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Closed incompressible surfaces in the complements of positive knots

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This paper is dedicated to Professor Shin'ichi Suzuki for his 60th birthday

Abstract. We show that any closed incompressible surface in the complement of a positive knot is algebraically non-split from the knot, positive knots cannot bound non-free incompressible Seifert surfaces and that the splittability and the primeness of positive knots and links can be seen from their positive diagrams.

Mathematics Subject Classification (2000). 57M25.

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1. Introduction

A knot K in the 3-sphere S^3 is called *positive* if it has an oriented diagram all crossings of which are positive crossings. For a closed surface F in $S^3 - K$, we define the *order* $o(F;K)$ of F for K as follows ([5]). Let $i : F \to S^3 - K$ be the inclusion map and let $i_* : H_1(F) \to H_1(S^3 - K)$ be the induced homomorphism. Since $\mathrm{Im}(i_*)$ is a subgroup of $H_1(S^3 - K) = \mathbb{Z}\langle\text{meridian}\rangle$, there is an integer m such that $\mathrm{Im}(i_*) = m\mathbb{Z}$. Then we define $o(S;K) = m$.

The positive knot complements have the following special properties.

Theorem 1.1. *Any closed incompressible surface in a positive knot complement has non-zero order.*

A Seifert surface F for a knot is said to be *free* if $\pi_1(S^3 - F)$ is a free group. In [5, Theorem 1.1], it is shown that a knot bounds a non-free incompressible Seifert surface if and only if there exists a closed incompressible surface in the

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knot complement whose order is equal to zero. Therefore, Theorem 1.1 gives us the next corollary.

Corollary 1.2. *Positive knots cannot bound non-free incompressible Seifert surfaces.*

Although positive links which have connected positive diagrams are non-split because they have positive linking numbers, we can give another geometrical proof of this fact.

Theorem 1.3. *Positive links are non-split if their positive diagrams are connected.*

Positive diagrams of positive knots or links also tell us their primeness. We say that a knot or link diagram $\tilde{K}$ on the 2-sphere S is *prime* if for any loop l in S intersecting $\tilde{K}$ in 2 points, l bounds a disk intersecting $\tilde{K}$ in an arc.

Theorem 1.4. *Non-trivial positive knots or links are prime if their positive diagrams are connected and prime.*

Remark 1.5. The referee suggested that one can show that: *A non-trivial positive link is prime iff its positive diagram is connected and prime*, with the addition of the assumption that the positive link projections contain no nugatory crossings. In fact, the converse of Theorem 1.3 and 1.4 is true, but it needs [2, Theorem 3].

There are other results about determining when a link projection represents a non-split or prime link.

For the splittability,

- alternating links ([1, Theorem 10.2], [4, Theorem 1 (a)]);
- almost alternating links ([6]);
- homogeneous links ([2, Corollary 3.1]).

For the primeness,

- alternating links ([4, Theorem 1 (b)]);
- positive braids ([3, 1.2 Theorem]).

2. Proof of Theorem 1.1 and 1.3

Theorem 1.1 and 1.3 follow from the next Theorem.

Theorem 2.1. *Let K be a positive knot or link in the 3-sphere S^3 and F a closed incompressible surface in the complement of K. Then one of the following conclusions hold.*

(1) *There exists a loop l in F such that $lk(l, K) \neq 0$.*
(2) *F is a splitting sphere for K, and any positive diagram of K is disconnected.*

Henceforth, we shall prove Theorem 2.1.

Let S be a 2-sphere in S^3 and $p : S^3 - \{2 \text{ points}\} \cong S \times R \to S$ a projection. Put K so that $p(K)$ is a positive diagram. As usual way, we express K in a bridge presentation. Thus we have the following data (see Figure 1).

- $S^3 = B^+ \cup_S B^-$ (S decomposes S^3 into two 3-balls).
- $K = K^+ \cup_S K^-$, where $K^\pm \subset B^\pm$ (S cuts K into over bridges and under bridges).
- $K^\pm = K_1^\pm \cup K_2^\pm \cup \ldots K_n^\pm$ (K is presented as n over bridges and n under bridges).
- $D^\pm = D_1^\pm \cup D_2^\pm \cup \ldots D_n^\pm$ (each $K_i^\pm \cup p(K_i^\pm)$ bounds a disk $D_i^\pm$ such that $p(D_i^\pm) = p(K_i^\pm)$).

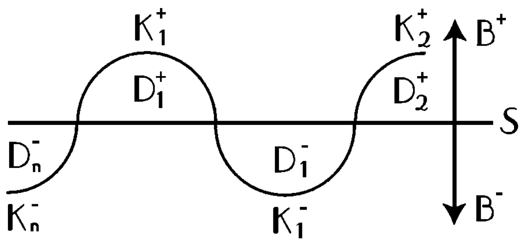

Figure 1. View from level surface

We take n minimal over all bridge presentations of $p(K)$.

Lemma 2.2. *We may assume that:*
(a) *$F \cap D^- = \emptyset$,*
(b) *$F \cap B^-$ consists of disks,*
(c) *$F \cap D^+$ consists of arcs, and*
(d) *any component of $F \cap B^+ - D^+$ is a disk.*

Proof. (a): Simply push out F near D^- into B^+.

(b): If there exists a component of $F \cap B^-$ which is not a disk, then $F \cap B^-$ has a compressing disk E in $B - N(D^-)$ since $B - N(D^-)$ is a 3-ball. By the incompressibility of F in $S^3 - K$, ∂E bounds a disk in F. Then by cutting and pasting F along E, we have a new incompressible surface F' and a sphere F''. Replace F with F' and continue this operation.

(c): Suppose there exists a loop component of $F \cap D^+$ and let E be an innermost disk in D^+. Then the similar argument to (b) passes by using E.

(d): If there exists a component of $F \cap B^+ - D^+$ which is not a disk, then $F \cap B^+ - D^+$ has a compressing disk E in $B^+ - D^+$. By using E, we can show (d) similarly. □

We take a 2-tuple lexicographically ordered complexity measure $(|F \cap B^-|, |F \cap D^+|)$ minimal. Note that the complexity measure is not $(0, *)$. For $(0, *)$, F fails to be incompressible in $S^3 - K$ since (B^+, K^+) is a trivial tangle. If the complexity measure is $(1, 0)$, then we have the conclusion (2).

Hereafter, we suppose that the complexity $(|F \cap B^-|, |F \cap D^+|) \geq (1, 1)$.

Then we obtain a connected graph G in F by regarding $F \cap B^-$ and $F \cap D^+$ as vertices and edges respectively. Note that every vertex has a positive even valency by the construction.

An arc α_j of $F \cap D_i^+$ divides D_i^+ into two disks δ_j and δ'_j, where δ'_j contains K_i^+. Put $\beta_j = \delta_j \cap S$. We may assume that $p(\alpha_j) = p(\delta_j) = \beta_j$ for all α_j. We assign an orientation endowed from K_i to α_j and β_j naturally (see Figure 2).

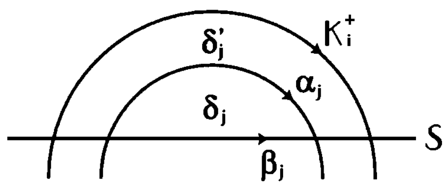

Figure 2. α_j and β_j have the orientation

Lemma 2.3. *For any arc α_j of $F \cap D_i^+$, $\beta_j \cap p(K^-) \neq \emptyset$.*

Proof. Suppose that there exists an arc α_j of $F \cap D_i^+$ such that $\beta_j \cap p(K^-) = \emptyset$. By exchanging α_j if necessary, we may assume that α_j is outermost in D_i^+, that is, $\operatorname{int} \delta_j \cap F = \emptyset$. If α_j connects different vertices, then a ∂-compression of F along δ_j reduces the complexity. Otherwise, α_j incidents a single vertex, say D_k^-. We perform a ∂-compression of F along δ_j, and obtain an annulus A consisting of the disk D_k^- and the resultant band b. Since we chose an outermost arc α_j and $\beta_j \cap p(K^-) = \emptyset$, there exists a compressing disk for A in $B^- - K^-$. By retaking F along the compressing disk, we can reduce the complexity. In both cases, there is a contradiction in the assumption the complexity is minimal. □

Now we pay attention to a face f of G in F. A *corner* is a subarc of $\partial(F \cap B^-) - (F \cap D^+)$. The *cycle* ∂f for f is a loop consisting of edges and corners such that it bounds f. The edges have orientations as previously mentioned.

Lemma 2.4 (The cycle lemma)**.** *For any face f, the cycle ∂f can not be oriented.*

Proof. Suppose that there is a face f such that ∂f can be oriented. Then, since no corner of ∂f intersects $p(K)$, and by Lemma 2.3, $p(\partial f)$ has non-zero intersection number with $p(K^-)$ on S. Figure 3 illustrates the projection of f and K^- on S. This is a contradiction. □

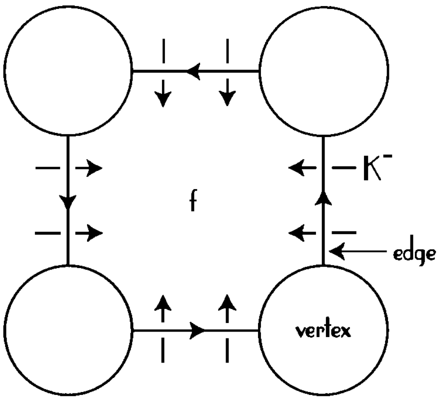

Figure 3. $p(\partial f)$ has non-zero intersection number

For each face f of G and any point in the interior of any edge of ∂f, we can find an arc γ on f satisfying the following property.

(*) γ connects two edges of ∂f whose orientations are different in ∂f.

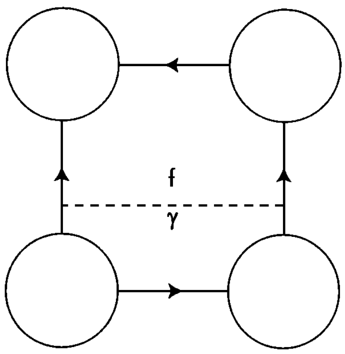

Figure 4. γ with the property (*)

Lemma 2.4 assures the existence of such an arc γ.

To find a loop l on F with $lk(l, K) \neq 0$, we depart a point in the interior of any edge of G, trace arcs with the property (*), and will arrive at the face on which we have walked. Connecting these arcs, we will obtain an oriented loop l in $F \cap B^+$ with a suitable orientation such that l has a positive intersection number with edges of G on F. Thus we got an oriented loop l in F which has non-zero linking number with K. Since any loop in a splitting sphere is contractible in $S^3 - K$, we have the conclusion (1).

This completes the proof of Theorem 2.1.

3. Proof of Theorem 1.4

Let K be a positive knot or link in S^3 and F be a decomposing sphere for K. We put K and F as the proof of Theorem 2.1 except that two points p_1 and p_2 of $F \cap K$ are in $\operatorname{int} B^+$ or $\operatorname{int} B^-$. Note that p_1 and p_2 can not be the ends of a single arc of $F \cap D^\pm$ because the tangle $(B^\pm, K^\pm)$ is trivial and F is a decomposing sphere. Hence, there are two arcs e_1 and e_2 of $F \cap D^\pm$ whose ends contain p_1 and p_2 respectively. We deform F by an isotopy relative to K so that $p(e_i) = p(p_i)$ $(i = 1, 2)$. We take the number of bridges n minimal.

Thus we have the following data in addition to the data in the proof of Theorem 2.1.

- $F \cap K = p_1 \cup p_2 \subset \operatorname{int} B^\pm$.
- $F \cap D^\pm \supset e_i \supset p_i$ $(i = 1, 2)$.
- $p(e_i) = p(p_i)$ $(i = 1, 2)$.

Lemma 3.1. *We may assume that:*
(a) $F \cap D^- \subset e_1 \cup e_2$,
(b) $F \cap B^-$ *consists of disks,*
(c) $F \cap D^+$ *consists of arcs, and*
(d) *any component of* $F \cap B^+ - D^+$ *is a disk.*

Proof. This can be done by an isotopy of F since Theorem 1.3 assures us that $S^3 - K$ is irreducible. □

We take a 2-tuple lexicographically ordered complexity measure $(|F \cap B^-|, |(F \cap D^+) - (e_1 \cup e_2)|)$ minimal. Then we obtain a connected graph G in F by regarding $F \cap B^-$ and $(F \cap D^+) - (e_1 \cup e_2)$ as vertices and edges respectively. Corners of each face of G may contain two points $\partial e_1 - p_1$ and $\partial e_2 - p_2$. Note that the complexity measure is not $(0, *)$, otherwise F is not a decomposing sphere since $(B^\pm, K^\pm)$ is a trivial tangle. If the complexity measure is $(1, 0)$, then $F \cap S$ gives a desired loop since $p(e_i) = p(p_i)$ $(i = 1, 2)$.

Lemma 3.2. *For any arc* α_j *of* $(F \cap D^+) - (e_1 \cup e_2)$*,* $\beta_j \cap p(K^-) \neq \emptyset$*.*

Proof. This can be done by the same argument to Lemma 2.3. □

Hereafter, we assume that $\tilde{K}$ is prime.

Lemma 3.3. *There is no vertex of* G *with valency* 1.

Proof. Suppose that there is a vertex V with valency 1. Then only one edge α incident to V, and hence exactly one of e_1 and e_2 is attached to V or contained in V. Thus ∂V intersects $\tilde{K}$ in two points. Since $\tilde{K}$ is prime, ∂V bounds a disk E in S which intersects $p(K)$ in an unknotted arc. In the former case, $p(K) \cap E$ lies under a subarc of K^+ by the minimality of the number of bridges n. Then by an isotopy of F along the 3-ball which is bounded by $V \cup E$, we can reduce the complexity. See Figure 5. In the latter case, E intersects K in one point, and $V \cup E$ bounds a pair of a 3-ball and an unknotted subarc of K^- by the minimality of n. Then an isotopy of F along the pair can reduce the complexity. See Figure 6. In both cases, there is a contradiction in the assumption the complexity is minimal. □

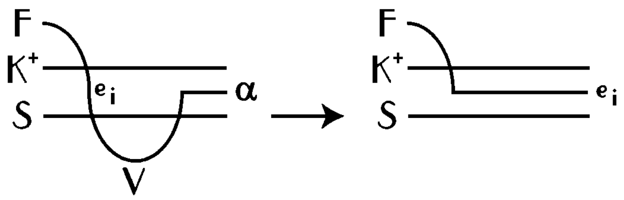

Figure 5. Isotopy of F along the 3-ball

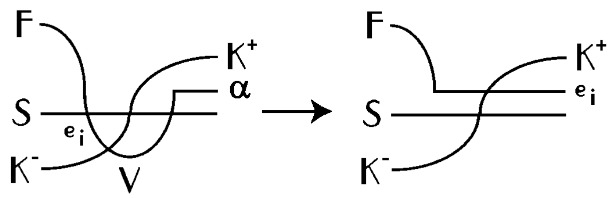

Figure 6. Isotopy of F along the pair

Lemma 3.4. *There is no face* f *of* G *in* F *such that* ∂f *is a loop of* G*.*

Proof. Suppose there exists a face f as Lemma 3.4. Then ∂f consists of an edge α of G and a subarc γ of the boundary of a vertex V of G. By Lemma 3.2, $p(\alpha)$ intersects $p(K^-)$. Moreover, since the loop $\gamma \cup p(\alpha)$ bounds a disk E in S, $|p(\alpha) \cap p(K^-)| = 1$ and γ meets exactly one of e_1 and e_2, say e_1. Thus a loop $l = \partial N(\partial E; E) - \partial E$ intersects $\tilde{K}$ in two points. Since $\tilde{K}$ is prime, $\operatorname{int} E$ intersects $p(K)$ in an embedded arc. Then, there are two possibilities for e_1, $e_1 \subset f$ or $e_1 \subset V$. In the former case, $f \cup E$ bounds a pair of a 3-ball and an unknotted arc, and an isotopy of F along the pair eliminates α. In the latter case, $f \cup E$ bounds a 3-ball, and an isotopy of F along the 3-ball eliminates α. These contradict the minimality of the complexity. □

Hence we have a condition that:

- G has at least two vertices,
- every vertex has valency at least two, and
- all faces of G in F are disks.

Next, we pay attention to a face of G in F.

Lemma 3.5. *For any face f, the cycle ∂f can not be oriented.*

Proof. If all corners of f do not meet $e_1 \cup e_2$, then this is same to Lemma 2.4.

If exactly one corner of f meets e_1 or e_2 at one point, then f and some K_i^+ have the intersection number ± 1, or a vertex which meets f along the corner intersects some K_k^- in one point. Since $p(\partial f)$ and $p(K^-) \cap p(K_i^+)$ must have the intersection number zero, ∂f is bounded by a loop of G consisting of a vertex and an edge α, and $p(\alpha)$ intersects $p(K^-)$ in one point. Then Lemma 3.4 gives the conclusion.

If some corners of f meet both e_1 and e_2, then the corners of f have the intersection number zero with $p(K)$ because F and K have the intersection number zero. In such a situation, we have a contradiction same as the proof of Lemma 2.4. □

By Lemma 3.5, starting a face f of G in F whose closure is a disk, we can get a loop l in $F - K$ with $|lk(l, K)| \geq 2$. But this is impossible because any loop in $F - K$ is null-homotopic in $S^3 - K$ or has linking number ± 1 with K. This finishes the proof of Theorem 1.4.

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