Rev. Mat. Iberoam. 38 (2022), no. 2, 615–634
DOI 10.4171/RMI/1297

Localized regularity of planar maps of finite distortion

Olli Hirviniemi, István Prause and Eero Saksman

Abstract. In this article we study fine regularity properties for mappings of finite distortion. Our main theorems yield strongly localized regularity results in the borderline case in the class of maps of exponentially integrable distortion. Analogues of such results were known earlier in the case of quasiconformal mappings. Moreover, we study regularity for maps whose distortion has better than exponential integrability.

1. Introduction

Let $f:\Omega \to \mathbb{C}$ be a function, where $\Omega \subset \mathbb{C}$ is a domain. We say that f is a (homeomorphic and orientation preserving) mapping of finite distortion if the following conditions are satisfied:

(i) $f \in W^{1,1}_{\mathrm{loc}}(\Omega)$.

(ii) $f:\Omega \to f(\Omega)$ is a homeomorphism with $J_f \geq 0$ a.e.

(iii) $|Df|^2 = K_f(z) J(z,f)$ for a.e. $z \in \mathbb{C}$, where K_f is a measurable function that is finite almost everywhere.

In an analogous way, one may define mappings of finite distortion on subdomains of $\mathbb{R}^d$. In this article we only consider mappings of finite distortion on the plane. A planar mapping of finite distortion satisfies a Beltrami equation

$$\overline{\partial} f(z) = \mu_f(z)\, \partial f(z),$$

where μ_f is a measurable function with $|\mu_f(z)| < 1$ for a.e. z. One has

$$|\mu_f(z)| = \frac{K_f(z)-1}{K_f(z)+1}\cdot$$

Here and henceforth we employ the standard notation

$$\overline{\partial} := \frac{d}{d\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y) \quad \text{and} \quad \partial := \frac{d}{dz} = \frac{1}{2}(\partial_x - i\partial_y).$$

2020 Mathematics Subject Classification: 30C65.
Keywords: Mappings of finite distortion, localized regularity, exponentially integrable distortion.

An important subclass is formed by mappings of finite exponential distortion which have the property that for some positive constant $p > 0$ one has

$$e^{pK_f(z)} \in L^1_{\mathrm{loc}}. \tag{1.1}$$

A natural version of the measurable Riemann mapping theorem, the Stoilow factorization theorem, and many other basic features of the standard quasiconformal theory generalise to these classes. For a good account of the theory we refer the reader to Chapter 20 of [4]. Improving on earlier results [9] (which has result valid also in n dimensions), it was shown in [3] that for a mapping of exponentially integrable distortion satisfying (1.1) there is the regularity

$$|Df|^2 \log^{\beta}(e + |Df|) \in L^1_{\mathrm{loc}} \quad \text{for } \beta < p - 1, \tag{1.2}$$

and this is not necessarily true for $\beta = p - 1$. Note that in the above results one is interested only in the local regularity of mappings of finite distortion with exponentially integrable distortion. Similarly, in our work it is enough to consider only the regularity of principal maps near the origin since local regularity results may then be transferred by the Stoilow factorisation theorem to maps that are defined on subdomains. Let us recall that a principal map $f : \mathbb{C} \to \mathbb{C}$ is conformal (i.e., $\mu_f(z) = 0$) outside the unit disk $\mathbb{D}$ and $f(z) = z + O(1/|z|)$ near infinity.

For (standard) K-quasiconformal maps (i.e., $K_f(z) \le K < \infty$, where K is a constant), the optimal area distortion [2] implies that $Df \in L^p_{\mathrm{loc}}$ for $p < 2K/(K-1)$, but this fails in general in the borderline case $p = 2K/(K-1)$. There is a substitute (Corollary 13.2.5 in [4]) in the form of inclusion in the weak space $Df \in L^{2K/(K-1),\infty}_{\mathrm{loc}}$. Another kind of result in the borderline case was given in Theorem 3.5 of [5], stating that a K-quasiconformal map satisfies

$$(K - K_f(z))|Df|^{2K/(K-1)} \in L^1_{\mathrm{loc}}, \tag{1.3}$$

which gives *strongly localized regularity* information on the map, especially we have $\int_{K_f \le K-\varepsilon} |Df|^{2K/(K-1)} < \infty$ for all $\varepsilon > 0$. For further basic results on planar maps of exponentially integrable distortion we refer e.g. to [6, 10, 12, 14, 15] and the references therein.

The principal aim of the present note is to establish a strongly localized regularity result for mappings of finite distortion analogous to (1.3). Our main result states the following.

Theorem 1.1. *Assume that f is a planar (homeomorphic) mapping of exponentially integrable distortion satisfying (1.1) with $p = 1$. Then it holds that*

$$\int_A \frac{1}{\log^{4+\varepsilon}(e + K_f)} |Df|^2 < \infty \quad \text{for any } \varepsilon > 0 \tag{1.4}$$

for any compact subset $A \subset \Omega$.

Arguments in the proof of Theorem 1.1 also give the following result.

Theorem 1.2. *With f a mapping of integrable distortion satisfying (1.1) for some $p > 0$, we have*

$$|Df|^2 \log^{p-1}(e + |Df|)[\log\log(|Df| + 10)]^{-(1+3p+\varepsilon)} \in L^1_{\mathrm{loc}} \quad \text{for any } \varepsilon > 0.$$

In the radial case, one can improve both Theorems 1.1 and 1.2:

Theorem 1.3. (i) *Let $f:\mathbb{C}\to\mathbb{C}$ be a planar and radial homeomorphic mapping of finite distortion with exponentially integrable distortion satisfying* (1.1) *with $p=1$. Then*

$$\int_A \frac{1}{\log^{1+\varepsilon}(e+K_f)}|Df|^2<\infty \quad \text{for any } \varepsilon>0$$

and any compact subset $A\subset\mathbb{C}$.
(ii) *For radial maps of p-integrable distortion with $p>0$ we have*

$$|Df|^2\log(e+|Df|)^{p-1}\log\log(|Df|+10)^{-(1+\varepsilon)}\in L^1_{\mathrm{loc}} \quad \text{for any } \varepsilon>0.$$

Our next result considers mappings that in a sense lie in between mappings of exponentially integrable distortion and standard quasiconformal maps.

Theorem 1.4. *Assume that f is a planar homeomorphic mapping of finite distortion satisfying the integrability*

$$e^{(K_f(z))^\alpha}\in L^1_{\mathrm{loc}}$$

for some $\alpha>1$. Then, for any $\beta<1-1/\alpha$, it holds that

$$\int_{\mathbb{D}}|Df|^2\exp\big(\log^\beta(e+|Df|)\big)<\infty. \tag{1.5}$$

The result is optimal in the sense that the conclusion fails for $\beta>1-1/\alpha$.

Sharpness of the previous theorem is shown by the map

$$f(z)=c_\alpha\frac{z}{|z|}\exp\big(-\tfrac{2}{1-1/\alpha}\log^{1-1/\alpha}(e+1/|z|)\big)$$

for $|z|<1$, and identity outside $\mathbb{D}$. We expect that the extremal maps for Theorems 1.1 and 1.2 are also given by radial maps, so it is natural to state:

Conjecture 1.5. *The conclusions of Theorem 1.3 remain true without assuming that the map is radial.*

Theorem 1.3 is sharp up to the possible borderline case. For any $0<\varepsilon<1$, we can choose $g_\varepsilon:\mathbb{C}\to\mathbb{C}$ to be

$$g_\varepsilon(z):=\frac{z}{|z|}\Big[\log\Big(e+\frac{1}{|z|}\Big)\Big]^{-p/2}\Big[\log\log\Big(e+\frac{1}{|z|}\Big)\Big]^{-\varepsilon/2} \quad \text{for } |z|<1$$

and $g_\varepsilon(z):=cz$ elsewhere for some constant c. Then one directly verifies that g_ε is a (radial) mapping of finite distortion satisfying (1.1) with p, but we have for general p,

$$\int_{\mathbb{D}}|D(g_\varepsilon)|^2\log(e+|D(g_\varepsilon)|)^{p-1}\log\log(|D(g_\varepsilon)|+10)^{-1+\varepsilon}=\infty,$$

and for $p=1$ we have

$$\int_{\mathbb{D}}\frac{1}{\log^{1-\varepsilon}(e+K_{g_\varepsilon})}|D(g_\varepsilon)|^2=\infty.$$

Section 2 below contains the proof of Theorems 1.1 and 1.2 assuming the quantitative estimate of Lemma 2.2. Next, Section 3 gives careful quantitative estimates for the decay of the Neumann series associated with the Beltrami equation. Then in Section 4 we are ready to accomplish the proof of Lemma 2.2, and also to complete the proof of Theorem 1.4. Finally, Section 5 treats the case of radial mappings, i.e., Theorem 1.3.

2. Proof of Theorems 1.1 and 1.2

In this section we prove Theorem 1.1 as well as Theorem 1.2 assuming Lemma 2.2, whose proof we provide later. It is useful to note the general comparison for mappings of finite distortion $|\partial f| \le |Df| \le 2|\partial f|$.

Proof of Theorem 1.1. Our basic assumption is that f is a principal mapping of finite distortion with

$$\int_{\mathbb{D}} e^{K_f} \le \widetilde{C} < \infty, \tag{2.1}$$

and we denote by μ the Beltrami coefficient of f. However, we first consider the class of quasiconformal f that satisfy (2.1) with a fixed $\widetilde{C}$. After obtaining uniform estimates for this class, we then at the end of the proof use approximation to deduce results for maps of finite distortion.

We next fix $0 < \varepsilon < 1/2$ and, for any w with $0 \le \Re w \le 1$, we let f_w be the unique principal solution to the Beltrami equation

$$\bar{\partial} f_w(z) = \nu_w(z)\, \partial f_w(z),$$

where

$$\nu_w(z) := \frac{\mu(z)}{|\mu(z)|}\, |\mu(z)|^{w+\varepsilon}.$$

A main idea in the proof is to consider the functions

$$g_w = (1-|\mu|)^{(1-w)/2}\, \partial f_w$$

and apply the analytic interpolation theorem, or actually a very special case of it that reduces to a vector-valued Phragmén–Lindelöf type maximum principle.

To accomplish this, note that since the dependence $w \mapsto \nu_w$ is analytic, we deduce by the Ahlfors–Bers theorem that the dependence $w \mapsto f_w$ (as an $L^2(\mathbb{D})$-valued function) is analytic over the closed strip, and hence also g depends analytically on w. Especially, the map $w \mapsto g_w$ is continuous in the strip and analytic in the interior. Moreover, by a standard application of the Neumann series and the definition of g we see that $\|g_w\|_{L^2(\mathbb{D})} \le C(\widetilde{C})$ for all w in the closed strip $\{0 \le \Re w \le 1\}$. Fix $h \in C_0^\infty(\mathbb{D})$ with $\|h\|_{L^2(\mathbb{D})} = 1$. A fortiori, the function $w \mapsto \int_{\mathbb{D}} g_w(z)h(z)\,dm(z)$ is a continuous and bounded analytic function in the closed strip and analytic in the interior. If we denote

$$\widetilde{M}_r := \sup_{\Re w = r} \Big| \int_{\mathbb{D}} g_w(z)h(z)\,dm(z) \Big|,$$

and

$$M_r := \sup_{\Re w = r} \|g_w\|_{L^2(\mathbb{D})},$$

then we have, by a classical version of the Hadamard three lines theorem, that

$$\widetilde{M}_\theta \leq \widetilde{M}_0^{1-\theta} \widetilde{M}_1^{\theta} \leq M_0^{1-\theta} M_1^{\theta}.$$

Since $h \in C_0^\infty(\mathbb{D})$ is arbitrary, we in fact have for any $\theta \in (0,1)$,

$$M_\theta \leq M_0^{1-\theta} M_1^{\theta}. \tag{2.2}$$

In order to continue the proof we need several auxiliary results.

Lemma 2.1. *For any w with $\Re w = 0$ we have $\int_{\mathbb{D}} |g_w|^2 \leq C\varepsilon^{-1}$, with a universal constant C. In particular, $M_0 \leq C_0 \varepsilon^{-1/2}$.*

Proof. As f_w is a quasiconformal principal mapping, we obtain by the Bieberbach area formula,

$$\int_{\mathbb{D}} J(z, f_w) = |f(\mathbb{D})| \leq \pi.$$

To use this, note first that as $J(z,h) = |\partial h|^2 - |\overline{\partial} h|^2$, we have by the definition of g for any w with $\Re w = 0$,

$$\int_{\mathbb{D}} |g_w|^2 = \int_{\mathbb{D}} (1-|\mu|)|\partial f_w|^2 = \int_{\mathbb{D}} \frac{1-|\mu|}{1-|\nu_w|^2} J(z, f_w) \leq \int_{\mathbb{D}} \frac{1-|\mu|}{1-|\mu|^{2\varepsilon}} J(z, f_w).$$

As $x \mapsto x^{2\varepsilon}$ is a concave function whose derivative at $x = 1$ equals 2ε, we have $x^{2\varepsilon} \leq 1 + 2\varepsilon(x-1)$ for all $x > 0$. This implies that

$$\frac{1-|\mu|}{1-|\mu|^{2\varepsilon}} \leq \frac{1-|\mu|}{2\varepsilon(1-|\mu|)} = \frac{1}{2\varepsilon},$$

finishing the proof. ∎

Lemma 2.2. *For any w with $\Re w = 1$, it holds that $\int_{\mathbb{D}} |g_w|^2 \leq C\varepsilon^{-4}$. The constant C depends only on $\widetilde{C}$ in (2.1). In particular, $M_1 \leq C_0 \varepsilon^{-2}$.*

We postpone the proof of this lemma to Section 4 as it needs more preparation, especially one needs to carefully check the dependence of constants in certain arguments of [3].

In order to continue the proof, we choose $\theta = 1 - \varepsilon$ in (2.2) and note that $f_{1-\varepsilon} = f$ in order to obtain

$$\int_{\mathbb{D}} (1-|\mu(z)|)^{\varepsilon} |\partial f|^2 \, dm(z) = \int_{\mathbb{D}} (1-|\mu(z)|)^{\varepsilon} |\partial f_{1-\varepsilon}(z)|^2 \, dm(z) \leq \frac{C}{\varepsilon^4}. \tag{2.3}$$

Up to now we have considered the case where f is quasiconformal and satisfies (2.1). We then choose a sequence of quasiconformal maps that converge to f locally uniformly and satisfy the condition (2.1). In order to find such a sequence one may e.g. use the factorization (see Corollary 4.4 in [3]) $f = g \circ h$, where g is (say) 5-quasiconformal and $\int_{\mathbb{D}} e^{5K_h} \leq \widetilde{C} + \pi e^5$. In this situation one may approximate h by quasiconformal maps h_n by truncating its dilatation in a standard way, and one defines $f_n := g \circ h_n$, and then the sequence f_n satisfies (2.1) with possibly slightly increased $\widetilde{C}$, but uniformly in n. That we have the convergence $h_n \to h$ locally uniformly is deduced by the fact that in this regime the Neumann-series of h_n converges in L^2 with uniform bounds for kth term, and clearly we have convergence in L^2 for each individual term of the Neumann-series. Thus $Dh_n \to Dh$ in L^2_{loc}, which implies local convergence in VMO for the maps h_n, and the uniform convergence then follows by the known uniform modulus of continuity estimates for mappings of exponentially integrable distortion.

Since $f_n \to f$ locally uniformly, we obtain that $f_n \to f$ in the sense of distributions, and hence $\partial f_n \to \partial f$ in the sense of distributions. Let us then fix $p < 2$. We have by (1.2) that $\int_B |\partial f_n|^p \le C$ uniformly in n for any fixed ball, and the same inequality holds also for f instead of f_n. This verifies (by using the density of test functions in L^q) that the convergence in distributions upgrades to weak convergence $\partial f_n \xrightarrow{w} \partial f$ in $L^p(\mathbb{D})$. This immediately implies the weak convergence in $L^p(\mathbb{D})$ of $(1-|\mu(z)|)^{\varepsilon/p}\partial f_n$ to $(1-|\mu(z)|)^{\varepsilon/p}\partial f$, and we obtain by the basic properties of weak L^p-convergence and the uniform estimate (2.3) that

$$\int_{\mathbb{D}} (1-|\mu(z)|)^{\varepsilon}|\partial f|^p\, dm(z) \le \liminf_{n\to\infty} \int_{\mathbb{D}} (1-|\mu(z)|)^{\varepsilon}\, |\partial f_n|^p\, dm(z)$$
$$\le \liminf_{n\to\infty} \int_{\mathbb{D}} (1-|\mu(z)|)^{\varepsilon}\, |\partial f_n|^2\, dm(z) + \pi \le \frac{C'}{\varepsilon^4}\cdot$$

By letting $p \nearrow 2$ we finally obtain for the general f the desired inequality

$$\int_{\mathbb{D}} (1-|\mu(z)|)^{\varepsilon}\, |\partial f|^2 \le \frac{C'}{\varepsilon^4}, \tag{2.4}$$

and again, the constant C' in (2.4) does not depend on ε.

The inequality (2.4) already provides a non-trivial localization result because we may consider small values of ε. However, as we have all values $\varepsilon \in (0, 1/2)$ at our disposal, the result can be improved on by invoking the following observation:

Lemma 2.3. *Let h and W be non-negative functions on $\mathbb{D}$, with $W(z) \le 1$ for all z. Let also $\varepsilon_0 \in (0, 1/2)$, $\alpha, C > 0$ be positive constants. Assume that for any $0 < \varepsilon < \varepsilon_0$ we have*

$$\int_{\mathbb{D}} (W(z))^{\varepsilon}\, h(z)\, dm(z) \le \frac{C}{\varepsilon^{\alpha}}\cdot \tag{2.5}$$

Then there is a constant $C_1 = C_1(\varepsilon_0, \alpha, C)$ such that for $0 < \eta \le 1$,

$$\int_{\mathbb{D}} \frac{1}{\left(\log\left(e + \frac{1}{W(z)}\right)\right)^{\alpha+\eta}}\, h(z)\, dm(z) \le \frac{C_1}{\eta}\cdot$$

Proof. The assumption remains valid if W is replaced by $\min(W, 1/2)$, and the conclusion obtained in this case yields the original one, in view of the assumption. We may hence assume that $W(z) \in [0, 1/2]$ for all z. From (2.5) it immediately follows that if $0 < \eta \le 1$, then

$$\int_0^{\varepsilon_0} \varepsilon^{\alpha-1+\eta} \int_{\mathbb{D}} (W(z))^{\varepsilon} h(z)\, dm(z)\, d\varepsilon \le C \int_0^{\varepsilon_0} \varepsilon^{\eta-1}\, d\varepsilon = \frac{C}{\eta}\, \varepsilon_0^{\eta}.$$

On the other hand, we can use Fubini's theorem to conclude that

$$\int_0^{\varepsilon_0} \varepsilon^{\alpha-1+\eta} \int_{\mathbb{D}} (W(z))^{\varepsilon} h(z)\, dm(z)\, d\varepsilon = \int_{\mathbb{D}} h(z) \int_0^{\varepsilon_0} \varepsilon^{\alpha-1+\eta} (W(z))^{\varepsilon}\, d\varepsilon\, dm(z).$$

For those z with $W(z) = 0$, the inner integral is 0. Let now $0 < a := W(z) \leq 1/2$. Then the inner integral is equal to

$$\begin{aligned}\int_0^{\varepsilon_0} x^{\alpha+\eta-1} a^x\, dx &= \frac{1}{\left(\log\left(\frac{1}{a}\right)\right)^{\alpha+\eta-1}} \int_0^{\varepsilon_0} \left(\log\left(\frac{1}{a}\right) x\right)^{\alpha+\eta-1} e^{-(\log(\frac{1}{a})x)}\, dx \\ &= \frac{1}{\left(\log\left(\frac{1}{a}\right)\right)^{\alpha+\eta}} \int_0^{\varepsilon_0 \log(\frac{1}{a})} s^{\alpha+\eta-1} e^{-s}\, ds.\end{aligned}$$

The last integral factor approaches $\Gamma(\alpha + \eta)$ uniformly on $\eta \in [0, 1]$ as $a \to 0$. The positive function $\phi\colon (0, 1/2] \times [0, 1] \to \mathbb{R}$,

$$\phi(a, \eta) := \frac{\left(\log\left(e + \frac{1}{a}\right)\right)^{\alpha+\eta}}{\left(\log\left(\frac{1}{a}\right)\right)^{\alpha+\eta}} \int_0^{\log(\frac{1}{a})\varepsilon_0} s^{\alpha+\eta-1} e^{-s}\, ds,$$

extends therefore to a continuous positive function on $[0, 1/2] \times [0, 1]$. Therefore there is a positive constant $c > 0$ so that for all z we have

$$\int_0^{\varepsilon_0} \varepsilon^{\alpha-1+\eta} (W(z))^{\varepsilon} d\varepsilon \geq \frac{c}{\left(\log\left(e + \frac{1}{W(z)}\right)\right)^{\alpha+\eta}},$$

which finishes the proof. ■

Theorem 1.1 is obtained by applying Lemma 2.3 in conjunction with inequality (2.4) using the choices $W(z) := (1 - |\mu(z)|)$ and $\alpha = 4$. ■

Remark. Generalizing Theorem 1.1 for values $p \neq 1$ appears to require interpolating in Orlicz space settings instead of L^2 with suitable counterparts of Lemmas 2.1 and 2.2. We have not attempted to carry out the necessary details for the generalisations since it would considerably increase the technicality of the paper.

Proof of Theorem 1.2. Following the argument of the proof of Lemma 2.2 and keeping track of the dependence of constant factors, we obtain under the assumption (1.1) that, instead of the result stated in Lemma 2.2, we obtain for general p that

$$\int_{\mathbb{D}} |Df|^2 \log(e + |Df|)^{p-1} \log^{-\varepsilon}\left(e + |Df|\right) \leq C\varepsilon^{-(1+3p)}.$$

Then, as before, the statement follows by an application of Lemma 2.3. ■

Remark. We note that one may apply Lemma 2.3 again directly on the result stated in Theorem 1.1, and this yields that the integral

$$\int_A \frac{1}{\log^{4+\varepsilon}(e + K_f)} |Df|^2$$

is bounded by C/ε. Thus taking $\alpha = 1$ in Lemma 2.3 we obtain a statement of the form

$$\int_{\mathbb{A}} \frac{1}{\log^4(e+K_f)(\log\log(10+K_f))^{1+\varepsilon}} |Df|^2 < \infty \quad \text{for any } \varepsilon > 0.$$

An industrious reader may refine this result by iterating the lemma, obtaining estimates for weights with more iterations of logarithms.

3. Decay of the Neumann series

For the proof of Lemma 2.2 we need quantitative versions of several auxiliary results in [3]. In this section we establish decay estimates for the Neumann series that suffice both for Theorem 1.1 and for Theorem 1.4. Our proof follows rather closely the ideas of [3, 8], but keeping track of the dependence of the constants is somewhat non-trivial even in the case $\alpha = 1$ which relates to that considered in [3, 8].

The Beurling operator $\mathcal{S}$ is the singular integral

$$\mathcal{S}\phi(z) := -\frac{1}{\pi}\int_{\mathbb{C}} \frac{\phi(\tau)}{(z-\tau)^2}\, d\tau.$$

Recall that in the context of quasiconformal mappings, the classical Beltrami equation in $W^{1,2}_{\mathrm{loc}}(\mathbb{C})$,

$$\bar\partial f(z) = \mu(z)\,\partial f(z),$$

has a unique principal solution $f(z) = z + O(1/z)$ –for this and other basic facts on quasiconformal maps we refer the reader to [1,4]. We can use the identity $\partial f - 1 = \mathcal{S}(\bar\partial f)$ to write the Beltrami equation equivalently for $\omega = \bar\partial f$:

$$\omega(z) = \mu(z)(\mathcal{S}\omega(z) + 1),$$

which is solved by the Neumann series

$$\omega = (\mathbb{I} - \mu\mathcal{S})^{-1}\mu = \mu + \mu\mathcal{S}\mu + \mu\mathcal{S}\mu\mathcal{S}\mu + \cdots.$$

The series converges absolutely when $|\mu(z)| \le k < 1$ almost everywhere because $\mathcal{S}$ is a unitary operator in $L^2(\mathbb{C})$. This is no longer true if only $|\mu(z)| < 1$, but we have as substitute the estimates of Lemma 3.1. We state here a refined (in the case $\alpha = 1$) and generalized (for $\alpha > 1$) version of Theorem 3.1 in [3] needed for our purposes. Its proof is adapted from the original proof in [3].

Lemma 3.1. *Let $|\mu(z)| < 1$ almost everywhere, with $\mu(z) \equiv 0$ for $|z| > 1$. Assume that the distortion function $K(z) = \frac{1+|\mu(z)|}{1-|\mu(z)|}$ satisfies*

$$e^{K^\alpha} \in L^p(\mathbb{D}), \quad \textit{for some } p > 0 \textit{ and } \alpha \ge 1.$$

In case $\alpha > 1$ we have, for every $p > 0$, $\beta \in [p/2, p)$ and $n \in \mathbb{N}$,

$$\int_{\mathbb{C}} |(\mu\mathcal{S})^n\mu|^2 \le C\exp\Big(-2(\beta/2)^{1/\alpha}\frac{1}{1-1/\alpha}\big((n+\beta/4+1)^{1-1/\alpha} - (\beta/4+1)^{1-1/\alpha}\big)\Big).$$

where by denoting $\delta := \frac{(p-\beta)^2}{\beta(p+\beta)}$, $\widetilde{C} := \frac{8p}{p-\beta}\big(\int_{\mathbb{D}} e^{pK^\alpha}\big)^{(p-\beta)/2p}$, $b := (\beta/2)^{1/\alpha}$, *and* $B := \max\big(\frac{b}{1-1/\alpha}((2b/\delta)^{\alpha-1} - (1+\beta/4)^{1-1/\alpha}), 0\big)$ *we have*

$$C := (4\delta^{-2}\widetilde{C}e^{2B} + 1). \tag{3.1}$$

In the case $\alpha = 1$ *one instead has*

$$\int_{\mathbb{C}} |(\mu\mathcal{S})^n\mu|^2 \le C_0\Big(\frac{n+\beta/4+1}{\beta/4+1}\Big)^{-\beta}, \quad n \in \mathbb{N},$$

where

$$C_0 := 12^{\beta+3}(p/\beta-1)^{-(5+2\beta)}\Big(\int_{\mathbb{D}} e^{pK}\Big)^{\frac{1}{2}(1-\beta/p)}. \tag{3.2}$$

Proof. We first note that a simple computation shows that the case $\alpha = 1$ follows from the case $\alpha > 1$ by first assuming that $\|\mu\|_\infty < 1$ and letting $\alpha \to 1^+$ in estimate (3.1). Hence we may assume that $\alpha > 1$ and start by fixing $0 < \beta < p$. For $n \in \mathbb{N}$, divide the unit disk into two sets,

$$B_n = \Big\{z \in \mathbb{D} : |\mu(z)| > 1 - \frac{2\beta^{1/\alpha}}{(4n)^{1/\alpha} + \beta^{1/\alpha}}\Big\} \quad \text{and} \quad G_n = \mathbb{D} \setminus B_n.$$

By Chebychev's inequality,

$$|B_n| \le \Big(\int_{\mathbb{D}} e^{pK^\alpha}\Big)e^{-4np/\beta}.$$

The terms of the Neumann series $\psi_n = (\mu\mathcal{S})^n\mu$ and the auxiliary terms g_n are obtained inductively:

$$\psi_n = \mu\mathcal{S}(\psi_{n-1}), \quad \psi_0 = \mu, \quad \text{and} \quad g_n = \chi_{G_n}\mu\mathcal{S}(g_{n-1}), \quad g_0 = \mu.$$

For g_n we can estimate by using the fact that $\mathcal{S}$ is L^2-isometry to see that

$$\|g_n\|_{L^2}^2 = \int_{G_n} |\mu S(g_{n-1})|^2 \le \Big(1 - \frac{2\beta^{1/\alpha}}{(4n)^{1/\alpha} + \beta^{1/\alpha}}\Big)^2 \|g_{n-1}\|_{L^2}^2,$$

and therefore

$$\begin{aligned}\|g_n\|_{L^2} &\le \prod_{j=1}^n \Big(1 - \frac{2\beta^{1/\alpha}}{(4j)^{1/\alpha} + \beta^{1/\alpha}}\Big)\|g_0\|_{L^2} \\ &= \exp\Big(\sum_{j=1}^n \log\Big(1 - \frac{2\beta^{1/\alpha}}{(4j)^{1/\alpha} + \beta^{1/\alpha}}\Big)\Big)\|\mu\|_{L^2}.\end{aligned}$$

As $\log(1-x) \le -x$ for $x < 1$, and $\|\mu\|_{L^2} \le \sqrt{\pi}$, it follows that

$$\begin{aligned}\|g_n\|_{L^2} &\le \exp\Big(-2^{1-2/\alpha}\beta^{1/\alpha}\sum_{j=1}^n \frac{1}{j^{1/\alpha} + (\beta/4)^{1/\alpha}}\Big)\pi^{1/2} \\ &\le \exp\Big(-2^{-1/\alpha}\beta^{1/\alpha}\sum_{j=1}^n \frac{1}{(j+\beta/4)^{1/\alpha}}\Big)\pi^{1/2},\end{aligned}$$

where we applied the inequality $(j)^{1/\alpha} + (\beta/4)^{1/\alpha} \le 2^{1-1/\alpha}(j + \beta/4)^{1/\alpha}$. The sum inside the exponential can be estimated by an integral:

(3.3)
$$\sum_{j=1}^{n} \frac{1}{(j+\beta/4)^{1/\alpha}} \ge \int_1^{n+1} \frac{dx}{(x+\beta/4)^{1/\alpha}} = \frac{(n+1+\beta/4)^{1-1/\alpha} - (1+\beta/4)^{1-1/\alpha}}{(1-1/\alpha)},$$

so that

$$\|g_n\|_{L^2} \le \exp\Big(-2^{-1/\alpha}\beta^{1/\alpha} \frac{(n+1+\beta/4)^{1-1/\alpha} - (1+\beta/4)^{1-1/\alpha}}{1-1/\alpha}\Big). \tag{3.4}$$

The difference of ψ_n and g_n is

$$\psi_n - g_n = \chi_{G_n}\,\mu\mathcal{S}(\psi_{n-1} - g_{n-1}) + \chi_{B_n}\,\mu\mathcal{S}(\psi_{n-1}).$$

For the norms, this gives

$$\|\psi_n - g_n\|_{L^2} \le \Big(1 - \frac{2\beta^{1/\alpha}}{(4j)^{1/\alpha} + \beta^{1/\alpha}}\Big)\|\psi_{n-1} - g_{n-1}\|_{L^2} + \sqrt{R(n)}$$

with

$$R(n) = \|\chi_{B_n}\,\mu\mathcal{S}(\psi_{n-1})\|_{L^2}^2 = \int_{B_n} |(\mu\mathcal{S})^n\mu|^2.$$

By induction and estimating like in (3.3) we deduce

$$\begin{aligned}
\|\psi_n - g_n\|_{L^2} &\le \Big(1 - \frac{2\beta^{1/\alpha}}{(4n)^{1/\alpha} + \beta^{1/\alpha}}\Big)\|\psi_{n-1} - g_{n-1}\|_{L^2} + \sqrt{R(n)} \\
&\le \sum_{j=1}^{n} \sqrt{R(j)} \prod_{k=j+1}^{n} \Big(1 - \frac{2\beta^{1/\alpha}}{(4k)^{1/\alpha} + \beta^{1/\alpha}}\Big) \\
&= \sum_{j=1}^{n} \sqrt{R(j)} \exp\Big(\sum_{k=j+1}^{n} \log\Big(1 - \frac{2\beta^{1/\alpha}}{(4k)^{1/\alpha} + \beta^{1/\alpha}}\Big)\Big) \\
&\le \sum_{j=1}^{n} \sqrt{R(j)} \exp\Big(-2^{1-2/\alpha}\beta^{1/\alpha} \sum_{k=j+1}^{n} \frac{1}{k^{1/\alpha} + (\beta/4)^{1/\alpha}}\Big) \\
&\le \sum_{j=1}^{n} \sqrt{R(j)} \exp\Big(-2^{-1/\alpha}\beta^{1/\alpha} \frac{(n+1+\beta/4)^{1-1/\alpha} - (j+1+\beta/4)^{1-1/\alpha}}{1-1/\alpha}\Big) \\
&= \exp\Big(-2^{-1/\alpha}\beta^{1/\alpha} \frac{1}{1-1/\alpha}\big((n+1)^{1-1/\alpha} - (1+\beta/4)^{1-1/\alpha}\big)\Big) \qquad (3.5)\\
&\quad \times \sum_{j=1}^{n} \exp\Big(2^{-1/\alpha}\beta^{1/\alpha} \frac{1}{1-1/\alpha}\big((j+1+\beta/4)^{1-1/\alpha} - (1+\beta/4)^{1-1/\alpha}\big)\Big)\sqrt{R(j)}.
\end{aligned}$$

We next recall that in [3] the Astala area distortion result $|f^\lambda(E)| \le \pi M|E|^{1/M}$ for quasiconformal maps was used to estimate $R(n)$ by considering the solution $f = f^\lambda$ to the Beltrami equation

$$\bar\partial f = \lambda\mu\,\partial f,$$

with $|\lambda| < 1$ via expressing the term $(\mu\mathcal{S})^n\mu$ of the Neumann series

$$\overline{\partial} f^\lambda = \sum_{n=0}^{\infty} \lambda^{n+1} (\mu\mathcal{S})^n \mu$$

by a Cauchy integral

$$(\mu\mathcal{S})^n\mu = \frac{1}{2\pi i} \int_{|\lambda|=\rho} \frac{1}{\lambda^{n+2}} \overline{\partial} f^\lambda \, d\lambda,$$

multiplying by the characteristic function χ_{B_n} and by forcing the Jabobian to appear under the integral. This yielded (see [3], p. 8) for any μ with just $\|\mu\|_\infty \le 1$, any $M > 1$, and any $E \subset \mathbb{D}$ the general estimates

$$\|\chi_E\, \psi_n\|_{L^2}^2 \le \pi \Big(\frac{M+1}{M-1}\Big)^{2n} \frac{(M+1)^2}{4} |E|^{1/M} \quad \text{and} \tag{3.6}$$

$$\|\chi_E\, \mathcal{S}(\psi_n)\|_{L^2}^2 \le \pi \Big(\frac{M+1}{M-1}\Big)^{2n+2} \frac{(M+1)^2}{4} |E|^{1/M}. \tag{3.7}$$

Choosing $E = B_n$ this yields

$$\begin{aligned} \sqrt{R(n)} &\le \sqrt{\pi} \left(\frac{M+1}{M-1}\right)^n \frac{(M+1)}{2} |B_n|^{1/2M} \\ &\le \sqrt{\pi} \left(\frac{M+1}{M-1}\right)^n \frac{(M+1)}{2} \left(\int_{\mathbb{D}} e^{pK^\alpha}\right)^{1/2M} e^{-2np/(\beta M)}. \end{aligned}$$

In our situation we may actually slightly improve this by invoking the Eremenko and Hamilton form of the area distortion estimate stating, for any measurable $E \subset \mathbb{D}$, the inequality

$$|g(E)| \le M^{1/M} \pi^{1-1/M} |E|^{1/M} \le \pi e^{1/(\pi e)} |E|^{1/M}. \tag{3.8}$$

This leads to

$$\sqrt{R(n)} \le \sqrt{\pi} e^{1/(2\pi e)} \Big(\frac{M+1}{M-1}\Big)^n \sqrt{\frac{(M+1)^2}{4M}} \Big(\int_{\mathbb{D}} e^{pK^\alpha}\Big)^{1/2M} e^{-2np/(\beta M)}.$$

We want to choose $M > 1$ in order to force $R(n)$ to decay exponentially. For that we need to have

$$\log\Big(\frac{M+1}{M-1}\Big) - \frac{2}{M}\frac{p}{\beta} \le -\delta < 0.$$

Choose $M = 2p/(p-\beta)$ and estimate

$$\begin{aligned} \log\Big(\frac{M+1}{M-1}\Big) - \frac{2}{M}\frac{p}{\beta} &\le \frac{2}{M-1} - \frac{2}{M}\frac{p}{\beta} = \frac{2(p-\beta)}{p+\beta} - \frac{2(p-\beta)}{2\beta} \\ &= -\frac{(p-\beta)^2}{\beta(p+\beta)} =: -\delta. \end{aligned}$$

Noting that $(M+1)^2/(4M) \leq M$ and $\sqrt{\pi}e^{1/(2\pi e)} \leq 2$, this yields

$$\sqrt{R(n)} \leq \widetilde{C} e^{-\delta n} \quad \text{with} \quad \widetilde{C} := \sqrt{\frac{8p}{p-\beta}} \Big(\int_{\mathbb{D}} e^{pK^\alpha} \Big)^{(p-\beta)/4p}.$$

Hence, if we denote $b := 2^{-1/\alpha}\beta^{1/\alpha}$, we obtain

$$\sum_{j=1}^{n} \exp\Big(2^{-1/\alpha}\beta^{1/\alpha}\frac{1}{1-1/\alpha}\big((j+1+\beta/4)^{1-1/\alpha} - (1+\beta/4)^{1-1/\alpha}\big)\Big)\sqrt{R(j)}$$
$$\leq \widetilde{C} e^{\widetilde{B}} \sum_{j=1}^{\infty} e^{-j\delta/2} \leq 2\delta^{-1}\widetilde{C} e^{\widetilde{B}},$$

where $\widetilde{B} := \sup_{j\geq 1}\big(\frac{b}{1-1/\alpha}\big((j+1+\beta/4)^{1-1/\alpha} - (1+\beta/4)^{1-1/\alpha}\big) - (\delta/2)j\big)$. An elementary computation where one simply differentiates with respect to j shows that

$$\widetilde{B} \leq B := \max\Big(\frac{b}{1-1/\alpha}\big((2b/\delta)^{\alpha-1} - (1+\beta/4)^{1-1/\alpha}\big), 0\Big).$$

In view of (3.5) we then obtain

$$\|\psi_n - g_n\|_{L^2}$$
$$\leq 2\delta^{-1}\widetilde{C} e^{B} \exp\Big(-2^{-1/\alpha}\beta^{1/\alpha}\frac{1}{1-1/\alpha}\big((n+1+\beta/4)^{1-1/\alpha} - (\beta/4)^{1-1/\alpha}\big)\Big).$$

Together with (3.4) this proves the lemma. ∎

4. Proof of Lemma 2.2 and Theorem 1.4

We start with an area distortion result that generalizes Corollary 3.2 and Theorem 5.1 in [3] to the range $\alpha \geq 1$. In case $\alpha = 1$ we need to keep careful track of the constants, which is somewhat non-trivial in this situation, and hence for the readers sake we give the details here although the basic idea of the proof follows that in [3, 8]. Thus for $\alpha = 1$ the novelty of the statement as compared to Theorem 5.1 in [3] is in a precise estimation of the dependencies of the constant terms. This is a crucial technical ingredient needed for our main results.

Proposition 4.1. *Let μ and $0 < \beta < p$ and f be as in Lemma 3.1.*

(i) *In case $\alpha > 1$, we have the area distortion estimate*

$$|f(E)| \leq c \exp\big(-c' \log^{1-1/\alpha}(e + 1/|E|)\big) \tag{4.1}$$

with some constants $c, c' > 0$.

(ii) *In case $\alpha = 1$, under the additional assumption $1/2 < \beta < p < 4$, it holds that*

$$|f(E)| \leq A_2|E|^{\delta/24} + A_2\delta^{-3\beta}\log^{-\beta}(e + 1/|E|)\Big(\int_{\mathbb{D}} e^{pK}\Big)^{1/2}, \quad E \subset \mathbb{D}, \tag{4.2}$$

where we denoted $\delta := p - \beta$ and A_2 is a universal constant.

Proof of Proposition 4.1. We start by observing that our maps are Sobolev homeomorphisms that satisfy Lusin's condition $\mathcal{N}$. Especially, we obtain (using the notation of the previous section)

$$|f(E)| = \int_E |\partial f|^2 - |\bar\partial f|^2 \le 2|E| + 2\int_E |\partial f - 1|^2 = 2|E| + 2\|\chi_E(\partial f - 1)\|_2^2$$

$$\le 2|E| + 2\Big(\sum_{n=0}^{\infty} \|\chi_E S\psi_n\|_2\Big)^2. \tag{4.3}$$

We estimate the last written sum in two parts, and fix to that end an index $m \ge 1$ that will be specified later on. First of all, using (3.6) with $M = 3$ yields

$$\sum_{n=0}^{m-1} \|\chi_E S\psi_n\|_2 \le \sum_{n=0}^{m-1} \sqrt{\pi}\, 2^{n+2}\, |E|^{1/6} \le 2^{m+3}|E|^{1/6}. \tag{4.4}$$

In case $\alpha > 1$ we choose $\beta = p/2$ in Lemma 3.1 and obtain with small work the estimate

$$\sum_{n=m}^{\infty} \|\chi_E S\psi_n\|_2 \le c_1 \exp(-c_2 m^{1-1/\alpha}).$$

Here and later, the c_j's are constants that may depend on β, p, α, and whose exact value is of no interest to us. By choosing $m = \lfloor 2 + \frac{1}{12\log 2}\log 1/|E|\rfloor$, we obtain in view of (4.3) and the previous estimates

$$|f(E)| \le 2|E| + c_3|E|^{1/12} + 4c_1 \exp\big(-c_4 \log^{1-1/\alpha}(e + 1/|E|)\big),$$

which proves part (i).

In case (ii) we have $\alpha = 1$. In this case we first assume that $2 < \beta < p$ and an application of Lemma 3.1 yields in this case

$$\Big(\sum_{n=m}^{\infty} \|\chi_E \psi_n \mu\|_2\Big)^2 \le C_0\Big(\sum_{n=m}^{\infty}\Big(\frac{n+\beta/4+1}{\beta/4+1}\Big)^{-\beta/2}\Big)^2 \le \frac{4C_0(\beta/4+1)^2}{(\beta-2)^2}\Big(\frac{m+\beta/4}{\beta/4+1}\Big)^{2-\beta}$$

where the expression for the constant $C_0 = C_0(\beta, p)$ is given in (3.2). In view of (4.4) we thus have

$$|f(E)| \le 2|E| + 2^{2m+7}|E|^{1/3} + \frac{8C_0(\beta/4+1)^2}{(\beta-2)^2}\Big(\frac{m+\beta/4}{\beta/4+1}\Big)^{2-\beta}.$$

Choosing $m = \big\lceil \frac{1}{12\log 2}\log(1 + 1/|E|)\big\rceil$ and noting that $|E| \le 4|E|^{1/6}$ and $12\log 2 \le 9$ yields

$$|f(E)| \le 2000|E|^{1/6} + \frac{8C_0(\beta/4+1)^2}{(\beta-2)^2}\Big(\frac{\log((1/|E|+1)^{1/9}) + \beta/4}{1+\beta/4}\Big)^{2-\beta} \tag{4.5}$$

Our next step is to apply G. David's factorisation trick to improve the above bound and extend it to all values of p. We assume thus that f is as in the statement of the proposition (with the general assumption $1/2 < \beta < p < 4$) and recall from [3] that for any $M \geq 1$ we may factorise f as $f = g \circ F$, where g and F are principal mappings, g is M-quasiconformal and F satisfies

$$I_M := \int_{\mathbb{D}} e^{pMK(z,F)} \leq e^M \int_{\mathbb{D}} e^{pK(z,f)}.$$

Denote $\beta_0 := (p+\beta)/2$, and $M = 2/(\beta_0 - \beta) = 4/(p-\beta) \geq 1$. We will apply (4.5) with parameters $(M\beta_0, Mp)$ instead of (β, p) in order to estimate $|F(E)|$. This is possible since by the assumption $p < 4$ we have $2 < 2 + \beta M = \beta_0 M < pM$. Thus,

$$|F(E)| \leq 2000|E|^{1/6} + \frac{8C_0(M\beta_0, Mp, I_M)(M\beta_0/4+1)^2}{(M\beta_0-2)^2}\Big(\frac{\log((1+1/|E|)^{1/9}) + M\beta_0/4}{1+M\beta_0/4}\Big)^{2-M\beta_0}.$$

Above, the notation $C_0(M\beta_0, Mp, I_M)$ recalls the dependences of the constant C_0. As g is a principal quasiconformal mapping, we obtain from the standard area distortion estimate (3.8),

$$|f(E)| = |g \circ F(E)| \leq 4|F(E)|^{1/M}.$$

By noting that $2 - M\beta_0 = -M\beta$, and $M\beta_0/4 = \frac{1}{2}\frac{p+\beta}{p-\beta} > (p/\beta - 1)^{-1}$, combing the last two inequalities leads to

$$|f(E)| \leq 8000|E|^{1/6M} + 4 \cdot 8^{1/M}\Big(\frac{C_0(M\beta_0, Mp, I_m)(M\beta_0/4+1)^2}{(M\beta)^2}\Big)^{1/M} \times \Big(\frac{(\log(1+1/|E|))^{1/6} + (p/\beta-1)^{-1}}{1+(p/\beta-1)^{-1}}\Big)^{-\beta}. \tag{4.6}$$

Here, since $Mp/M\beta_0 - 1 = (p-\beta)/(p+\beta) \geq (p-\beta)/2p$ and $(p/\beta_0 - 1)/2M \leq 1/2$ we obtain by recalling (3.2) and easy estimates,

$$8^{1/M}\Big(\frac{C_0(M\beta_0, Mp, I_m)(M\beta_0/4+1)^2}{(M\beta)^2}\Big)^{1/M} \leq A_1(p/\beta-1)^{-2\beta}\Big(\int_{\mathbb{D}} e^{pK}\Big)^{1/2},$$

where A_1 is an absolute constant. In the simplification we applied our assumption on the range of p and β and observed that $(p-\beta)^{-(p-\beta)}$ has a universal upper bound. We also observe in (4.6) that $(p/\beta - 1)^{\beta}$ has a universal upper bound, and by increasing A_1 we may replace $\log(1+1/|E|)^{1/6}$ by $\log(1+1/|E|)$. In addition, in our situation $p/\beta - 1 \asymp (p-\beta)$. Combining these estimates completes the proof of part (ii). ■

We then turn to the goals stated in the title of this section. As expected, we will first estimate the integrability of the Jacobian using the estimates for area-distortion we just proved. For that purpose we will first state a general lemma that yields (essentially optimal) integrability estimates from estimates of area distortion.

Lemma 4.2. *Assume that f is a principal mapping of finite distortion and $g\colon [0,\pi) \to [0,\infty)$ is concave with $g(0) = 0$, satisfying for any measurable subsets $E \subset B(0,1)$ the area distortion estimate*

$$|f(E)| \leq g(|E|). \tag{4.7}$$

Then for any convex and increasing H on $[0,\infty)$ it holds that

$$\int_{B(0,1)} H(J_f(z))\, dA(z) \leq \int_0^{\pi} H(g'(t))\, dt.$$

Proof. Let us denote by $h\colon (0,\pi) \to \mathbb{R}_+$ the decreasing rearrangement of J_f. By assuming first that g is differentiable on $(0,\pi)$, our assumption may be rewritten as

$$\int_0^x h(t)\, dt \leq \int_0^x g'(t)\, dt \quad \text{for all } x \in (0,\pi).$$

The statement now follows from a continuous version of the Hardy–Littlewood Pólya (or Karamata) inequality, see Theorem 2.1 in [7] or [11]. ■

Proof of Theorem 1.4. It follows from Lemma 4.2 and Proposition 4.1(i) that in our situation the higher integrability of J_f is at least as good as that of the derivative h' on the interval $(0,\pi)$, where

$$h(x) := \exp\big(-c' \log^{1-1/\alpha}(e+1/x)\big)$$

on the interval $(0,\pi)$. Namely, h is clearly decreasing near the origin which is enough for us in order to apply Lemma 4.2. We may safely leave to the reader to check that $\phi(h')$ is integrable near the origin with $\phi(y) := y \exp(\log^{\beta}(e+y))$ for $\beta < 1-1/\alpha$. In other words, we have

$$\int_{\mathbb{D}} J_f \exp(\log^{\beta}(e+J_f)) < \infty \quad \text{for } \beta < 1-\alpha^{-1}.$$

By recalling that $|Df|^2 = KJ_f$, the stated integrability of the derivative follows immediately by the elementary inequality

$$xy \exp\big(\log^{\beta'}(e+xy)\big) \leq C\big(\exp(px^{\alpha}) + y\exp\big(\log^{\beta}(e+y)\big)\big), \quad x,y \geq 1.$$

for any $0 < \beta' < \beta < 1$ and $p > 0$, and where $C = C(p,\beta,\beta',\alpha)$. The latter inequality follows easily by examining separately the cases $x < \exp((1/2)\log^{\beta}(e+y))$ and $x \geq \exp((1/2)\log^{\beta}(e+y))$. ■

Proof of Lemma 2.2. Easy estimates that just apply differentiation show that the function

$$x \mapsto (1+\delta\log(1+1/x))^{-\beta}$$

is concave for $x > 0$ as soon as $\delta < (1+\beta)^{-1}$, which in our situation holds at least if $\delta < 1/5$. We now fix $p = 1+2\varepsilon$, $\beta = 1+\varepsilon$, with $\varepsilon \in (0,1/10)$ in Proposition 4.1 (ii) and note that Lemma 4.2 yields the integrability

$$\int_{\mathbb{D}} J(z,f)\log(e+J(z,f)) \leq \int_0^{\pi} h'(x)\log(e+h'(x))\, dx,$$

where $h(x) := A_2 x^{\varepsilon/24} + A_2\varepsilon^{-3-3\varepsilon}(\log(1+1/x))^{-1-\varepsilon}\big(\int_{\mathbb{D}} e^{pK}\big)^{1/2}$. Hence, if we denote $A_3 := A_2'\big(\int_{\mathbb{D}} e^{pK}\big)^{1/2}$ with another universal constant A_2', we have

$$h'(x) \leq A_3\Big(\varepsilon x^{-1+\varepsilon/24} + \frac{\varepsilon^{-3}}{x}\big(\log(1+1/x)\big)^{-2-\varepsilon}\Big),$$

Obviously, $\log h'(x) \le \log(A_3) + 3\log(1/\varepsilon) + 3\log(10/x)$, so that noting that $\int_0^\pi h'(x)\,dx = h(\pi) \le 10A_3\varepsilon^{-3}$ we obtain that

$$\begin{aligned}\int_{\mathbb{D}} J(z,f)\log(e+J(z,f)) &\le \int_0^\pi h'(x)\log(e+h'(x))dx \\ &\le 10A_3\varepsilon^{-3}\big(\log(A_3)+\log(1/\varepsilon)\big) \\ &\quad + 3A_3\int_0^\pi \Big(\varepsilon x^{-1+\varepsilon/24} + \frac{\varepsilon^{-3}}{x}\big(\log(1+1/x)\big)^{-2-\varepsilon}\Big)\log(10/x)\,dx \\ &\le A_3\log(A_3)10^6\varepsilon^{-4}.\end{aligned}$$

In the estimation of the last written integral we noted that

$$\int_0^\pi \varepsilon x^{-1+\varepsilon/100}\log(10/x)\,dx \le 10^5\varepsilon^{-1},$$

and estimated the second integral from the above by

$$2\log(20)\Big(\varepsilon^{-3}\int_0^{1/2}\log(1/x)^{-1-\varepsilon}\frac{dx}{x} + 3\varepsilon^{-3}\Big) \le 40\varepsilon^{-4}.$$

We next note the well-known inequality stating that for any $\varepsilon \in (0,1)$ and reals $x, y > 0$ it holds that

$$xy \le x\log(e+x) + e^{(1+\varepsilon)y}$$

(one simply checks that is true for $\varepsilon = 0$). The choice $x = J_j(z)$, $y = K := K_f(z)$, and integration over $\mathbb{D}$ finally yields that

$$\begin{aligned}\int_{\mathbb{D}}|Df|^2 &\le A_4\Big(\int_{\mathbb{D}} e^{(1+\varepsilon)K}\Big)^{1/2}\log\Big(\int_{\mathbb{D}} e^{(1+\varepsilon)K}\Big)\varepsilon^{-4} + \int_{\mathbb{D}} e^{(1+\varepsilon)K} \\ &\le A_5\varepsilon^{-4}\int_{\mathbb{D}} e^{(1+\varepsilon)K},\end{aligned} \tag{4.8}$$

where A_4 and A_5 are universal constants.

We are now ready to complete the proof of Lemma 2.2. To that end we need to establish for any w with $\Re w = 1$ the key estimate

$$\int_{\mathbb{D}}|g_w|^2 \le \frac{C}{\varepsilon^4},$$

with constant C does not depend on ε. Note that this estimate implies the bound $M_1 \le C_1/\varepsilon^2$ for some constant C_1. Moreover, estimating the integrability of $|g_w|$ reduces to that of $|\partial(f_w)|$ because $|g_w| = |\partial(f_w)|$ a.e. since we have $\Re w = 1$.

Let us first estimate the distortion $K(z, f_w)$. Assume that $\varepsilon \in (0, 1/2)$ and consider the function $r(x) := 1 - x^{1+\varepsilon} - (1+\varepsilon/2)(1-x)$. We claim that $r(x) \ge 0$ for $x \in [1/2, 1]$. As r is concave with $r(1) = 0$ and $r'(1) = -\varepsilon/2 < 0$, it is enough to check that $r(1/2) \ge 0$, or equivalently that $1 + \varepsilon/2 \le 2 - 2^{-\varepsilon}$. In turn this follows from the concavity of $\varepsilon \mapsto R(\varepsilon) := 2 - 2^{-\varepsilon} - (1+\varepsilon/2)$ and by noting that $R(0) = R(1/2) = 0$.

We thus have that $1-|\nu(z)|^{1+\varepsilon} \geq (1+\varepsilon/2)(1-|\nu(z)|)$ assuming that $|\nu(z)| \geq 1/2$, and we may estimate the distortion as follows:

$$\begin{aligned} K(z,f_w) &= \frac{1+|\nu(z)|^{1+\varepsilon}}{1-|\nu(z)|^{1+\varepsilon}} \\ &= -1+\frac{2}{1-|\nu(z)|^{1+\varepsilon}} \leq -1+\frac{2}{\min((1+\varepsilon/2)(1-|\nu(z)|),1-1/2^{1+\varepsilon})} \\ &\leq -1+\frac{2}{(1+\varepsilon/2)(1-|\nu(z)|)}+\frac{2}{1-1/2^{1+\varepsilon}} \leq 3+\frac{2}{(1+\varepsilon/2)(1-|\nu(z)|)} \\ &= \frac{1}{1+\varepsilon/2}\Big(-1+\frac{2}{1-|\nu(z)|}\Big)+3+\frac{1}{1+\varepsilon/2} \leq \frac{K(z,f)}{1+\varepsilon/2}+4. \end{aligned}$$

It follows that

$$\int_{\mathbb{D}} e^{(1+\varepsilon/2)K(z,f_w)} \leq \int_{\mathbb{D}} e^{K(z,f)+4+2\varepsilon} \leq e^5 \int_{\mathbb{D}} e^{K(z,f)}.$$

In conclusion, an application of inequality (4.8) (with $\varepsilon/4$ in place of ε) yields the desired uniform bound

$$\int_{\mathbb{D}} |\partial(f_w)|^2 \leq \int_{\mathbb{D}} |Df_w|^2 \leq \frac{C}{\varepsilon^4}. \qquad \blacksquare$$

5. Proof of Theorem 1.3

Throughout this section we assume that $f:\mathbb{D}\to\mathbb{D}$ is a radial homeomorphism of the form

$$f(z) = \frac{z}{|z|}\phi(|z|),$$

where $\phi: [0,1] \to [0,1]$ is an increasing homeomorphism. We also assume that f is a map with exponentially integrable distortion (satisfying (1.1)). By the Lusin condition, this implies that ϕ is absolutely continuous, and the assumed exponential integrability of K_f can expressed as

$$\int_0^1 \Big(e^{p\frac{r\phi'(r)}{\phi(r)}} + e^{p\frac{\phi(r)}{r\phi'(r)}}\Big) r\,dr = C_0 < \infty. \tag{5.1}$$

Our aim is to first prove an area distortion estimate for these maps.

Proposition 5.1. *Let $f:\mathbb{D}\to\mathbb{D}$ be a radial homeomorphism of exponentially integrable distortion (see (1.1)). Then for any measurable subset $E \subset \mathbb{D}$,*

$$|f(E)| \leq C\big(\log(1+1/|E|)\big)^{-p}.$$

The constant $C = C(p, C_0)$ is uniform for fixed C_0 and $p \in [1,2]$.

Proof. We shall denote by C constants whose actual size if of no interest to us, and their value may change from line to line. We call the set $E \subset \mathbb{D}$ 'radial' if one has that $z \in E$ if and only if $|z| \in E$. By a standard approximation argument, it is enough to prove the claim in the case where E is a disjoint union of sets of the form $\{a < |z| < b,\ \alpha_0 < \arg(z) < \alpha_1\}$, and this case is easily reduced to the case of radial sets. Thus, we may assume that $E = \{|z| \in F, \}$ where $F \subset (0,1)$ is a disjoint union of open intervals.

For $n = 1, 2, \ldots$, we denote the dyadic annuli $A_n := \{2^{-n} \le |z| \le 2^{1-n}\}$. Our first goal is to estimate $\phi(r)$ from the above. To that end, fix $n \ge 1$ and note that by (5.1) and Jensen's inequality applied on the probability measure $r^{-1}dr$ on (e^{-n}, e^{1-n}) and on the convex function $x \mapsto e^{p/x}$ yields

$$\log\big(\phi(e^{1-n})/\phi(e^{-n})\big) = \int_{e^{-n}}^{e^{1-n}} \frac{r\phi'(r)}{\phi(r)}\frac{dr}{r} \ge p\Big(\log\Big(\int_{e^{-n}}^{e^{1-n}} \exp\Big(p\frac{\phi(r)}{r\phi'(r)}\Big)\frac{dr}{r}\Big)\Big)^{-1}$$
$$\ge p\Big(\log\Big(e^{2n}\int_{e^{-n}}^{e^{1-n}} \exp\Big(p\frac{\phi(r)}{r\phi'(r)}\Big)r dr\Big)\Big)^{-1} \ge \frac{p}{\log C_0 + 2n}.$$

Applying this for first n annuli yields

$$\phi(e^{-n}) \le \exp\Big(\sum_{k=1}^{n} \frac{p}{\log C_0 + 2n}\Big) \le C n^{-p/2}. \tag{5.2}$$

We next produce a very crude estimate of area distortion for radial sets $E \subset A_n$. Write $E = \{|z| \in F, \}$ where $F \subset (e^{-n}, e^{1-n})$ and note that $|F| \le e^n|E|$. Let us observe first that (5.1), Jensen's inequality and the convexity of the map $x \mapsto \exp(p\sqrt{x+1})$ on $[0,\infty)$ yield that

$$\int_{e^{-n}}^{e^{1-n}} \Big(\frac{r\phi'(r)}{\phi(r)}\Big)^2\frac{dr}{r} \le \Big(\frac{1}{p}\log\Big(\int_{e^{-n}}^{e^{1-n}} \exp\Big(p\sqrt{\Big(\frac{r\phi'(r)}{\phi(r)}\Big)^2+1}\Big)\frac{dr}{r}\Big)\Big)^2 - 1$$
$$\le \Big(\frac{1}{p}\log\Big(e^{2n}\int_{e^{-n}}^{e^{1-n}} \exp\Big(p\frac{r\phi'(r)}{\phi(r)} + p\Big)r dr\Big)\Big)^2 - 1$$
$$\le p^{-2}(\log C_0 + 2n + p)^2 \ \le\ Cn^2.$$

We may then compute using the above estimate, the bound (5.2) and Cauchy–Schwarz to obtain, for radial subsets of $E \subset A_n$,

$$|f(E)| = 2\pi\int_F \phi(r)\phi'(r)dr \le 2\pi(\phi(e^{n+1}))^2\int_F \frac{r\phi'(r)}{\phi(r)}\frac{dr}{r}$$
$$\le \frac{C}{n^p}\sqrt{\int_F \frac{dr}{r}}\sqrt{\int_{e^{-n}}^{e^{1-n}}\Big(\frac{r\phi'(r)}{\phi(r)}\Big)^2\frac{dr}{r}}$$
$$\le \frac{C}{n^p}\sqrt{|F|}e^{n/2}n \ \le Cn^{1-p}e^n\sqrt{E}. \tag{5.3}$$

We finally observe that in the general case we may assume that $|E| = e^{-4N}$ for some integer $N \geq 1$. By using the estimates (5.2) and (5.3), it follows that

$$|f(E)| \leq |f(\{|z| \leq e^{-N}\})| + \sum_{n=1}^{N} |f(E \cap A_n)| \leq \pi(\phi(e^{-N}))^2 + C \sum_{n=1}^{N} e^n n^{1-p} \sqrt{e^{-4N}}$$
$$\leq \frac{C}{N^p} + Ne^{-N} \leq \frac{C'}{4N^p},$$

as was to be shown. ∎

Proof of Theorem 1.3. One simply applies the area distortion estimate we just proved and obtains the analogue of (4.8) now with term $1/\varepsilon$ instead of $1/\varepsilon^4$. The first part follows then directly from Lemma 2.3. Similarly, part (ii) follows by keeping the track of the dependence of constant factors under this area distortion estimate. ∎

Acknowledgements. We are grateful for the anonymous referees for careful reading of the paper and their thoughtful comments that led to substantial improvements in the presentation of the paper.

Funding. The work was supported by the Finnish Academy CoE 'Analysis and Dynamics', and the Finnish Academy projects 1266182, 1303765, and 1309940.

References

[1] Ahlfors, L. V.: *Lectures on quasiconformal mappings*. Van Nostrand Mathematical Studies 10, Van Nostrand, Toronto, Ont.-New York-London, 1966; reprinted in University Lecture Series 38, American Mathematical Society, Providence, RI, 2006.

[2] Astala, K.: Area distortion of quasiconformal mappings. *Acta Math.* **173** (1994), no. 1, 37–60.

[3] Astala, K., Gill, J. T., Rohde, S. and Saksman, E.: Optimal regularity for planar mappings of finite distortion. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **27** (2010), no. 1, 1–19.

[4] Astala, K., Iwaniec, T. and Martin, G. J.: *Elliptic partial differential equations and quasiconformal mappings in the plane*. Princeton Mathematical Series 48, Princeton University Press, Princeton, NJ, 2009.

[5] Astala, K., Iwaniec, T., Prause, I. and Saksman, E.: Burkholder integrals, Morrey's problem and quasiconformal mappings. *J. Amer. Math. Soc.* **25** (2012), no. 2, 507–531.

[6] Brakalova, M. A. and Jenkins, J. A.: On solutions of the Beltrami equation. *J. Anal. Math.* **76** (1998), 67–92.

[7] Chong, K. M.: Some extensions of a theorem of Hardy, Littlewood and Pólya and their applications. *Canadian J. Math.* **26** (1974), 1321–1340.

[8] David, G.: Solutions de l'equation de Beltrami avec $\|\mu\|_\infty = 1$. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **13** (1988), no. 1, 25–70.

[9] Faraco, D., Koskela, P. and Zhong, X.: Mappings of finite distortion: The degree of regularity. *Adv. Math.* **190** (2005), no. 2, 300–318.

[10] Guo, C.-Y., Koskela, P. and Takkinen, J.: Generalized quasidisks and conformality. *Publ. Mat.* **58** (2014), no. 1, 193–212.

[11] Hardy, G. H., Littlewood, J. E. and Pólya, G.: *Inequalities*. Cambridge University Press, 1934.

[12] Iwaniec, T., Koskela, P. and Martin, G.: Mappings of BMO-distortion and Beltrami-type operators. *J. Anal. Math.* **88** (2002), 337–381.

[13] Iwaniec, T., Koskela, P., Martin, G. and Sbordone, C.: Mappings of finite distortion: $L^n \log L$ integrability. *J. London Math. Soc. (2)* **67** (2003), no. 2, 123–136.

[14] Koskela, P.: Planar mappings of finite distortion. *Comput. Methods Funct. Theory* **10** (2010), no. 2, 663–678.

[15] Ryazanov, V., Srebro, U. and Yakubov, E.: BMO-quasiconformal mappings. *J. Anal. Math.* **83** (2001), 1–20.

Received June 15, 2020; revised April 15, 2021. Published online July 14, 2021.

Olli Hirviniemi
Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68,
00014 University of Helsinki, Finland;
olli.hirviniemi@helsinki.fi

István Prause
Department of Physics and Mathematics, University of Eastern Finland, P.O. Box 111,
80101 Joensuu, Finland;
istvan.prause@uef.fi

Eero Saksman
Department of Mathematics and Statistics, University of Helsinki, P.O. Box 68,
00014 University of Helsinki, Finland;
eero.saksman@helsinki.fi