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Commensurability in Artin groups of spherical type

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Abstract. We give an almost complete classification of Artin groups of spherical type up to commensurability. Let A and A' be two Artin groups of spherical type, and let $A_1, \dots, A_p$ (respectively, $A'_1, \dots, A'_q$) be the irreducible components of A (respectively, A'). We show that A and A' are commensurable if and only if $p = q$ and, up to permutation of the indices, A_i and A'_i are commensurable for every i. We prove that, if two Artin groups of spherical type are commensurable, then they have the same rank. For a fixed n, we give a complete classification of the irreducible Artin groups of rank n that are commensurable with the group of type A_n. Note that there are six remaining comparisons of pairs of groups to get the complete classification of Artin groups of spherical type up to commensurability, two of which have been done by Ignat Soroko after the first version of the present paper.

1. Introduction

We start by recalling the definitions of Coxeter groups and Artin groups. Let S be a finite set. A *Coxeter matrix* over S is a square matrix $M = (m_{s,t})_{s,t \in S}$ indexed by the elements of S, having coefficients in $\mathbb{N} \cup \{\infty\}$, and satisfying $m_{s,s} = 1$ for every $s \in S$, and $m_{s,t} = m_{t,s} \geq 2$ for every $s, t \in S$, $s \neq t$. This matrix is represented by a labeled graph Γ, called *Coxeter graph* and defined by the following data. The set of vertices of Γ is S. Two vertices $s, t \in S$, $s \neq t$, are connected by an edge if $m_{s,t} \geq 3$, and this edge is labeled with $m_{s,t}$ if $m_{s,t} \geq 4$.

If $s, t \in S$ and m is an integer ≥ 2, we denote by $\Pi(s,t,m)$ the word $sts\cdots$ of length m. In other words, $\Pi(s,t,m) = (st)^{m/2}$ if m is even and $\Pi(s,t,m) = (st)^{(m-1)/2}s$ if m is odd. Let Γ be the Coxeter graph associated to such a Coxeter matrix. The *Artin group* associated to Γ is the group $A = A[\Gamma]$ defined by the following presentation:

$$A[\Gamma] = \langle S \mid \Pi(s,t,m_{s,t}) = \Pi(t,s,m_{s,t}),\ \text{for } s,t \in S,\ s \neq t,\ m_{s,t} \neq \infty \rangle\,.$$

The *Coxeter group* $W = W[\Gamma]$ of Γ is the quotient of $A[\Gamma]$ by the relations $s^2 = 1$, $s \in S$. We say that Γ is of *spherical type* if $W[\Gamma]$ is finite.

Let $\Gamma_1, \dots, \Gamma_p$ be the connected components of Γ and, for $i \in \{1, \dots, p\}$, let S_i be the set of vertices of Γ_i, A_i the subgroup of A generated by S_i, and W_i the subgroup

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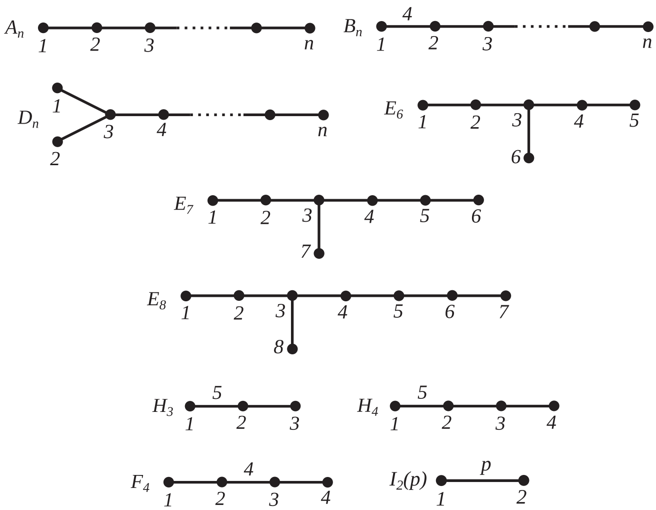

Figure 1. Coxeter graphs of spherical type.

of W generated by S_i. We can easily check that A_i is the Artin group of Γ_i and W_i is the Coxeter group of Γ_i for every i, and that $A = A_1 \times \cdots \times A_p$ and $W = W_1 \times \cdots \times W_p$. In particular, Γ has spherical type if and only if Γ_i has spherical type for every $i \in \{1, \dots, p\}$. The classification of Coxeter graphs of spherical type has been known for a long time and it is given in the following theorem.

Theorem 1.1 ([14]). *A Coxeter graph Γ is connected and has spherical type if and only if it is isomorphic to one of the graphs A_n $(n \ge 1)$, B_n $(n \ge 2)$, D_n $(n \ge 4)$, E_n $(n \in \{6,7,8\})$, F_4, H_3, H_4 and $I_2(p)$ $(p \ge 5)$ represented in Figure 1.*

Actually, this classification is also the classification of Artin groups of spherical type up to isomorphism because, by Theorem 1.1 in [31], two Artin groups of spherical type are isomorphic if and only if their associated Coxeter graphs are isomorphic. It is then natural to ask if such a result remains valid when changing the word “isomorphic” by “commensurable”. The answer has been known for a long time: it is *no* because the Artin groups associated to A_n and B_n are commensurable (see Lemma 6.1) and they are not isomorphic by Theorem 1.1 in [31]. However, the classification of Artin groups of spherical type up to commensurability was a very open question before this article. For instance, no example of two non-commensurable Artin groups of spherical type having the same rank was known before. This article almost gives the entire classification of Artin groups of spherical type up to commensurability, meaning that there are only 6 comparisons of groups that we do not treat. Two of them have been solved by Ignat Soroko [39] after the first version of the present paper.

We recall that two groups G_1 and G_2 are *commensurable* if there are two finite index subgroups H_1 of G_1 and H_2 of G_2 such that H_1 is isomorphic to H_2. The study of commensurability is useful when studying virtual properties of groups. There is also a strong relationship between commensurable groups and quasi-isometric groups. In particular, for a finitely generated group G endowed with any word metric, the inclusion map of a finite index subgroup in G is a quasi-isometry. This implies that, if two finitely generated groups are commensurable, then they are also quasi-isometric. The converse implication is true only under certain conditions.

The *commensurator* (also called abstract commensurator) of a group G will be denoted by $\mathrm{Com}(G)$. We recall its definition. Let $\widetilde{\mathrm{Com}}(G)$ be the set of triples (U, V, f) where U and V are finite index subgroups of G, and $f: U \to V$ is an isomorphism. Let $\sim$ be the equivalence relation on $\widetilde{\mathrm{Com}}(G)$ such that $(U, V, f) \sim (U', V', f')$ if there is a finite index subgroup W of $U \cap U'$ such that $f(\alpha) = f'(\alpha)$ for every $\alpha \in W$. Hence we define $\mathrm{Com}(G)$ as $\widetilde{\mathrm{Com}}(G)/\sim$, and the group operation is induced by the composition. We can easily show that, if A and B are two commensurable groups, then $\mathrm{Com}(A)$ and $\mathrm{Com}(B)$ are isomorphic. Commensurators are in general difficult to compute. Fortunately, the commensurator of the Artin group associated to A_n (the braid group) is well understood [12, 26] and it is indeed used to prove the results in this paper.

So far, the results regarding commensurability for Artin groups in general are quite limited. In [13], the author studies commensurability for Artin groups of large type (each $m_{s,t} \geq 3$ for $s \neq t$) associated to triangle-free connected Coxeter graphs having at least three vertices. In the last years, the research on this topic has been focused on right-angled Artin groups (RAAGs). A RAAG is an Artin group whose only relations in its presentation are commutations. It is often represented by a *commutation graph*, Υ, which is defined by the following data. The set of vertices of Υ is the set of standard generators of the group. Two vertices are connected by an edge if and only if the corresponding generators commute. Apart from the classifications made for free and free-abelian groups, commensurability studies are made for RAAGs with commutation graphs Υ in the following cases:

- Υ is connected, triangle-free and square-free without vertices of degree one [22];
- Υ is star-rigid with no induced 4-cycles and the outer automorphism of the Artin group is finite [17];
- Υ is a tree of diameter ≤ 4 [3, 10];
- Υ is a path graph [11]. In this work they also compared these commensurability classes to the ones of RAAGs defined by trees of diameter 4.

Remark. The results of this paper, notably part (3) of Theorem 2.4, are being used in a paper in preparation of Ursula Hamenstädt [18] to refute a conjecture made by Kontsevich and Zorich [23]. We fix a tuple of non-negative integers $d = (p_1, p_2, \dots, p_k)$ and consider the vector space of holomorphic one-forms of a Riemann surface with genus g bigger or equal to 2. We denote by M_d the moduli space of these one-forms having zeros $x_1, x_2, \dots, x_k$ with multiplicity $p_1, p_2, \dots, p_k$, respectively. The conjecture says that each connected component of M_d has homotopy type $K(G, 1)$, where G is a group commensurable to some mapping class group. Hamenstädt uses the results in [28] to show that there are components in genus 3 that are classifying spaces for the quotients of the Artin groups $A[E_6]$ and $A[E_7]$ by their centers. She proves that the only mapping class group which could be commensurable to $A[E_6]/Z(A[E_6])$ is the quotient of the braid group on 7 strands by its center, that is, $A[A_6]/Z(A[A_6])$. By Proposition 3.1, the non-commensurability of $A[E_6]/Z(A[E_6])$ and $A[A_6]/Z(A[A_6])$ is equivalent to the non-commensurability of $A[E_6]$ and $A[A_6]$. These components provide a counterexample to the conjecture.

2. Statements

Recall that our aim is to partially classify the Artin groups of spherical type up to commensurability. Our starting point is the following result, which can be easily proven. It allows to reduce the question to the case where both Coxeter graphs have the same number of vertices.

Proposition 2.1. *Let Γ and Ω be two Coxeter graphs of spherical type. If $A[\Gamma]$ and $A[\Omega]$ are commensurable, then Γ and Ω have the same number of vertices.*

Proof. Suppose that $A[\Gamma]$ and $A[\Omega]$ are commensurable. Let n be the number of vertices of Γ and let m be the number of vertices of Ω. We know that the cohomological dimension of $A[\Gamma]$ is n and the cohomological dimension of $A[\Omega]$ is m (see Proposition 3.1 in [31]). As every finite index subgroup of $A[\Gamma]$ has the same cohomological dimension as $A[\Gamma]$ and every finite index subgroup of $A[\Omega]$ has the same cohomological dimension as $A[\Omega]$, we have $n = m$. ■

In Section 5 we will prove the following result, which allows to reduce our problem to the study of two connected Coxeter graphs having the same number of vertices.

Theorem 2.2. *Let Γ and Ω be two Coxeter graphs of spherical type. Let $\Gamma_1, \dots, \Gamma_p$ be the connected components of Γ and let $\Omega_1, \dots, \Omega_q$ be the connected components of Ω. Then $A[\Gamma]$ and $A[\Omega]$ are commensurable if and only if $p = q$ and $A[\Gamma_i]$ and $A[\Omega_i]$ are commensurable for every $i \in \{1, \dots, p\}$, up to permutation of the indices.*

Let G be a group. A subgroup H of G is a *direct factor* of G is there is a subgroup K of G such that $G = H \times K$. We say that G is *indecomposable* if G does not have any non-trivial proper direct factor. We say that G is *strongly indecomposable* if G is infinite and every finite index subgroup H of G is indecomposable. A *strong Remak decomposition* of G is a finite index subgroup H of G with a direct product decomposition $H = H_1 \times \cdots \times H_p$ such that H_i is strongly indecomposable for every $i \in \{1, \dots, p\}$. Two strong Remak decompositions of G, $H = H_1 \times \cdots \times H_p$ and $H' = H'_1 \times \cdots \times H'_q$, are said to be *equivalent* if $p = q$ and H_i and H'_i are commensurable for every $i \in \{1, \dots, p\}$, up to permutation of the indices.

The center of a group G will be denoted by $Z(G)$. If Γ is a connected Coxeter graph of spherical type then, thanks to [8] and [16], the center of $A[\Gamma]$ is a cyclic infinite group. The quotient $A[\Gamma]/Z(A[\Gamma])$ will be denoted by $\overline{A[\Gamma]}$ and it will play an important role in our study. Moreover, we denote by $\theta\colon A[\Gamma] \to W[\Gamma]$ the canonical projection and by $\mathrm{CA}[\Gamma]$ the kernel of θ. As before, we let $\overline{\mathrm{CA}[\Gamma]} = \mathrm{CA}[\Gamma]/Z(\mathrm{CA}[\Gamma])$. In Section 3, we will prove that $Z(\mathrm{CA}[\Gamma]) \simeq \mathbb{Z}$ and $\mathrm{CA}[\Gamma] \simeq \overline{\mathrm{CA}[\Gamma]} \times Z(\mathrm{CA}[\Gamma])$ (see Proposition 3.1). If Γ is reduced to a single vertex, then $\mathrm{CA}[\Gamma] = Z(\mathrm{CA}[\Gamma]) \simeq \mathbb{Z}$ and $\overline{\mathrm{CA}[\Gamma]} = \{1\}$. Otherwise $\overline{\mathrm{CA}[\Gamma]} \neq \{1\}$.

The proof of Theorem 2.2 is based on the following result, which will be proven in Section 4.

Theorem 2.3.

(1) *Let* Γ *be a connected Coxeter graph of spherical type which is not reduced to a single vertex. Then* $\overline{\mathrm{CA}[\Gamma]}$ *is strongly indecomposable.*

(2) *Let* Γ *be a Coxeter graph of spherical type and let* $\Gamma_1, \dots, \Gamma_p$ *be its connected components. We suppose that each* $\Gamma_1, \dots, \Gamma_k$ *has at least two vertices and each of* $\Gamma_{k+1}, \dots, \Gamma_p$ *is reduced to a single vertex. Then*

$$CA[\Gamma] = \overline{\mathrm{CA}[\Gamma_1]} \times \cdots \times \overline{\mathrm{CA}[\Gamma_k]} \times Z(\mathrm{CA}[\Gamma_1]) \times \cdots \times Z(\mathrm{CA}[\Gamma_p])$$

is a strong Remak decomposition of $A[\Gamma]$*, and it is unique up to equivalence.*

A similar result for Coxeter groups is obtained in [35]. In order to finish the classification, we just need to compare the Artin groups associated to connected Coxeter graphs of spherical type with the same number of vertices. In Section 6 we prove the following result, which compares every group of this type with the corresponding Artin group of type A_n.

Theorem 2.4.

(1) *Let* $n \geq 2$*. Then* $A[A_n]$ *and* $A[B_n]$ *are commensurable.*

(2) *Let* $n \geq 4$*. Then* $A[A_n]$ *and* $A[D_n]$ *are not commensurable.*

(3) *Let* $n \in \{6, 7, 8\}$*. Then* $A[A_n]$ *and* $A[E_n]$ *are not commensurable.*

(4) $A[A_4]$ *and* $A[F_4]$ *are not commensurable.*

(5) *Let* $n \in \{3, 4\}$*. Then* $A[A_n]$ *and* $A[H_n]$ *are not commensurable.*

(6) *Let* $p \geq 5$*. Then* $A[A_2]$ *and* $A[I_2(p)]$ *are commensurable.*

The strategy of the proof of this theorem is the following. We use direct proofs to show part (1) and part (6). Using the fact that the abstract commensurator of $\overline{A[A_n]}$ is known to be a mapping class group of a punctured sphere (see [12]), we show that, if $A[\Gamma]$ is commensurable with $A[A_n]$, then there is a homomorphism $\varphi\colon \overline{A[\Gamma]} \to \mathfrak{S}_{n+2} \times \{\pm 1\}$ whose kernel has no generalized torsion. Then, in parts (2) to (5), in order to prove that $A[\Gamma]$ and $A[A_n]$ are not commensurable, we check in each case that the kernel of every homomorphism $\varphi\colon \overline{A[\Gamma]} \to \mathfrak{S}_{n+2} \times \{\pm 1\}$ has generalized torsion.

The description of $\overline{A[D_4]}$ as the pure mapping class group of the three times punctured torus [24] has been recently used by Soroko [39] to apply the same techniques presented in this article to show that $A[D_4]$ is not commensurable with $A[F_4]$ and $A[H_4]$. For the remaining cases, we have no hint on how to describe the abstract commensurator of one of the two groups, and this is needed in our argument. So, the following cases remain open:

- For $n = 6, 7, 8$, we do not know if $A[D_n]$ and $A[E_n]$ are commensurable.
- For $n = 4$, we do not know if $A[F_4]$ and $A[H_4]$ are commensurable.

3. A technical and useful result

This section is devoted to some technical results (see Proposition 3.1) that will be the key to prove the main theorems of the forthcoming sections. These results are also interesting by themselves.

Let Γ be a Coxeter graph of spherical type. The *Artin monoid* associated to Γ is the monoid $A[\Gamma]^+$ having the same presentation as $A[\Gamma]$, that is,

$$A[\Gamma]^+ = \langle S \mid \Pi(s,t,m_{s,t}) = \Pi(t,s,m_{s,t}) \text{ for } s,t \in S,\ s \neq t,\ m_{s,t} \neq \infty \rangle^+ .$$

By [8] (see also [34]), $A[\Gamma]^+$ naturally injects in $A[\Gamma]$. We define a partial order $\le_L$ on $A[\Gamma]$ by $\alpha \le_L \beta$ if $\alpha^{-1}\beta \in A[\Gamma]^+$. Also by [8], the ordered set $(A[\Gamma], \le_L)$ is a lattice. We denote by $\wedge_L$ and $\vee_L$ the lattice operations in $(A[\Gamma], \le_L)$. In this case, the *Garside element* of $A[\Gamma]$ is defined as $\Delta = \vee_L S$. Again by [8] and [16] we know that, if Γ is connected, then the center of $A[\Gamma]$ is infinite and cyclic, and it is generated by an element δ of the form $\delta = \Delta^\kappa$, where $\kappa \in \{1,2\}$. This element δ will be called the *standard generator* of $Z(A[\Gamma])$. We can also express δ as follows. Let $S = \{s_1, \dots, s_n\}$. Then, by [8], $\delta = (s_1 s_2 \cdots s_n)^{h/2}$ if $\kappa = 1$ and $\delta = (s_1 s_2 \cdots s_n)^h$ if $\kappa = 2$, where h is the Coxeter number of Γ, that is, the order of $s_1 s_2 \cdots s_n$ in the associated Coxeter group. These equalities do not depend on the choice when numbering the elements of S.

Let Γ be a connected Coxeter graph of spherical type. Let $z \colon A[\Gamma] \to \mathbb{Z}$ be the homomorphism such that $z(s) = 1$ for every $s \in S$. Hence, considering the later expression of δ, we have that $z(\delta) > 0$. The quotient $A[\Gamma]/Z(A[\Gamma])$ is denoted by $\overline{A[\Gamma]}$. Moreover, recall that $\theta \colon A[\Gamma] \to W[\Gamma]$ is the canonical projection, $\mathrm{CA}[\Gamma]$ is the kernel of θ, and $\overline{\mathrm{CA}[\Gamma]} = \mathrm{CA}[\Gamma]/Z(\mathrm{CA}[\Gamma])$.

Proposition 3.1. *Let Γ and Ω be two connected Coxeter graphs of spherical type.*

(1) *If U is a finite index subgroup of $A[\Gamma]$, then $Z(U) = Z(A[\Gamma]) \cap U$. In particular, $Z(U)$ is an infinite cyclic group.*

(2) *We have* $\mathrm{CA}[\Gamma] \simeq \overline{\mathrm{CA}[\Gamma]} \times Z(\mathrm{CA}[\Gamma]) \simeq \overline{\mathrm{CA}[\Gamma]} \times \mathbb{Z}$.

(3) *$A[\Gamma]$ and $A[\Omega]$ are commensurable if and only if $\overline{A[\Gamma]}$ and $\overline{A[\Omega]}$ are commensurable.*

(4) *The group $\overline{A[\Gamma]}$ injects in its commensurator* $\mathrm{Com}(\overline{A[\Gamma]})$.

Proof. (1) Let U be a finite index subgroup of $A[\Gamma]$. The inclusion $Z(A[\Gamma]) \cap U \subset Z(U)$ is obvious. We need to show $Z(U) \subset Z(A[\Gamma]) \cap U$. Let $\alpha \in Z(U)$ and $s \in S$. As U is a finite index subgroup, there is $k \ge 1$ such that $s^k \in U$. Then $\alpha s^k \alpha^{-1} = s^k$ and, by Corollary 5.3 in [32], $\alpha s \alpha^{-1} = s$. This proves that α belongs to $Z(A[\Gamma])$. To see that $Z(U)$ is infinite cyclic, notice that $Z(U)$ is a finite index subgroup of $Z(A[\Gamma])$, which is infinite cyclic because Γ is connected.

(2) Let $V = \oplus_{s \in S} \mathbb{R} e_s$ be a real vector space with a basis in one-to-one correspondence with S. By [5], $W = W[\Gamma]$ has a faithful linear representation $\rho \colon W \to \mathrm{GL}(V)$ and $\rho(W)$ is generated by reflections. We denote by $\mathcal{H}$ the set of reflection hyperplanes of W. We let $V_\mathbb{C} = \mathbb{C} \otimes V$ and $H_\mathbb{C} = \mathbb{C} \otimes H$ for every $H \in \mathcal{H}$. Let also

$$M = V_\mathbb{C} \setminus \Big(\bigcup_{H \in \mathcal{H}} H_\mathbb{C} \Big) .$$

Notice that M is a connected manifold of dimension $2|S|$. By [7], $\pi_1(M) = \mathrm{CA}[\Gamma]$.

Let $h\colon V_{\mathbb{C}} \setminus \{0\} \to \mathbb{P}V_{\mathbb{C}}$ be the Hopf fibration. Let $\overline{M} = h(M)$ and denote by $h_{\mathcal{H}}\colon M \to \overline{M}$ the restriction of h to M. Recall that the fiber of $h_{\mathcal{H}}$ is $\mathbb{C}^*$. As $\mathcal{H}$ is non-empty, we know that $h_{\mathcal{H}}$ is topologically a trivial fibration (see Proposition 5.1 in [30]). In other words, M is homeomorphic to $\overline{M} \times \mathbb{C}^*$, hence $\mathrm{CA}[\Gamma] = \pi_1(M) \simeq \pi_1(\overline{M}) \times \mathbb{Z}$. From this decomposition it follows that $Z(\mathrm{CA}[\Gamma]) \simeq Z(\pi_1(\overline{M})) \times \mathbb{Z}$. But, thanks to part (1), $Z(\mathrm{CA}[\Gamma])$ is isomorphic to $\mathbb{Z}$, which does not have any non-trivial direct product decomposition, hence $Z(\pi_1(\overline{M})) = 1$, $\pi_1(\overline{M}) \simeq \mathrm{CA}[\Gamma]/Z(\mathrm{CA}[\Gamma]) = \overline{\mathrm{CA}[\Gamma]}$, and $\mathrm{CA}[\Gamma] \simeq \overline{\mathrm{CA}[\Gamma]} \times \mathbb{Z} = \overline{\mathrm{CA}[\Gamma]} \times Z(\mathrm{CA}[\Gamma])$.

(3) Suppose that $A[\Gamma]$ and $A[\Omega]$ are commensurable. There is a finite index subgroup U of $A[\Gamma]$ and a finite index subgroup V of $A[\Omega]$ such that U is isomorphic to V. Let $\pi\colon A[\Gamma] \to \overline{A[\Gamma]}$ and $\pi'\colon A[\Omega] \to \overline{A[\Omega]}$ be the corresponding canonical projections. Then $\pi(U) = U/(Z(A[\Gamma]) \cap U)$ is a finite index subgroup of $\overline{A[\Gamma]}$, $\pi'(V) = V/(Z(A[\Omega]) \cap V)$ is a finite index subgroup of $\overline{A[\Omega]}$, and, by part (1), we have $\pi(U) = U/Z(U)$ and $\pi'(V) = V/Z(V)$. Hence $\pi(U)$ is isomorphic to $\pi'(V)$. Therefore, $\overline{A[\Gamma]}$ and $\overline{A[\Omega]}$ are commensurable.

Suppose that $\overline{A[\Gamma]}$ and $\overline{A[\Omega]}$ are commensurable. By part (1), we know that $Z(\mathrm{CA}[\Gamma]) = \mathrm{CA}[\Gamma] \cap Z(A[\Gamma])$, and then $\pi(\mathrm{CA}[\Gamma]) = \overline{\mathrm{CA}[\Gamma]}$ and $\overline{\mathrm{CA}[\Gamma]}$ is a finite index subgroup of $\overline{A[\Gamma]}$. Likewise, $\overline{\mathrm{CA}[\Omega]}$ is a finite index subgroup of $\overline{A[\Omega]}$, then $\overline{\mathrm{CA}[\Gamma]}$ and $\overline{\mathrm{CA}[\Omega]}$ are commensurable. This means that there are finite index subgroups $\bar{U}$ of $\overline{\mathrm{CA}[\Gamma]}$ and $\bar{V}$ of $\overline{\mathrm{CA}[\Omega]}$ such that $\bar{U}$ and $\bar{V}$ are isomorphic. By part (2), $\mathrm{CA}[\Gamma] = \overline{\mathrm{CA}[\Gamma]} \times \mathbb{Z}$ and $\mathrm{CA}[\Omega] = \overline{\mathrm{CA}[\Omega]} \times \mathbb{Z}$. Let $U = \bar{U} \times \mathbb{Z} \subset \mathrm{CA}[\Gamma]$ and $V = \bar{V} \times \mathbb{Z} \subset \mathrm{CA}[\Omega]$. Hence U is a finite index subgroup of $\mathrm{CA}[\Gamma]$, V is a finite index subgroup of $\mathrm{CA}[\Omega]$, and U and V are isomorphic. Thus, $\mathrm{CA}[\Gamma]$ and $\mathrm{CA}[\Omega]$ are commensurable, so $A[\Gamma]$ and $A[\Omega]$ are commensurable.

(4) If G is a group and $\alpha \in G$, we denote by $c_\alpha\colon G \to G$, $\beta \mapsto \alpha\beta\alpha^{-1}$, the conjugation by α. Then we have a homomorphism $\iota_G\colon G \to \mathrm{Com}(G)$ sending α to the class of (G, G, c_α). Let $\iota = \iota_{\overline{A[\Gamma]}}\colon \overline{A[\Gamma]} \to \mathrm{Com}(\overline{A[\Gamma]})$, and let $\alpha \in A[\Gamma]$ be such that $\pi(\alpha) \in \mathrm{Ker}(\iota)$, where $\pi\colon A[\Gamma] \to \overline{A[\Gamma]}$ is the corresponding canonical projection. There is a finite index subgroup $\bar{U}$ of $\overline{A[\Gamma]}$ such that $\pi(\alpha)\,\pi(\beta)\,\pi(\alpha^{-1}) = \pi(\beta)$ for every $\beta \in \pi^{-1}(\bar{U})$. Let $s \in S$. As $\bar{U}$ has finite index in $\overline{A[\Gamma]}$, there is $k \geq 1$ such that $\pi(s^k) \in \bar{U}$. We have $\pi(\alpha)\,\pi(s^k)\,\pi(\alpha^{-1}) = \pi(s^k)$, so $\pi(\alpha s^k \alpha^{-1} s^{-k}) = 1$ and then $\alpha s^k \alpha^{-1} s^{-k} \in \mathrm{Ker}(\pi) = Z(A[\Gamma]) = \langle \delta \rangle$. Hence there is $\ell \in \mathbb{Z}$ such that $\alpha s^k \alpha^{-1} s^{-k} = \delta^\ell$. Recall that $z\colon A[\Gamma] \to \mathbb{Z}$ is the homomorphism sending every element of S to 1 and $z(\delta) > 0$. Then $0 = z(\alpha s^k \alpha^{-1} s^{-k}) = z(\delta^\ell) = \ell\, z(\delta)$ and $z(\delta) > 0$, having that $\ell = 0$ and $\alpha s^k \alpha^{-1} = s^k$. By Corollary 5.3 in [31], it follows that $\alpha s \alpha^{-1} = s$. This shows that α belongs to $Z(A[\Gamma])$, so $\pi(\alpha) = 1$ and ι is injective. ∎

The proof of the following corollary is completely and explicitly included in the proof of the proposition above.

Corollary 3.2. *Let Γ be a connected Coxeter graph of spherical type. Then $Z(\mathrm{CA}[\Gamma])$ is an infinite cyclic group. On the other hand, $\overline{\mathrm{CA}[\Gamma]}$ can be viewed as a subgroup of $\overline{A[\Gamma]}$, it has finite index in $\overline{A[\Gamma]}$, and its center is trivial.*

4. Strong Remak decomposition

In this section, we denote by Γ a Coxeter graph of spherical type associated to a Coxeter matrix $M = (m_{s,t})_{s,t \in S}$. Recall that our aim is to show Theorem 2.3.

Let G be a group and E be a subset of G. Recall that the *normalizer* of E in G is $N_G(E) = \{\alpha \in G \mid \alpha E \alpha^{-1} = E\}$ and the *centralizer* of E in G is defined as $Z_G(E) = \{\alpha \in G \mid \alpha e \alpha^{-1} = e \text{ for every } e \in E\}$. If $E = \{e\}$, we just write $Z_G(e) = Z_G(\{e\})$ to refer to the centralizer of e. We also recall that the center of G is denoted by $Z(G)$.

Lemma 4.1. *Suppose that Γ is connected and different from a single vertex. Let U be a finite index subgroup of $\overline{\mathrm{CA}[\Gamma]}$. Then $Z(U) = Z_{\overline{\mathrm{CA}[\Gamma]}}(U) = \{1\}$.*

Proof. We just need to show that $Z_{\overline{\mathrm{CA}[\Gamma]}}(U) = \{1\}$, because $Z(U) \subset Z_{\overline{\mathrm{CA}[\Gamma]}}(U)$. This follows directly from the fourth statement of Proposition 3.1. Indeed, as U has finite index, any α element in $Z(U)$ is sent to the class $(G, G, 1)$ via the homomorphism ι of the aforementioned proposition. Hence α sits in the kernel of ι, which is trivial. ■

Lemma 4.2. *Let G be a group, let G_1, G_2 be two subgroups of G such that $G = G_1 \times G_2$, and let H be a subgroup of G. Then*

$$Z_G(H) = (Z_G(H) \cap G_1) \times (Z_G(H) \cap G_2).$$

Proof. The inclusion $(Z_G(H) \cap G_1) \times (Z_G(H) \cap G_2) \subset Z_G(H)$ is obvious. Then we just need to show that $Z_G(H) \subset (Z_G(H) \cap G_1) \times (Z_G(H) \cap G_2)$. Let $\alpha \in Z_G(H)$ and $\gamma \in H$. We write $\alpha = (\alpha_1, \alpha_2)$ and $\gamma = (\gamma_1, \gamma_2)$ with $\alpha_1, \gamma_1 \in G_1$ and $\alpha_2, \gamma_2 \in G_2$. We have $1 = \alpha\gamma\alpha^{-1}\gamma^{-1} = (\alpha_1\gamma_1\alpha_1^{-1}\gamma_1^{-1}, \alpha_2\gamma_2\alpha_2^{-1}\gamma_2^{-1})$, hence $\alpha_1\gamma_1\alpha_1^{-1}\gamma_1^{-1} = 1$. Moreover, $\alpha_1\gamma_2\alpha_1^{-1}\gamma_2^{-1} = 1$, because $\alpha_1 \in G_1$ and $\gamma_2 \in G_2$, so $\alpha_1\gamma\alpha_1^{-1}\gamma^{-1} = 1$. Thus, $\alpha_1 \in (Z_G(H) \cap G_1)$. Analogously, we can prove that $\alpha_2 \in (Z_G(H) \cap G_2)$. ■

Proof of Theorem 2.3. We suppose that Γ is connected and different from a single vertex. Let U be a finite index subgroup of $\overline{\mathrm{CA}[\Gamma]}$. Let U_1, U_2 be two subgroups of U such that $U = U_1 \times U_2$ and let $\tilde{U} = U \times Z(\mathrm{CA}[\Gamma])$, which is included in $\overline{\mathrm{CA}[\Gamma]} \times Z(\mathrm{CA}[\Gamma]) = \mathrm{CA}[\Gamma]$. Let $\tilde{U}_1 = U_1$ and $\tilde{U}_2 = U_2 \times Z(\mathrm{CA}[\Gamma])$, having $\tilde{U} = \tilde{U}_1 \times \tilde{U}_2$. As $\mathrm{CA}[\Gamma]$ has finite index in $A[\Gamma]$, $\tilde{U}$ has finite index in $A[\Gamma]$ and, by applying Theorem 5B of [29], we know that either $\tilde{U}_1 \subset Z(A[\Gamma])$ or $\tilde{U}_2 \subset Z(A[\Gamma])$. Also by Proposition 3.1, we have that $Z(A[\Gamma]) \cap \tilde{U} = Z(\tilde{U}) \subset Z(A[\Gamma]) \cap \mathrm{CA}[\Gamma] = Z(\mathrm{CA}[\Gamma])$. Then $\tilde{U}_1 \subset Z(\mathrm{CA}[\Gamma])$ or $\tilde{U}_2 \subset Z(\mathrm{CA}[\Gamma])$, so $U_1 = \{1\}$ or $U_2 = \{1\}$. This shows the first part of the theorem. We still have to prove the second part.

Let $\Gamma_1, \dots, \Gamma_p$ be the connected components of Γ. We suppose that every $\Gamma_1, \dots, \Gamma_k$ has at least two vertices and that each of $\Gamma_{k+1}, \dots, \Gamma_p$ is reduced to a single vertex. We have that $\mathrm{CA}[\Gamma] = \mathrm{CA}[\Gamma_1] \times \cdots \times \mathrm{CA}[\Gamma_p]$. By Proposition 3.1, $\mathrm{CA}[\Gamma_i] = \overline{\mathrm{CA}[\Gamma_i]} \times Z(\mathrm{CA}[\Gamma_i])$ for every $i \in \{1, \dots, k\}$ and $\mathrm{CA}[\Gamma_i] = Z(\mathrm{CA}[\Gamma_i])$ for every $i \in \{k+1, \dots, p\}$, hence

$$\mathrm{CA}[\Gamma] = \overline{\mathrm{CA}[\Gamma_1]} \times \cdots \times \overline{\mathrm{CA}[\Gamma_k]} \times Z(\mathrm{CA}[\Gamma_1]) \times \cdots \times Z(\mathrm{CA}[\Gamma_p])\,. \tag{4.1}$$

Then, $\overline{\mathrm{CA}[\Gamma_i]}$ is strongly indecomposable for every $i \in \{1, \dots, k\}$. Moreover, $Z(\mathrm{CA}[\Gamma_i])$ is strongly indecomposable, because we have that $Z(\mathrm{CA}[\Gamma_i]) \simeq \mathbb{Z}$, for every $i \in \{1, \dots, p\}$.

Therefore, (4.1) is a strong Remak decomposition of $A[\Gamma]$. Now, we take a strong Remak decomposition of $A[\Gamma]$ of the form $H = H_1 \times \cdots \times H_m$ and turn to prove that it is equivalent to (4.1).

Claim 1. We can assume that $H \subset \mathrm{CA}[\Gamma]$.

Proof of Claim 1. Let $H_i' = H_i \cap \mathrm{CA}[\Gamma]$ for every $i \in \{1, \dots, m\}$ and $H' = H_1' \times \cdots \times H_m'$. Since $\mathrm{CA}[\Gamma]$ is a finite index subgroup of $A[\Gamma]$, H_i' has finite index in H_i for every $i \in \{1, \dots, m\}$. This means that H' has finite index in H and therefore H' has finite index in $A[\Gamma]$. As H_i is strongly indecomposable and H_i' has finite index in H_i, H_i' is strongly indecomposable, for every $i \in \{1, \dots, m\}$. Then $H' = H_1' \times \cdots \times H_m'$ is a strong Remak decomposition of $A[\Gamma]$. By construction, this decomposition is equivalent to $H = H_1 \times \cdots \times H_m$ and H' is included in $\mathrm{CA}[\Gamma]$. This finishes the proof of Claim 1.

Let $\tilde{B} = Z(\mathrm{CA}[\Gamma_1]) \times \cdots \times Z(\mathrm{CA}[\Gamma_p]) \simeq \mathbb{Z}^p$ and $B = H \cap \tilde{B}$. Set $K_i = H \cap \overline{\mathrm{CA}[\Gamma_i]}$ for every $i \in \{1, \dots, k\}$. As H has finite index in $\mathrm{CA}[\Gamma]$, K_i has finite index in $\overline{\mathrm{CA}[\Gamma_i]}$ for every $i \in \{1, \dots, k\}$ and B has finite index in $\tilde{B}$.

Claim 2. We have that $Z(H) = B$.

Proof of Claim 2. Let $\alpha \in Z(H) \subset \mathrm{CA}[\Gamma]$. Then, by Lemma 4.2, α can be expressed as $\alpha = \alpha_1 \cdots \alpha_k \beta$, where $\alpha_i \in \overline{\mathrm{CA}[\Gamma_i]} \cap Z_{\mathrm{CA}[\Gamma]}(H)$ for every $i \in \{1, \dots, k\}$ and $\beta \in \tilde{B}$. Since $K_i \subset H$, α_i commutes with every element in K_i, so $\alpha_i \in Z_{\overline{\mathrm{CA}[\Gamma_i]}}(K_i)$. By Lemma 4.1, $Z_{\overline{\mathrm{CA}[\Gamma_i]}}(K_i) = \{1\}$, hence $\alpha_i = 1$. Therefore, $\alpha = \beta \in \tilde{B} \cap H = B$. This proves that $Z(H) \subset B$. To see $B \subset Z(H)$, just notice that $B = Z(\mathrm{CA}[\Gamma]) \cap H \subset Z(H)$, because $H \subset \mathrm{CA}[\Gamma]$ by Claim 1. This finishes the proof of Claim 2.

Let $\hat{K}_i = K_1 \times \cdots \times K_{i-1} \times K_{i+1} \times \cdots \times K_k \times B$ and $L_i = (\overline{\mathrm{CA}[\Gamma_i]} \times \tilde{B}) \cap H$, for every $i \in \{1, \dots, k\}$.

Claim 3. Let $i \in \{1, \dots, k\}$. Then $Z_H(\hat{K}_i) = L_i$ and $L_i = (L_i \cap H_1) \times \cdots \times (L_i \cap H_m)$.

Proof of Claim 3. By Lemma 4.2, we have that

$$Z_{\mathrm{CA}[\Gamma]}(\hat{K}_i) = (Z_{\mathrm{CA}[\Gamma]}(\hat{K}_i) \cap \overline{\mathrm{CA}[\Gamma_1]}) \times \cdots \times (Z_{\mathrm{CA}[\Gamma]}(\hat{K}_i) \cap \overline{\mathrm{CA}[\Gamma_k]}) \times (Z_{\mathrm{CA}[\Gamma]}(\hat{K}_i) \cap \tilde{B}).$$

Let $j \in \{1, \dots, k\}$ be such that $j \neq i$. Then, by Lemma 4.1, $(Z_{\mathrm{CA}[\Gamma]}(\hat{K}_i) \cap \overline{\mathrm{CA}[\Gamma_j]}) \subset Z_{\overline{\mathrm{CA}[\Gamma_j]}}(K_j) = \{1\}$. Moreover, $(Z_{\mathrm{CA}[\Gamma]}(\hat{K}_i) \cap \overline{\mathrm{CA}[\Gamma_i]}) = \overline{\mathrm{CA}[\Gamma_i]}$ and $(Z_{\mathrm{CA}[\Gamma]}(\hat{K}_i) \cap \tilde{B}) = \tilde{B}$. Therefore $Z_{\mathrm{CA}[\Gamma]}(\hat{K}_i) = \overline{\mathrm{CA}[\Gamma_i]} \times \tilde{B}$ and $Z_H(\hat{K}_i) = Z_{\mathrm{CA}[\Gamma]}(\hat{K}_i) \cap H = L_i$. Finally, by Lemma 4.2, $L_i = Z_H(\hat{K}_i) = (L_i \cap H_1) \times \cdots \times (L_i \cap H_m)$. This finishes the proof of Claim 3.

Claim 4. Let $i \in \{1, \dots, k\}$. Then $Z(L_i) = B$ and L_i/B is strongly indecomposable. Also, there is $\chi(i) \in \{1, \dots, m\}$ such that $L_i/B = (L_i \cap H_{\chi(i)})/Z(H_{\chi(i)})$ and $L_i \cap H_j = Z(H_j)$ if $j \neq \chi(i)$.

Proof of Claim 4. Since $B \subset Z(\mathrm{CA}[\Gamma])$ and $B \subset L_i \subset H$, we have $B \subset Z(L_i)$. Now, take $\alpha \in Z(L_i)$. As L_i is a subgroup of $\overline{\mathrm{CA}[\Gamma_i]} \times \tilde{B}$, by Lemma 4.2 we can write α in the form $\alpha = \alpha_i \beta$, where $\alpha_i \in \overline{\mathrm{CA}[\Gamma_i]} \cap Z_{CA[\Gamma]}(L_i)$ and $\beta \in \tilde{B}$. Since $K_i \subset L_i$, α_i commutes with every element in K_i, so $\alpha_i \in Z_{\overline{\mathrm{CA}[\Gamma_i]}}(K_i)$. By Lemma 4.1, $Z_{\overline{\mathrm{CA}[\Gamma_i]}}(K_i) = \{1\}$, hence $\alpha_i = 1$. Therefore, $\alpha = \beta \in \tilde{B} \cap H = B$ and then $Z(L_i) \subset B$.

Let $\pi: \overline{\mathrm{CA}[\Gamma_i]} \times \tilde{B} \to \overline{\mathrm{CA}[\Gamma_i]}$ be the projection homomorphism and let π' be the restriction of π to L_i. Then $\mathrm{Ker}(\pi') = \mathrm{Ker}(\pi) \cap L_i = \tilde{B} \cap L_i \subset Z(L_i) = B$. On the other hand, $B \subset \tilde{B} \cap L_i$, hence $\mathrm{Ker}(\pi') = B$. Using the first isomorphism theorem, we have that $L_i/B \simeq \pi(L_i)$. As L_i has finite index in $\overline{\mathrm{CA}[\Gamma_i]} \times \tilde{B}$, $\pi(L_i)$ has finite index in $\overline{\mathrm{CA}[\Gamma_i]}$, which is strongly indecomposable. This implies that L_i/B is strongly indecomposable.

By Claim 3, we have that $L_i = (L_i \cap H_1) \times \cdots \times (L_i \cap H_m)$. If we quotient this equality by $B = Z(H) = Z(H_1) \times \cdots \times Z(H_m)$, we get $L_i/B = (L_i \cap H_1)/Z(H_1) \times \cdots \times (L_i \cap H_m)/Z(H_m)$. We already know that L_i/B is strongly indecomposable, so there is $\chi(i) \in \{1, \dots, m\}$ such that $L_i/B = (L_i \cap H_{\chi(i)})/Z(H_{\chi(i)})$ and $(L_i \cap H_j)/Z(H_j) = \{1\}$ if $j \neq \chi(i)$. This implies that $L_i \cap H_j \subset Z(H_j)$ if $j \neq \chi(i)$. Finally, notice that $Z(H_i) \subset B \subset L_i$. This proves Claim 4.

For $j \in \{1, \dots, m\}$, we denote by $f_j: H \to H_j$ the projection of H on H_j. Let $K = K_1 \times \cdots \times K_k \times B$. For $i \in \{1, \dots, k\}$ we denote by $g_i: K \to K_i$ the projection of K on K_i, and we denote by $h: K \to B$ the projection of K on B. Notice that, since $K_i \times B$ is a subgroup of L_i, K_i is a subgroup of L_i/B. Then, K_i injects into L_i and into L_i/B, which by Claim 4 is isomorphic to $(L_i \cap H_{\chi(i)})/Z(H_{\chi(i)})$. This means that the composition

$$K_i \hookrightarrow L_i \xrightarrow{f_{\chi(i)}} L_i \cap H_{\chi(i)}$$

has to be injective. In other words, the restriction $f_{\chi(i)}|_{K_i} : K_i \to H_{\chi(i)}$ is injective. Also by Claim 4, if $j \neq \chi(i)$,

$$K_i \hookrightarrow L_i \xrightarrow{f_j} L_i \cap H_j = Z(H_j)$$

Hence, we have $f_j(K_i) \subset Z(H_j)$.

Let $\psi_i: K_i \to B$ be the map defined by $\psi_i(\alpha) = \prod_{j \neq \chi(i)} f_j(\alpha)^{-1}$. As B is an abelian group, ψ_i is a well-defined homomorphism. Let $\psi: K \to B$ be the map defined by $\psi(\alpha) = \prod_{i=1}^{k} (\psi_i \circ g_i)(\alpha)$. Then, again, ψ is a well-defined homomorphism because B is abelian. Also, notice that $\psi(\beta) = 1$ for every $\beta \in B$. If $\varphi: K \to K$ is the map defined by $\varphi(\alpha) = \alpha\psi(\alpha)$, it is clear that φ is a homomorphism. In addition, as $\psi(\beta) = 1$ for every $\beta \in B$, φ is invertible and φ^{-1} is defined by $\varphi^{-1}(\alpha) = \alpha\, \psi(\alpha)^{-1}$.

Claim 5. For every $i \in \{1, \dots, k\}$ we have $\varphi(K_i) \subset H_{\chi(i)}$.

Proof of Claim 5. Let $i \in \{1, \dots, k\}$ and $\alpha \in K_i$. For $\ell \in \{1, \dots, k\}$, $\ell \neq i$, we have that $g_\ell(\alpha) = 1$, then $(\psi_\ell \circ g_\ell)(\alpha) = 1$ and $\psi(\alpha) = \psi_i(\alpha)$. Moreover, $\alpha = \prod_{j=1}^{m} f_j(\alpha)$, having $\varphi(\alpha) = f_{\chi(i)}(\alpha) \in H_{\chi(i)}$. This finishes the proof of Claim 5.

Up to applying φ, we can assume that $K_i \subset H_{\chi(i)}$ for every $i \in \{1, \dots, k\}$.

Claim 6.

(1) For every $i \in \{1, \dots, k\}$ we have $f_{\chi(i)}(K) = K_i$ and $f_{\chi(i)}(\hat{K}_i) = \{1\}$. Moreover, K_i is a finite index subgroup of $H_{\chi(i)}$.

(2) For every $i, j \in \{1, \dots, k\}$, $i \neq j$, we have $\chi(i) \neq \chi(j)$.

Proof of Claim 6. As K has finite index in H, $f_{\chi(i)}(K)$ has finite index in $H_{\chi(i)}$. Notice also that $f_{\chi(i)}(K_i) = K_i$ because $K_i \subset H_{\chi(i)}$. We have that $K = K_i \times \hat{K}_i$, so $f_{\chi(i)}(K) = f_{\chi(i)}(K_i)\, f_{\chi(i)}(\hat{K}_i) = K_i\, f_{\chi(i)}(\hat{K}_i)$ and $[K_i, f_{\chi(i)}(\hat{K}_i)] = \{1\}$. Moreover, by Lemma 4.1, $K_i \cap f_{\chi(i)}(\hat{K}_i) \subset Z(K_i) = \{1\}$. Hence, $f_{\chi(i)}(K) \simeq K_i \times f_{\chi(i)}(\hat{K}_i)$. As $H_{\chi(i)}$ is strongly indecomposable, we have that $f_{\chi(i)}(\hat{K}_i) = \{1\}$ and $f_{\chi(i)}(K) = K_i$. On the other hand, let $j \in \{1, \dots, k\}$ be such that $j \neq i$. As K_j is a subgroup of $\hat{K}_i$, we have that $f_{\chi(i)}(K_j) = \{1\} \neq K_j$, and then $\chi(i) \neq \chi(j)$. This finishes the proof of Claim 6.

By the results from above, $m \geq k$ and we can suppose that $\chi(i) = i$ for every $i \in \{1, \dots, k\}$, up to renumbering the H_i's. Recall that we have that $B = Z(H) = Z(H_1) \times \cdots \times Z(H_m)$ and $Z(H_i) = f_i(B)$ for every $i \in \{1, \dots, m\}$. If $i \in \{1, \dots, k\}$, then $B \subset \hat{K}_i$, so by Claim 6, $\{1\} = f_i(B) = Z(H_i)$. Hence, $B = Z(H_{k+1}) \times \cdots \times Z(H_m)$. We also have that $K = K_1 \times \cdots \times K_k \times B$ is a subgroup of $K_1 \times \cdots \times K_k \times H_{k+1} \times \cdots \times H_m$, and that K is a finite index subgroup of $H = H_1 \times \cdots \times H_k \times H_{k+1} \times \cdots \times H_m$. Therefore B has finite index in $H_{k+1} \times \cdots \times H_m$.

For $j \in \{k+1, \dots, m\}$, we let $B_j = B \cap H_j$. As B is a finite index subgroup of $H_{k+1} \times \cdots \times H_m$, B_j has finite index in H_j. In addition, as H_j is strongly indecomposable, B_j is indecomposable. The group B_j is a subgroup of $B \simeq \mathbb{Z}^p$ and it is indecomposable, so $B_j \simeq \mathbb{Z}$. Let $B' = B_{k+1} \times \cdots \times B_m$. Then B' is a finite index subgroup of $H_{k+1} \times \cdots \times H_m$ and B' has finite index in B. As $B' \simeq \mathbb{Z}^{m-k}$ and $B \simeq \mathbb{Z}^p$, it follows that $m - k = p$.

For $i \in \{1, \dots, k\}$, K_i is a finite index subgroup of both H_i and $\overline{\mathrm{CA}[\Gamma_i]}$, so H_i and $\overline{\mathrm{CA}[\Gamma_i]}$ are commensurable. Moreover, for every $i \in \{k+1, \dots, m\}$, we have that $Z(\mathrm{CA}(\Gamma_{i-k})) \simeq \mathbb{Z} \simeq B_i$, and B_i is a finite index subgroup of H_i. Therefore $Z(\mathrm{CA}[\Gamma_{i-k}])$ and H_i are commensurable. ∎

5. Reduction to the connected case

Theorem 2.2 is a consequence of Theorem 2.3 and the following proposition.

Proposition 5.1. *Let G and G' be two infinite groups. We suppose that G (respectively, G') has a unique strong Remak decomposition up to equivalence, $H = H_1 \times \cdots \times H_p$ (respectively, $H' = H'_1 \times \cdots \times H'_q$). Then G is commensurable with G' if and only if $p = q$ and, up to permutation of the factors, H_i is commensurable with H'_i for every $i \in \{1, \dots, p\}$.*

Proof. Suppose that G and G' are commensurable. There is a finite index subgroup K of G and a finite index subgroup K' of G' such that $K \simeq K'$. Let $\varphi\colon K \to K'$ be an isomorphism between K and K'. For every $i \in \{1, \dots, p\}$ we take $K_i = K \cap H_i$ and $U = K_1 \times \cdots \times K_p$. As K has finite index in G, K_i has finite index in H_i. It follows that U is a finite index subgroup of H (and of G) and $U = K_1 \times \cdots \times K_p$ is a strong Remak decomposition of G. The group U is a finite index subgroup of K, so $\varphi(U) = \varphi(K_1) \times \cdots \times \varphi(K_p)$ is a finite index subgroup of $\varphi(K) = K'$ and then also a finite index subgroup of G'. The subgroups $\varphi(K_i)$ ($i \in \{1, \dots, p\}$) are strongly indecomposable, hence $\varphi(U) = \varphi(K_1) \times \cdots \times \varphi(K_p)$ is a strong Remak decomposition of G'. As G' has only one decomposition of that form (up to equivalence), we have that $p = q$ and $\varphi(K_i)$ is commensurable with H_i' for every $i \in \{1, \dots, p\}$, up to permutation of the factors. Also, as $K_i \simeq \varphi(K_i)$ is a finite index subgroup of H_i, it follows that H_i and H_i' are commensurable for every $i \in \{1, \dots, p\}$.

Suppose that $p = q$ and that H_i is commensurable with H_i' for every $i \in \{1, \dots, p\}$. There is a finite index subgroup K_i of H_i and a finite index subgroup K_i' of H_i' such that $K_i \simeq K_i'$. Take $U = K_1 \times \cdots \times K_p$ and $U' = K_1' \times \cdots \times K_p'$. As K_i has finite index in H_i for every i, the subgroup U has finite index in H and also has finite index in G. Analogously, U' has finite index in G'. It is obvious that U and U' are isomorphic. Therefore, G and G' are commensurable. ∎

Proof of Theorem 2.2. Let Γ, Ω be two Coxeter graphs of spherical type. Let $\Gamma_1, \dots, \Gamma_p$ be the connected components of Γ and $\Omega_1, \dots, \Omega_q$ be the connected components of Ω. If $p = q$ and $A[\Gamma_i]$ and $A[\Omega_i]$ are commensurable for every $i \in \{1, \dots, p\}$, then it is clear that $A[\Gamma]$ and $A[\Omega]$ are commensurable. Then suppose that $A[\Gamma]$ and $A[\Omega]$ are commensurable. We need to show that $p = q$ and that $A[\Gamma_i]$ and $A[\Omega_i]$ are commensurable for every $i \in \{1, \dots, p\}$ up to permutation of the indices.

Suppose that every $\Gamma_1, \dots, \Gamma_k$ has at least two vertices and that each of $\Gamma_{k+1}, \dots, \Gamma_p$ is reduced to a single vertex. Analogously, suppose that every $\Omega_1, \dots, \Omega_\ell$ has at least two vertices and that each of $\Omega_{\ell+1}, \dots, \Omega_q$ is reduced to a single vertex. By Theorem 2.3, $\mathrm{CA}[\Gamma] = \overline{\mathrm{CA}[\Gamma_1]} \times \cdots \times \overline{\mathrm{CA}[\Gamma_k]} \times Z(\mathrm{CA}[\Gamma_1]) \times \cdots \times Z(\mathrm{CA}[\Gamma_p])$ and $\mathrm{CA}[\Omega] = \overline{\mathrm{CA}[\Omega_1]} \times \cdots \times \overline{\mathrm{CA}[\Omega_\ell]} \times Z(\mathrm{CA}[\Omega_1]) \times \cdots \times Z(\mathrm{CA}[\Omega_q])$ are strong Remak decompositions of $A[\Gamma]$ and $A[\Omega]$, respectively, and they are unique up to equivalence. Let $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, q\}$. Let U be a finite index subgroup of $\overline{\mathrm{CA}[\Gamma_i]}$ and let V be a finite index subgroup of $Z(\mathrm{CA}[\Omega_j])$. By Lemma 4.1, we have that $Z(U) = \{1\}$. Moreover, V is a finite index subgroup of $Z(\mathrm{CA}[\Omega_j]) \simeq \mathbb{Z}$, hence $V \simeq \mathbb{Z}$. Then, U and V are not isomorphic. This shows $\overline{\mathrm{CA}[\Gamma_i]}$ and $Z(\mathrm{CA}[\Omega_j])$ are not commensurable.

By applying Proposition 5.1, we know that $k = \ell$, $p = q$ and that $\overline{\mathrm{CA}[\Gamma_i]}$ and $\overline{\mathrm{CA}[\Omega_i]}$ are commensurable for every $i \in \{1, \dots, k\}$, up to permutation of the indices. Let $i \in \{1, \dots, k\}$. As $\overline{\mathrm{CA}[\Gamma_i]}$ and $\overline{\mathrm{CA}[\Omega_i]}$ are commensurable, by Corollary 3.2, $\overline{A[\Gamma_i]}$ and $\overline{A[\Omega_i]}$ are commensurable. Then, by Proposition 3.1, $A[\Gamma_i]$ and $A[\Omega_i]$ are commensurable. Let $i \in \{k+1, \dots, p\}$. Thus, $A[\Gamma_i] \simeq \mathbb{Z} \simeq A[\Omega_i]$, having that $A[\Gamma_i]$ and $A[\Omega_i]$ are commensurable. ∎

Remark 5.2. An alternative proof of Theorem 2.2, based on Theorem B of [21], has been communicated to us by one of the referees. The idea is as follows. Consider the decomposition (4.1) given in the proof of Theorem 2.3. We can show using [9], [38] and Proposition 4.2 in [21] that each $\overline{\mathrm{CA}[\Gamma_i]}$ is of coarse type I. It follows by Theorem B in [21] that the decomposition (4.1) is unique up to quasi-isometry. To pass from quasi-isometry to commensurability we have to apply again Theorem B in [21] in the following manner. Let Γ and Ω be two Coxeter graphs of spherical type. We assume that $\mathrm{CA}[\Gamma]$ and $\mathrm{CA}[\Omega]$ are commensurable and we consider the same decompositions of $\mathrm{CA}[\Gamma]$ and $\mathrm{CA}[\Omega]$ as in the above proof. Then, there exist finite index subgroups U of $\mathrm{CA}[\Gamma]$ and V of $\mathrm{CA}[\Omega]$ and an isomorphism $f : U \to V$. We extend f to a quasi-isometry $f : \mathrm{CA}[\Gamma] \to \mathrm{CA}[\Omega]$. By applying Theorem B in [21], we obtain that $p = q$, $k = \ell$ and, for each $i \in \{i, \dots, k\}$ there exists $j \in \{1, \dots, k\}$ such that the composition $\overline{\mathrm{CA}[\Gamma_i]} \to \mathrm{CA}[\Gamma] \to \mathrm{CA}[\Omega] \to \overline{\mathrm{CA}[\Omega_j]}$ is a quasi-isometry, where the first map is the inclusion, the second map is f, and the third map is the projection. This map restricted to $U \cap \overline{\mathrm{CA}[\Gamma_i]}$ is an injective homomorphism whose image must be of finite index in $\mathrm{CA}[\Omega_j]$.

6. Comparison with the Artin group of type A_n

Let $n \in \mathbb{N}$, $n \geq 2$, and let Γ be a connected Coxeter graph of spherical type with n vertices. Recall that the aim of this section is to determine whether $A[\Gamma]$ and $A[A_n]$ are commensurable or not. We start with the cases where $A[\Gamma]$ and $A[A_n]$ are commensurable.

Lemma 6.1. *Let $n \geq 2$. Then $A[A_n]$ and $A[B_n]$ are commensurable.*

Proof. Let $\theta\colon A[A_n] \to W[A_n]$ be the quotient homomorphism and let H be the subgroup of $W[A_n]$ generated by $\{s_2, \dots, s_n\}$. By [25], $\theta^{-1}(H)$ is isomorphic to $A[B_n]$. It has finite index in $A[A_n]$ because $W[A_n]$ is finite, hence $A[A_n]$ and $A[B_n]$ are commensurable. ∎

Lemma 6.2. *Let $p \geq 5$. Then $A[A_2]$ and $A[I_2(p)]$ are commensurable.*

Proof. Let $\Gamma = I_2(p)$. Then $A[\Gamma] = \langle s, t \mid \Pi(s, t, p) = \Pi(t, s, p) \rangle$. We consider the construction of the proof of Proposition 3.1 (2). Let $V = \mathbb{R}e_s \oplus \mathbb{R}e_t$. By [5], $W = W[\Gamma]$ has a faithful linear representation $\rho\colon W \to \mathrm{GL}(V)$, and $\rho(W)$ is generated by reflections. In our case, W is the dihedral group of order $2p$ and $\rho\colon W \to \mathrm{GL}(V)$ is the standard representation of W. Let $\mathcal{H}$ be the set of reflection lines of W. Take $V_{\mathbb{C}} = \mathbb{C} \otimes V$, $H_{\mathbb{C}} = \mathbb{C} \otimes H$ for every $H \in \mathcal{H}$, and $M = V_{\mathbb{C}} \setminus (\cup_{H \in \mathcal{H}} H_{\mathbb{C}})$. Let $h\colon V_{\mathbb{C}} \setminus \{0\} \to \mathbb{P}V_{\mathbb{C}}$ be the Hopf fibration and $\overline{M} = h(M)$. Thanks to the proof of Proposition 3.1 (2), we know that $\pi_1(\overline{M}) = \overline{\mathrm{CA}[\Gamma]}$.

In this case, $\mathbb{P}V_{\mathbb{C}}$ is the complex projective line and $\overline{M}$ is the complement of $|\mathcal{H}| = p$ points in $\mathbb{P}V_{\mathbb{C}}$, hence $\overline{\mathrm{CA}[\Gamma]} = \pi_1(\overline{M})$ is isomorphic to the free group F_{p-1} of rank $p-1$. Analogously, $\overline{\mathrm{CA}[A_2]}$ is isomorphic to F_2. As F_{p-1} is isomorphic to a finite index subgroup of F_2, it follows that $\overline{\mathrm{CA}[\Gamma]}$ and $\overline{\mathrm{CA}[A_2]}$ are commensurable. By Corollary 3.2, we have that $\overline{A[A_2]}$ and $\overline{A[\Gamma]}$ are commensurable. Therefore, by Proposition 3.1, $A[A_2]$ and $A[\Gamma]$ are commensurable. ∎

Let $\Sigma = \Sigma_{g,b}$ be the orientable surface of genus g with b boundary components. Let $\mathcal{P}_n$ be a collection of n different points in the interior of Σ. Recall that the *mapping class group* of the pair $(\Sigma, \mathcal{P}_n)$, denoted by $\mathcal{M}(\Sigma, \mathcal{P}_n)$, is the group of isotopy classes of homeomorphisms $h\colon \Sigma \to \Sigma$ that preserve the orientation, fix the boundary of Σ pointwise and preserve $\mathcal{P}_n$ setwise. The *extended mapping class group* of the pair $(\Sigma, \mathcal{P}_n)$, denoted by $\mathcal{M}^*(\Sigma, \mathcal{P}_n)$, is the group of isotopy classes of homeomorphisms $h\colon \Sigma \to \Sigma$ that fix the boundary of Σ pointwise and preserve $\mathcal{P}_n$ setwise. Notice that, if the surface Σ has non-empty boundary, the homeomorphisms fixing this boundary pointwise cannot change the orientation of Σ and we have $\mathcal{M}^*(\Sigma, \mathcal{P}_n) = \mathcal{M}(\Sigma, \mathcal{P}_n)$. Otherwise, $\mathcal{M}(\Sigma, \mathcal{P}_n)$ has index 2 in $\mathcal{M}^*(\Sigma, \mathcal{P}_n)$.

Denote by $\mathfrak{S}_n$ the permutation group of $\{1, \dots, n\}$. The action of $\mathcal{M}^*(\Sigma, \mathcal{P}_n)$ on $\mathcal{P}_n$ induces a homomorphism $\theta'\colon \mathcal{M}^*(\Sigma, \mathcal{P}_n) \to \mathfrak{S}_n$, whose kernel is denoted by $\mathcal{PM}^*(\Sigma, \mathcal{P}_n)$.

On the other hand, we can define another homomorphism $\omega\colon \mathcal{M}^*(\Sigma,\mathcal{P}_n)\to\{\pm 1\}$ sending an element $h\in\mathcal{M}^*(\Sigma,\mathcal{P}_n)$ to 1 if it preserves the orientation and to -1 otherwise. Notice that the kernel of ω is $\mathcal{M}(\Sigma,\mathcal{P}_n)$. These two homomorphisms lead to the construction of the homomorphism $\hat\theta\colon \mathcal{M}^*(\Sigma,\mathcal{P}_n)\to\mathfrak{S}_n\times\{\pm 1\}$ defined by $h\mapsto(\theta'(h),\omega(h))$. The kernel of $\hat\theta$ is called the *pure mapping class group* of the pair $(\Sigma,\mathcal{P}_n)$ and it is denoted by $\mathcal{PM}(\Sigma,\mathcal{P}_n)$.

These mapping class groups and the problem that we are studying are related by the following theorem.

Theorem 6.3 ([12]). *Let $\Sigma=\Sigma_{0,0}$ and let $\mathcal{P}_{n+2}$ be a family of $n+2$ points in Σ. Then* $\mathrm{Com}(\overline{A[A_n]})\simeq\mathcal{M}^*(\Sigma,\mathcal{P}_{n+2})$.

Lemma 6.4. *Let $\Sigma=\Sigma_{0,0}$ and let $\mathcal{P}_{n+2}$ be a family of $n+2$ points in Σ. Then* $\mathrm{Ker}(\hat\theta)=\mathcal{PM}(\Sigma,\mathcal{P}_{n+2})\simeq\overline{\mathrm{CA}[A_n]}$.

Proof. Let $\mathcal{B}_{n+1}$ be the braid group on $n+1$ strands. By [1], $\mathcal{M}(\Sigma_{0,1},\mathcal{P}_{n+1})=\mathcal{B}_{n+1}=A[A_n]$ and $\mathcal{PM}(\Sigma_{0,1},\mathcal{P}_{n+1})=\mathrm{CA}[A_n]$. Let δ be the standard generator of $Z(A[A_n])$. It is well known that $\delta\in\mathrm{CA}[A_n]$ and $Z(A[A_n])=Z(\mathrm{CA}[A_n])=\langle\delta\rangle$. Notice that δ, seen as an element of $\mathcal{PM}(\Sigma_{0,1},\mathcal{P}_{n+1})$, is the Dehn twist about the boundary component of $\Sigma_{0,1}$. Then $\overline{\mathrm{CA}[A_n]}=\mathcal{PM}(\Sigma_{0,1},\mathcal{P}_{n+1})/\langle\delta\rangle=\mathcal{PM}(\Sigma_{0,0},\mathcal{P}_{n+2})$; see, for example [36]. ∎

Let G be a group. We say that an element $\alpha\in G$ is a *generalized torsion* element if there are $p\ge 1$ and $\beta_1,\dots,\beta_p\in G$ such that $(\beta_1\alpha\beta_1^{-1})(\beta_2\alpha\beta_2^{-1})\cdots(\beta_p\alpha\beta_p^{-1})=1$. We say that G has *generalized torsion* if it contains a non-trivial generalized torsion element. For most of our cases, the criterion we will use to show that $A[\Gamma]$ and $A[A_n]$ are not commensurable is given by the following two results.

Lemma 6.5. *Let Γ be a connected Coxeter graph of spherical type with n vertices. Let $\Phi\colon\overline{A[\Gamma]}\to\mathcal{M}^*(\Sigma_{0,0},\mathcal{P}_{n+2})$ be a homomorphism and set $\varphi=\hat\theta\circ\Phi:\overline{A[\Gamma]}\to\mathfrak{S}_{n+2}\times\{\pm 1\}$. If $\mathrm{Ker}(\varphi)$ has generalized torsion, then Φ is not injective.*

Proof. Assume that Φ is injective and that $\mathrm{Ker}(\varphi)$ has generalized torsion. As $\overline{\mathrm{CA}[A_n]}=\mathrm{Ker}(\hat\theta)$, the homomorphism Φ induces an injective homomorphism $\Phi'\colon\mathrm{Ker}(\varphi)\to\overline{\mathrm{CA}[A_n]}$. Recall that a group is called *biorderable* if it admits a total ordering invariant under left-multiplication and right-multiplication. We know by [37] that $\mathrm{CA}[A_n]$ is biorderable. By Proposition 3.1, $\overline{\mathrm{CA}[A_n]}$ is a subgroup of $\mathrm{CA}[A_n]$, hence $\overline{\mathrm{CA}[A_n]}$ is also biorderable, having that $\mathrm{Ker}(\varphi)$ is biorderable. However, a non-trivial biorderable group has no generalized torsion [37]. This is a contradiction. ∎

Corollary 6.6. *Let Γ be a connected Coxeter graph of spherical type with n vertices. If the kernel of every homomorphism $\varphi\colon\overline{A[\Gamma]}\to\mathfrak{S}_{n+2}\times\{\pm 1\}$ has generalized torsion, then $A[\Gamma]$ and $A[A_n]$ are not commensurable.*

Proof. Assume that $A[\Gamma]$ and $A[A_n]$ are commensurable. By Proposition 3.1, $\overline{A[\Gamma]}$ injects in $\mathrm{Com}(\overline{A[\Gamma]})$. Again by Proposition 3.1, $\overline{A[\Gamma]}$ and $\overline{A[A_n]}$ are commensurable, so we have $\mathrm{Com}(\overline{A[\Gamma]})\simeq\mathrm{Com}(\overline{A[A_n]})$. Moreover, by Theorem 6.3, we know that $\mathrm{Com}(\overline{A[A_n]})=\mathcal{M}^*(\Sigma_{0,0},\mathcal{P}_{n+2})$. Then, there is an injective homomorphism $\Phi\colon\overline{A[\Gamma]}\to\mathcal{M}^*(\Sigma_{0,0},\mathcal{P}_{n+2})$.

Let $\varphi = \hat{\theta} \circ \Phi : \overline{A[\Gamma]} \to \mathfrak{S}_{n+2} \times \{\pm 1\}$. Therefore, by Lemma 6.5, $\mathrm{Ker}(\varphi)$ does not have generalized torsion, having a contradiction. ■

From here, in order to finish our proof, for each considered Coxeter graph Γ and each homomorphism $\varphi\colon \overline{A[\Gamma]} \to \mathfrak{S}_{n+2} \times \{\pm 1\}$, we show an element in $\mathrm{Ker}(\varphi)$ having generalized torsion. To find such elements we apply the following strategy, well known to some experts. Let G be a group. An element $\beta \in G$ is *quasi-central* if there exists $n \ge 1$ such that β^n lies in the center of G. Assume that β is a quasi-central element and that α is an element of G which does not commute with β. Then $\alpha\beta\alpha^{-1}\beta^{-1}$ is a generalized torsion non-trivial element of G. The quasi-central elements of the Artin groups of spherical type are well-understood, and we look for quasi-central elements in standard parabolic subgroups that belong to the kernel of φ to find our generalized torsion elements.

We will use the following notations and definitions. For a group G and $\alpha \in G$ we denote by $c_\alpha\colon G \to G$, $\beta \mapsto \alpha\beta\alpha^{-1}$, the conjugation by α. We say that two homomorphisms $\varphi_1, \varphi_2\colon G \to H$ are *conjugate* if there is $\alpha \in H$ such that $\varphi_2 = c_\alpha \circ \varphi_1$. Moreover, a homomorphism $\varphi\colon G \to H$ is said to be *cyclic* if the image of φ is a cyclic subgroup of H.

Lemma 6.7. *The groups $A[D_4]$ and $A[A_4]$ are not commensurable.*

Proof. Let $\varphi\colon \overline{A[D_4]} \to \mathfrak{S}_6 \times \{\pm 1\}$ be a homomorphism written in the form $\varphi = \varphi_1 \times \varphi_2$, where $\varphi_1\colon \overline{A[D_4]} \to \mathfrak{S}_6$ and $\varphi_2\colon \overline{A[D_4]} \to \{\pm 1\}$ are two homomorphisms. By Corollary 6.6, we just need to show that $\mathrm{Ker}(\varphi)$ has generalized torsion. Denote by s_1, s_2, s_3, s_4 the standard generators of $A[D_4]$ numbered as in Figure 1. We also denote by $\pi\colon A[D_4] \to \overline{A[D_4]}$ the quotient homomorphism and $\bar{s}_i = \pi(s_i)$ for every $i \in \{1,2,3,4\}$. Notice that φ_2 is always cyclic since its image is contained in $\{\pm 1\}$, which is a cyclic group. Then, the relations $s_j s_3 s_j = s_3 s_j s_3$, for $j \in \{1,2,4\}$, imply that there is $\epsilon \in \{\pm 1\}$ such that $\varphi_2(\bar{s}_i) = \epsilon$ for every $i \in \{1,2,3,4\}$.

Firstly, suppose that φ_1 is cyclic. Let $\alpha = \bar{s}_1\bar{s}_2^{-1}$ and $\beta = \bar{s}_3\bar{s}_2\bar{s}_1\bar{s}_3\bar{s}_1^{-4}$. Then $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha \ne 1$, and $\alpha\beta\alpha\beta^{-1} = 1$, having that $\mathrm{Ker}(\varphi)$ has generalized torsion. Now, suppose that φ_1 is not cyclic. A direct computation using the software SageMath (see code in [15]) shows that there are 14400 non-cyclic homomorphisms from $\overline{A[D_4]}$ to $\mathfrak{S}_6$ divided into 40 conjugacy classes. By using the same software, we check that in every case we have either $\varphi_1(\bar{s}_1) = \varphi_1(\bar{s}_2)$ or $\varphi_1(\bar{s}_1) = \varphi_1(\bar{s}_4)$ or $\varphi_1(\bar{s}_2) = \varphi_1(\bar{s}_4)$. Then we can assume without loss of generality that $\varphi_1(\bar{s}_1) = \varphi_1(\bar{s}_2)$. In this case we have 8640 homomorphisms satisfying our conditions that are divided into 24 conjugacy classes. Let $\beta = \bar{s}_1\bar{s}_3\bar{s}_2\bar{s}_1\bar{s}_3\bar{s}_1$ and $\alpha = \bar{s}_1\bar{s}_2^{-1}$. Note that they both belong to $\mathrm{Ker}(\varphi_2)$. We check that $\varphi_1(\beta) = 1$ in every case. Moreover, as $\varphi_1(\bar{s}_1) = \varphi_1(\bar{s}_2)$, we also have that $\varphi_1(\alpha) = 1$. Therefore, $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha \ne 1$ and $\alpha\beta\alpha\beta^{-1} = 1$ and $\mathrm{Ker}(\varphi)$ has generalized torsion. ■

Lemma 6.8. *Let $n \ge 5$. Then $A[D_n]$ and $A[A_n]$ are not commensurable.*

Proof. We denote by $s_1, \dots, s_n$ the standard generators of $A[D_n]$ numbered as in Figure 1. We also let $t_i = (i, i+1) \in \mathfrak{S}_{n+2}$ for every $i \in \{1, \dots, n+1\}$. Let $\zeta\colon A[D_n] \to \mathfrak{S}_{n+2}$ be the homomorphism defined by $\zeta(s_1) = \zeta(s_2) = t_1$ and $\zeta(s_i) = t_{i-1}$ for every $i \in \{3, \dots, n\}$.

Moreover, for $n = 6$, let $\nu\colon A[D_6] \to \mathfrak{S}_8$ be the homomorphism defined by

$$\nu(s_1) = \nu(s_2) = (1,2)(3,4)(5,6),\ \nu(s_3) = (2,3)(1,5)(4,6),\ \nu(s_4) = (1,3)(2,4)(5,6),$$
$$\nu(s_5) = (1,2)(3,5)(4,6),\ \nu(s_6) = (2,3)(1,4)(5,6).$$

Claim. Let $\psi\colon A[D_n] \to \mathfrak{S}_{n+2}$ be a homomorphism. Then, we have one of the following situations, up to conjugation:

(1) ψ is cyclic,

(2) $\psi = \zeta$,

(3) $n = 6$ and $\psi = \nu$.

Proof of the Claim. Let $s_1', \dots, s_{n-1}'$ be the standard generators of $A[A_{n-1}]$. Let $\zeta'\colon A[A_{n-1}] \to \mathfrak{S}_{n+2}$ be the homomorphism defined by $\zeta'(s_i') = t_i$ for every $i \in \{1, \dots, n-1\}$. For $n = 6$, let $\nu'\colon A[A_5] \to \mathfrak{S}_8$ be the homomorphism defined by

$$\nu'(s_1') = (1,2)(3,4)(5,6)\,,\ \nu'(s_2') = (2,3)(1,5)(4,6)\,,\ \nu'(s_3') = (1,3)(2,4)(5,6)\,,$$
$$\nu'(s_4') = (1,2)(3,5)(4,6)\,,\ \nu'(s_5') = (2,3)(1,4)(5,6)\,.$$

Let $\iota\colon A[A_{n-1}] \to A[D_n]$ be the homomorphism sending s_i' to s_{i+1} for every $i \in \{1, \dots, n-1\}$, and let $\psi' = \psi \circ \iota\colon A[A_{n-1}] \to \mathfrak{S}_{n+2}$. By Theorem 1 in [2] and Theorems A and E in [27], we have one of the following possibilities, up to conjugation:

(1) ψ' is cyclic,

(2) $\psi' = \zeta'$,

(3) $n = 6$ and $\psi' = \nu'$.

First assume that ψ' is cyclic. Then there is $w \in \mathfrak{S}_{n+2}$ such that $\psi'(s_i') = \psi(s_{i+1}) = w$ for every $i \in \{1, \dots, n-1\}$. Let $\gamma = s_1 s_3 s_2 s_1 s_3 s_1$. We have $\gamma s_2 \gamma^{-1} = s_1$ and $\gamma s_3 \gamma^{-1} = s_3$, hence $w = \psi(s_3) = \psi(\gamma s_3 \gamma^{-1}) = \psi(\gamma s_2 \gamma^{-1}) = \psi(s_1)$. Thus, ψ is cyclic.

Now suppose that $\psi' = \zeta'$. We have $\psi(s_{i+1}) = \psi'(s_i') = t_i$ for every $i \in \{1, \dots, n-1\}$. Let $u = \psi(s_1)$. As u commutes with $\psi(s_i) = t_{i-1}$ for every $i \geq 4$, it follows that $u(k) = k$ for every $k \in \{3, 4, \dots, n\}$. Moreover, as u commutes with $t_1 = \psi(s_2)$, we have that $u \in E = \{1, t_1, t_{n+1}, t_1 t_{n+1}\}$. The only element u of E satisfying $u t_2 u = t_2 u t_2$ is $u = t_1$, hence $u = t_1$ and $\psi = \zeta$.

Assume that $n = 6$ and $\psi' = \nu'$. Let

$$u_1 = (1,2)(3,4)(5,6)\,,\ u_2 = (2,3)(1,5)(4,6)\,,\ u_3 = (1,3)(2,4)(5,6)\,,$$
$$u_4 = (1,2)(3,5)(4,6)\,,\ u_5 = (2,3)(1,4)(5,6)\,.$$

A direct computation with the software SageMath (see code in [15]) shows that the only element $v \in \mathfrak{S}_8$ satisfying $v u_1 = u_1 v$, $v u_2 v = u_2 v u_2$, $v u_3 = u_3 v$, $v u_4 = u_4 v$, $v u_5 = u_5 v$ is $v = u_1$, hence $\psi = \nu$. This finishes the proof of the claim.

Let Δ be the Garside element of $A[D_n]$. Let δ be the standard generator of $Z(A[D_n])$. By Lemma 5.1 in [33], $\Delta = (s_n \cdots s_3 s_2 s_1 s_3 \cdots s_n) \cdots (s_3 s_2 s_1 s_3)(s_2 s_1)$. Moreover, $\delta = \Delta$ if n is even, and $\delta = \Delta^2$ si n is odd. Notice that $\zeta(\Delta) = 1$, so $\zeta(\delta) = 1$. It follows that ζ induces a homomorphism $\bar\zeta\colon \overline{A[D_n]} \to \mathfrak{S}_{n+2}$. Similarly, if $n = 6$, $\nu(\Delta) = 1$ and

$\nu(\delta) = 1$, then ν induces a homomorphism $\bar{\nu}\colon \overline{A[D_6]} \to \mathfrak{S}_8$. Let $\varphi_1\colon \overline{A[D_n]} \to \mathfrak{S}_{n+2}$ be a homomorphism. Then, by the results from above and the claim, we have one of the following three possibilities, up to conjugation:

(1) φ_1 is cyclic,

(2) $\varphi_1 = \bar{\zeta}$,

(3) $n = 6$ and $\varphi_1 = \bar{\nu}$.

Let $\varphi\colon \overline{A[D_n]} \to \mathfrak{S}_{n+2} \times \{\pm 1\}$ be a homomorphism written in the form $\varphi = \varphi_1 \times \varphi_2$, where $\varphi_1\colon \overline{A[D_n]} \to \mathfrak{S}_{n+2}$ and $\varphi_2\colon \overline{A[D_n]} \to \{\pm 1\}$ are homomorphisms. By Corollary 6.6, we just need to show that $\mathrm{Ker}(\varphi)$ has generalized torsion. Denote by $\pi\colon A[D_n] \to \overline{A[D_n]}$ the quotient homomorphism and $\bar{s}_i = \pi(s_i)$ for every $i \in \{1, \dots, n\}$. Here again, φ_2 is always cyclic since its image is contained in $\{\pm 1\}$.

Suppose that φ_1 is cyclic. Let $\alpha = \bar{s}_1 \bar{s}_2^{-1}$ and $\beta = \bar{s}_3 \bar{s}_2 \bar{s}_1 \bar{s}_3 \bar{s}_1^{-4}$. In this case $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha \neq 1$ and $\alpha\beta\alpha\beta^{-1} = 1$, and then $\mathrm{Ker}(\varphi)$ has generalized torsion. Assume either $\varphi = \bar{\zeta}$ or $n = 6$ and $\varphi = \bar{\nu}$. Let $\alpha = \bar{s}_1 \bar{s}_2^{-1}$ and $\beta = \bar{s}_1 \bar{s}_3 \bar{s}_2 \bar{s}_1 \bar{s}_3 \bar{s}_1$. In both cases $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha \neq 1$ and $\alpha\beta\alpha\beta^{-1} = 1$, hence $\mathrm{Ker}(\varphi)$ has generalized torsion. ■

Lemma 6.9. *Let $n \in \{6, 7, 8\}$. Then $A[E_n]$ and $A[A_n]$ are not commensurable.*

Proof. We denote by $s_1, \dots, s_n$ the standard generators of $A[E_n]$ numbered as in Figure 1. We also let $t_i = (i, i+1) \in \mathfrak{S}_{n+2}$ for every $i \in \{1, \dots, n+1\}$.

Claim. Every homomorphism $\psi\colon A[E_n] \to \mathfrak{S}_{n+2}$ is cyclic.

Proof of the Claim. Denote by $s'_1, \dots, s'_{n-1}$ the standard generators of $A[A_{n-1}]$. Let $\zeta'\colon A[A_{n-1}] \to \mathfrak{S}_{n+2}$ be the homomorphism defined, for every $i \in \{1, \dots, n-1\}$, by $\zeta'(s'_i) = t_i$. For $n = 6$, let $\nu'\colon A[A_5] \to \mathfrak{S}_8$ be the homomorphism defined by

$$\nu'(s'_1) = (1,2)(3,4)(5,6)\,,\ \nu'(s'_2) = (2,3)(1,5)(4,6)\,,\ \nu'(s'_3) = (1,3)(2,4)(5,6)\,,$$
$$\nu'(s'_4) = (1,2)(3,5)(4,6)\,,\ \nu'(s'_5) = (2,3)(1,4)(5,6)\,.$$

Let $\iota\colon A[A_{n-1}] \to A[E_n]$ be the homomorphism sending s'_i to s_i for $i = 1, \dots, n-1$, and let $\psi' = \psi \circ \iota\colon A[A_{n-1}] \to \mathfrak{S}_{n+2}$. By Theorem 1 in [2] and Theorems A and E of [27], we have one of the following possibilities, up to conjugation:

(1) ψ' is cyclic,

(2) $\psi' = \zeta'$,

(3) $n = 6$ and $\psi' = \nu'$.

First suppose that ψ' is cyclic. Then there is $w \in \mathfrak{S}_{n+2}$ such that $\psi'(s'_i) = \psi(s_i) = w$ for every $i \in \{1, \dots, n-1\}$. Let $\gamma = s_2 s_3 s_n s_2 s_3 s_2$. We have $\gamma s_2 \gamma^{-1} = s_n$ and $\gamma s_3 \gamma^{-1} = s_3$, hence $w = \psi(s_3) = \psi(\gamma s_3 \gamma^{-1}) = \psi(\gamma s_2 \gamma^{-1}) = \psi(s_n)$. Then ψ is cyclic.

Now assume that $\psi' = \zeta'$. In this case we have $\psi(s_i) = \psi'(s'_i) = t_i$ for $i = 1, \dots, n-1$. Let $u = \psi(s_n)$. As u commutes with $\psi(s_i) = t_i$ for every $i \in \{1, 2, 4, \dots, n-1\}$, it follows that $u(k) = k$ for every $k \in \{1, 2, 3, 4, 5, \dots, n\}$, so $u \in E = \{1, t_{n+1}\}$. But there is no element u of E satisfying $ut_3u = t_3ut_3$, so we cannot have $\psi' = \zeta'$.

Finally, assume $n = 6$ and $\psi' = \nu'$. Let

$$u_1 = (1,2)(3,4)(5,6)\,,\ u_2 = (2,3)(1,5)(4,6)\,,\ u_3 = (1,3)(2,4)(5,6)\,,$$
$$u_4 = (1,2)(3,5)(4,6)\,,\ u_5 = (2,3)(1,4)(5,6)\,.$$

A direct computation with the software SageMath (see code in [15]) shows that there is no element $v \in \mathfrak{S}_8$ satisfying $vu_1 = u_1v$, $vu_2 = u_2v$, $vu_3v = u_3vu_3$, $vu_4 = u_4v$ and $vu_5 = u_5v$, hence we cannot have $n = 6$ and $\psi' = \nu'$. This finishes the proof of the claim.

Denote by $\pi\colon A[E_n] \to \overline{A[E_n]}$ the quotient homomorphism and let $\bar{s}_i = \pi(s_i)$ for every $i \in \{1, \dots, n\}$. Let $\varphi\colon \overline{A[E_n]} \to \mathfrak{S}_{n+2} \times \{\pm 1\}$ be a homomorphism written in the form $\varphi = \varphi_1 \times \varphi_2$, where $\varphi_1\colon \overline{A[E_n]} \to \mathfrak{S}_{n+2}$ and $\varphi_2\colon \overline{A[E_n]} \to \{\pm 1\}$ are homomorphisms. By the claim, $\varphi_1 \circ \pi\colon A[E_n] \to \mathfrak{S}_{n+2}$ is cyclic, hence φ_1 is also cyclic. On the other hand, φ_2 is cyclic since the image of φ_2 is contained in $\{\pm 1\}$. Let $\alpha = \bar{s}_2\bar{s}_n^{-1}$ and $\beta = \bar{s}_3\bar{s}_n\bar{s}_2\bar{s}_3\bar{s}_2^{-4}$. We have that $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha \neq 1$ and $\alpha\beta\alpha\beta^{-1} = 1$, and then $\mathrm{Ker}(\varphi)$ has generalized torsion. By Corollary 6.6, it follows that $A[A_n]$ and $A[E_n]$ are not commensurable. ∎

Lemma 6.10. *The groups $A[F_4]$ and $A[A_4]$ are not commensurable.*

Proof. Let $\varphi\colon \overline{A[F_4]} \to \mathfrak{S}_6 \times \{\pm 1\}$ be a homomorphism written in the form $\varphi = \varphi_1 \times \varphi_2$, where $\varphi_1\colon \overline{A[F_4]} \to \mathfrak{S}_6$ and $\varphi_2\colon \overline{A[F_4]} \to \{\pm 1\}$ are two homomorphisms. By Corollary 6.6, we just need to show that $\mathrm{Ker}(\varphi)$ has generalized torsion. We denote by s_1, s_2, s_3, s_4 the standard generators of $A[F_4]$ numbered as in Figure 1. We also denote by $\pi\colon A[F_4] \to \overline{A[F_4]}$ the quotient homomorphism and $\bar{s}_i = \pi(s_i)$ for every $i \in \{1, 2, 3, 4\}$. Notice that the relation $s_1s_2s_1 = s_2s_1s_2$ implies $\varphi_2(\bar{s}_1) = \varphi_2(\bar{s}_2)$. Analogously, $\varphi_2(\bar{s}_3) = \varphi_2(\bar{s}_4)$.

If g is an element of a group, we denote by $\mathrm{ord}(g)$ the order of g. Let $i \in \{1, 2, 3, 4\}$. As $\varphi_1(\bar{s}_i) \in \mathfrak{S}_6$, we have that $\mathrm{ord}(\varphi_1(\bar{s}_i)) \in \{1, 2, 3, 4, 5, 6\}$. It follows that $\mathrm{ord}(\varphi(\bar{s}_i)) \in \{1, 2, 3, 4, 5, 6, 10\}$. Suppose that $\varphi(\bar{s}_1) = \varphi(\bar{s}_2)$ and $\mathrm{ord}(\varphi(\bar{s}_1)) \in \{1, 2, 4\}$. Let $\alpha = \bar{s}_1\bar{s}_2^{-1}$ and $\beta = (\bar{s}_1\bar{s}_2)^4$. In this case $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha \neq 1$ and $\alpha(\beta\alpha\beta^{-1})(\beta^2\alpha\beta^{-2}) = 1$, and then $\mathrm{Ker}(\varphi)$ has generalized torsion. Now assume that $\varphi(\bar{s}_1) = \varphi(\bar{s}_2)$ and $\mathrm{ord}(\varphi(\bar{s}_1)) = 3$. If we let $\alpha = \bar{s}_1\bar{s}_2^{-1}$, $\beta = \bar{s}_1\bar{s}_2\bar{s}_1$, then $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha \neq 1$, $\alpha(\beta\alpha\beta^{-1}) = 1$, and $\mathrm{Ker}(\varphi)$ has generalized torsion. Now suppose that $\varphi(\bar{s}_1) = \varphi(\bar{s}_2)$ and $\mathrm{ord}(\varphi(\bar{s}_1)) \in \{5, 10\}$. We let $\alpha = \bar{s}_1\bar{s}_2^{-1}$ and $\beta = (\bar{s}_1\bar{s}_2)^{10}$. Then $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha(\beta\alpha\beta^{-1})(\beta^2\alpha\beta^{-2}) = 1$, and $\mathrm{Ker}(\varphi)$ has generalized torsion.

By the reasoning above we can assume that, if $\varphi(\bar{s}_1) = \varphi(\bar{s}_2)$, then $\mathrm{ord}(\varphi(\bar{s}_1)) = 6$. We can also suppose that, if $\varphi(\bar{s}_3) = \varphi(\bar{s}_4)$, then $\mathrm{ord}(\varphi(\bar{s}_3)) = 6$.

Suppose that $\varphi(\bar{s}_1) = \varphi(\bar{s}_2)$ and $\varphi(\bar{s}_3) = \varphi(\bar{s}_4)$. Then we also have $\mathrm{ord}(\varphi(\bar{s}_1)) = \mathrm{ord}(\varphi(\bar{s}_3)) = 6$. If $\varphi_1(\bar{s}_1) = \varphi_1(\bar{s}_2)$ and $\varphi_1(\bar{s}_3) = \varphi_1(\bar{s}_4)$ are both of order 3, then $\varphi_2(\bar{s}_1) = \varphi_2(\bar{s}_2) = \varphi_2(\bar{s}_3) = \varphi_2(\bar{s}_4) = -1$. In this case, we let $\alpha = \bar{s}_1\bar{s}_2^{-1}$ and $\beta = \bar{s}_1\bar{s}_2\bar{s}_1\bar{s}_4^3$, having $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha \neq 1$ and $\alpha(\beta\alpha\beta^{-1}) = 1$. Hence $\mathrm{Ker}(\varphi)$ has generalized torsion. We can then assume that $\varphi_1(\bar{s}_1)$ or $\varphi_1(\bar{s}_3)$ is of order 6, say that $\varphi_1(\bar{s}_1)$ has order 6. Then $\varphi_1(\bar{s}_1)$ is conjugate to $(1, 2, 3, 4, 5, 6)$ or to $(1, 2, 3)(4, 5)$ in $\mathfrak{S}_6$. In both cases it follows that the centralizer of $\varphi_1(\bar{s}_1)$ in $\mathfrak{S}_6$ is a cyclic group of order 6 generated by $\varphi_1(\bar{s}_1)$. As $\varphi_1(\bar{s}_3)$ belongs to this centralizer and it has order 3 or 6, there is $k \in \{1, 2, -1, -2\}$ such that $\varphi_1(\bar{s}_3) = \varphi_1(\bar{s}_4) = \varphi_1(\bar{s}_1)^k$. We let $\alpha = \bar{s}_3\bar{s}_4^{-1}$ and $\beta = \bar{s}_3\bar{s}_4\bar{s}_1^{-2k}$. Then, $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha \neq 1$, and $\alpha(\beta\alpha\beta^{-1})(\beta^2\alpha\beta^{-2}) = 1$, having generalized torsion in $\mathrm{Ker}(\varphi)$.

By [8], the standard generator of the center of $A[F_4]$ coincides with its Garside element and equals $(s_1s_2s_3s_4)^{h/2}$, where h is the Coxeter number of F_4. As $h = 12$ (see page 80 of [19]), we have $\delta = \Delta = (s_1s_2s_3s_4)^6$. Let $\hat\alpha_0 = (s_1s_2s_3s_4)^3$. Recall that $z\colon A[F_4] \to \mathbb{Z}$ is the homomorphism sending s_i to 1 for every $i \in \{1,2,3,4\}$. As $z(\delta) = 24$, we have $z(Z(A[F_4])) = 24\mathbb{Z}$, so $\hat\alpha_0 \notin Z(A[F_4])$ because $z(\hat\alpha_0) = 12$. On the other hand, $\hat\alpha_0^2 = \delta$, so $\hat\alpha_0^2 \in Z(A[F_4])$. Let $\alpha_0 = \pi(\hat\alpha_0)$. Then $\alpha_0 \neq 1$ and $\alpha_0^2 = 1$. In the remaining cases, we will show that $\alpha_0 \in \operatorname{Ker}(\varphi)$, which will immediately imply that $\operatorname{Ker}(\varphi)$ has (generalized) torsion.

Suppose that $\varphi(\bar s_1) \neq \varphi(\bar s_2)$ and $\varphi(\bar s_3) = \varphi(\bar s_4)$ (hence $\operatorname{ord}(\varphi(\bar s_3)) = 6$). Let E_1 be the set of triples (u_1,u_2,u_3) of elements of $\mathfrak{S}_6$ such that $u_1u_2u_1 = u_2u_1u_2$, $u_1u_3 = u_3u_1$, $u_2u_3 = u_3u_2$, $u_1 \neq u_2$ and $\operatorname{ord}(u_3) \in \{3,6\}$. Another direct computation with SageMath (see code in [15]) shows that E_1 has 1440 elements divided into 6 conjugacy classes. Again with SageMath, we compute a set E_1^0 of representatives of the conjugacy classes in E_1 and we get

$$\begin{aligned} E_1^0 = \big\{&\big((1,2),(2,3),(4,5,6)\big), \big((1,2,3,4,5,6),(1,6,3,2,5,4),(1,3,5)(2,4,6)\big),\\ &\big((1,2,3,4,5,6),(1,6,3,2,5,4),(1,5,3)(2,6,4)\big),\\ &\big((1,4)(2,5)(3,6),(1,2)(3,4)(5,6),(1,3,5)(2,4,6)\big),\\ &\big((2,3)(4,5,6),(1,2)(4,5,6),(4,5,6)\big), \big((2,3)(4,5,6),(1,2)(4,5,6),(4,6,5)\big)\big\}. \end{aligned}$$

We check with a direct computation that $(u_1u_2u_3^2)^3 = 1$ for every $(u_1,u_2,u_3) \in E_1^0$. Up to conjugation, we can suppose that $(\varphi_1(\bar s_1), \varphi_1(\bar s_2), \varphi_1(\bar s_3)) = (u_1,u_2,u_3) \in E_1^0$. Then, as $(u_1u_2u_3^2)^3 = 1$, we have $\varphi_1(\alpha_0) = 1$. It is obvious that $\varphi_2(\alpha_0) = 1$. So, $\varphi(\alpha_0) = 1$ and $\operatorname{Ker}(\varphi)$ has (generalized) torsion.

Suppose that $\varphi(\bar s_1) \neq \varphi(\bar s_2)$ and $\varphi(\bar s_3) \neq \varphi(\bar s_4)$. Let E_2 be the set of quadruples (u_1,u_2,u_3,u_4) of elements of $\mathfrak{S}_6$ such that $u_1u_2u_1 = u_2u_1u_2$, $u_1u_3 = u_3u_1$, $u_1u_4 = u_4u_1$, $u_2u_3u_2u_3 = u_3u_2u_3u_2$, $u_2u_4 = u_4u_2$, $u_3u_4u_3 = u_4u_3u_4$, $u_1 \neq u_2$ and $u_3 \neq u_4$. A direct computation with SageMath (see code in [15]) shows that E_2 has 1440 elements divided into 2 conjugacy classes. Again with SageMath, we compute a set E_2^0 of representatives of the conjugacy classes in E_2 and we get

$$\begin{aligned} E_2^0 = \big\{&\big((1,2),(2,3),(5,6),(4,5)\big),\\ &\big((1,4)(2,5)(3,6),(1,2)(3,4)(5,6),(1,4)(2,3)(5,6),(1,6)(2,5)(3,4)\big)\big\}. \end{aligned}$$

We check by a direct computation that $(u_1u_2u_3u_4)^3 = 1$ for every $(u_1,u_2,u_3,u_4) \in E_2^0$. Up to conjugation, we can suppose that $(\varphi_1(\bar s_1), \varphi_1(\bar s_2), \varphi_1(\bar s_3), \varphi_1(\bar s_4)) = (u_1,u_2,u_3,u_4) \in E_2^0$. Then, as $(u_1u_2u_3u_4)^3 = 1$, we have $\varphi_1(\alpha_0) = 1$. It is clear that $\varphi_2(\alpha_0) = 1$. Then $\varphi(\alpha_0) = 1$ and $\operatorname{Ker}(\varphi)$ has (generalized) torsion. ∎

Lemma 6.11. *The groups $A[H_4]$ and $A[A_4]$ are not commensurable.*

Proof. We denote by s_1,s_2,s_3,s_4 the standard generators of $A[H_4]$ numbered as in Figure 1. We also consider the quotient homomorphism $\pi\colon A[H_4] \to \overline{A[H_4]}$ and $\bar s_i = \pi(s_i)$ for every $i \in \{1,2,3,4\}$. Let $\varphi\colon \overline{A[H_4]} \to \mathfrak{S}_6 \times \{\pm 1\}$ be a homomorphism written in the form $\varphi = \varphi_1 \times \varphi_2$ where $\varphi_1\colon \overline{A[H_4]} \to \mathfrak{S}_6$ and $\varphi_2 : \overline{A[H_4]} \to \{\pm 1\}$ are two homomorphisms. Notice that the relations $s_3s_4s_3 = s_4s_3s_4$, $s_2s_3s_2 = s_3s_2s_3$, $s_1s_2s_1s_2s_1 = s_2s_1s_2s_1s_2$

imply $\varphi_2(\bar{s}_1) = \varphi_2(\bar{s}_2) = \varphi_2(\bar{s}_3) = \varphi_2(\bar{s}_4)$. Then φ_2 is always cyclic. For φ_1, a direct computation with SageMath (see code in [15]) shows that there are 720 homomorphisms from $\overline{A[H_4]}$ to $\mathfrak{S}_6$, all of them cyclic. We let $\alpha = \bar{s}_3^{-1}\bar{s}_4$ and $\beta = \bar{s}_3\bar{s}_4\bar{s}_3\bar{s}_1^{-3}$. They both belong to $\mathrm{Ker}(\varphi)$ and they satisfy $\alpha\beta\alpha\beta^{-1} = 1$. Therefore, $\mathrm{Ker}(\varphi)$ has generalized torsion and, by Corollary 6.6, $A[A_4]$ and $A[H_4]$ are not commensurable. ∎

Our last issue is to compare $A[H_3]$ and $A[A_3]$. In this case we cannot apply Corollary 6.6, as we have done with the previous cases. Indeed, we can find homomorphisms sending $\overline{A[H_3]}$ to $\mathfrak{S}_5$ whose kernel does not have generalized torsion:

Lemma 6.12. *Let s_1, s_2, s_3 be the standard generators of $A[H_3]$ numbered as in Figure 1, let $\pi\colon A[H_3] \to \overline{A[H_3]}$ be the quotient homomorphism, and, for each $i \in \{1,2,3\}$, let $\bar{s}_i = \pi(s_i)$. Let $\zeta\colon \overline{A[H_3]} \to \mathfrak{S}_5$ be the homomorphism defined by*

$$\zeta(\bar{s}_1) = (2,4)(3,5)\,, \quad \zeta(\bar{s}_2) = (1,2)(4,5)\,, \quad \zeta(\bar{s}_3) = (2,3)(4,5)\,.$$

Then $\mathrm{Ker}(\zeta)$ *does not have generalized torsion.*

Proof. By [8], $Z(A[H_3])$ is an infinite cyclic group generated by $\delta = (s_1s_2s_3)^5$. Let $u_1 = (2,4)(3,5)$, $u_2 = (1,2)(4,5)$ and $u_3 = (2,3)(4,5)$. A direct computation shows that we have the relations $u_1u_2u_1u_2u_1 = u_2u_1u_2u_1u_2$, $u_1u_3 = u_3u_1$, $u_2u_3u_2 = u_3u_2u_3$ and $(u_1u_2u_3)^5 = 1$, hence ζ is well-defined. We are going to prove that $\mathrm{Ker}(\zeta) = \overline{\mathrm{CA}[H_3]}$. As $\overline{\mathrm{CA}[H_3]}$ embeds into $\mathrm{CA}[H_3]$ by Proposition 3.1 and $\mathrm{CA}[H_3]$ has no generalized torsion by Theorem 3 in [29], it will follow that $\mathrm{Ker}(\zeta)$ has no generalized torsion.

Let H be the subgroup of $\mathfrak{S}_5$ generated by $\{u_1, u_2, u_3\}$. A direct computation with SageMath (see code in [15]) shows that $|H| = 60$. As $u_1^2 = u_2^2 = u_3^2 = 1$ and $\overline{\mathrm{CA}[H_3]}$ is the normal subgroup of $\overline{A[H_3]}$ generated by $\{\bar{s}_1^2, \bar{s}_2^2, \bar{s}_3^2\}$, we have $\overline{\mathrm{CA}[H_3]} \subset \mathrm{Ker}(\zeta)$. Then, to show that $\mathrm{Ker}(\zeta) = \overline{\mathrm{CA}[H_3]}$, we just need to prove that $|\overline{A[H_3]}/\overline{\mathrm{CA}[H_3]}| = 60$. It is well known that $|A[H_3]/\mathrm{CA}[H_3]| = |W[H_3]| = 120$ (See Page 46 of [19]). The projection $\pi\colon A[H_3] \to \overline{A[H_3]}$ induces a surjective homomorphism $\bar{\pi}\colon A[H_3]/\mathrm{CA}[H_3] \to \overline{A[H_3]}/\overline{\mathrm{CA}[H_3]}$ whose kernel is the cyclic group generated by the class $[\delta]$ of δ. We have that $\delta = \Delta$ is the Garside element of $A[H_3]$, so $\delta \notin \mathrm{CA}[H_3]$. However, $\delta^2 = \Delta^2 \in \mathrm{CA}[H_3]$, hence $\mathrm{Ker}(\bar{\pi})$ is a cyclic group $\langle[\delta]\rangle$ of order 2, having $|\overline{A[H_3]}/\overline{\mathrm{CA}[H_3]}| = |A[H_3]/\mathrm{CA}[H_3]|/2 = 60$. ∎

Let Σ be a closed surface and $\mathcal{P}_n$ be a collection of n different points in Σ. With such a pair $(\Sigma, \mathcal{P}_n)$ we can associate a simplicial complex called the *curve complex* of $(\Sigma, \mathcal{P}_n)$, denoted by $\mathcal{C}(\Sigma, \mathcal{P}_n)$. The vertices of $\mathcal{C}(\Sigma, \mathcal{P}_n)$ are the isotopy classes of simple closed curves on $\Sigma \setminus \mathcal{P}_n$ that are non-degenerate. Non-degenerate means that the curve does not bound a disk embedded in Σ containing 0 or 1 point of $\mathcal{P}_n$. Every n-simplex is formed by $n+1$ classes having representatives that are pairwise disjoint. We say that a mapping class $f \in \mathcal{M}^*(\Sigma, \mathcal{P}_n)$ is *pseudo-Anosov* if $f^n(\alpha) \neq \alpha$ for every $\alpha \in \mathcal{C}(\Sigma, \mathcal{P}_n)$ and every $n \in \mathbb{Z} \setminus \{0\}$. We say that f is *periodic* if it has finite order. The following lemma finishes the proof of Theorem 2.4.

Lemma 6.13. *The groups $A[H_3]$ and $A[A_3]$ are not commensurable.*

Proof. Recall that, by Proposition 3.1, we need to prove that $\overline{A[H_3]}$ and $\overline{A[A_3]}$ are not commensurable, and, to do this, it is enough to prove that $\mathrm{Com}(\overline{A[H_3]})$ and $\mathrm{Com}(\overline{A[A_3]})$ are not isomorphic. Also by Proposition 3.1, $\overline{A[H_3]}$ injects in $\mathrm{Com}(\overline{A[H_3]})$ and recall that $\mathrm{Com}(\overline{A[A_3]})$ and $\mathcal{M}^*(\Sigma_{0,0}, \mathcal{P}_5)$ are isomorphic by Theorem 6.3. Then, to prove our lemma it suffices to prove that there is no injective homomorphism from $\overline{A[H_3]}$ to $\mathcal{M}^*(\Sigma_{0,0}, \mathcal{P}_5)$.

Let $\Phi\colon \overline{A[H_3]} \to \mathcal{M}^*(\Sigma_{0,0}, \mathcal{P}_5)$ be a homomorphism. Recall $\hat{\theta}$ and θ' defined right before Theorem 6.3 and consider $\varphi = \hat{\theta} \circ \Phi\colon \overline{A[H_3]} \to \mathfrak{S}_5 \times \{\pm 1\}$ being of the form $\varphi = \varphi_1 \times \varphi_2$, where $\varphi_1 = \theta' \circ \Phi : \overline{A[H_3]} \to \mathfrak{S}_5$ and $\varphi_2 = \omega \circ \Phi : \overline{A[H_3]} \to \{\pm 1\}$. We denote by s_1, s_2, s_3 the standard generators of $A[H_3]$ numbered as in Figure 1. Moreover, we let $\pi\colon A[H_3] \to \overline{A[H_3]}$ be the quotient homomorphism and $\bar{s}_i = \pi(s_i)$ for every $i \in \{1, 2, 3\}$.

Notice that the relations $s_2 s_3 s_2 = s_3 s_2 s_3$ and $s_1 s_2 s_1 s_2 s_1 = s_2 s_1 s_2 s_1 s_2$ imply $\varphi_2(\bar{s}_2) = \varphi_2(\bar{s}_3)$ and $\varphi_2(\bar{s}_1) = \varphi_2(\bar{s}_2)$. Notice also that the standard generator of the center of $A[H_3]$ is $\delta = (s_1 s_2 s_3)^5$, hence $(\bar{s}_1 \bar{s}_2 \bar{s}_3)^5 = 1$. Let $\epsilon = \varphi_2(\bar{s}_1) = \varphi_2(\bar{s}_2) = \varphi_2(\bar{s}_3) \in \{\pm 1\}$. Then $1 = \varphi_2(1) = \varphi_2((\bar{s}_1 \bar{s}_2 \bar{s}_3)^5) = \epsilon^{15}$, having that $\epsilon = 1$.

Suppose that φ_1 is cyclic, that is, there is $w \in \mathfrak{S}_5$ such that $\varphi_1(\bar{s}_1) = \varphi_1(\bar{s}_2) = \varphi_1(\bar{s}_3) = w$. We denote by $\mathrm{ord}(w)$ the order of w. As $w \in \mathfrak{S}_5$, we have $\mathrm{ord}(w) \in \{1, 2, 3, 4, 5, 6\}$. On the other hand, as $(\bar{s}_1 \bar{s}_2 \bar{s}_3)^5 = 1$, we have $w^{15} = 1$, hence $\mathrm{ord}(w)$ divides 15. Thus, $\mathrm{ord}(w) \in \{1, 3, 5\}$. Now, we let $\alpha = \bar{s}_2 \bar{s}_3^{-1}$ and $\beta = (\bar{s}_2 \bar{s}_3 \bar{s}_2)^5$. Then, $\alpha, \beta \in \mathrm{Ker}(\varphi)$, $\alpha \neq 1$, $\alpha\beta\alpha\beta^{-1} = 1$, and $\mathrm{Ker}(\varphi)$ has generalized torsion. By Lemma 6.5, Φ is not injective.

Suppose that φ_1 is not cyclic. Consider the two homomorphisms $\zeta_1, \zeta_2 : \overline{A[H_3]} \to \mathfrak{S}_5$ defined by

$$\zeta_1(\bar{s}_1) = (1,2,3,4,5)\,,\ \zeta_1(\bar{s}_2) = (1,4,2,3,5)\,,\ \zeta_1(\bar{s}_3) = (1,5,4,3,2)\,,$$
$$\zeta_2(\bar{s}_1) = (2,4)(3,5)\,,\ \zeta_2(\bar{s}_2) = (1,2)(4,5)\,,\ \zeta_2(\bar{s}_3) = (2,3)(4,5)\,.$$

A direct computation with SageMath (see code in [15]) shows that every non-cyclic homomorphism from $\overline{A[H_3]}$ to $\mathfrak{S}_5$ is conjugate to either ζ_1 or ζ_2. We can then suppose that $\varphi_1 \in \{\zeta_1, \zeta_2\}$.

If $\varphi_1 = \zeta_1$, by Proposition 9.4 in [6] and Lemma 5.9 in [4] it follows that $\Phi(\bar{s}_1)$ is periodic or pseudo-Anosov. If $\Phi(\bar{s}_1)$ is periodic, then there is an integer $k \geq 1$ such that $\Phi(\bar{s}_1)^k = \mathrm{id}$, hence $\bar{s}_1^k$ is a non-trivial element of $\mathrm{Ker}(\Phi)$ and Φ is not injective. Suppose that $\Phi(\bar{s}_1)$ is pseudo-Anosov. As $\Phi((\bar{s}_1\bar{s}_2)^5)$ is in the centralizer of $\Phi(\bar{s}_1)$ in $\mathcal{M}^*(\Sigma_{0,0}, \mathcal{P}_5)$ and the centralizer of a pseudo-Anosov element is virtually cyclic (see Lemma 8.13 in [20]), there are integers $k, \ell \in \mathbb{Z}$, $\ell \neq 0$, such that $\Phi(\bar{s}_1)^k = \Phi((\bar{s}_1\bar{s}_2)^5)^\ell$. Let $\alpha = (\bar{s}_1\bar{s}_2)^{5\ell}\bar{s}_1^{-k}$. Then α is a non-trivial element of $\mathrm{Ker}(\Phi)$ and Φ is not injective.

Suppose that $\varphi_1 = \zeta_2$. Then $\varphi_1(\bar{s}_1\bar{s}_2) = (1,4,3,5,2)$, and again by Proposition 9.4 in [6] and Lemma 5.9 in [4], $\Phi(\bar{s}_1\bar{s}_2)$ is periodic or pseudo-Anosov. If $\Phi(\bar{s}_1\bar{s}_2)$ is periodic, there is an integer $k \geq 1$ such that $\Phi(\bar{s}_1\bar{s}_2)^k = \mathrm{id}$ and $\alpha = (\bar{s}_1\bar{s}_2)^k$ is a non-trivial element belonging to the kernel of Φ. This means that Φ is not injective. If $\Phi(\bar{s}_1\bar{s}_2)$ is pseudo-Anosov, then $\Phi((\bar{s}_1\bar{s}_2)^5) = \Phi(\bar{s}_1\bar{s}_2)^5$ is also pseudo-Anosov and $\Phi(\bar{s}_1)$ is in the centralizer of $\Phi((\bar{s}_1\bar{s}_2)^5)$ in $\mathcal{M}^*(\Sigma_{0,0}, \mathcal{P}_5)$, which is virtually cyclic. Hence there are integers $k, \ell \in \mathbb{Z}$, $\ell \neq 0$, such that $\Phi(\bar{s}_1)^\ell = \Phi((\bar{s}_1\bar{s}_2)^5)^k$. Let $\alpha = (\bar{s}_1\bar{s}_2)^{5k}\bar{s}_1^{-\ell}$. Therefore, α is a non-trivial element of $\mathrm{Ker}(\Phi)$ and Φ is not injective. ■

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