > t\})\, dt.$$

2020 Mathematics Subject Classification: Primary 31B15, 31C45; Secondary 42B99.
Keywords: Nonlinear potential theory, capacity, maximal function, capacitary strong type inequality, Choquet integral.

One of the fundamental results of potential theory is the following Maz'ya's capacitary inequality, originally obtained by Maz'ya, and subsequently extended by Adams, Dahlberg, and Hansson:

$$\int_{\mathbb{R}^n} (G_\alpha * f)^s \, d\mathrm{Cap}_{\alpha,s} \leq A \int_{\mathbb{R}^n} f^s \, dx,$$

which holds for any nonnegative Lebesgue measurable function f. See, e.g., [3], [9] and [7], and in particular, see Section 2.3.1 and the historical comments in Section 2.3.13 of [7]. This kind of capacitary inequalities and their many applications are discussed in Chapters 2, 3 and 11 of [7].

In [2], Adams conjectured (in the context of Riesz capacities and Riesz potentials) that another capacitary strong type inequality

$$\int_{\mathbb{R}^n} (G_\alpha * f) \, d\mathrm{Cap}_{\alpha,s} \leq A \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} \, dx \tag{1.1}$$

holds for any nonnegative Lebesgue measurable function f (see equation (3.11) in [2]). (The integral $\int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} dx$ is understood as ∞ whenever $f = \infty$ on a set of positive Lebesgue measure. In the case $f \equiv 0$, it is understood as 0). Moreover, he essentially showed for the corresponding Riesz capacities and potentials that this is true provided α is an *integer* in $(0, n)$ (see page 23 in [2]). However, we observed that his argument does not appear to work for Bessel capacities and Bessel potentials as in (1.1) even with integers $\alpha \in (0, n)$.

One of the main purposes of this note is to verify (1.1) for any real $\alpha > 0$.

Theorem 1.1. *Let $\alpha > 0$ and $s > 1$ be such that $\alpha s \leq n$. There exists a constant $A > 0$ such that* (1.1) *holds for any nonnegative Lebesgue measurable function f.*

Our proof of (1.1) is also applicable to the setting of Riesz capacities and potentials, and thereby extends the above mentioned results of [2] to all real $\alpha \in (0, n)$.

Our approach to (1.1) is based mainly in our recent work [11] in which predual spaces to a Sobolev multiplier type space were considered. For $\alpha > 0$, $s > 1$, and $p > 1$, let $M_p^{\alpha,s} = M_p^{\alpha,s}(\mathbb{R}^n)$ be the Banach space of functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that the trace inequality

$$\left(\int_{\mathbb{R}^n} (G_\alpha * h)^s |f|^p \, dx \right)^{1/p} \leq A \|h\|_{L^s(\mathbb{R}^n)}^{s/p} \tag{1.2}$$

holds for all nonnegative $h \in L^s(\mathbb{R}^n)$. A norm of a function $f \in M_p^{\alpha,s}$ can be defined as

$$\|f\|_{M_p^{\alpha,s}} := \sup_K \left(\frac{\int_K |f(x)|^p \, dx}{\mathrm{Cap}_{\alpha,s}(K)} \right)^{1/p}, \tag{1.3}$$

where the supremum is taken over all compact sets $K \subset \mathbb{R}^n$ with non-zero capacity. Note that the right-hand side of (1.3) is known to be equivalent to the least possible constant A in (1.2) (see [3, 9]).

In [11], we showed that a predual of $M_p^{\alpha,s}$ is its Köthe dual space $(M_p^{\alpha,s})'$ defined by

$$(M_p^{\alpha,s})' = \left\{ \text{measurable functions } f : \sup \int |fg| \, dx < +\infty \right\},$$

where the supremum is taken over all functions g in the unit ball of $M_p^{\alpha,s}$. The norm of $f \in (M_p^{\alpha,s})'$ is defined as the above supremum. Thus we have

$$[(M_p^{\alpha,s})']^* = M_p^{\alpha,s},$$

with equality of norms. Various characterizations of $(M_p^{\alpha,s})'$ can be found in [11]. For our purpose here the case $p = s' = s/(s-1)$ is of special interest. In particular, as mentioned in Remark 2.10 in [11], it follows from [6, 10] that the space $M_{s'}^{\alpha,s}$ is an intrinsic space associated to the nonlinear integral equation

$$u = G_\alpha * (u^{s'}) + f \quad \text{a.e.}$$

Another important observation in [11] is the following equivalence:

$$\int_{\mathbb{R}^n} |u| \, d\mathrm{Cap}_{\alpha,s} \simeq \gamma_{\alpha,s}(u), \tag{1.4}$$

which holds for all q.e. defined functions u in $\mathbb{R}^n$. Here the functional $\gamma_{\alpha,s}(\cdot)$ is defined for each q.e. defined function u by

$$\gamma_{\alpha,s}(u) := \inf\left\{ \int f^s \, dx : 0 \le f \in L^s(\mathbb{R}^n) \text{ and } G_\alpha * f \ge |u|^{1/s} \text{ q.e.}\right\}.$$

Note that $\gamma_{\alpha,s}(tu) = |t|\gamma_{\alpha,s}(u)$ for all $t \in \mathbb{R}$ and moreover $\gamma_{\alpha,s}(u_1 + u_2) \le \gamma_{\alpha,s}(u_1) + \gamma_{\alpha,s}(u_2)$ (see [11]). On the other hand, the Choquet integral $\int_{\mathbb{R}^n} |\cdot| \, d\mathrm{Cap}_{\alpha,s}$ is known to be subadditive only for $s = 2$ and $0 < \alpha \le 1$. In particular, the set of all q.e. defined functions u in $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} |u| \, d\mathrm{Cap}_{\alpha,s} < +\infty$ is a normable space. An argument as in the proof of Proposition 2.3 in [11] can be used to show that this space is complete.

As a consequence of (1.4) and the proof of Theorem 1.1, in this paper we obtain two other characterizations for the Choquet integral. For a q.e. defined function u in $\mathbb{R}^n$, we denote by $\lambda_{\alpha,s}(u)$ and $\beta_{\alpha,s}$, $\alpha > 0$, $s > 1$, the following quantities:

$$\lambda_{\alpha,s}(u) := \inf\left\{ \|f\|_{(M_{s'}^{\alpha,s})'} : 0 \le f \in (M_{s'}^{\alpha,s})' \text{ and } G_\alpha * f \ge |u| \text{ q.e.}\right\}$$

and

$$\beta_{\alpha,s}(u) := \inf\left\{ \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} \, dx : f \ge 0,\ G_\alpha * f \ge |u| \text{ q.e.}\right\}.$$

Theorem 1.2. *Let $\alpha > 0$ and $s > 1$ be such that $\alpha s \le n$. For any q.e. defined function u in $\mathbb{R}^n$ it holds that*

$$\int_{\mathbb{R}^n} |u| \, d\mathrm{Cap}_{\alpha,s} \simeq \lambda_{\alpha,s}(u) \simeq \beta_{\alpha,s}(u). \tag{1.5}$$

In particular, we have

$$\mathrm{Cap}_{\alpha,s}(E) \simeq \lambda_{\alpha,s}(\chi_E) \simeq \beta_{\alpha,s}(\chi_E)$$

for any set $E \subset \mathbb{R}^n$.

To discuss a consequence of Theorem 1.2, we now recall that the (center) local Hardy–Littlewood maximal function is defined for each $f \in L^1_{\rm loc}(\mathbb{R}^n)$ by

$$\mathbf{M}^{\rm loc} f(x) = \sup_{0<r\leq 1} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y)|\, dy.$$

for every $x \in \mathbb{R}^n$.

Theorem 1.3. *Let $\alpha > 0$ and $s > 1$ be such that $\alpha s \leq n$. For any $q > (n-\alpha)/n$ and any measurable and q.e. defined function f, we have*

$$\int_{\mathbb{R}^n} (\mathbf{M}^{\rm loc} f)^q \, d\mathrm{Cap}_{\alpha,s} \leq A(n,\alpha,s,q) \int_{\mathbb{R}^n} |f|^q \, d\mathrm{Cap}_{\alpha,s}.$$

An interesting aspect of Theorem 1.3 is that the power q is allowed to be strictly less than 1. Moreover, here we do not assume any continuity assumption on f. See [1], Theorem 7.5 in [4], and [12] for some related results.

Finally, we remark that Theorems 1.1, 1.2, and 1.3 also hold in the homogeneous setting provided $\alpha \in (0,n)$, $s > 1$, and Bessel potentials and capacities are replaced by the corresponding Riesz potentials and capacities. Moreover, in the homogeneous setting the local Hardy–Littlewood maximal function $\mathbf{M}^{\rm loc}$ can be replaced by the larger standard Hardy–Littlewood maximal function.

Recall that the Riesz kernel I_α, $\alpha \in (0,n)$, is defined as the inverse Fourier transform of $|\xi|^\alpha$ (in the distributional sense), and explicitly we have $I_\alpha(x) = \gamma(n,\alpha)|x|^{\alpha-n}$, where $\gamma(n,\alpha) = \Gamma(\frac{n-\alpha}{2})/[\pi^{n/2} 2^\alpha \Gamma(\alpha/2)]$. The Riesz potential of a nonnegative measure μ is defined by the convolution $I_\alpha * \mu$. For $\alpha \in (0,n)$ and $s > 1$, the Riesz capacity $\mathrm{cap}_{\alpha,s}$ is defined for each set $E \subset \mathbb{R}^n$ by

$$\mathrm{cap}_{\alpha,s}(E) := \inf\left\{ \|f\|^s_{L^s(\mathbb{R}^n)} : f \geq 0, I_\alpha * f \geq 1 \text{ on } E \right\}.$$

This capacity is the capacity associated to the homogeneous Sobolev space $\dot{H}^{\alpha,s}$ (see Section 9 in [11]).

Notation. The characteristic function of a set $E \subset \mathbb{R}^n$ is denoted by χ_E. For two quantities A and B, we write $A \simeq B$ to mean that there exist positive constants c_1 and c_2 such that $c_1 A \leq B \leq c_2 A$.

2. Proof of Theorem 1.1

Proof of Theorem 1.1. Let $L^1(C)$ denote the space of quasicontinuous function f in $\mathbb{R}^n$ such that

$$\|f\|_{L^1(C)} := \int_{\mathbb{R}^n} |f|\, d\mathrm{Cap}_{\alpha,s} < +\infty.$$

Recall a function f is said to be quasicontinuous (with respect to $\mathrm{Cap}_{\alpha,s}$) if for any $\epsilon > 0$ there exists an open set O such that $\mathrm{Cap}_{\alpha,s}(O) < \epsilon$ and f is continuous in $O^c := \mathbb{R}^n \setminus O$. It is known that the dual of $L^1(C)$ can be identify with the space $\mathfrak{M}^{\alpha,s} = \mathfrak{M}^{\alpha,s}(\mathbb{R}^n)$ which consists of locally finite signed measures μ in $\mathbb{R}^n$ such that the norm $\|\mu\|_{\mathfrak{M}^{\alpha,s}} < +\infty$ (see Theorem 2.4 in [11]). Here we define

$$\|\mu\|_{\mathfrak{M}^{\alpha,s}} := \sup_K \frac{|\mu|(K)}{\mathrm{Cap}_{\alpha,s}(K)},$$

where the supremum is taken over all compact sets $K \subset \mathbb{R}^n$ such that $\mathrm{Cap}_{\alpha,s}(K) \neq 0$.

In view of (1.4), $L^1(C)$ is normable and thus it follows from the Hahn–Banach theorem that for any $u \in L^1(C)$ we have

$$\|u\|_{L^1(C)} \simeq \sup\Big\{\Big|\int u\,d\mu\Big| : \|\mu\|_{\mathfrak{M}^{\alpha,s}} \le 1\Big\}. \tag{2.1}$$

Let f be a nonnegative measurable and bounded function with compact support. Applying (2.1) with $u = G_\alpha * f$, we have

$$\begin{aligned}\int_{\mathbb{R}^n} G_\alpha * f \, d\mathrm{Cap}_{\alpha,s} &\le A \sup_{\|\mu\|_{\mathfrak{M}^{\alpha,s}} \le 1} \int G_\alpha * f \, d|\mu| = A \sup_{\|\mu\|_{\mathfrak{M}^{\alpha,s}} \le 1} \int (G_\alpha * |\mu|) f \, dx \\ &\le A \, \|f\|_{(M^{\alpha,s}_{s'})'} \sup_{\|\mu\|_{\mathfrak{M}^{\alpha,s}} \le 1} \|G_\alpha * |\mu|\|_{M^{\alpha,s}_{s'}} \le A \, \|f\|_{(M^{\alpha,s}_{s'})'},\end{aligned}$$

where the last inequality follows from Theorem 1.2 in [10]. By density (see Remark 3.3 in [11]), we see that the inequality

$$\int_{\mathbb{R}^n} G_\alpha * f \, d\mathrm{Cap}_{\alpha,s} \le A \, \|f\|_{(M^{\alpha,s}_{s'})'} \tag{2.2}$$

holds for any nonnegative function $f \in (M^{\alpha,s}_{s'})'$.

In proving (1.1) we may assume that $\int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} \, dx < +\infty$ and hence f is finite a.e. by our convention. In this case we must have that $f \in (M^{\alpha,s}_{s'})'$. Indeed, for any $g \in M^{\alpha,s}_{s'}$ such that $\|g\|_{M^{\alpha,s}_{s'}} \le 1$, by Remark 2.10 in [11] and [6], there exists a nonnegative function $u \in L^{s'}_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$u = G_\alpha * (u^{s'}) + \frac{|g|}{M} \quad \text{a.e.}$$

for a constant $M > 0$ independent of g and u. Thus, as in [5] (see also [6]), we have

$$\begin{aligned}\int_{\mathbb{R}^n} f|g| \, dx &= M \int_{\mathbb{R}^n} f(u - G_\alpha * (u^{s'})) \, dx = M \int_{\mathbb{R}^n} (fu - u^{s'} G_\alpha * f) \, dx \\ &= M \int_{\mathbb{R}^n} G_\alpha * f \Big(u \frac{f}{G_\alpha * f} - u^{s'}\Big) \, dx \\ &\le M s^{-s} (s-1)^{s-1} \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} \, dx,\end{aligned} \tag{2.3}$$

where we used the Young inequality $ab - a^{s'}/s' \le b^s/s$, $a, b \ge 0$, in the last inequality. Thus taking the supremum over $g \in M^{\alpha,s}_{s'}$ such that $\|g\|_{M^{\alpha,s}_{s'}} \le 1$ in (2.3), we find

$$\|f\|_{(M^{\alpha,s}_{s'})'} \le A \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s} \, dx < +\infty. \tag{2.4}$$

Finally, combining (2.2) with (2.4) we obtain (1.1) as desired. ∎

Remark 2.1. We remark that (1.1) and (2.2) are indeed equivalent. On one hand, the proof above shows that (2.2) implies (1.1). On the other hand, (1.1) implies that

$$\int_{\mathbb{R}^n} G_\alpha * f \, d\mathrm{Cap}_{\alpha,s} \leq A \, \|f\|_{KV}$$

for any nonnegative measurable function f. Here we define

$$\|f\|_{KV} := \inf\left\{ \int_{\mathbb{R}^n} h^s (G_\alpha * h)^{1-s} \, dx : h \geq |f| \text{ a.e.} \right\}.$$

($\|f\|_{KV}$ is understood as ∞ if there is no measurable function h such that $h \geq |f|$ a.e. and $\int_{\mathbb{R}^n} h^s (G_\alpha * h)^{1-s} \, dx < +\infty$.) As we observe in Remark 2.10 in [11], the two-sided bound $\|f\|_{(M^{\alpha,s}_{s'})'} \simeq \|f\|_{KV}$ follows from [6, 10]. Thus (1.1) implies (2.2).

3. Proof of Theorem 1.2

In order to prove Theorem 1.2, we first prove the following "integration by parts" lemma.

Lemma 3.1. *Let $\alpha > 0$ and $s > 1$ be such that $\alpha s \leq n$. Suppose that μ is a nonnegative measure such that the diameter of* supp(μ) *is less than* 1. *Then there is a constant $A = A(n, \alpha, s) > 0$ such that, for $f = (G_\alpha * \mu)^{s'-1}$, we have*

$$(G_\alpha * f)^s \leq A \, G_\alpha * [f (G_\alpha * f)^{s-1}]$$

pointwise everywhere in $\mathbb{R}^n$.

Remark 3.2. For Riesz potentials, this lemma has been established for all $f \geq 0$ in [14] (see also [6, 13]). In our setting, which deals with Bessel potentials, it is necessary to require μ to have compact support.

Proof of Lemma 3.1. Without loss of generality, we may assume that supp$(\mu) \subset B_{1/2}(0)$. With $f = (G_\alpha * \mu)^{s'-1}$, we write $f = f_1 + f_2$, where

$$f_1 = f \chi_{B_3(0)} \quad \text{and} \quad f_2 = f \chi_{B_3(0)^c} \quad (\text{with } B_3(0)^c = \mathbb{R}^n \setminus B_3(0)).$$

Then

$$(G_\alpha * f)^s \leq A [(G_\alpha * f_1)^s + (G_\alpha * f_2)^s]. \tag{3.1}$$

We shall use the following pointwise two-sided estimates for G_α (see, e.g., Section 1.2.4 in [3]):

$$G_\alpha(x) \simeq |x|^{\alpha-n}, \quad \forall |x| \leq 15, (0 < \alpha < n). \tag{3.2}$$

and

$$G_\alpha(x) \simeq G_\alpha(x+y), \quad \forall |x| \geq 3, |y| \leq 1, (\alpha > 0). \tag{3.3}$$

We mention that (3.3) follows from the asymptotic behavior G_α near infinity that can be found, e.g., in equation (1.2.24) in [3].

We now write

$$[G_\alpha * f_1(x)]^s = \int_{|y|\le 3} G_\alpha(x-y) f(y) \Big[\int_{|z|\le 3} G_\alpha(x-z) f(z)\, dz \Big]^{s-1} dy.$$

Thus if $|x| \ge 10$, then $|x-z| \ge 7 \ge |y-z|$, which yields that

$$G_\alpha(x-z) \le G_\alpha(y-z).$$

Therefore, we get

$$[G_\alpha * f_1(x)]^s \le G_\alpha * [f(G_\alpha * f)^{s-1}](x)$$

in the case $|x| \ge 10$.

On the other hand, if $|x| < 10$, then for $|y| \le 3$ by (3.2) we have

$$G_\alpha(x-y) \simeq |x-y|^{\alpha-n}.$$

Thus applying Lemma 2.1 in [13] we obtain

$$[G_\alpha * f_1(x)]^s \le A\, G_\alpha * [f_1(G_\alpha * f_1)^{s-1}](x) \le A\, G_\alpha * [f(G_\alpha * f)^{s-1}](x)$$

in the case $|x| < 10$.

Combining these two estimates we get that

$$[G_\alpha * f_1(x)]^s \le A\, G_\alpha * [f(G_\alpha * f)^{s-1}](x), \quad \forall x \in \mathbb{R}^n. \tag{3.4}$$

To estimate $[G_\alpha * f_2(x)]^s$ we first observe the following bound:

$$f_2(x) \le A\, G_\alpha * f(x), \quad \forall x \in \mathbb{R}^n. \tag{3.5}$$

Inequality (3.5) is trivial when $|x| < 3$. On the other hand, for $|x| \ge 3$, we have by (3.3),

$$(f_2(x))^{s-1} = \int_{|y|<1/2} G_\alpha(x-y) d\mu(y) \le A \int_{|y|<1/2} G_\alpha(x) d\mu(y) = A\, \|\mu\|\, G_\alpha(x).$$

Note that for $|y-x| < 1/2$ and $|x| \ge 3$, by (3.3) we have

$$f(y)^{s-1} = \int_{|z|<1/2} G_\alpha(y-z) d\mu(z) \ge c_0\, G_\alpha(x)\, \|\mu\|,$$

and so, for $|x| \ge 3$,

$$\begin{aligned} G_\alpha * f(x) &\ge \int_{|y-x|<1/2} G_\alpha(x-y) f(y)\, dy \\ &\ge \int_{|y-x|<1/2} G_\alpha(x-y)(c_0\, G_\alpha(x)\, \|\mu\|)^{s'-1}\, dy \\ &\ge c\, (\|\mu\|\, G_\alpha(x))^{s'-1} \ge c_1 f_2(x). \end{aligned}$$

Thus (3.5) is verified. Now by Hölder's inequality and (3.5) we have

$$[G_\alpha * f_2]^s \le A\, G_\alpha * (f_2^s) \le A_1 G_\alpha * [f(G_\alpha * f)^{s-1}]. \tag{3.6}$$

At this point, combining (3.1), (3.4), and (3.6), we obtain the lemma. ■

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let u be a q.e. defined function in $\mathbb{R}^n$. Suppose that f is a nonnegative measurable function such that $G_\alpha * f \geq |u|$ quasi-everywhere. Then by (2.2) and (2.4) it follows that

$$\int_{\mathbb{R}^n} |u|\, d\mathrm{Cap}_{\alpha,s} \leq \int_{\mathbb{R}^n} G_\alpha * f\, d\mathrm{Cap}_{\alpha,s} \leq A_1 \|f\|_{(M^{\alpha,s}_{s'})'} \leq A_2 \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{1-s}\, dx.$$

Now taking the infimum over such f we arrive at

$$\int_{\mathbb{R}^n} |u|\, d\mathrm{Cap}_{\alpha,s} \lesssim \lambda_{\alpha,s}(u) \lesssim \beta_{\alpha,s}(u).$$

Thus to complete the proof, it is left to show that

$$\beta_{\alpha,s}(u) \lesssim \int_{\mathbb{R}^n} |u|\, d\mathrm{Cap}_{\alpha,s}. \tag{3.7}$$

To this end, we first show (3.7) for $u = \chi_E$, where E is any set such that $\mathrm{Cap}_{\alpha,s}(E) > 0$ and the diameter of E is less than 1. By Theorems 2.5.6 and 2.6.3 in [3] one can find a nonnegative measure $\mu = \mu^E$ with $\mathrm{supp}(\mu) \subset \overline{E}$ (called capacitary measure for E) such that the function $V^E = G_\alpha * ((G_\alpha * \mu)^{s'-1})$ satisfies the following properties:

$$\mu^E(\overline{E}) = \mathrm{Cap}_{\alpha,s}(E) = \int_{\mathbb{R}^n} V^E\, d\mu^E = \int_{\mathbb{R}^n} (G_\alpha * \mu^E)^{s'}\, dx,$$

and

$$V^E \geq 1 \quad \text{quasieverywhere on } E.$$

Let $f = (G_\alpha * \mu)^{s'-1}$. By Lemma 3.1, we have

$$\chi_E \leq (V^E)^s = (G_\alpha * f)^s \leq A\, G_\alpha * [f (G_\alpha * f)^{s-1}] \quad \text{q.e.}$$

Thus,

$$\begin{aligned} \beta_{\alpha,s}(\chi_E) &\leq A \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{(s-1)s} \left\{G_\alpha * [f (G_\alpha * f)^{s-1}]\right\}^{1-s} dx \\ &\leq A \int_{\mathbb{R}^n} f^s (G_\alpha * f)^{(s-1)s} (G_\alpha * f)^{(1-s)s}\, dx \\ &= A \int_{\mathbb{R}^n} f^s\, dx = A \int_{\mathbb{R}^n} (G_\alpha * \mu)^{s'}\, dx = A\, \mathrm{Cap}_{\alpha,s}(E), \end{aligned}$$

as desired.

We now let $\{\mathcal{B}^j\}_{j\geq 0}$ be a covering of $\mathbb{R}^n$ by open balls with unit diameter. This covering is chosen in such a way that it has a finite multiplicity depending only on n. We shall use the following quasi-additivity of $\mathrm{Cap}_{\alpha,s}$:

$$\sum_{j\geq 0} \mathrm{Cap}_{\alpha,s}(E \cap \mathcal{B}^j) \leq M\, \mathrm{Cap}_{\alpha,s}(E) \tag{3.8}$$

for any set $E \subset \mathbb{R}^n$. For compact sets E, a proof of (3.8) can be found in Proposition 3.1.5 in [9]. The same proof also works for any set E provided one uses Corollary 2.6.8 in [3].

In proving (3.7), we may assume $\int_{\mathbb{R}^n} |u|\, d\mathrm{Cap}_{\alpha,s} < +\infty$. Let $E_k = \{2^{k-1} < |u| \le 2^k\}$, and let $E_{j,k} = E_k \cap \mathcal{B}^j$ for $k \in \mathbb{Z}$ and $j \ge 0$. We have

$$\beta_{\alpha,s}(u) = \beta_{\alpha,s}\Big(\sum_{k\in\mathbb{Z}} |u|\chi_{E_k}\Big) \le \beta_{\alpha,s}\Big(\sum_{k\in\mathbb{Z}}\sum_{j\ge 0} |u|\chi_{E_{j,k}}\Big). \tag{3.9}$$

For $k \in \mathbb{Z}$ and $j \ge 0$, let

$$f_{j,k} = (G_\alpha * \mu^{E_{j,k}})^{s'-1} \quad \text{and} \quad F_{j,k} = f_{j,k}(G_\alpha * f_{j,k})^{s-1}.$$

By the above argument, we have

$$G_\alpha * (2^k F_{jk}) \ge c\,|u|\,\chi_{E_{j,k}} \quad \text{q.e.}$$

and

$$\int_{\mathbb{R}^n} (2^k F_{jk})^s (G_\alpha * (2^k F_{j,k}))^{1-s}\, dx \le A\, 2^k\, \mathrm{Cap}_{\alpha,s}(E_{jk}).$$

By (2.4), this gives

$$\|2^k F_{j,k}\|_{(M^{\alpha,s}_{s'})'} \le A\, 2^k\, \mathrm{Cap}_{\alpha,s}(E_{jk}). \tag{3.10}$$

Set $F = \sup_{j,k} 2^k F_{j,k}$. Then we have $(G_\alpha * F)^{1-s} \le (G_\alpha * (2^k F_{j,k}))^{1-s}$ for any $k \in \mathbb{Z}$ and $j \ge 0$. Moreover,

$$G_\alpha * F \ge c \sum_{k\in\mathbb{Z}} |u|\,\chi_{E_k} \ge c_1 \sum_{k\in\mathbb{Z}}\sum_{j\ge 0} |u|\,\chi_{E_{j,k}} \quad \text{q.e.}$$

due to the finite multiplicity of $\{\mathcal{B}^j\}_{j\ge 0}$. Also, it follows from (3.8) and (3.10) that

$$\begin{aligned}\|F\|_{(M^{\alpha,s}_{s'})'} &\le C_1 \sum_{k\in\mathbb{Z}}\sum_{j\ge 0} 2^k\, \mathrm{Cap}_{\alpha,s}(E_{j,k}) \le C_2 \sum_{k\in\mathbb{Z}} 2^k\, \mathrm{Cap}_{\alpha,s}(E_k)\\ &\le A \int_{\mathbb{R}^n} |u|\, d\mathrm{Cap}_{\alpha,s} < +\infty.\end{aligned}$$

In particular, F is finite a.e. and thus there is a set N such that $|N| = 0$ and

$$\mathbb{R}^n = \bigcup_{k\in\mathbb{Z},\, j\ge 0} \{0 < F \le 2^{k+1} F_{j,k}\} \bigcup \{F = 0\} \bigcup N.$$

Thus we find

$$\begin{aligned}\beta_{\alpha,s}\Big(\sum_{k\in\mathbb{Z}}\sum_{j\ge 0} |u|\chi_{E_{j,k}}\Big) &\le A \int_{\mathbb{R}^n} F^s (G_\alpha * F)^{1-s}\, dx\\ &\le A \sum_{k\in\mathbb{Z}}\sum_{j\ge 0} \int_{\{0<F\le 2^{k+1}F_{j,k}\}} F^s (G_\alpha * F)^{1-s}\, dx\\ &\le A \sum_{k\in\mathbb{Z}}\sum_{j\ge 0} \int_{\mathbb{R}^n} (2^k F_{j,k})^s (G_\alpha * (2^k F_{j,k}))^{1-s}\, dx\\ &\le A \sum_{k\in\mathbb{Z}}\sum_{j\ge 0} 2^k\, \mathrm{Cap}_{\alpha,s}(E_{jk}) \le C \int_{\mathbb{R}^n} |u|\, d\mathrm{Cap}_{\alpha,s}.\end{aligned}$$

Inequality (3.7) now follows from (3.9) and the last bound. This completes the proof of the theorem. ∎

Remark 3.3. For Riesz potentials $I_\alpha * f$ and Riesz capacities $\mathrm{cap}_{\alpha,s}$, where $\alpha \in (0,n)$ and $s > 1$, the corresponding bound (3.7) can be obtained using (1.4) and the pointwise bound

$$(I_\alpha * f)^s \le A I_\alpha * [f(I_\alpha * f)^{s-1}], \tag{3.11}$$

which holds for all nonnegative measurable functions f (see [13, 14]). Indeed, for any $f \ge 0$ such that $I_\alpha * f \ge |u|^{1/s}$ q.e., by (3.11) we have $A I_\alpha * [f(I_\alpha * f)^{s-1}] \ge |u|$ q.e., and thus again by (3.11),

$$\beta_{\alpha,s}(u) \le A \int_{\mathbb{R}^n} f^s (I_\alpha * f)^{(s-1)s} I_\alpha * [f(I_\alpha * f)^{s-1}]^{1-s}\, dx \le A \int_{\mathbb{R}^n} f^s\, dx.$$

Minimizing over such f and recalling (1.4), we get the corresponding bound (3.7) as desired.

4. Proof of Theorem 1.3

Proof of Theorem 1.3. By Theorem 1.2, we have

$$\int_{\mathbb{R}^n} |f|^q \, d\mathrm{Cap}_{\alpha,s} \simeq \inf\Big\{ \int_{\mathbb{R}^n} h^s (G_\alpha * h)^{1-s}\, dx : h \ge 0, (G_\alpha * h)^{1/q} \ge |f| \text{ q.e.}\Big\}.$$

On the other hand, for any $h \ge 0$ and $(G_\alpha * h)^{1/q} \ge |f|$ q.e., by Theorem 3.1 in [11] we have

$$\mathbf{M}^{\mathrm{loc}} f \le \mathbf{M}^{\mathrm{loc}}[(G_\alpha * h)^{1/q}] \le A (G_\alpha * h)^{1/q}$$

pointwise everywhere, provided $q > (n-\alpha)/n$. Thus

$$\begin{aligned} \int_{\mathbb{R}^n} |f|^q \, d\mathrm{Cap}_{\alpha,s} &\ge c \inf\Big\{ \int_{\mathbb{R}^n} g^s (G_\alpha * g)^{1-s}\, dx : g \ge 0, (G_\alpha * g)^{1/q} \ge \mathbf{M}^{\mathrm{loc}} f \text{ q.e.}\Big\} \\ &\simeq \int_{\mathbb{R}^n} (\mathbf{M}^{\mathrm{loc}} f)^q \, d\mathrm{Cap}_{\alpha,s}. \end{aligned}$$

This completes the proof of the theorem. ∎

Funding. N. C. Phuc is supported in part by Simons Foundation, award number 426071.

References

[1] Adams, D.: A note on Choquet integrals with respect to Hausdorff capacity. In *Function spaces and applications (Lund, 1986)*, 115–124. Lecture Notes in Math. 1302, Springer, Berlin, 1988.

[2] Adams, D.: Choquet integrals in potential theory. *Publ. Mat.* **42** (1998), no. 1, 3–66.

[3] Adams, D. and Hedberg, L.: *Function spaces and potential theory*. Grundlehren der Mathematischen Wissenschaften 314, Springer-Verlag, Berlin, 1996.

[4] Adams, D. and Xiao, J.: Nonlinear analysis on Morrey spaces and their capacities. *Indiana Univ. Math. J.* **53** (2004), no. 6, 1629–1663.

[5] Baras, P. and Pierre, M.: Critère d'existence de solutions positives pour des équations semi-linéaires non monotones. *Ann. Inst. H. Poincaré, Analyse Non Linéaire* **2** (1985), no. 3, 185–212.

[6] Kalton, N. J. and Verbitsky, I. E.: Nonlinear equations and weighted norm inequalities. *Trans. Amer. Math. Soc.* **351** (1999), no. 9, 3441–3497.

[7] Maz'ya, V.: *Sobolev spaces with applications to elliptic partial differential equations*. Second, revised and augmented edition. Grundlehren der Mathematischen Wissenschaften 342, Springer, Heidelberg, 2011.

[8] Maz'ya, V. and Havin, V.: A nonlinear potential theory. *Uspehi Mat. Nauk* **27** (1972), no. 6, 67–138 (in Russian). English translation: *Russ. Math. Surv.* **27** (1972), 71–148.

[9] Maz'ya, V. and Shaposhnikova, T.: *Theory of Sobolev multipliers. With applications to differential and integral operators*. Grundlehren der Mathematischen Wissenschaften 337, Springer-Verlag, Berlin, 2009.

[10] Maz'ya, V. and Verbitsky, I.: Capacitary inequalities for fractional integrals, with applications to partial differential equations and Sobolev multipliers. *Ark. Mat.* **33** (1995), no. 1, 81–115.

[11] Ooi, K. and Phuc, N.: Characterizations of predual spaces to a class of Sobolev multiplier type spaces. *J. Funct. Anal.* **282** (2022), no. 6, Paper no. 109348, 52 pp.

[12] Orobitg, J. and Verdera, J.: Choquet integrals, Hausdorff content and the Hardy–Littlewood maximal operator. *Bull. London Math. Soc.* **30** (1998), no. 2, 145–150.

[13] Verbitsky, I. E.: Nonlinear potentials and trace inequalities. In *The Maz'ya anniversary collection, Vol. 2 (Rostock, 1998)*, 323–343. Oper. Theory Adv. Appl. 110, Birkhäuser, Basel, 1999.

[14] Verbitsky, I. E. and Wheeden, R. L.: Weighted norm inequalities for integral operators. *Trans. Amer. Math. Soc.* **350** (1998), no. 8, 3371–3391.

Received May 22, 2020. Published online June 22, 2021.

Keng Hao Ooi
Department of Mathematics, Louisiana State University, 303 Lockett Hall, Baton Rouge, LA 70803, USA;
kooi1@lsu.edu

Nguyen Cong Phuc
Department of Mathematics, Louisiana State University, 303 Lockett Hall, Baton Rouge, LA 70803, USA;
pcnguyen@math.lsu.edu