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Scattering for critical radial Neumann waves outside a ball

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Abstract. We show that the solutions of the three-dimensional critical defocusing nonlinear wave equation with Neumann boundary conditions outside a ball and radial initial data scatter. This is to our knowledge the first result of scattering for a nonlinear wave equation with Neumann boundary conditions. Our proof uses the scheme of concentration-compactness/rigidity introduced by Kenig and Merle, extending it to our setup, together with the so-called channels of energy method to rule out compact-flow solutions. We also obtain, for the focusing equation, the same exact scattering/blow-up dichotomy below the energy of the ground-state as in $\mathbb{R}^3$.

1. Introduction

This work concerns the energy-critical wave equation outside an obstacle of $\mathbb{R}^3$ with Neumann boundary condition:

$$\partial_t^2 u - \Delta u + \iota u^5 = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\partial_n u = 0 \quad \text{in } \partial\Omega, \tag{1.2}$$

$$\vec{u}_{\restriction t=0} = (u_0, u_1) \in \dot{H}^1(\Omega) \times L^2(\Omega), \tag{1.3}$$

where $\Omega = \mathbb{R}^3 \setminus K$, K is a compact subset of $\mathbb{R}^3$ with smooth boundary, $\partial_n u$ is the normal derivative of u on the boundary $\partial\Omega$ of Ω, $\vec{u}$ denotes $(u, \partial_t u)$, and $\iota \in \{\pm 1\}$. In our main result we will treat the case where K is the unit ball of $\mathbb{R}^3$ and the initial data (u_0, u_1) is assumed to be radial.

Problem (1.1)–(1.3) is locally well-posed (see [11]). The energy

$$\mathcal{E}(\vec{u}(t)) = \frac{1}{2}\int_\Omega |\nabla u(t,x)|^2\,dx + \frac{1}{2}\int_\Omega |\partial_t u(t,x)|^2\,dx + \frac{\iota}{6}\int_\Omega u^6(t,x)\,dx$$

is conserved. When $\iota = 1$ (defocusing case), the energy yields a uniform bound of the norm of the solution in $\dot{H}^1 \times L^2$ and solutions are expected to be global and to scatter to linear solutions (see definition below). When $\iota = -1$ (focusing case), one can easily construct,

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using the differential equation $u'' = u^5$ and finite-speed of propagation, solutions with initial data in $\dot{H}^1 \times L^2$ that blow up in finite time.

We first consider the defocusing case $\iota = 1$. When there is no obstacle ($\Omega = \mathbb{R}^3$), global existence was obtained for smooth radial data by Struwe [46], and extended to smooth non-radial data by Grillakis [24]. Global existence for data in the energy space was then proved by Shatah and Struwe [42]. Bahouri and Shatah [4] have shown that any solution u to the defocusing equation scatters to a linear solution, i.e., there exists a solution u_L of the free wave equation

$$\partial_t^2 u_L - \Delta u_L = 0 \tag{1.4}$$

on $\mathbb{R} \times \mathbb{R}^3$ such that

$$\lim_{t \to +\infty} \|\vec{u}(t) - \vec{u}_L(t)\|_{\dot{H} \times L^2} = 0.$$

The scattering is proved as a consequence of the fact that the L^6 norm of the solution goes to 0, which is obtained by multipliers techniques involving integration by parts on the wave cone $\{|x| \leq |t|\}$.

The equation (1.1) with Dirichlet boundary condition:

$$u_{\restriction \partial\Omega} = 0, \quad \vec{u}_{\restriction t=0} = (u_0, u_1) \in \dot{H}_0^1(\Omega) \times L^2(\Omega), \tag{1.5}$$

where $\dot{H}_0^1(\Omega) = \{f \in H^1(\Omega) : f_{\restriction \partial\Omega} = 0\}$, was studied in several articles. The global well-posedness is proved in [10]. The local well-posedness follows from a local-in-time Strichartz estimate, which is a direct consequence of a spectral projector estimate of Smith and Sogge [45]. The global well-posedness is obtained by the same arguments as in the case without obstacle, observing that the boundary term appearing in the integration by parts can be dealt with a commutator estimate.

The asymptotic behaviour of equations (1.1) and (1.4) with Dirichlet boundary conditions (1.5) is not known in general, and depends on geometrical assumptions on the obstacle. When K is non-trapping, for the linear equation (1.4), Morawetz, Ralston and Strauss [38] proved the exponential decay of the local energy in odd dimensions, polynomial in even dimensions, for compactly supported initial data. A related estimate is the integrability of the local energy, introduced in [8],

$$\|(\chi u, \chi \partial_t u)\|_{L^2(\mathbb{R}, H^1 \times L^2)} \lesssim_\chi \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}, \tag{1.6}$$

where χ is an arbitrary smooth compactly supported function. In odd space dimensions, the exponential decay of the local energy was first used in [44] to show global-in-time Strichartz estimates. This result was then extended independently to all space dimensions in [8] and [35]. The general argument of Burq [8] shows that (1.6), together with the local-in-time Strichartz estimates, imply global Strichartz estimates. When the obstacle K is moreover assumed to be star-shaped, the same computation as in the article of Bahouri and Shatah [4] yields that any solution scatters to a linear solution. The only difference with the case without obstacle is that boundary terms appear in the integration by parts. The key point is that when Ω is star-shaped and u satisfies Dirichlet boundary conditions, these boundary terms come with a good sign, so that the proof is still valid in this case. This argument can be extended to illuminated obstacles, that are generalisations of star-shaped obstacles, as done in [1], [2], adapting the multiplier so that the boundary term has the right sign, and in [31], showing that it decays to zero. However, the case of a general non-trapping obstacle seems at the moment out of hand due to the rigidity of the Morawetz multiplier arguments used for now.

Much less is known in the case of Neumann boundary conditions (defined precisely in Section 2.2 below). Note that these boundary conditions are more challenging than the Dirichlet boundary conditions, as they do not make sense in the energy space. Also, the strong Huygens principle is lost in this case (see Proposition 2.4).

Local-in-time Strichartz inequalities for the linear wave equation with Neumann boundary condition were obtained by Blair, Smith and Sogge [5], and global existence for equation (1.1)–(1.2), with $\iota = 1$, by Burq and Planchon [11]. Exponential decay of the local energy in the three-dimensional case was shown by [37]. Combined with the local-in-time Strichartz estimates [5], this should lead to global in time Strichartz estimates by the arguments of [44]. We give a direct proof of (1.6) (see Proposition 2.6) when the obstacle is the unit ball and the solution is radial, which implies global Strichartz estimates by the main result of [8].

The asymptotic behaviour of the solutions of the nonlinear equation (1.1)–(1.3) was to our knowledge not previously investigated. Assuming the global Strichartz estimates for the linear wave equation, the proof of scattering in [4] does not work anymore since the boundary terms appearing in the integration by parts do not seem to have any specific signs and cannot be controlled.

The main result of this article is that the scattering to a linear solution holds for the defocusing wave equation with Neumann boundary conditions, when K is the unit ball of $\mathbb{R}^3$ and (u_0, u_1) is radially symmetric. We thus consider the equation

$$\partial_t^2 u - \Delta u + u^5 = 0 \quad \text{in } \mathbb{R}^3 \setminus B(0,1), \tag{1.7}$$

$$\partial_r u = 0 \quad \text{for } r = 1, \tag{1.8}$$

$$\vec{u}\restriction_{t=0} = (u_0, u_1) \in \mathcal{H}(B^c), \tag{1.9}$$

where $B(0,1)$ is the unit ball of $\mathbb{R}^3$ and $\mathcal{H}(B^c)$ is the space of radial functions in $(\dot{H}^1 \times L^2)(\mathbb{R}^3 \setminus B(0,1))$, and the corresponding linear wave equation

$$\partial_t^2 u_L - \Delta u_L = 0 \quad \text{in } \mathbb{R}^3 \setminus B(0,1), \tag{1.10}$$

with the boundary condition (1.8).

Theorem 1.1. *Let u be a solution of (1.7) with Neumann boundary condition (1.8) and initial data (1.9). Then u is global and there exists a solution u_L of (1.10), (1.8), with initial data in $\mathcal{H}(B^c)$, such that*

$$\lim_{t \to +\infty} \|\vec{u}(t) - \vec{u}_L(t)\|_{\mathcal{H}(B^c)} = 0.$$

Our proof uses and extends the by now standard compactness/rigidity scheme introduced by Kenig and Merle in [27], [28] to study the focusing energy-critical Schrödinger and wave equations on $\mathbb{R}^N$. The compactness step consists in constructing, in a contradiction argument, a global nonzero solution u_c of (1.7), (1.8) such that $\{\vec{u}_c(t) : t \in \mathbb{R}\}$ has compact closure in $\mathcal{H}(B^c)$. The essential tool of this construction is a profile decomposition (in the spirit of the one introduced by Bahouri and Gérard [3] on the whole space), describing the defect of compactness of the Strichartz inequality $\|u_L\|_{L^5(\mathbb{R},L^{10})} \lesssim \|(u_0,u_1)\|_{\mathcal{H}(B^c)}$ for solutions of (1.10), (1.8). We construct this profile decomposition, which is new for the wave equation with Neumann boundary conditions, in Section 4. In this decomposition, the linear wave equation on the whole space appears as a limiting equation for dilating profiles, as shown in Section 3. The knowledge of the fact that any solution of the defocusing equation on the whole space scatters is essential to rule out these profiles and obtain the critical solution u_c, constructed in Section 5.

The second step of the proof (the rigidity argument), carried out in Section 6, consists in ruling out the existence of the critical solution. Since no monotonicity formula is available due to the Neumann boundary condition, we use the channels of energy method introduced in [12], [13] to classify solutions of the focusing energy-critical wave equation on $\mathbb{R}^3$. Using this method, we prove that u_c must be independent of time, a contradiction with the well-known fact that there is no stationary solution of (1.7) with boundary conditions (1.8) in $\dot{H}^1$. This idea was first used in the context of the supercritical wave equation in [14].

Our method also gives scattering for solutions of the focusing wave equation:

$$\partial_t^2 u - \Delta u - u^5 = 0 \quad \text{in } \mathbb{R}^3 \setminus B(0,1), \tag{1.11}$$

with Neumann boundary condition (1.8) below a natural energy threshold. Let W be the ground state of the equation on $\mathbb{R}^3$:

$$W = (1+|x|^2/3)^{-1/2},$$

and recall that W is (up to scaling and sign change) the unique radial, stationary solution of $-\Delta W = W^5$, see [41], [21]. Denote by $\mathcal{E}_{\mathbb{R}^3}(W,0)$ the energy of the solution $(W,0)$ on the whole space $\mathbb{R}^3$:

$$\mathcal{E}_{\mathbb{R}^3}(W,0) := \frac{1}{2}\int_{\mathbb{R}^3} |\nabla W|^2 - \frac{1}{6}\int_{\mathbb{R}^3} W^6.$$

Then we have the following.

Theorem 1.2. *Let u be a solution of* (1.11), (1.8) *with initial data* (1.9). *Assume*

$$\mathcal{E}(u_0,u_1) < \mathcal{E}_{\mathbb{R}^3}(W,0), \qquad \int_{\mathbb{R}^3\setminus B(0,1)} |\nabla u_0(x)|^2\,dx < \int_{\mathbb{R}^3} |\nabla W(x)|^2\,dx.$$

Then u is global,

$$\int_{\mathbb{R}^3\setminus B(0,1)} |\nabla u(t,x)|^2\,dx < \int_{\mathbb{R}^3} |\nabla W(x)|^2\,dx \quad \textit{for all } t\in\mathbb{R},$$

and u scatters to a linear solution.

Finally, we have exactly the same dichotomy as in $\mathbb{R}^3$ for the solutions below the energy threshold $\mathcal{E}_{\mathbb{R}^3}(W,0)$. Indeed, with the same proof as in [28], one obtains:

Theorem 1.3. *Let u be a solution of* (1.11), (1.8) *with initial data* (1.9). *Assume*

$$\mathcal{E}(u_0,u_1) < \mathcal{E}_{\mathbb{R}^3}(W,0), \qquad \int_{\mathbb{R}^3\setminus B(0,1)} |\nabla u_0|^2\,dx > \int_{\mathbb{R}^3} |\nabla W|^2\,dx.$$

Then u blows up in finite time.

Noting that by variational arguments (see Proposition 7.1), using that W is a maximizer to the critical Sobolev inequilaity on $\mathbb{R}^3$, one cannot have $\mathcal{E}(u_0, u_1) < \mathcal{E}_{\mathbb{R}^3}(W, 0)$ and $\int_{\{|x|>1\}} |\nabla u_0|^2 = \int_{\mathbb{R}^3} |\nabla W|^2$, we see that Theorems 1.2 and 1.3 describe all solutions of (1.11), (1.8) such that $\mathcal{E}(u_0, u_1) < \mathcal{E}_{\mathbb{R}^3}(W, 0)$. Let us also mention that Theorems 1.2 and 1.3 cannot be generalized to non-symmetric solutions or other domains than the exterior of a ball, see Section 7.4.

We finally give a few more references related to this problem. The linear wave equation outside an obstacle was initiated by Morawetz in [36], and considered in the 60s and 70s by Lax and Phillips, Morawetz, Ralston and Strauss, and many other contributors; for an extensive discussion, see, for example [38], and references therein.

For resolvent estimates in general non-trapping geometries, leading, in particular, to (1.6), see [7] and references therein. For a general discussion about local energy decay estimates, one can look at the recent paper [6].

A profile decomposition for the wave equation outside a convex obstacle with Dirichlet boundary condition was constructed in [19].

The focusing nonlinear wave equation with a superquintic nonlinearity outside the unit ball of $\mathbb{R}^3$ with Dirichlet boundary conditions was considered in [17].

The nonlinear Schrödinger equation outside a non-trapping obstacle with Dirichlet boundary conditions was first considered in [9]. The scattering for the three-dimensional defocusing cubic Schrödinger equation outside a star-shaped obstacle was shown by Planchon and Vega in [39] and, for the analogous equation in two space dimensions, in [40]. The energy-critical case outside a strictly convex obstacle in three dimensions was treated in [29]. A scattering result for a nonlinear Schrödinger equation in a model case of weakly trapping geometry can be found in [32]. To our knowledge, there is no work on the nonlinear Schrödinger equation outside an obstacle with Neumann boundary conditions.

Notations

We will use the following notations:

- If u is a function of time and space, $\vec{u}$ is understood to be $(u, \partial_t u)$.
- Conversely, if $\vec{u} \in \mathcal{H}(B^c)$, u is understood to be the first component of $\vec{u}$.
- $B(0, R)$ is the ball centered in 0 of radius R, $B = B(0, 1)$, and $B^c := \mathbb{R}^3 \backslash B(0, 1)$ is the domain we are interested in.
- $S_{\mathbb{R}^3}$ and S_N are the linear flow of the wave equation, respectively, in $\mathbb{R}^3$ and in B^c with Neumann boundary condition. If (u_0, u_1) is the initial data, we will denote by $S_N(t)(u_0, u_1)$ or $(S_N(u_0, u_1))(t)$ the solution of (1.8), (1.9), (1.10) at time t, and by $(S_N(u_0, u_1))(t, r)$ the solution at time t, with location $x = |r|$. We use similar notations for $S_{\mathbb{R}^3}$, and the flows $\mathcal{S}_N$ and $\mathcal{S}_{\mathbb{R}^3}$ defined below. The arrowed versions $\vec{S}_{\mathbb{R}^3}$ and $\vec{S}_N$ denote the flows together with their first temporal derivative.
- $\mathcal{S}_{\mathbb{R}^3}$ and $\mathcal{S}_N$ are the corresponding nonlinear flows for the defocusing energy critical wave equation (1.1).
- We will make the following convention: if $(u_0, u_1) \in \mathcal{H}(\mathbb{R}^3)$, $S_N(t)(u_0, u_1)$ and $\mathcal{S}_N(t)(u_0, u_1)$ will denote the flows applied to the restriction of (u_0, u_1) to B^c.

- Throughout the paper, we often deal with solutions of linear and nonlinear equations both in B^c with Neumann boundary conditions and in $\mathbb{R}^3$. In such situations, the letter u has been chosen for the Neumann solutions, whereas v stands for $\mathbb{R}^3$ solutions.
- $L^p L^q := L^p(\mathbb{R}, L^q(B^c))$.
- $\dot{H}^1(B^c)$ is the space of radial functions $f \in L^6(B^c)$ such that $|\nabla f| \in L^2(B^c)$.
- $\mathcal{H}(B^c)$ is the space of radial functions in $\dot{H}^1(B^c) \times L^2(B^c)$.
- Finally, $\mathcal{H}(\mathbb{R}^3)$ is the space of radial functions in $\dot{H}^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

2. Preliminaries

2.1. The functional setting

Definition 2.1. We define the extension operator $\mathcal{P}$ from $\dot{H}^1(B^c)$ to $\dot{H}^1_{\mathrm{rad}}(\mathbb{R}^3)$ by

$$\mathcal{P}u(r) := \begin{cases} u(r), & r \geq 1, \\ u(1), & r < 1, \end{cases}$$

which is well defined since, by the radial Sobolev embedding, for $u \in \dot{H}^1(B^c)$, the function $r \mapsto u(r)$ is continuous on $[1, \infty)$. Similarly, we define the extension operator $\vec{\mathcal{P}}$ from $\mathcal{H}(B^c)$ to $\mathcal{H}(\mathbb{R}^3)$ by

$$\vec{\mathcal{P}}(f, g)(r) := \begin{cases} (f(r), g(r)), & r \geq 1, \\ (f(1), 0), & r < 1. \end{cases}$$

Lemma 2.2. *For $u \in \dot{H}^1(B^c)$, we have*

$$\int_1^\infty u(r)^2\, dr \leq 4 \int_1^\infty u'(r)^2 r^2\, dr, \tag{2.1}$$

in particular, for any compact $K \subset B^c$,

$$\|u\|_{L^2(K)} \lesssim \|u\|_{\dot{H}^1}. \tag{2.2}$$

Moreover,

$$|u(1)| \lesssim \|u\|_{\dot{H}^1(B^c)}. \tag{2.3}$$

Proof. Integrating by parts, we get

$$2 \int_1^\infty r\, u(r) u'(r)\, dr = -\int_1^\infty u(r)^2\, dr - u(1)^2. \tag{2.4}$$

Note that the integration by parts is justified by approximating u by smooth compactly supported functions. Thus,

$$\int_1^\infty u(r)^2\, dr \leq -2 \int_1^\infty r\, u(r) u'(r)\, dr,$$

and (2.1) follows by the Cauchy–Schwarz inequality. The estimate (2.2) follows immediately. Bounding the left-hand side of (2.4) by the Cauchy–Schwarz inequality, and using (2.1), we obtain (2.3). ∎

Remark 2.3. With the same proof, one can generalize (2.3) to $|u(r)| \lesssim \|u\|_{\dot{H}^1(B^c)}$. This implies readily that a radial solution of the defocusing critical wave equation with Neumann boundary condition (1.7), (1.8) is uniformly bounded, thus global (giving a short proof of the result of [11] in the radial case). Similarly, any radial solution of the focusing equation (1.11), (1.8) that is bounded in $\mathcal{H}(B^c)$, is global.

2.2. Linear estimates

We first define precisely the wave equation with Neumann boundary conditions (see, e.g., [25] for a general study of wave equations with boundary conditions).

We recall that the Neumann Laplace operator Δ_N is a self-adjoint operator on $L^2(B^c)$ with domain

$$D(\Delta_N) = \{u \in H^2(B^c) : \partial_N u_{\restriction \partial\Omega=0} = 0\}.$$

Consider the operator $\mathcal{A}_N$ on $H^1(B^c) \times L^2(B^c)$, with

$$D(\mathcal{A}_N) = H^2(B^c) \times H^1(B^c), \quad \mathcal{A}_N(u_0, u_1) = (u_1, \Delta_N u_0).$$

Then $\mathcal{A}_N$ is closed and densely-defined, and thus generates a contraction semi-group. The solution $u(t,r) = (S_N(u_0,u_1))(t,r)$ of the linear wave equation with Neumann boundary condition is defined as $(u, \partial_t u) = e^{t\mathcal{A}_N}(u_0, u_1)$. If $(u_0, u_1) \in D(\mathcal{A}_N^3)$, then u is C^2 and satisfies the linear wave equation with Neumann boundary condition in the classical sense.

In the present radial case, we can derive an explicit formula for the linear flow:

Proposition 2.4 (The linear group). *For any $(u_0, u_1) \in \mathcal{H}(B^c)$, we have, for almost every $r \geq 1$ and $t \in \mathbb{R}$, also for every $r \geq 1$ and $t \in \mathbb{R}$ if we have additionally $(u_0, u_1) \in C^1 \times C^0$,*

$$(S_N(u_0,u_1))(t,r) = \frac{1}{r}(\varphi_+(r-t) + \varphi_-(r+t)), \tag{2.5}$$

where, denoting $(\zeta_0, \zeta_1) := (ru_0, ru_1)$, for $s \geq 1$,

$$\varphi_+(s) = \frac{1}{2}\zeta_0(s) - \frac{1}{2}\int_1^s \zeta_1(\sigma)\,d\sigma, \quad \varphi_-(s) = \frac{1}{2}\zeta_0(s) + \frac{1}{2}\int_1^s \zeta_1(\sigma)\,d\sigma, \tag{2.6}$$

and, for $s \in (-\infty, 1]$,

$$\begin{aligned}\varphi_+(s) = {} & \int_1^{2-s} e^{s+\sigma-2}(\zeta_0'(\sigma) + \zeta_1(\sigma))\,d\sigma - \frac{1}{2}\zeta_0(2-s) \\ & - \frac{1}{2}\int_1^{2-s} \zeta_1(\sigma)\,d\sigma + e^{s-1}\zeta_0(1),\end{aligned} \tag{2.7}$$

$$\begin{aligned}\varphi_-(s) = {} & \int_1^{2-s} e^{s+\sigma-2}(\zeta_0'(\sigma) - \zeta_1(\sigma))\,d\sigma - \frac{1}{2}\zeta_0(2-s) \\ & + \frac{1}{2}\int_1^{2-s} \zeta_1(\sigma)\,d\sigma + e^{s-1}\zeta_0(1).\end{aligned} \tag{2.8}$$

Moreover, for $f \in L^1(\mathbb{R}, L^2(\mathbb{R}^3, B^c))$ *radial, we have, for* $t \geq 0$ *and* $r - t < 1$,

$$\begin{aligned}
(2.9)\quad & \int_0^t (S_N(0, f(\tau)))(t-\tau, r)\,d\tau \\
&= \frac{1}{r}\int_0^{1-r+t}\left(\int_1^{2-r+t-\tau} e^{r-t+\tau+\sigma-2}\sigma f(\tau,\sigma)\,d\sigma + \int_{2-r+t-\tau}^{r+t-\tau}\sigma f(\tau,\sigma)\,d\sigma\right)d\tau \\
&\quad + \frac{1}{2r}\int_{1-r+t}^{t}\int_{r-t+\tau}^{r+t-\tau}\sigma f(\tau,\sigma)\,d\sigma\,d\tau.
\end{aligned}$$

Proof. Observe that, arguing by density, it suffices to consider $(u_0, u_1) \in D(\mathcal{A}_N^3)$, for which $\partial_r(S_N(u_0,u_1))(1,t) = 0$ for all t. Let us denote $\zeta(t,r) = rS_N(u_0,u_1)(t,r)$. Then ζ is the solution of the one dimensional problem

$$\partial_t^2\zeta - \partial_r^2\zeta = 0, \tag{2.10}$$

$$\partial_r\zeta - \zeta_{\restriction r=1} = 0 \quad \text{for all } t \neq 0, \tag{2.11}$$

$$\zeta_{\restriction t=0} = (ru_0, ru_1). \tag{2.12}$$

By (2.10), $\zeta(r) = \varphi_+(r-t) + \varphi_-(r+t)$. The boundary condition (2.11) gives

$$\varphi_+'(1-t) + \varphi_-'(1+t) = \varphi_+(1-t) + \varphi_-(1+t) \quad \text{for all } t, \tag{2.13}$$

and the initial condition (2.12) gives (2.6). Then integrating (2.13) for $t \geq 0$ gives (2.7), and integrating it for $t \leq 0$ gives (2.8). The identity (2.9) is then a straightforward computation. ∎

As a first consequence of Proposition 2.4, we prove that any radial solution of the linear wave equation on B^c with Neumann boundary conditions is asymptotically close to a solution of the linear wave equation on $\mathbb{R}^3$.

Proposition 2.5. *Let* $(u_0, u_1) \in \mathcal{H}(B^c)$. *Then*

$$\lim_{t\to\infty}\left\|\frac{1}{|x|}S_N(t)(u_0,u_1)\right\|_{L^2(B^c)} = 0, \tag{2.14}$$

and there exists $(v_0, v_1) \in \mathcal{H}(\mathbb{R}^3)$ *such that*

$$\lim_{t\to+\infty}\left\|\vec{S}_N(t)(u_0,u_1) - \vec{S}_{\mathbb{R}^3}(t)(v_0,v_1)\right\|_{\mathcal{H}(B^c)} = 0. \tag{2.15}$$

Proof. The proof is divided into two steps.

Step 1. We first prove (2.14). By a straightforward density argument and the conservation of the energy, we can assume that (u_0, u_1) is smooth and compactly supported. We let φ_+ and φ_- be as in Proposition 2.4, and

$$\bar{\varphi}_+(s) = \varphi_+(s) + \frac{1}{2}\int_1^{+\infty}\zeta_1(\sigma)\,d\sigma, \quad \bar{\varphi}_-(s) = \varphi_-(s) - \frac{1}{2}\int_1^{+\infty}\zeta_1(\sigma)\,d\sigma.$$

By (2.5),

$$(S_N(u_0,u_1))(t,r) = \frac{1}{r}(\bar{\varphi}_+(r-t) + \bar{\varphi}_-(r+t)). \tag{2.16}$$

We claim that there exists $C > 0$ (depending on u) such that

$$|\bar{\varphi}_+(s)| + |\bar{\varphi}_-(s)| \le Ce^s \mathbb{1}_{s\le C}. \tag{2.17}$$

Note that (2.16) and (2.17) imply easily (2.14). Using that ζ_0, ζ_0' and ζ_1 are bounded and compactly supported, the bound of $\bar{\varphi}_+$ in (2.17) follows from the fact that if $s \ge 1$,

$$\bar{\varphi}_+(s) = \frac{1}{2}\zeta_0(s) + \frac{1}{2}\int_s^{+\infty} \zeta_1(\sigma)\,d\sigma,$$

and if $s < 1$,

$$\bar{\varphi}_+(s) = \int_1^{2-s} e^{s+\sigma-2}(\zeta_0'(\sigma) + \zeta_1(\sigma))\,d\sigma - \frac{1}{2}\zeta_0(2-s) + \frac{1}{2}\int_{2-s}^{+\infty} \zeta_1(\sigma)\,d\sigma + e^{s-1}\zeta_0(1).$$

The proof of the bound of $\bar{\varphi}_-$ in (2.16) is very similar and we omit it.

Step 2. We next prove that there exists $(v_0, v_1) \in \mathcal{H}(\mathbb{R}^3)$ such that (2.15) holds. We recall (see, e.g., Theorem 2.1 of [15]) that for any $G \in L^2(\mathbb{R})$, there exists a radial solution $v(t) = S_{\mathbb{R}^3}(t)(v_0, v_1)$ of the linear wave equation on $\mathbb{R}^3$ such that

$$\lim_{t\to\infty} \int_0^{+\infty} |r\partial_t v(t,r) - G(r-t)|^2\,dr = 0, \tag{2.18}$$

$$\lim_{t\to\infty} \int_0^{+\infty} |r\partial_r v(t,r) + G(r-t)|^2\,dr = 0. \tag{2.19}$$

Let $u(t,r) = (S_N(u_0,u_1))(t,r)$ and let $\varphi_+(s)$ be as in Proposition 2.4. We will prove that $\varphi_+' \in L^2(\mathbb{R})$ and that

$$\lim_{t\to\infty} \int_1^{+\infty} |r\partial_t u(t,r) + \varphi_+'(r-t)|^2\,dr = 0, \tag{2.20}$$

$$\lim_{t\to\infty} \int_1^{+\infty} |r\partial_r u(t,r) - \varphi_+'(r-t)|^2\,dr = 0. \tag{2.21}$$

Letting (v_0, v_1) be such that (2.18) and (2.19) are satisfied with $G = -\varphi_+'$, we see that (2.20) and (2.21) imply the desired conclusion (2.15).

By the definition of φ_+, we have

$$\varphi_+'(s) = \begin{cases} \frac{1}{2}\zeta_0'(s) - \frac{1}{2}\zeta_1(s) & \text{if } s \ge 1, \\ -\frac{1}{2}\zeta_0'(2-s) - \frac{1}{2}\zeta_1(2-s) + e^{s-1}u_0(1) & \text{if } s \le 1, \end{cases}$$

where $(\zeta_0, \zeta_1) = (ru_0, ru_1)$. Since ζ_0' and ζ_1 are in $L^2([1,+\infty))$, we obtain $\varphi_+' \in L^2(\mathbb{R})$. The same proof yields $\varphi_-' \in L^2(\mathbb{R})$. By Proposition 2.4,

$$\partial_t u(t,r) = \frac{1}{r}(-\varphi_+'(r-t) + \varphi_-'(r+t)),$$

and thus

$$\int_1^{+\infty} |r\partial_t u(t,r) + \varphi_+'(r-t)|^2\,dr = \int_1^{+\infty} |\varphi_-'(t+r)|^2\,dr \xrightarrow{r\to\infty} 0.$$

Similarly,

$$\partial_r u(t,r) = \frac{1}{r}(\varphi_+'(r-t) + \varphi_-'(r+t)) - \frac{1}{r}u(t,r),$$

and thus, using (2.14),

$$\begin{aligned}\int_1^{+\infty} &|r\partial_r u(t,r) - \varphi_+'(r-t)|^2\,dr\\ &\leq 2\int_1^{+\infty}|\varphi_-'(t+r)|^2\,dr + 2\int_1^{+\infty}|u(t,r)|^2\,dr \xrightarrow{r\to\infty} 0.\end{aligned}$$

This concludes the proof. ■

Another consequence of Proposition 2.4 is the local decay of energy.

Proposition 2.6 (Local energy decay). *Let $\chi \in C_c^\infty$. For any $(u_0,u_1) \in \mathcal{H}(B^c)$,*

$$\|(\chi u, \chi\partial_t u)\|_{L^2(\mathbb{R},\mathcal{H}(B^c))} \lesssim_\chi \|(u_0,u_1)\|_{\mathcal{H}(B^c)},$$

where $u = S_N(u_0,u_1)$.

Proof. Let $\zeta(t,r) := ru(t,r)$ and let $R > 0$ be arbitrary. Note that

$$\int_1^R r^2((\partial_r u)^2 + u^2 + (\partial_t u)^2)\,dr \lesssim_R \int_1^R ((\partial_r\zeta)^2 + \zeta^2 + (\partial_t\zeta)^2)\,dr,$$

thus, to obtain the proposition, it is sufficient to show that

$$\int_{-\infty}^{\infty}\int_1^R ((\partial_r\zeta)^2 + \zeta^2 + (\partial_t\zeta)^2)\,dr\,dt \lesssim_R \|u_0\|^2_{\dot{H}^1(B^c)} + \|u_1\|^2_{L^2}. \tag{2.22}$$

To this end, observe that, by conservation of energy,

$$\int_{-R+1}^{R-1}\int_1^R ((\partial_r\zeta)^2 + \zeta^2 + (\partial_t\zeta)^2)\,dr\,dt \lesssim_R \|u_0\|^2_{\dot{H}^1(B^c)} + \|u_1\|^2_{L^2}, \tag{2.23}$$

where we used (2.2) to bound the u^2 term. Thus, it suffices to bound the integrals $\int_{t\geq R-1}$ and $\int_{t\leq -R+1}$. We will deal with the first one; the proof of the bound for the second one is similar. Thus, let us suppose that $t \geq R-1$. In particular, $t \geq r-1$, so , by Proposition 2.4, for such t's, for all $r \in [1,R]$, ζ reads

$$\begin{aligned}\zeta(t,r) = \int_1^{2-r+t} & e^{r-t+\sigma-2}(\zeta_0'(\sigma) + \zeta_1(\sigma))\,d\sigma + \frac{1}{2}\int_{2-r+t}^{r+t}\zeta_1(\sigma)\,d\sigma\\ &+ \frac{1}{2}\zeta_0(r+t) - \frac{1}{2}\zeta_0(2-r+t) + e^{r-t-1}\zeta_0(1).\end{aligned}$$

Thus, we have, for $t \geq R-1$ and $1 \leq r \leq R$,

$$\begin{aligned}&(\partial_r\zeta(t,r))^2 + (\partial_t\zeta)^2 + \zeta^2\\ &\lesssim_R \left(\int_1^{2-r+t} e^{\sigma-t}(\zeta_0'(\sigma) + \zeta_1(\sigma))\,d\sigma\right)^2 + \left(\int_{2-r+t}^{r+t}(\zeta_0'(\sigma) + \zeta_1(\sigma))\,d\sigma\right)^2\\ &\quad + \zeta_0'(2-r+t)^2 + \zeta_0'(r+t)^2 + \zeta_1(r+t)^2 + \zeta_1(2-r+t)^2 + e^{-2t}\zeta_0(1)^2.\end{aligned} \tag{2.24}$$

By the Cauchy–Schwarz inequality,

$$\begin{aligned}
&\Big(\int_1^{2-r+t} e^{\sigma-t}(\zeta_0'(\sigma)+\zeta_1(\sigma))\,d\sigma\Big)^2 \\
&\quad\le \Big(\int_1^{2-r+t} e^{\sigma-t}\,d\sigma\Big)\Big(\int_1^{2-r+t} e^{\sigma-t}(\zeta_0'(\sigma)+\zeta_1(\sigma))^2\,d\sigma\Big) \\
&\quad\lesssim_R \int_1^{2-r+t} e^{\sigma-t}(\zeta_0'(\sigma)^2+\zeta_1(\sigma)^2)\,d\sigma,
\end{aligned}$$

and therefore,

$$\begin{aligned}
&\int_{R-1}^{\infty}\int_1^R \Big(\int_1^{2-r+t} e^{\sigma-t}(\zeta_0'(\sigma)+\zeta_1(\sigma))\,d\sigma\Big)^2\,d\sigma\,dr\,dt \\
&\quad\lesssim_R \int_{R-1}^{\infty}\int_1^R\int_1^{2-r+t} e^{\sigma-t}(\zeta_0'(\sigma)^2+\zeta_1(\sigma)^2)\,d\sigma\,dr\,dt \\
&\quad\lesssim_R \int_{R-1}^{\infty}\int_1^{\infty} e^{\sigma-t}(\zeta_0'(\sigma)^2+\zeta_1(\sigma)^2)\mathbb{1}_{\sigma\le 2+t}\,d\sigma\,dt \\
&\quad= \int_1^{\infty}\Big(\int_{R-1}^{\infty} e^{\sigma-t}\mathbb{1}_{\sigma\le 2+t}\,dt\Big)(\zeta_0'(\sigma)^2+\zeta_1(\sigma)^2)\,d\sigma \\
&\quad\le \int_1^{\infty}\Big(\int_{\sigma-2}^{\infty} e^{\sigma-t}\,dt\Big)(\zeta_0'(\sigma)^2+\zeta_1(\sigma)^2)\,d\sigma \\
&\quad\lesssim \int_1^{\infty}(\zeta_0'(\sigma)^2+\zeta_1(\sigma)^2)\,d\sigma \\
&\quad\lesssim \|u_0\|^2_{\dot H^1(B^c)}+\|u_1\|^2_{L^2},
\end{aligned}$$

where we used (2.1) and the Cauchy–Schwarz inequality to obtain the last bound. As $\int_{2-r+t}^{r+t} d\sigma \lesssim_R 1$, the term coming from the second term in the first line of (2.24) is handled in the same way. Moreover,

$$\begin{aligned}
\int_{R-1}^{\infty}\int_1^R \zeta_0'(2-r+t)^2\,dr\,dt &= \int_1^R\int_{R-1}^{\infty}\zeta_0'(2-r+t)^2\,dt\,dr \\
&\le R\int_1^{\infty}\zeta_0'(s)^2\,ds \lesssim_R \|u_0\|^2_{\dot H^1(B^c)},
\end{aligned}$$

and all the terms of the second line of (2.24) are dealt with similarly. Finally, the fact that, by (2.3),

$$\zeta_0(1)^2 = u_0(1)^2 \lesssim \|u_0\|^2_{\dot H^1(B^c)}$$

permits to handle the term coming from the third line of (2.24). We just showed that

$$\int_{R-1}^{+\infty}\int_1^R (\partial_r\zeta)^2+(\partial_t\zeta)^2+\zeta^2\,dr\,dt \lesssim_R \|u_0\|^2_{\dot H^1(B^c)}+\|u_1\|^2_{L^2}.$$

Dealing with the part $\int_{-\infty}^{-R+1}$ in the same way and using (2.23), the estimate (2.22) on ζ and hence the proposition follow. ∎

The integrability of the local energy allows us to obtain the following crucial global Strichartz estimates for the Neumann flow.

Proposition 2.7 (Strichartz estimates for Neumann flow). *For any couple (p,q) verifying*

$$\frac{1}{p}+\frac{3}{q}=\frac{1}{2},\quad \frac{3}{p}+\frac{2}{q}\leq 1,\quad 2<p\leq\infty,\quad q<\infty, \tag{2.25}$$

there exists a constant $C>0$ such that for all $(u_0,u_1)\in\mathcal{H}(B^c)$ and $f\in L^1(\mathbb{R},L^2_{\mathrm{rad}}(r\geq 1))$, if u verifies

$$\partial_t^2 u-\Delta_N u=f\quad \text{in } B^c,\qquad \partial_n u=0\quad \text{on } \partial B(0,1),\qquad \vec{u}_{\restriction t=0}=(u_0,u_1),$$

then, for all $T>0$,

$$\|u\|_{L^p([-T,T],L^q(r\geq 1))}\leq C(\|(u_0,u_1)\|_{\mathcal{H}(B^c)}+\|f\|_{L^1([-T,T],L^2(r\geq 1))}).$$

Proof. The main result of [8] shows that the local energy decay of Proposition 2.6 combined with local in time Strichartz estimates imply global in time ones. Such local estimates where shown by [5] for the above range of couples (p,q), hence the proposition follows. ∎

Let us also recall the Strichartz estimates in $\mathbb{R}^3$.

Proposition 2.8 (Strichartz estimates in $\mathbb{R}^3$, [22], [23], [26], [34]). *For any couple (p,q) verifying*

$$\frac{1}{p}+\frac{3}{q}=\frac{1}{2},\quad \frac{1}{p}+\frac{1}{q}\leq \frac{1}{2},\quad 2<p\leq\infty,\quad q<\infty, \tag{2.26}$$

there exists a constant $C>0$ such that for all $(u_0,u_1)\in\mathcal{H}(\mathbb{R}^3)$ and $f\in L^1(\mathbb{R},L^2(\mathbb{R}^3))$, if v verifies

$$\partial_t^2 v-\Delta v=f,\quad \vec{v}_{\restriction t=0}=(v_0,v_1),$$

then, for all $T>0$,

$$\|v\|_{L^p([-T,T],L^q(\mathbb{R}^3))}\leq C(\|(v_0,v_1)\|_{\mathcal{H}(\mathbb{R}^3)}+\|f\|_{L^1([-T,T],L^2(\mathbb{R}^3))}).$$

Remark 2.9. Observe the loss in the range of admissible couples (2.25) in Proposition 2.7 compared to the free case (2.26). This is because we used the local-in-time Strichartz estimates of [5], valid in a general geometrical setup. It is likely that the above Strichartz estimates, outside a ball, could be extended to the full range of couples (2.26), using for the local-in-time estimates a construction similar to the one done in [43] for Dirichlet boundary conditions. However, the range of exponents (2.25) is sufficient for our analysis and we do not pursue this question here.

As a last consequence of the explicit formula for the linear group given by Proposition 2.4, we have the following.

Lemma 2.10. *Let $(u_0,u_1)\in(C^1\times C^0)\cap\mathcal{H}(B^c)$.*

(1) *We have*

$$S_N(\cdot)(u_0,u_1)\in C^0(\mathbb{R}\times B^c)\cap C^1(\{t\pm r\neq 1\}),$$

with

$$\partial_r(S_N(u_0,u_1))(1,t)=0\quad \textit{for all } t\neq 0.$$

(2) *If, in addition, $f \in L^1(\mathbb{R}, L^2(B^c))$ is radial and continuous and u is defined by*

$$u(t) := S_N(t)(u_0, u_1) + \int_0^t S_N(t-\tau)(0, f(\tau))\, d\tau,$$

then $u \in C^0(\mathbb{R} \times B^c) \cap C^1(\{t \pm r \neq 1\})$ and

$$\partial_r u(1,t) = 0 \quad \text{for all } t \neq 0.$$

Proof. The explicit formulas (2.5)–(2.8) give the first part of the lemma, and (2.9) then gives the second part for $t > 0$. The case $t < 0$ is given by a similar computation. ∎

2.3. Perturbative theory

Definition 2.11. We say that a solution u of the nonlinear wave equation (1.7) with Neumann boundary conditions (1.8) *scatters in the future* when there exists a solution u_L of the linear wave equation (1.10) with Neumann boundary conditions (1.8) such that

$$\lim_{t \to +\infty} \|\vec{u}(t) - \vec{u}_L(t)\|_{\mathcal{H}(B^c)} = 0.$$

We define similarly *scattering in the past*. We say that the solution *scatters* when it scatters both in the future and in the past.

In a classical way, we have the following.

Proposition 2.12. *Let $(u_0, u_1) \in \mathcal{H}(B^c)$ and $u(t) = \mathcal{S}_N(t)(\vec{u}_0)$. Then*

$$u \in L^5([0,+\infty), L^{10}) \implies u \text{ scatters in the future}. \tag{2.27}$$

A similar property holds in the past. Moreover, there exists $\epsilon_0 > 0$ such that, for any $(u_0, u_1) \in \mathcal{H}(B^c)$,

$$\|(u_0,u_1)\|_{\mathcal{H}(B^c)} \leq \epsilon_0 \implies \mathcal{S}_N(\cdot)(u_0,u_1) \in L^5 L^{10}, \tag{2.28}$$

and $\mathcal{S}_N(\cdot)(u_0,u_1)$ scatters. In addition, for any $(u_0,u_1) \in \mathcal{H}(B^c)$, there exists a solution $U^\pm \in L^5(\mathbb{R}_\pm, L^{10})$ of (1.7)–(1.8) such that

$$\|\vec{U}^\pm(t) - \vec{S}_N(t)(u_0,u_1)\|_{\mathcal{H}(B^c)} \to 0 \quad \text{as } t \to \pm\infty. \tag{2.29}$$

Sketch of proof. Observe that (5, 10) is Strichartz-admissible in the sense of Proposition 2.7. The properties (2.27) and (2.28) are then classical consequences of the global in time Strichartz estimates. Finally, (2.29) can be proved by a fixed point argument using the Strichartz estimates. ∎

Proposition 2.13 (Perturbation). *For any $M > 0$, there exists $\epsilon(M) > 0$ such that, for any $0 < \epsilon \leq \epsilon(M)$, and all (u_0,u_1), $(\tilde{u}_0, \tilde{u}_1) \in \mathcal{H}(B^c)$, $e \in L^1L^2$ and $u \in L^5L^{10}$ verifying*

$$\|u\|_{L^5L^{10}} \leq M, \quad \|S_N(\cdot)((u_0,u_1) - (\tilde{u}_0,\tilde{u}_1))\|_{L^5L^{10}} \leq \epsilon, \quad \|e\|_{L^1L^2} \leq \epsilon,$$

if $u, \tilde{u}$ are solutions of

$$\begin{cases} \partial_t^2 u - \Delta_N u = -u^5 \quad \text{in } B(0,1)^c, \\ \vec{u}_{\restriction t=0} = (u_0, u_1), \\ \partial_n u = 0 \quad \text{on } \partial B(0,1), \end{cases} \qquad \begin{cases} \partial_t^2 \tilde{u} - \Delta_N \tilde{u} = -\tilde{u}^5 + e \quad \text{in } B(0,1)^c, \\ \vec{\tilde{u}}_{\restriction t=0} = (\tilde{u}_0, \tilde{u}_1), \\ \partial_n \tilde{u} = 0 \quad \text{on } \partial B(0,1), \end{cases}$$

then $\tilde{u} \in L^5 L^{10}$ and we have

$$\|u - \tilde{u}\|_{L^5 L^{10}} \lesssim \epsilon.$$

In addition, the same statement holds for the corresponding equations in $\mathbb{R}^3$.

Proof. The proof is classical and similar to Proposition 4.7 of [18]; we give it for completeness. Let us denote $w = u - \tilde{u}$. Then w is solution of

$$\partial_t^2 w - \Delta_N w = -u^5 + \tilde{u}^5 - e, \quad \vec{w}_{\restriction t=0} = (u_0, u_1) - (\tilde{u}_0, \tilde{u}_1).$$

Let $T > 0$. By the Strichartz inequality for the Neumann flow (Proposition 2.7) applied to w, we get, with an implicit constant independent of T,

$$\begin{aligned} \|u - \tilde{u}\|_{L^5(-T,T)L^{10}} &\lesssim \|\tilde{u}^5 - u^5\|_{L^1(-T,T)L^2} + \|e\|_{L^1L^2} \\ &\quad + \|S_N(\cdot)((u_0,u_1) - (\tilde{u}_0,\tilde{u}_1))\|_{L^5L^{10}} \\ &\lesssim \||u - \tilde{u}|(|u|^4 + |u - \tilde{u}|^4)\|_{L^1((-T,T)L^2)} + \|e\|_{L^1L^2} \\ &\quad + \|S_N(\cdot)((u_0,u_1) - (\tilde{u}_0,\tilde{u}_1))\|_{L^5L^{10}} \\ &\leq C\bigg(\int_{-T}^{T} \|u - \tilde{u}\|_{L^{10}} \|u\|_{L^{10}}^4 + \|u - \tilde{u}\|_{L^5((-T,T),L^{10})}^5 \\ &\quad + \|e\|_{L^1L^2} + \|S_N(\cdot)((u_0,u_1) - (\tilde{u}_0,\tilde{u}_1))\|_{L^5L^{10}} \bigg). \end{aligned}$$

We apply the Grönwall-type lemma of [18] (Lemma 8.1), with

$$\varphi = \|u - \tilde{u}\|_{L^{10}}, \quad \gamma = 5, \quad f = C\|u\|_{L^{10}}^4, \quad \beta = 1,$$
$$\eta = C\big(\|u - \tilde{u}\|_{L^5((-T,T),L^{10})}^5 + \|e\|_{L^1L^2} + \|S_N(\cdot)((u_0,u_1) - (\tilde{u}_0,\tilde{u}_1))\|_{L^5L^{10}}\big).$$

We obtain, for all $T > 0$,

$$\begin{aligned} \|u - \tilde{u}\|_{L^5((-T,T),L^{10})} \leq \big(&\|e\|_{L^1L^2} + \|S_N(\cdot)((u_0,u_1) - (\tilde{u}_0,\tilde{u}_1))\|_{L^5L^{10}} \\ &+ \|u - \tilde{u}\|_{L^5((-T,T),L^{10})}^5\big) \times \Phi(CM^4), \end{aligned}$$

where $\Phi(s) = 2\Gamma(3 + 2s)$, with Γ being the Gamma function. Let $C_M := 6\Phi(CM^4)$ and let $\epsilon(M) > 0$ be sufficiently small so that, for any $\epsilon \leq \epsilon(M)$,

$$\epsilon^5 C_M^5 \leq \epsilon \implies \epsilon \leq 1/C_M^{5/4}.$$

Then, given $T > 0$ so that $\|u - \tilde{u}\|_{L^5((-T,T),L^{10})} \leq C_M\epsilon$, we have

$$\|u - \tilde{u}\|_{L^5((-T,T),L^{10})} \leq \Phi(CM^4)(2\epsilon + C_M^5\epsilon^5),$$

and thus $\|u - \tilde{u}\|_{L^5((-T,T),L^{10})} \leq 3\Phi(CM^4)\epsilon \leq \frac{1}{2}C_M\epsilon$. It easily follows that we can make T goes to infinity, thus $\|u - \tilde{u}\|_{L^5(\mathbb{R},L^{10})} \leq \frac{1}{2}C_M\epsilon$ and the lemma follows. The same proof works for the problem in $\mathbb{R}^3$ using the corresponding Strichartz estimates. ∎

3. Comparison between Neumann and $\mathbb{R}^3$ evolutions for dilating profiles

Let us introduce the following notation for the scaling associated to the equation

Definition 3.1. For $\lambda > 0$, σ_λ denotes the rescaling on $\dot{H}^1(\mathbb{R}^3)$, given by

$$\sigma_\lambda(f) = \frac{1}{\lambda^{1/2}} f\Big(\frac{\cdot}{\lambda}\Big),$$

and on $\mathcal{H}(\mathbb{R}^3)$, given by

$$\sigma_\lambda(f, g) := \bigg(\frac{1}{\lambda^{1/2}} f\Big(\frac{\cdot}{\lambda}\Big), \frac{1}{\lambda^{3/2}} g\Big(\frac{\cdot}{\lambda}\Big)\bigg).$$

The aim of this section is to show that a dilating profile ($\lambda \to \infty$) does not see the obstacle, in the sense that for such profiles, the associated Neumann and $\mathbb{R}^3$ evolutions are asymptotically the same.

Lemma 3.2 (Comparison of linear evolutions for dilating profiles). *Let $\vec{\psi} \in \mathcal{H}(\mathbb{R}^3)$, let $f \in L^1(\mathbb{R}, L^2(\mathbb{R}^3))$ be radial, let $(\lambda_n)_{n\geq 1}$ be a sequence of positive real numbers such that $\lambda_n \to +\infty$, let $(t_n)_{n\geq 1}$ be a sequence of times, let v be the solution in the sense of Duhamel of*

$$\partial_t^2 v - \Delta v = f \quad \textit{in } \mathbb{R}^3, \qquad \vec{v}_{\restriction t=0} = \vec{\psi},$$

and let $v_n := \sigma_{\lambda_n} v$. Finally, let

$$f_n := \frac{1}{\lambda_n^{5/2}} f\Big(\frac{\cdot - t_n}{\lambda_n}, \frac{\cdot}{\lambda_n}\Big)$$

and let u_n be the solution in the sense of Duhamel of

$$\partial_t^2 u_n - \Delta u_n = f_n \quad \textit{in } B^c, \qquad \partial_r u_n = 0 \quad \textit{for } r = 1, \qquad \vec{u}_{n\restriction t=-t_n} = \vec{v}_{n\restriction t=-t_n}.$$

Then, as n goes to infinity,

$$\sup_{t\in\mathbb{R}} \|u_n(t) - v_n(t)\|_{\mathcal{H}(B^c)} \to 0 \tag{3.1}$$

and

$$\|u_n - v_n\|_{L^5 L^{10}} \to 0. \tag{3.2}$$

Proof. Observe that, by interpolation, it suffices to obtain (3.1). Indeed, if (3.1) holds, by Sobolev embedding, we have $\|u_n - v_n\|_{L^\infty L^6} \to 0$, and then (3.2) follows by Hölder's inequality, the Minkowski inequality, the Strichartz estimates for both flows (Propositions 2.7 and 2.8) and the conservation of energy. Moreover, arguing by density, we can assume that $\vec{\psi}$ and f are smooth and compactly supported. We will argue in three steps:

(1) $t_n = 0$ for all n and $f = 0$,

(2) $t_n = 0$ for all n and $\vec{\psi} = \vec{0}$,

(3) general case.

Step 1: $t_n = 0$ and $f = 0$. We have, using the equations satisfied by u_n and v_n,

$$\begin{aligned}\text{(3.3)}\qquad &\frac{d}{dt}\Big(\frac{1}{2}\int_{B^c}|\nabla(u_n-v_n)|^2+\frac{1}{2}\int_{B^c}|\partial_t(u_n-v_n)|^2\Big)\\ &=-\int_{\partial B^c}\partial_r(u_n-v_n)\partial_t(u_n-v_n)\\ &=\int_{\partial B^c}\partial_r v_n\partial_t(u_n-v_n)=4\pi\partial_r v_n(t,1)\partial_t(u_n-v_n)(t,1).\end{aligned}$$

We now claim that, for large n,

$$|\partial_t v_n(t,1)|+|\partial_r v_n(t,1)|\lesssim\frac{1}{\lambda_n^{3/2}}\mathbb{1}_{[-C\lambda_n,C\lambda_n]},\tag{3.4}$$

$$|\partial_t u_n(t,1)|\lesssim\frac{1}{\lambda_n^{3/2}}+\frac{e^{-|t|}}{\lambda_n^{1/2}},\tag{3.5}$$

where the constant $C>0$ and the implicit constants depend on $\vec{\varphi}$. Observe that integrating (3.3), (3.4) and (3.5) give (3.1).

Let us first show (3.4). Observe that

$$v_n(t,x)=\frac{1}{\lambda_n^{1/2}}v\Big(\frac{t}{\lambda_n},\frac{x}{\lambda_n}\Big),$$

where $v:=S_{\mathbb{R}^3}(t)\vec{\psi}$. As $\vec{\psi}\in C_c^\infty$, $\vec{v}$ is bounded in any Sobolev space $H^\sigma(\mathbb{R}^3)\times H^{\sigma-1}(\mathbb{R}^3)$ for $\sigma\geq 1$. As a consequence,

$$|\partial_t v_n(t,1)|+|\partial_r v_n(t,1)|\lesssim\frac{1}{\lambda_n^{3/2}}.\tag{3.6}$$

Furthermore, by the strong Huygens principle, v is supported in $\{|t|\leq|x|+C\}$, and thus

$$v_n(t,1)=0\quad\text{for } |t|\geq 1+C\lambda_n.\tag{3.7}$$

Together with (3.6), (3.7) gives (3.4).

We now show (3.5). By Proposition 2.4, we have, for $t\geq 0$,

$$\partial_t u_n(t,1)=-\varphi'_{+,n}(1-t)+\varphi'_{-,n}(1+t),\tag{3.8}$$

where, denoting $\vec{\psi}=(\psi_0,\psi_1)$,

$$\varphi'_{-,n}(s)=\frac{1}{2}\Big(\frac{1}{\lambda_n^{3/2}}\psi_0'\Big(\frac{s}{\lambda_n}\Big)+\frac{1}{\lambda_n^{3/2}}\psi_1'\Big(\frac{s}{\lambda_n}\Big)\Big)\tag{3.9}$$

and

$$\begin{aligned}\text{(3.10)}\quad \varphi'_{+,n}(s)=&-\frac{1}{2}\Big(\frac{1}{\lambda_n^{3/2}}\psi_0'\Big(\frac{2-s}{\lambda_n}\Big)+\frac{1}{\lambda_n^{3/2}}\psi_1'\Big(\frac{2-s}{\lambda_n}\Big)\Big)\\ &+\frac{1}{\lambda_n^{3/2}}\int_1^{2-s}e^{s+\sigma-2}\Big(\psi_0'\Big(\frac{\sigma}{\lambda_n}\Big)+\psi_1'\Big(\frac{\sigma}{\lambda_n}\Big)\Big)\,d\sigma+e^{s-1}\frac{1}{\lambda_n^{1/2}}\psi_0\Big(\frac{1}{\lambda_n}\Big).\end{aligned}$$

This last identity (3.10), with (3.8) and (3.9), gives (3.5) for $t\geq 0$. The argument for $t\leq 0$ is similar and step 1 follows.

Step 2: $t_n = 0$ and $\vec{\psi} = \vec{0}$. As in the first step, we have

$$\begin{aligned}(3.11)\qquad &\frac{d}{dt}\Big(\frac{1}{2}\int_{B^c}|\nabla(u_n - v_n)|^2 + \frac{1}{2}\int_{B^c}|\partial_t(u_n - v_n)|^2\Big)\\ &\quad = 4\pi\,\partial_r v_n(t,1)\partial_t(u_n - v_n)(t,1).\end{aligned}$$

Let us show that

$$(3.12)\qquad |\partial_t v_n(t,1)| + |\partial_r v_n(t,1)| \lesssim \frac{1}{\lambda_n^{3/2}}\,\mathbb{1}_{[-C\lambda_n, C\lambda_n]},$$

$$(3.13)\qquad |\partial_t u_n(t,1)| \lesssim \frac{1}{\lambda_n^{7/2}}\,t^2,$$

which, together with (3.11), implies (3.1).

We first show (3.12). We have

$$v_n(t,x) = \frac{1}{\lambda_n^{1/2}} v\Big(\frac{t}{\lambda_n}, \frac{x}{\lambda_n}\Big),$$

where $v := S_{\mathbb{R}^3}(t)\vec{\psi}$. As $\partial_t v$ and $\partial_r v$ are bounded,

$$(3.14)\qquad |\partial_t v_n(t,1)| + |\partial_r v_n(t,1)| \lesssim \frac{1}{\lambda_n^{3/2}}.$$

In addition, as we assumed f to be compactly supported in time and space,

$$v_n(t,1) = 0 \quad \text{for } |t| \geq 1 + C\lambda_n,$$

which, with (3.14), gives (3.12).

In order to prove (3.13), we will need the following.

Claim. Let $f \in C^0(\mathbb{R} \times B^c)$ be radial and bounded, that is, for all $(t,x) \in \mathbb{R} \times B^c$, $|f(t,x)| \leq M$, and let w be the solution of

$$\partial_t^2 w - \Delta w = f \quad \text{in } B^c, \qquad \partial_r w = 0 \quad \text{for } r = 1, \qquad \vec{w}_{\restriction t=0} = \vec{0}.$$

Then we have

$$|w(t,x)| \leq \frac{1}{2}Mt^2 \quad \text{for all } (t,x) \in \mathbb{R} \times B^c.$$

To obtain (3.13) from the claim, we apply it to $w := \partial_t u_n$, observing that as u_n is a regular solution, $\partial_t u_n$ is in $C^0(\mathbb{R}, D(-\Delta_N))$, and thus satisfies Neumann boundary conditions. Let us now prove the claim to achieve the proof of step 2. Let

$$z(t,r) := \frac{1}{2}Mt^2 - w(t,r).$$

By the formulas of Proposition 2.4, we see that if u_1 is positive for $t \geq 0$, then so is $S_N(0,u_1)(t,r)$. Thus, by the Duhamel formula, as $(\partial_t^2 - \Delta)z \geq 0$, we have $z \geq 0$ for $t \geq 0$, from which, we obtain $w \leq \frac{1}{2}Mt^2$ for $t \geq 0$. Considering $\tilde{z}(,r) := \frac{1}{2}Mt^2 + w(t,r)$, we obtain as well $-w \leq \frac{1}{2}Mt^2$ for $t \geq 0$. The negative times are obtained in a similar fashion.

Step 3: general case. By the two first steps, we obtain the case $t_n = 0$. Now, let w_n be the solution of the Neumann problem with initial condition at $t = 0$

$$\partial_t^2 w_n - \Delta w_n = f_n \quad \text{in } B^c, \qquad \partial_r w_n = 0 \quad \text{for } r = 1, \qquad \vec{w}_{n\restriction t=0} = \vec{v}_{n\restriction t=0}.$$

By the case $t_n = 0$, we have, as $n \to \infty$,

$$\sup_{t\in\mathbb{R}} \|w_n - v_n\|_{\mathcal{H}(B^c)} \to 0 \tag{3.15}$$

and, in particular, since by definition $u_n(-t_n) = v_n(-t_n)$,

$$\|w_n(-t_n) - u_n(-t_n)\|_{\mathcal{H}(B^c)} \to 0. \tag{3.16}$$

From (3.16), as $w_n - u_n$ is solution of the homogeneous linear wave equation with Neumann boundary conditions in B^c, it follows from conservation of energy that

$$\sup_{t\in\mathbb{R}} \|w_n - u_n\|_{\mathcal{H}(B^c)} = \|w_n(-t_n) - u_n(-t_n)\|_{\mathcal{H}(B^c)} \to 0. \tag{3.17}$$

The result (3.1) follows from (3.15) and (3.17). ■

The following lemma will play a key role in the comparison between the $\mathbb{R}^3$ and Neumann dynamics in the nonlinear profile decomposition introduced in Section 5 (see, in particular, (5.9)).

Lemma 3.3 (Comparison of nonlinear evolutions for dilating profiles). *Assume that $V \in L^5(\mathbb{R}, L^{10}(\mathbb{R}^3))$ is a solution of the critical defocusing nonlinear wave equation in $\mathbb{R}^3$, (i.e., (1.1) with $\Omega = \mathbb{R}^3$ and $\iota = 1$), $(\lambda_n)_n$ is a sequence of positive real numbers such that $\lambda_n \to +\infty$, and $(t_n)_n \in \mathbb{R}^{\mathbb{N}}$. We denote*

$$V_n(t,x) := \frac{1}{\lambda_n^{1/2}} V\Big(\frac{t-t_n}{\lambda_n}, \frac{x}{\lambda_n}\Big) = \mathcal{S}_{\mathbb{R}^3}(t)\sigma_{\lambda_n}\Big(\vec{V}\Big(\frac{-t_n}{\lambda_n}\Big)\Big)$$

and let U_n be the solution of the nonlinear Neumann problem

$$\partial_t^2 U_n - \Delta U_n + U_n^5 = 0 \quad \text{in } B^c, \qquad \partial_r U_n = 0 \quad \text{for } r = 1, \qquad \vec{U}_{n\restriction t=0} = \vec{V}_{n\restriction t=0}.$$

Then

$$\limsup_{n\in\mathbb{N}} \|U_n\|_{L^5 L^{10}} < \infty$$

and, as $n \to \infty$,

$$\sup_{t\in\mathbb{R}} \|\vec{U}_n(t) - \vec{V}_n(t)\|_{\mathcal{H}(B^c)} + \|U_n - V_n\|_{L^5 L^{10}} \to 0.$$

Remark 3.4. The conclusion of the proposition implies

$$\lim_{n\to\infty} \Big\| \vec{\mathcal{S}}_{\mathbb{R}^3}(t)\sigma_{\lambda_n}\Big(\vec{V}\Big(\frac{-t_n}{\lambda_n}\Big)\Big) - \vec{\mathcal{S}}_N(t)\sigma_{\lambda_n}\Big(\vec{V}\Big(\frac{-t_n}{\lambda_n}\Big)\Big)\Big\|_{\mathcal{H}(B^c)} = 0.$$

Proof. Observe that, by energy estimates, it suffices to show $\|U_n - V_n\|_{L^5L^{10}} \to 0$. Let Z_n be the solution of the nonlinear Neumann problem

$$\partial_t^2 Z_n - \Delta Z_n + V_n^5 = 0 \quad \text{in } B^c, \qquad \partial_r Z_n = 0 \quad \text{for } r = 1, \qquad \vec{Z}_{n\restriction t=0} = \vec{U}_{n\restriction t=0}.$$

By Lemma 3.2 applied to $Z_n(\cdot + t_n)$ and $V_n(\cdot + t_n)$, we get

$$\|Z_n - V_n\|_{L^5L^{10}} \to 0. \tag{3.18}$$

Let $T > 0$ and observe that

$$\begin{cases} \partial_t^2(Z_n - U_n) + \Delta(Z_n - U_n) = U_n^5 - V_n^5 & \text{in } B^c, \\ \partial_r(Z_n - U_n) = 0 \quad \text{for } r = 1, \\ \vec{Z}_n - \vec{U}_{n\restriction t=0} = \vec{0}, \end{cases}$$

and therefore, we have, by the global Strichartz estimates for the Neumann flow (Proposition 2.7), together with the Hölder and Minkowski inequalities, with an implicit constant which is independent of $T > 0$,

$$\begin{aligned} \|Z_n - U_n\|_{L^5(-T,T)L^{10}} &\lesssim \|U_n^5 - V_n^5\|_{L^1(-T,T)L^2} \\ &\lesssim \int_{-T}^{T} \big[\|V_n(t)\|_{L^{10}}^4 \|U_n(t) - V_n(t)\|_{L^{10}} + \|U_n(t) - V_n(t)\|_{L^{10}}^5\big]\,dt \\ &\lesssim \int_{-T}^{T} \big[\|V_n(t)\|_{L^{10}}^4 \|Z_n(t) - U_n(t)\|_{L^{10}} + \|Z_n(t) - U_n(t)\|_{L^{10}}^5\big]\,dt + \epsilon_n(T), \end{aligned} \tag{3.19}$$

where we decomposed $U_n(t) - V_n(t) = U_n(t) - Z_n(t) + Z_n(t) - V_n(t)$ in the last line, and

$$\epsilon_n(T) = \int_{-T}^{T} \big[\|V_n(t) - Z_n(t)\|_{L^{10}}^5 + \|V_n(t)\|_{L^{10}}^4 \|V_n(t) - Z_n(t)\|_{L^{10}}\big]\,dt.$$

By Hölder's inequality and (3.18),

$$\begin{aligned} \epsilon_n' &:= \sup_{T>0} \epsilon_n(T) \\ &\leq \|V_n - Z_n\|_{L^5(\mathbb{R},L^{10})}^5 + \|V\|_{L^5(\mathbb{R},L^{10}(\mathbb{R}^3))}^4 \|V_n - Z_n\|_{L^5(\mathbb{R},L^{10})} \to 0. \end{aligned} \tag{3.20}$$

By (3.19), we have, with an implicit constant independent of T,

$$\begin{aligned} \|Z_n - U_n\|_{L^5(-T,T)L^{10}} \lesssim &\int_{-T}^{T} \|V_n(t)\|_{L^{10}}^4 \|Z_n(t) - U_n(t)\|_{L^{10}}\,dt \\ &+ \epsilon_n' + \|Z_n - U_n\|_{L^5(-T,T)L^{10}}^5. \end{aligned} \tag{3.21}$$

Now, $\|V_n\|_{L^{10}}^4 \in L^{5/4}(\mathbb{R})$ and $\|\|V_n\|_{L^{10}}^4\|_{L^{5/4}(\mathbb{R})} = \|V\|_{L^5L^{10}}^4$. Thus, by (3.21), using the Gronwall-type lemma of [18] (Lemma 8.1), for all $T > 0$, with $C > 0$ independent of $T > 0$, we get

$$\|Z_n - U_n\|_{L^5(-T,T)L^{10}} \leq C(\epsilon_n' + \|Z_n - U_n\|_{L^5(-T,T)L^{10}}^5). \tag{3.22}$$

Let $\epsilon > 0$ be small enough so that $2C\epsilon^5 \leq \frac{1}{2}\epsilon$, and let n be large enough so that $\epsilon'_n \leq \epsilon^5$. From (3.22), it follows that if T is such that $\|Z_n - U_n\|_{L^5(-T,T)L^{10}} \leq \epsilon$, then we have

$$\|Z_n - U_n\|_{L^5(-T,T)L^{10}} \leq \frac{1}{2}\epsilon.$$

We can therefore send T to infinity to obtain

$$\|Z_n - U_n\|_{L^5(\mathbb{R},L^{10})} \to 0,$$

and the lemma follows using (3.18). ∎

4. Linear profile decomposition

We recall that, by convention, if $(u_0, u_1) \in \mathcal{H}(\mathbb{R}^3)$, $S_N(t)(u_0, u_1)$ (resp. $\mathcal{S}_N(t)(u_0, u_1)$) denotes the flow of the linear (resp. nonlinear) wave equation with Neumann boundary condition applied to the *restriction* of (u_0, u_1) to B^c. The aim of this section is to show the following result.

Proposition 4.1 (Linear profile decomposition). *Let $(\vec{\phi}_n)_{n\geq 1}$ be a bounded sequence in $\mathcal{H}(B^c)$. Then, up to a subsequence, there exists sequences of real parameters $(t_{j,n})_{j,n\geq 1}$, $(\lambda_{j,n})_{j,n\geq 1}$ and a sequence $(\vec{\psi}^j)_{j\geq 1}$ of elements of $\mathcal{H}(\mathbb{R}^3)$ such that*

$$j \neq k \implies \lim_{n\to+\infty} \frac{|t_{j,n} - t_{k,n}|}{\lambda_{j,n}} + \left|\log \frac{\lambda_{j,n}}{\lambda_{k,n}}\right| = +\infty, \tag{4.1}$$

and there exists a partition $(J_{\mathrm{comp}}, J_{\mathrm{diff}})$ of $\mathbb{N}$ such that

$$j \in J_{\mathrm{comp}} \implies \lambda_{j,n} = 1 \quad \text{for all } n, \tag{4.2}$$

$$j \in J_{\mathrm{diff}} \implies \lambda_{j,n} \xrightarrow{n\to\infty} +\infty. \tag{4.3}$$

Moreover,

$$t_{j,n}/\lambda_{j,n} \to \pm\infty \text{ for all } j \quad \text{or} \quad t_{j,n} = 0 \text{ for all } n, \tag{4.4}$$

$$\vec{\phi}_n = \sum_{j=1}^{J} \vec{S}_N(-t_{j,n})\sigma_{\lambda_{j,n}}\vec{\psi}^j + \vec{w}_n^J \quad \text{for all } J \geq 1, \tag{4.5}$$

where the remainder enjoys the decay

$$\lim_{J\to+\infty} \limsup_{n\to+\infty} \|S_N(\cdot)\vec{w}_n^J\|_{L^5L^{10}} = 0. \tag{4.6}$$

In addition, this decomposition verifies the Pythagorean expansion

$$\|\vec{\phi}_n\|^2_{\mathcal{H}(B^c)} = \sum_{\substack{j\in J_{\mathrm{comp}}\\ 1\leq j\leq J}} \|\vec{\psi}^j\|^2_{\mathcal{H}(B^c)} + \sum_{\substack{j\in J_{\mathrm{diff}}\\ 1\leq j\leq J}} \|\vec{\psi}^j\|^2_{\mathcal{H}(\mathbb{R}^3)} + \|\vec{w}_n^J\|^2_{\mathcal{H}(B^c)} + o_n(1) \tag{4.7}$$

for all J, as well as the L^6 version of it:

$$\|\phi_n\|^6_{L^6} = \sum_{j=1}^{J} \|S_N(-t_{j,n})\sigma_{\lambda_{j,n}}\vec{\psi}^j\|^6_{L^6} + \|w_n^J\|^6_{L^6} + o_n(1) \quad \text{for all } J. \tag{4.8}$$

Recall from (2.1) the definition of the extension operator $\mathcal{P}$. Proposition 4.1 will be a consequence of the following.

Lemma 4.2. *Let $(f_n)_{n\geq 1}$ be a bounded sequence in $\dot{H}^1(B^c)$ such that for all sequence of real numbers $(\lambda_n)_{n\geq 1}$ verifying*

$$\lim_n \lambda_n = +\infty \quad \textit{or} \quad \lambda_n = 1 \textit{ for all } n,$$

we have, as n goes to infinity,

$$\lambda_n^{1/2}\mathcal{P}(f_n)(\lambda_n \cdot) \rightharpoonup 0 \quad \textit{in } \dot{H}^1(\mathbb{R}^3).$$

Then, up to a subsequence, as n goes to infinity,

$$\|f_n\|_{L^6(B^c)} \to 0.$$

Proof. As $(\mathcal{P}(f_n))_{n\geq 1}$ is a bounded sequence in $\dot{H}^1_{\mathrm{rad}}(\mathbb{R}^3)$, we may apply the elliptic profile decomposition of [20] and, up to a subsequence,

$$\mathcal{P}(f_n) = \sum_{j=1}^{J} \frac{1}{\lambda_{j,n}^{1/2}}\varphi_j\Big(\frac{\cdot}{\lambda_{j,n}}\Big) + w_n^J,$$

with

$$\lim_{J\to+\infty}\limsup_{n\to+\infty}\|w_n^J\|_{L^6} = 0.$$

Remark that

$$\varphi_j = \operatorname*{weak\,lim}_{n\to\infty} \lambda_{j,n}^{1/2}\mathcal{P}(f_n)(\lambda_{j,n}\cdot) \quad \text{in } \dot{H}^1(\mathbb{R}^3).$$

Thus, for all j such that $\liminf_n \lambda_{j,n} > 0$, we have $\varphi_j = 0$ by hypothesis. Indeed, in this case, extracting subsequences, we can assume that $\lambda_{j,n}$ has a limit $\lambda_\infty \in (0,\infty)\cup\{+\infty\}$. If this limit is finite, we may furthermore assume, rescaling φ_j if necessary, that $\lambda_{j,n} = 1$ for all n.

On the other hand, if j is such that $\lambda_{j,n} \to 0$ as $n \to \infty$, observe that

$$\lambda_{j,n}^{1/2}\mathcal{P}(f_n)(\lambda_{j,n}\cdot) = \lambda_{j,n}^{1/2} f_n(1) \quad \text{on } \{r \leq 1/\lambda_{j,n}\}.$$

By Lemma 2.2,

$$|f_n(1)| \lesssim \|f_n\|_{\dot{H}^1(B^c)},$$

which is bounded independently of n, and we deduce that $\lambda_{j,n}^{1/2}\mathcal{P}(f_n)(\lambda_n\cdot)$ goes to zero as n goes to infinity, uniformly on every compact of $\mathbb{R}^3$, and thus in the sense of distributions as well. By the uniqueness of the limit, we conclude that $\varphi_j = 0$. Therefore, $\varphi_j = 0$ for all j and the lemma follows. ∎

Before showing Proposition 4.1, let us observe that

Lemma 4.3. *Let $(\vec{R}_n)_n$ be a sequence in $\mathcal{H}(\mathbb{R}^3)$. For $j = 1, 2$, let $(t_{j,n})_n \in \mathbb{R}^{\mathbb{N}}$, and let $(\lambda_{j,n})_n \in (\mathbb{R}_+^*)^{\mathbb{N}}$ be such that*

$$\lim_{n\to\infty}\lambda_{j,n} = +\infty \quad \textit{or} \quad \lambda_{j,n} = 1 \textit{ for all } n. \tag{4.9}$$

(1) *If there exists M such that*

$$|t_{1,n}-t_{2,n}|+\left|\log\frac{\lambda_{1,n}}{\lambda_{2,n}}\right|\leq M \quad \textit{for all } n,$$

then, up to a subsequence, weakly in $\mathcal{H}(\mathbb{R}^3)$,

$$\vec{R}_n \rightharpoonup 0 \implies \sigma_{\lambda_{2,n}}^{-1}\vec{\mathcal{P}}(\vec{S}_N(t_{1,n}-t_{2,n})\sigma_{\lambda_{1,n}}\vec{R}_n)\rightharpoonup 0.$$

(2) *If*

$$\frac{|t_{1,n}-t_{2,n}|}{\lambda_{1,n}}+\left|\log\frac{\lambda_{1,n}}{\lambda_{2,n}}\right|\to+\infty,$$

then, for all $\vec{\psi}\in\mathcal{H}(\mathbb{R}^3)$, up to a subsequence, weakly in $\mathcal{H}(\mathbb{R}^3)$,

$$\sigma_{\lambda_{2,n}}^{-1}\vec{\mathcal{P}}(\vec{S}_N(t_{1,n}-t_{2,n})\sigma_{\lambda_{1,n}}\vec{\psi})\rightharpoonup 0.$$

Proof. Let us show the first point. Up to the extraction of a subsequence, we have

$$t_{1,n}-t_{2,n}\to\tau\in\mathbb{R},$$

and, additionally, either

$$(\lambda_{1,n},\lambda_{2,n})\to(+\infty,+\infty) \quad \text{or} \quad (\lambda_{1,n},\lambda_{2,n})=(1,1) \text{ for all } n.$$

In the first situation, Lemma 3.2 allows us to replace S_N by $S_{\mathbb{R}^3}$, for which the result is known. In the second situation, we have, for any test function $\vec{\xi}\in\mathcal{H}(B^c)$,

$$\sigma_{\lambda_{1,n}}^{-1}\vec{\mathcal{P}}(\vec{S}_N(t_{2,n}-t_{1,n})\sigma_{\lambda_{2,n}}\vec{\xi})\to\vec{\mathcal{P}}(\vec{S}_N(-\tau)\vec{\xi})$$

strongly in $\mathcal{H}(B^c)$, and the first point follows.

Let us now deal with the second point. We are in one of the three following situations:

(i) $\lambda_{1,n}\to\infty$,

(ii) for all n, $\lambda_{2,n}\to\infty$, $\lambda_{1,n}=1$ and there exists $M>0$ such that $|t_{1,n}-t_{2,n}|\leq M$,

(iii) for all n, $\lambda_{1,n}=1$ and $|t_{1,n}-t_{2,n}|\to\infty$.

In the situation (i), we can use again Lemma 3.2 to replace S_N by $S_{\mathbb{R}^3}$ and the result follows.

In the situation (ii), up to a subsequence, $\vec{\mathcal{P}}(\vec{S}_N(t_{1,n}-t_{2,n})\sigma_{\lambda_{1,n}}\vec{\psi})$ is converging strongly in $\mathcal{H}(\mathbb{R}^3)$:

$$\vec{\mathcal{P}}(\vec{S}_N(t_{1,n}-t_{2,n})\sigma_{\lambda_{1,n}}\vec{\psi})\to\vec{\xi}.$$

By a density argument, we can assume that $\vec{\xi}$ is smooth and compactly supported. Then, by the definition of the scaling σ,

$$\sigma_{\lambda_{2,n}}^{-1}\vec{\xi}(r)\to 0 \quad \text{for all } r\neq 0,$$

and the result follows.

In the situation (iii), we use this time Proposition 2.5 to compare the solution to a solution in $\mathbb{R}^3$, for which the result is known. ■

We are now in position to prove the main result of this section.

Proof of Proposition 4.1. We will first construct the profiles and the parameters by induction, so that the expansion (4.5) holds together with the orthogonality of the parameters (4.1), (4.2), (4.3), and the Pythagorean expansion (4.7), (4.8). Then, we will show the decay of the remainder (4.6).

For $\vec{\alpha} = (\vec{\alpha}_n)_n$ a bounded sequence in $\mathcal{H}(B^c)$, let us denote by $\Lambda(\vec{\alpha})$ the set of all $\vec{\psi} \in \mathcal{H}(\mathbb{R}^3)$ such that there exist an extraction $\{n_k\}_k$ and sequences $(\lambda_{n_k})_k \in (0,\infty)^{\mathbb{N}}$ and $(t_{n_k})_k \in \mathbb{R}^{\mathbb{N}}$ such that

$$\lim_{k\to\infty} \lambda_{n_k} = \infty \quad \text{or} \quad \lambda_{n_k} = 1 \text{ for all } k,$$
$$\vec{\psi} = \operatorname*{weak\,lim}_{k\to\infty}\big(\sigma^{-1}_{\lambda_{n_k}} \vec{\mathcal{P}}(\vec{S}_N(t_{n_k})\vec{\alpha}_{n_k})\big) \quad \text{in } \mathcal{H}(\mathbb{R}^3).$$

We denote

$$\eta(\vec{\alpha}) := \sup_{\vec{\psi}\in\Lambda(\vec{\alpha})} \|\vec{\psi}\|_{\mathcal{H}(\mathbb{R}^3)}, \tag{4.10}$$

and observe that, by the definition of $\vec{\mathcal{P}}$ and $\vec{\psi}$,

$$\|\vec{\psi}\|_{\mathcal{H}(\mathbb{R}^3)} = \|\vec{\psi}\|_{\mathcal{H}(B^c)} \quad \text{if } \vec{\psi} \text{ is associated with } \lambda_{n_k} = 1. \tag{4.11}$$

Extraction of the first profile. If $\eta((\vec{\phi}_n)_{n\geq 1}) = 0$, then the decomposition holds. Otherwise, there exists $\vec{\psi}^1 \in \mathcal{H}(\mathbb{R}^3)$ and $(\lambda_{1,n})_{n\geq 1} \in (\mathbb{R}^*_+)^{\mathbb{N}}$, $(t_{1,n})_{n\geq 1} \in \mathbb{R}^{\mathbb{N}}$, with $\lambda_{1,n} \to +\infty$ or $\lambda_{1,n} = 1$ for all n, such that, up to an extraction,

$$\vec{\psi}^1 = \operatorname*{weak\,lim}_{n\to\infty} \sigma^{-1}_{\lambda_{1,n}} \vec{\mathcal{P}}(\vec{S}_N(t_{1,n})\vec{\phi}_n) \quad \text{in } \mathcal{H}(\mathbb{R}^3) \tag{4.12}$$

and

$$\frac{1}{2}\,\eta((\vec{u}_n)_{n\geq 1}) \leq \|\vec{\psi}^1\|_{\mathcal{H}(\mathbb{R}^3)}.$$

Let us denote

$$\vec{w}^1_n := \vec{\phi}_n - \vec{S}_N(-t_{1,n})\sigma_{\lambda_{1,n}}\vec{\psi}^1. \tag{4.13}$$

Observe that, if $t_{1,n}/\lambda_{1,n}$ has a finite limit $\bar{\tau}_1$, we can harmlessly assume that $t_{1,n} = 0$ for all n. Indeed, if $\lambda_{1,n} = 1$ for all n, we see by (4.12) that

$$\vec{\mathcal{P}}(\vec{S}_N(-\bar{\tau}_1)(\vec{\psi}^1)) = \operatorname*{weak\,lim}_{n\to\infty} \vec{\mathcal{P}}(\vec{\phi}_n).$$

If $\lambda_{1,n} \to +\infty$, we have, by (4.12) and Lemma 3.2,

$$\begin{aligned}\vec{\psi}^1 = \operatorname*{weak\,lim}_{n\to\infty} \sigma^{-1}_{\lambda_{1,n}}(\vec{S}_{\mathbb{R}^3}(t_{1,n})\vec{\phi}_n) &= \operatorname*{weak\,lim}_{n\to\infty}(\vec{S}_{\mathbb{R}^3}(t_{1,n}/\lambda_{1,n})\sigma^{-1}_{\lambda_{1,n}}\vec{\phi}_n) \\ &= \operatorname*{weak\,lim}_{n\to\infty}(\vec{S}_{\mathbb{R}^3}(\bar{\tau}_1)\sigma^{-1}_{\lambda_{1,n}}\vec{\phi}_n).\end{aligned}$$

In both cases, we see that we can assume $t_{1,n} = 0$ by modifying the limiting profile $\vec{\psi}^1$.

Now, by the definition of $\vec{w}_n^1$ (see (4.13)) and the weak convergence (4.12), we have

$$
\begin{aligned}
(4.14)\quad \langle \vec{S}_N(-t_{1,n})\sigma_{\lambda_{1,n}}\vec{\psi}^1, \vec{w}_n^1\rangle_{\mathcal{H}(B^c)} &= \langle \sigma_{\lambda_{1,n}}\vec{\psi}^1, \vec{S}_N(t_{1,n})\vec{\phi}_n - \sigma_{\lambda_{1,n}}\vec{\psi}^1\rangle_{\mathcal{H}(B^c)} \\
&= \langle \sigma_{\lambda_{1,n}}\vec{\psi}^1, \vec{\mathcal{P}}(\vec{S}_N(t_{1,n})\vec{\phi}_n) - \sigma_{\lambda_{1,n}}\vec{\psi}^1\rangle_{\mathcal{H}(\mathbb{R}^3)} \\
&= \langle \vec{\psi}^1, \sigma_{\lambda_{1,n}}^{-1}\vec{\mathcal{P}}(\vec{S}_N(t_{1,n})\vec{\phi}_n) - \vec{\psi}^1\rangle_{\mathcal{H}(\mathbb{R}^3)} \xrightarrow{n\to\infty} 0
\end{aligned}
$$

and therefore

$$
(4.15)\qquad \|\vec{\phi}_n\|^2_{\mathcal{H}(B^c)} = \|\vec{S}_N(-t_{1,n})\sigma_{\lambda_{1,n}}\vec{\psi}^1\|^2_{\mathcal{H}(B^c)} + \|w_n^1\|^2_{\mathcal{H}(B^c)} + o_n(1).
$$

But, by conservation of energy,

$$
(4.16)\qquad \|\vec{S}_N(-t_{1,n})\sigma_{\lambda_{1,n}}\vec{\psi}^1\|^2_{\mathcal{H}(B^c)} = \|\sigma_{\lambda_{1,n}}\vec{\psi}^1\|^2_{\mathcal{H}(B^c)}.
$$

Now, remark that if $\lambda_{1,n} \to \infty$, then, as n goes to infinity, we have

$$
\|\sigma_{\lambda_{1,n}}\vec{\psi}^1\|^2_{\mathcal{H}(B(0,1))} \to 0,
$$

and thus, as $\sigma_{\lambda_{1,n}}$ is an isometry on $\mathcal{H}(\mathbb{R}^3)$,

$$
\begin{aligned}
(4.17)\qquad \|\sigma_{\lambda_{1,n}}\vec{\psi}^1\|^2_{\mathcal{H}(B^c)} &= \|\sigma_{\lambda_{1,n}}\vec{\psi}^1\|^2_{\mathcal{H}(\mathbb{R}^3)} + o_n(1) \\
&= \|\vec{\psi}^1\|^2_{\mathcal{H}(\mathbb{R}^3)} + o_n(1) \quad \text{as } \lambda_{1,n} \to \infty.
\end{aligned}
$$

Therefore, combining (4.17) with (4.15) and (4.16), the decomposition (4.5) with the Pythagorean expansion (4.7) holds at rank $J = 1$.

Let us now show the L^6 Pythagorean expansion (4.8).

First case: $t_{1,n} = 0$. Let

$$
f_n := \Big| \int |\phi_n|^6 - |\sigma_{\lambda_{1,n}}\psi^1|^6 - |w_n^1|^6 \Big|,
$$

and observe that, as for any $z, w \in \mathbb{R}$,

$$
\big| |z+w|^6 - |z|^6 - |w|^6 \big| \lesssim |z|\,|w|(|z|^4 + |w|^4),
$$

we have, by (4.13),

$$
f_n \lesssim \int |\sigma_{\lambda_{1,n}}\vec{\psi}^1|\,|w_n^1| g_n, \qquad g_n := |\sigma_{\lambda_{1,n}}\vec{\psi}^1|^4 + |w_n^1|^4.
$$

On the other hand, by Sobolev embedding, conservation of energy and scale invariance,

$$
\|\sigma_{\lambda_{1,n}}\psi^1\|_{L^6} \lesssim \|\sigma_{\lambda_{1,n}}\vec{\psi}^1\|_{\mathcal{H}(B^c)} \le \|\sigma_{\lambda_{1,n}}\vec{\psi}^1\|_{\mathcal{H}(\mathbb{R}^3)} = \|\psi^1\|_{\mathcal{H}(\mathbb{R}^3)}.
$$

Together with (4.13) and Sobolev embedding, it follows that $\sup_n \|g_n\|_{L^{3/2}} < \infty$, and we get, by Hölder's inequality,

$$
\begin{aligned}
(4.18)\qquad f_n &\lesssim \Big(\int_{B^c} |\sigma_{\lambda_{1,n}}\psi^1|^3 |w_n^1|^3 \Big)^{1/3} \\
&\le \Big(\int_{\mathbb{R}^3} |\sigma_{\lambda_{1,n}}\psi^1|^3 |\tilde{w}_n^1|^3 \Big)^{1/3} = \Big(\int_{\mathbb{R}^3} |\psi^1|^3 |\sigma_{\lambda_{1,n}^{-1}}\tilde{w}_n^1|^3 \Big)^{1/3},
\end{aligned}
$$

where $\vec{\tilde{w}}_n^1 := \vec{\mathcal{P}}\vec{\phi}_n - \sigma_{\lambda_{1,n}}\vec{\psi}^1$ extends the definition of $\vec{w}_n^1$ to $\mathbb{R}^3$ in the present case $t_{1,n} = 0$. Now, observe that, by (4.12) and (4.13), $\sigma_{\lambda_{1,n}^{-1}}\vec{\tilde{w}}_n^1 \rightharpoonup 0$ weakly in $\mathcal{H}(\mathbb{R}^3)$. By Rellich's theorem, for any compact $K \subset \mathbb{R}^3$, $\sigma_{\lambda_{1,n}^{-1}}\tilde{w}_n^1$ strongly converges to 0 in $L^4(K)$. It follows that $|\sigma_{\lambda_{1,n}^{-1}}\tilde{w}_n^1|^3$ converges strongly to 0 in $L^{4/3}(K)$. By Sobolev embedding, $|\sigma_{\lambda_{1,n}^{-1}}\tilde{w}_n^1|^3$ is bounded in $L^2(\mathbb{R}^3)$, thus has a weakly convergent subsequence in $L^2(\mathbb{R}^3)$. By uniqueness of the limit in the sense of distributions, this weak limit is zero and (4.8) follows from (4.18).

Second case: $t_{1,n}/\lambda_{1,n} \to \pm\infty$. In this case, we have

$$\|S_N(-t_{1,n}/\lambda_{1,n})\vec{\psi}^1\|_{L^6} \xrightarrow{n\to\infty} 0,$$

which can be proved easily from the corresponding property for the free flow $S_{\mathbb{R}^3}$, and Proposition 2.5. The L^6 Pythagorean expansion (4.8) follows immediately.

Extraction of the subsequent profiles. Let us show how to extract the second profile, the extraction of the Jth from the $(J-1)$th being the same for arbitrary $J \geq 2$. If $\eta(\vec{w}_n^1) = 0$, then we are done, otherwise, there exist $\vec{\psi}^2 \in \mathcal{H}(\mathbb{R}^3)$ and sequences $(\lambda_{2,n})_{n\geq 1} \in (\mathbb{R}_+^*)^{\mathbb{N}}$, $(t_{2,n})_{n\geq 1} \in \mathbb{R}^{\mathbb{N}}$, with $\lambda_{2,n} \to +\infty$ or $\lambda_{2,n} = 1$, such that

$$\vec{\psi}^2 = \operatorname{weak\,lim} \sigma_{\lambda_{2,n}}^{-1}\vec{\mathcal{P}}(\vec{S}_N(t_{2,n})\vec{w}_n^1) \quad \text{in } \mathcal{H}(\mathbb{R}^3) \tag{4.19}$$

and

$$\frac{1}{2}\,\eta((\vec{w}_n^1)_{n\geq 1}) \leq \|\vec{\psi}^2\|_{\mathcal{H}(\mathbb{R}^3)}.$$

We take

$$\begin{aligned} \vec{w}_n^2 &:= \vec{w}_n^1 - \vec{S}_N(-t_{2,n})\sigma_{\lambda_{2,n}}\vec{\psi}^2 \\ &= \vec{u}_n - \vec{S}_N(-t_{2,n})\sigma_{\lambda_{2,n}}\vec{\psi}^2 - \vec{S}_N(-t_{1,n})\sigma_{\lambda_{1,n}}\vec{\psi}^1. \end{aligned} \tag{4.20}$$

Let us first show the orthogonality condition (4.1). Denoting

$$\vec{r}_n^1 := \sigma_{\lambda_{1,n}}^{-1}\vec{\mathcal{P}}(\vec{S}_N(t_{1,n})\vec{w}_n^1) = \sigma_{\lambda_{1,n}}^{-1}\vec{\mathcal{P}}(\vec{S}_N(t_{1,n})\vec{u}_n) - \sigma_{\lambda_{1,n}^{-1}}\vec{\mathcal{P}}\sigma_{\lambda_{1,n}}\vec{\psi}^1,$$

by (4.12) and (4.13), we have

$$\vec{r}_n^1 \rightharpoonup 0 \quad \text{weakly in } \mathcal{H}(\mathbb{R}^3),$$

and, in addition, by (4.19),

$$\sigma_{\lambda_{2,n}}^{-1}\vec{\mathcal{P}}(\vec{S}_N(t_{2,n} - t_{1,n})\sigma_{\lambda_{1,n}}\vec{r}_n^1) \rightharpoonup \vec{\psi}^2 \neq 0.$$

Therefore, by Lemma 4.3, the orthogonality condition (4.1) for $(j,k) = (1,2)$ follows.

To show the Pythagorean expansion (4.7), using the arguments of the case $J = 1$, it suffices to show that the newly arising mixed term goes to zero, namely, that

$$\big\langle \vec{S}_N(-t_{2,n})\sigma_{\lambda_{2,n}}\vec{\psi}^2, \vec{S}_N(-t_{1,n})\sigma_{\lambda_{1,n}}\vec{\psi}^1\big\rangle_{\mathcal{H}(B^c)} \xrightarrow{n\to\infty} 0.$$

Noting that the left-hand side of the previous line equals

$$\langle \vec{\psi}^2, \sigma_{\lambda_{2,n}}^{-1} \vec{\mathcal{P}}(\vec{S}_N(t_{2,n} - t_{1,n})\sigma_{\lambda_{1,n}} \vec{\psi}^1) \rangle_{\mathcal{H}(B^c)},$$

the result follows by the orthogonality condition, together with Lemma 4.3.

Finally, (4.19) and (4.20) imply, by the exact same arguments as in the extraction of the first profile, that

$$\|w_n^1\|_{L^6}^6 = \|S_N(-t_{2,n})\sigma_{\lambda_{2,n}} \vec{\psi}^2\|_{L^6}^6 + \|w_n^2\|_{L^6}^6 + o_n(1),$$

from which the L^6 Pythagorean expansion (4.8) follows using the decomposition proved at the previous rank, which reads

$$\|\phi_n\|_{L^6}^6 = \|S_N(-t_{1,n})\sigma_{\lambda_{1,n}} \vec{\psi}^1\|_{L^6}^6 + \|w_n^1\|_{L^6}^6 + o_n(1).$$

Labeling. We define J_{diff} and J_{comp} as follows: if $\lambda_{j,n} = 1$ for all n, then $j \in J_{\mathrm{comp}}$, otherwise, $j \in J_{\mathrm{diff}}$.

Decay of the remainder. In order to obtain (4.6), it suffices to show that

$$\lim_{J \to +\infty} \limsup_{n \to +\infty} \|S_N(\cdot)\vec{w}_n^J\|_{L^\infty L^6} = 0. \tag{4.21}$$

Indeed, if (4.21) holds, the Strichartz estimates of Proposition 2.7 together with Hölder's inequality, conservation of energy, and the fact that, by the Pythagorean expansion (4.7),

$$\limsup_{n \to +\infty} \|\vec{w}_n^J\|_{\mathcal{H}(B^c)} \le \limsup_{n \to +\infty} \|\vec{\phi}_n\|_{\mathcal{H}(B^c)} \quad \text{for all } J$$

yield (4.6).

Let us show (4.21). To this end, observe that, by the Pythagorean expansion (4.7),

$$\sum_{j=1,\, j \in J_{\mathrm{comp}}}^{J} \|\vec{\psi}^j\|_{\mathcal{H}(B^c)}^2 + \sum_{j=1,\, j \in J_{\mathrm{diff}}}^{J} \|\vec{\psi}^j\|_{\mathcal{H}(\mathbb{R}^3)}^2 \le \limsup_{n \ge 1} \|\vec{\phi}_n\|_{\mathcal{H}(B^c)}^2 \quad \text{for all } J,$$

and thus both series in j are convergent. Since, by (4.11), the profiles are constructed in such a way that

$$\eta((\vec{w}_n^j)_{n \ge 1}) \le 2 \begin{cases} \|\vec{\psi}^j\|_{\mathcal{H}(\mathbb{R}^3)} & \text{if } j \in J_{\mathrm{diff}}, \\ \|\vec{\psi}^j\|_{\mathcal{H}(B^c)} & \text{if } j \in J_{\mathrm{comp}},\ j \ne 0, \end{cases}$$

it follows that

$$\eta((\vec{w}_n^J)_{n \ge 1}) \xrightarrow{J \to \infty} 0. \tag{4.22}$$

Arguing by contradiction, the $L^\infty L^6$ decay of $S_N(\cdot)\vec{w}_n^J$ follows by Lemma 4.2. Indeed, if the decay of the remainder (4.21) does not hold, by a diagonal argument, there exists $\epsilon_0 > 0$ and sequences $J_k \to +\infty$, $n_k \to +\infty$, and t_k such that

$$\eta((\vec{w}_{n_k}^{J_k})_k) = 0 \quad \text{and} \quad \|S_N(t_k)\vec{w}_{n_k}^{J_k}\|_{L^6(B^c)} \ge \epsilon_0 \quad \text{for all } k.$$

Using Lemma 4.2, it follows that there exists $\vec{\psi} \in \mathcal{H}(\mathbb{R}^3)$, $\vec{\psi} \neq 0$, and a sequence $(\lambda_k)_k$, with

$$\lim_k \lambda_k = \infty \quad \text{or} \quad \lambda_k = 1 \text{ for all } k,$$

such that, after extraction,

$$\sigma_{\lambda_k}^{-1} \vec{\mathcal{P}}(\vec{S}_N(t_k)\vec{w}_{n_k}^{J_k}) \rightharpoonup \vec{\psi} \quad \text{weakly in } \mathcal{H}(\mathbb{R}^3).$$

This contradicts the definition (4.10) of η and completes the proof of the proposition. ∎

5. Construction of a compact flow solution

Let us define the critical energy E_c by

$$E_c := \sup\{E > 0 : \mathcal{E}(\vec{u}) \leq E \text{ for all } \vec{u} \in \mathcal{H}(B^c) \implies \mathcal{S}_N(\cdot)\vec{u} \in L^5L^{10}\}, \tag{5.1}$$

where, for $\vec{u} \in \mathcal{H}(B^c)$, $\mathcal{E}$ is as before the conserved energy

$$\mathcal{E}(\vec{u}) := \frac{1}{2}\|\vec{u}\|_{\mathcal{H}(B^c)}^2 + \frac{1}{6}\|u\|_{L^6}^6.$$

Observe that $E_c > 0$ by Proposition 2.12. The aim of this section is to show the following.

Theorem 5.1. *If $E_c < +\infty$, then there exists $\vec{u}_c \in \mathcal{H}(B^c)$, $\vec{u}_c \neq \vec{0}$, such that the nonlinear flow $\{\vec{\mathcal{S}}_N(t)\vec{u}_c : t \in \mathbb{R}\}$ has a compact closure in $\mathcal{H}(B^c)$.*

Proof. If $E_c < +\infty$, let $\vec{u}_0^n$ be a minimising sequence for E_c, in the sense that

$$\mathcal{E}(\vec{u}_0^n) \geq E_c, \quad \lim_{n\to\infty} \mathcal{E}(\vec{u}_0^n) = E_c, \quad \mathcal{S}_N(\cdot)\vec{u}_0^n \notin L^5L^{10}. \tag{5.2}$$

Translating $u_n = \mathcal{S}_N(\cdot)\vec{u}_0^n$ in time if necessary, we may assume

$$\lim_{n\to\infty} \|u_n\|_{L^5((0,+\infty),L^{10})} = \lim_{n\to\infty} \|u_n\|_{L^5((-\infty,0),L^{10})} = +\infty, \tag{5.3}$$

where, by convention, $\|u_n\|_{L^5((-\infty,0),L^{10})} = +\infty$ if $u_n \notin L^5((-\infty,0), L^{10})$, and similarly for $L^5((0,\infty), L^{10})$. As $\vec{u}_0^n$ is bounded in $\mathcal{H}(B^c)$, we can, up to a subsequence, decompose it into profiles according to Proposition 4.1:

$$\vec{u}_0^n = \sum_{j=1}^{J} \vec{S}_N(-t_{j,n})\sigma_{\lambda_{j,n}}\vec{\psi}^j + \vec{w}_n^J. \tag{5.4}$$

To each profile $(\vec{\psi}^j, (\lambda_{j,n})_{n\geq 1}, (t_{j,n})_{n\geq 1})$, we associate a family of nonlinear Neumann profiles $(U_n^j)_{n\geq 1}$ and, additionally, for $j \in J_{\text{diff}}$, a free nonlinear profile V^j and its rescaled family $(V_n^j)_{n\geq 1}$, in the following way.

If $j \in J_{\text{comp}}$, i.e., $\lambda_{j,n} = 1$, let U^j be the only solution of the critical nonlinear wave equation with Neumann boundary conditions (1.7)–(1.8), given by Proposition 2.12, such that

$$\lim_{n\to\infty} \left\|\vec{U}^j(-t_{j,n}) - \vec{S}_N(-t_{j,n})\vec{\psi}^j\right\|_{\mathcal{H}(B^c)} = 0, \tag{5.5}$$

and we set

$$U_n^j(t) := U^j(t - t_{j,n}). \tag{5.6}$$

Notice that, if $-t_{j,n} \to \pm\infty$, $U^j \in L^5(\mathbb{R}_\pm, L^{10}(B^c))$, by construction.

If $j \in J_{\mathrm{diff}}$, i.e., $\lambda_{j,n} \to \infty$, then, by Lemma 3.2,

$$\lim_{n\to\infty} \big\| \vec{S}_N(-t_{j,n})\sigma_{\lambda_{j,n}}\vec{\psi}^j - \vec{S}_{\mathbb{R}^3}(-t_{j,n})\sigma_{\lambda_{j,n}}\vec{\psi}^j \big\|_{\mathcal{H}(B^c)} = 0.$$

Furthermore, by denoting $V_L^j(t) := S_{\mathbb{R}^3}(t)\psi^j(t)$, we have

$$S_{\mathbb{R}^3}(t - t_{j,n})\sigma_{\lambda_{j,n}}\psi^j = \frac{1}{\lambda_{j,n}^{1/2}} V_L^j\Big(\frac{t - t_{j,n}}{\lambda_{j,n}}, \frac{x}{\lambda_{j,n}}\Big).$$

We define the free nonlinear profile V^j as the unique solution of the critical nonlinear wave equation on $\mathbb{R}^3$ such that if $t_{j,n} = 0$ for all n, then we have $\vec{V}^j(0) = \psi^j$, and if $\lim_{n\to\infty} -t_{j,n}/\lambda_{j,n} = \pm\infty$, then $\lim_{t\to\pm\infty} \|\vec{V}^j(t) - \vec{V}_L^j(t)\|_{\mathcal{H}(B^c)} = 0$. In other words,

$$\lim_{n\to\pm\infty} \big\| \vec{V}^j(-t_{j,n}/\lambda_{j,n}) - \vec{V}_L^j(-t_{j,n}/\lambda_{j,n}) \big\|_{\mathcal{H}(B^c)} = 0. \tag{5.7}$$

Furthermore, we set

$$V_n^j(t) := \frac{1}{\lambda_{j,n}^{1/2}} V^j\Big(\frac{t - t_{j,n}}{\lambda_{j,n}}\Big),$$

and we then define the associated family of nonlinear Neumann profiles as

$$U_n^j(t) := \mathcal{S}_N(t)(\vec{V}_n^j(0)) = \mathcal{S}_N(t)\Big(\sigma_{\lambda_{j,n}}\Big(\vec{V}^j\Big(\frac{-t_{j,n}}{\lambda_{j,n}}\Big)\Big)\Big). \tag{5.8}$$

Observe that, as a solution of a defocusing nonlinear wave equation in $\mathbb{R}^3$, for which the scattering is well known, we have $V^j \in L^5L^{10}(\mathbb{R}^3)$. Furthermore, as $\vec{U}_n^j(0) = \vec{V}_n^j(0)$, Lemma 3.3 (used with $t_n = t_{j,n}$) yields

$$\sup_n \|U_n^j\|_{L^5(\mathbb{R},L^{10}(B^c))} < \infty \quad \text{for all } j \in J_{\mathrm{diff}}$$

and

$$\sup_t \|\vec{V}_n^j(t) - \vec{U}_n^j(t)\|_{\mathcal{H}(B^c)} + \|V_n^j - U_n^j\|_{L_t^5L_x^{10}} \xrightarrow{n\to\infty} 0 \quad \text{for all } j \in J_{\mathrm{diff}}. \tag{5.9}$$

Let us assume from now on, by contradiction, that the decomposition (5.4) has strictly more than one non trivial profile, i.e.,

$$J > 1. \tag{5.10}$$

Then, by the Pythagorean expansion (4.7), together with its L^6 version (4.8),

$$\limsup_{n\to\infty} \mathcal{E}(S_N(-t_{j,n})\vec{\psi}^j) < E_c \quad \text{for all } j \in J_{\mathrm{comp}}.$$

Hence, by (5.5), $\mathcal{E}(U^j) < E_C$, and $U^j \in L^5 L^{10}(B^c)$ by the definition of the critical energy. Summing up, we have

$$\begin{cases} U^j \in L^5(\mathbb{R}, L^{10}(B^c)) & \text{for all } j \in J_{\text{comp}}, \\ V^j \in L^5(\mathbb{R}, L^{10}(B^c)) & \text{for all } j \in J_{\text{diff}}. \end{cases} \tag{5.11}$$

Let $u_n := \mathcal{S}_N \vec{u}_0^n$. We will show the following nonlinear profile decomposition.

Proposition 5.2. *We have*

$$\begin{aligned} u_n(t) &= \sum_{1 \le j \le J} U_n^j(t) + R_n^J(t) \\ &= \sum_{\substack{j \in J_{\text{comp}} \\ 1 \le j \le J}} U_n^j(t) + \sum_{\substack{j \in J_{\text{diff}} \\ 1 \le j \le J}} V_n^j(t) + \tilde{R}_n^J(t) \quad \textit{for all } J, \end{aligned} \tag{5.12}$$

where

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|R_n^J\|_{L^5 L^{10}} = \lim_{J \to \infty} \limsup_{n \to \infty} \|\tilde{R}_n^J\|_{L^5 L^{10}} = 0.$$

To this end, let

$$\tilde{u}_n^J := \sum_{j=1}^{J} U_n^j + z_n^J, \tag{5.13}$$

where

$$z_n^J(t) := S_N(t) \vec{w}_n^J \tag{5.14}$$

verifies, by the decay of the remainder of the linear profile decomposition,

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|z_n^J\|_{L^5 L^{10}} = 0. \tag{5.15}$$

Observe that $\tilde{u}_n^J$ is solution in B^c of the following nonlinear wave equation with Neumann boundary conditions:

$$(\partial_t^2 - \Delta_N)\tilde{u}_n^J + (\tilde{u}_n^J)^5 = e_n^J, \quad \text{with } e_n^J := (\tilde{u}_n^J)^5 - \sum_{j=1}^{J} (U_n^j)^5. \tag{5.16}$$

Lemma 5.3. *We have*

$$\lim_{J \to \infty} \limsup_{n \to \infty} \|e_n^J\|_{L^1 L^2} = 0 \tag{5.17}$$

and

$$\vec{\tilde{u}}^J_{n \restriction t=0} = \vec{u}_n + \vec{\alpha}_n^J, \quad \lim_{J \to \infty} \limsup_{n \to \infty} \|S_N(\cdot)\alpha_n^J\|_{L^5 L^{10}} = 0. \tag{5.18}$$

Proof. We will first show (5.17). We have

$$|e_n^J| \lesssim_J \sum_{1 \le j \ne k \le J} |U_n^j|^4 |U_n^k| + |z_n^J|^5 + |z_n^J| \sum_{j=1}^{J} |U_n^j|^4. \tag{5.19}$$

Let us begin with the mixed terms $|U_n^j|^4|U_n^k|$. We start with the case $j,k \in J_{\mathrm{diff}}$. Notice that

$$|U_n^j|^4|U_n^k| \le |V_n^j|^4|V_n^k| + |U_n^j|^4|V_n^k - U_n^k| + |V_n^j|^4|V_n^k - U_n^k|^4,$$

thus, by Hölder's inequality, we get

$$\begin{aligned}\||U_n^j|^4|U_n^k|\|_{L^1L^2} \le \||V_n^j|^4|V_n^k|\|_{L^1L^2} + \|U_n^j\|_{L^5L^{10}}^4\|V_n^k - U_n^k\|_{L^5L^{10}} \\ + \|V_n^j\|_{L^5L^{10}}\|V_n^k - U_n^k\|_{L^5L^{10}}^4.\end{aligned} \tag{5.20}$$

On the one hand, as V_n^j and V_n^k are rescaled solutions of the defocusing critical nonlinear wave equation in $\mathbb{R}^3$ associated with orthogonal parameters, it is well known that, as n goes to infinity (see, for example, [3]),

$$\||V_n^j|^4|V_n^k|\|_{L^1L^2} \to 0. \tag{5.21}$$

On the other hand, as

$$\sup_n \|U_n^j\|_{L^5L^{10}} + \|V_n^j\|_{L^5L^{10}} < \infty,$$

it follows from (5.9) that

$$\|U_n^j\|_{L^5L^{10}}^4\|V_n^k - U_n^k\|_{L^5L^{10}} + \|V_n^j\|_{L^5L^{10}}\|V_n^k - U_n^k\|_{L^5L^{10}}^4 \to 0 \tag{5.22}$$

as n goes to infinity, and thus (5.20), combined with (5.21) and (5.22), gives

$$\||U_n^j|^4|U_n^k|\|_{L^1L^2} \to 0 \quad \text{for } j,k \in J_{\mathrm{diff}}. \tag{5.23}$$

Let us now assume that $j \in J_{\mathrm{comp}}$ and $k \in J_{\mathrm{diff}}$. We have, in the same way as before,

$$\||U_n^j|^4|U_n^k|\|_{L^1L^2} \le \||U_n^j|^4|V_n^k|\|_{L^1L^2} + \|U_n^j\|_{L^5L^{10}}^4\|V_n^k - U_n^k\|_{L^5L^{10}}. \tag{5.24}$$

On the one hand, we already saw that for $k \in J_{\mathrm{diff}}$,

$$\|U_n^j\|_{L^5L^{10}}^4\|V_n^k - U_n^k\|_{L^5L^{10}} \xrightarrow{n\to\infty} 0. \tag{5.25}$$

On the other hand, by Hölder's inequality and change of variables,

$$\begin{aligned}&\||U_n^j|^4|V_n^k|\|_{L^1L^2} \le \|U_n^j\|_{L^5L^{10}}^3\|V_n^k U_n^j\|_{L^{5/2}L^5} \\ &= \|U^j\|_{L^5L^{10}}^3 \frac{1}{\sqrt{\lambda_{k,n}}}\Big(\int\Big(\int_{r\ge1} U^j(t-t_{j,n},x)^5 V^k\Big(\frac{t-t_{k,n}}{\lambda_{k,n}},\frac{x}{\lambda_{k,n}}\Big)^5 dx\Big)^{1/2} dt\Big)^{2/5} \\ &= \|U^j\|_{L^5L^{10}}^3 \frac{1}{\sqrt{\lambda_{k,n}}}\Big(\int\Big(\int_{r\ge1} U^j(s,y)^5 V^k\Big(\frac{s+t_{j,n}-t_{k,n}}{\lambda_{k,n}},\frac{y}{\lambda_{k,n}}\Big)^5 dy\Big)^{1/2} ds\Big)^{2/5}.\end{aligned}$$

As the above expression is uniformly continuous in $V^k \in L^5L^{10}$, we can assume that V^k is continuous and compactly supported. Then we get

$$\||U_n^j|^4|V_n^k|\|_{L^1L^2} \lesssim \frac{1}{\sqrt{\lambda_{k,n}}} \to 0, \tag{5.26}$$

and thus, by (5.24), (5.25) and (5.26),

$$\||U_n^j|^4|U_n^k|\|_{L^1L^2} \to 0 \quad \text{for } j \in J_{\text{comp}},\ k \in J_{\text{diff}}. \tag{5.27}$$

In a similar fashion, we obtain

$$\||U_n^j|^4|U_n^k|\|_{L^1L^2} \to 0 \quad \text{for } k \in J_{\text{comp}},\ j \in J_{\text{diff}}. \tag{5.28}$$

To conclude with the mixed term $|U_n^j|^4|U_n^k|$, let us deal with the case $j,k \in J_{\text{comp}}$. Then

$$\||U_n^j|^4|U_n^k|\|_{L^1L^2} = \int\Big(\int_{r\geq 1} U^j(t-t_{j,n},x)^8 U^k(t-t_{k,n},x)^2\,dx\Big)^{1/2}dt. \tag{5.29}$$

By the orthogonality of the parameters,

$$|t_{j,n}-t_{k,n}| \to +\infty, \tag{5.30}$$

but, by the change of variable $s = t - t_{j,n}$, we obtain, from (5.29),

$$\||U_n^j|^4|U_n^k|\|_{L^1L^2} = \int\Big(\int_{r\geq 1} U^j(s,x)^8 U^k(s+t_{j,n}-t_{k,n},x)^2\,dx\Big)^{1/2}ds.$$

Again, as this expression is uniformly continuous in $(U^j,U^k) \in L^5L^{10}$, we may assume that both are continuous and compactly supported. But for such functions, the above expression vanishes for n large enough by (5.30). Thus, we have

$$\||U_n^j|^4|U_n^k|\|_{L^1L^2} \to 0 \quad \text{for } j,k \in J_{\text{comp}}. \tag{5.31}$$

We dealt with all the cases (5.23), (5.27), (5.28), (5.31) and showed that

$$\Big\|\sum_{1\leq j\neq k\leq J}|U_n^j|^4|U_n^k|\Big\|_{L^1L^2} \to 0 \quad \text{for all } J. \tag{5.32}$$

Finally, by the decay of the remainder (5.15),

$$\lim_{J\to\infty}\limsup_{n\to\infty}\||z_n^J|^5\|_{L^1L^2} = \lim_{J\to\infty}\limsup_{n\to\infty}\|z_n^J\|_{L^5L^{10}}^5 = 0 \tag{5.33}$$

and, moreover, by the Minkowski and Hölder inequalities,

$$\Big\||z_n^J|\sum_{j=1}^J|U_n^j|^4\Big\|_{L^1L^2} \leq \|z_n^J\|_{L^5L^{10}}\sum_{j=1}^J\|U_n^j\|_{L^5L^{10}}^4. \tag{5.34}$$

By (5.15),

$$\lim_{J\to\infty}\limsup_{n\to\infty}\Big\||z_n^J|\sum_{j=1}^J|U_n^j|^4\Big\|_{L^1L^2} = 0. \tag{5.35}$$

Combining (5.32), (5.33), (5.34) and (5.35), we thus proved the L^1L^2 decay of the error term e_n^J, that is, (5.17).

Let us now show (5.18). By the definition of $\tilde{u}_n^J$ (5.13), of the remainder (5.14) and of the modified profiles (5.8), (5.6), we have

$$\vec{\tilde{u}}_n^j(0) = \sum_{\substack{j\in J_{\text{comp}}\\ j\leq J}} \vec{U}^j(-t_{j,n}) + \sum_{\substack{j\in J_{\text{diff}}\\ j\leq J}} \sigma_{\lambda_{j,n}}\Big(\vec{V}^j\Big(\frac{-t_{j,n}}{\lambda_{j,n}}\Big)\Big) + \vec{w}_n^J. \tag{5.36}$$

As a consequence of the definition (5.5) of U^j for $j \in J_{\text{comp}}$, we have, in $\mathcal{H}(B^c)$, as n goes to infinity,

$$\vec{U}^j(-t_{j,n}) = \vec{S}_N(-t_{j,n})\vec{\psi}^j + o_n(1) \quad \text{for all } j \in J_{\text{comp}}. \tag{5.37}$$

Let us deal now with the first component of the diffusive profiles, the derivative component being handled in the same fashion. For $j \in J_{\text{diff}}$, by the definition (5.7), this first component verifies, in $\dot{H}^1$,

$$\begin{aligned}\sigma_{\lambda_{j,n}}\Big(V^j\Big(\frac{-t_{j,n}}{\lambda_{j,n}}\Big)\Big) &= \frac{1}{\lambda_{j,n}^{1/2}} V^j\Big(\frac{-t_{j,n}}{\lambda_{j,n}}, \frac{\cdot}{\lambda_{j,n}}\Big)\\ &= \frac{1}{\lambda_{j,n}^{1/2}} V_L^j\Big(\frac{-t_{j,n}}{\lambda_{j,n}}, \frac{\cdot}{\lambda_{j,n}}\Big) + o_n(1) = S_N(-t_{j,n})\sigma_{\lambda_{j,n}}\vec{\psi}^j + o_n(1),\end{aligned} \tag{5.38}$$

where at the last line we have used Lemma 3.2. This last expansion (5.38), together with the similar one for the derivative component, (5.36), (5.37), the linear profile decomposition (5.4) and the Strichartz estimates for the Neumann flow (Proposition 2.7) gives (5.18), and completes the proof of the lemma. ∎

The proof of the nonlinear profile decomposition follows.

Proof of Proposition 5.2. By (5.16), together with (5.17) and (5.18), the perturbative result of Proposition 2.13 gives, together with (5.15),

$$u_n = \tilde{u}_n^J + \tilde{R}_n^J,$$

with

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|\tilde{R}_n^J\|_{L^5L^{10}} = 0.$$

But (5.9) enables us to replace all the U_n^j by V_n^j for $j \in J_{\text{diff}}$ in the definition (5.13) of u_n^J and ends the proof of the nonlinear profile decomposition. ∎

We are now in position to end the proof of the theorem. Indeed, by Proposition 5.2 together with (5.11), u_n is in L^5L^{10} for n large enough, and (5.2) is contradicted. Therefore, the assumption (5.10) cannot hold, that is, $J = 1$ and there is only one non-trivial profile in the decomposition (5.2):

$$\vec{u}_0^n = S_N(-t_{1,n})\sigma_{\lambda_{1,n}}\vec{\psi}^1 + \vec{w}_n, \quad \|S_N(\cdot)\vec{w}_n\|_{L^5L^{10}} \to 0. \tag{5.39}$$

Let us show that it is the time-compact ($t_{1,n} = 0$), scaling-compact ($\lambda_{1,n} = 1$) one.

As noticed before, as the scattering in the free space $\mathbb{R}^3$ is well known, we have $V^j \in L^5L^{10}$ for any $j \in J_{\text{diff}}$. Therefore, if $1 \in J_{\text{diff}}$, the same proof as before yields the decomposition

$$u_n(t) = \frac{1}{\lambda_{1,n}^{1/2}} V^1\Big(\frac{t - t_{1,n}}{\lambda_{1,n}}, \frac{\cdot}{\lambda_{1,n}}\Big) + R_n(t), \tag{5.40}$$

with

$$\limsup_{n\to\infty} \|R_n\|_{L^5L^{10}} = 0, \tag{5.41}$$

proving that $u_n \in L^5L^{10}$, a contradiction. Thus, $1 \in J_{\text{comp}}$, i.e., $\lambda_{1,n} = 1$.

It remains to eliminate the case $t_{1,n} \to \pm\infty$. Recall that

$$\|u_n\|_{L^5((-\infty,0)L^{10})} \to \infty, \quad \|u_n\|_{L^5((0,+\infty)L^{10})} \to \infty. \tag{5.42}$$

Let us for example assume, by contradiction, that $t_{1,n} \to +\infty$. This implies

$$\lim_{n\to\infty} \|S_N(\cdot - t_{1,n})\vec{\psi}^1\|_{L^5((-\infty,0)L^{10})} = 0,$$

and, by the small data well-posedness theory, for large n, we obtain $u_n \in L^5((-\infty,0), L^{10})$ with

$$\lim_{n\to\infty} \|u_n\|_{L^5((-\infty,0),L^{10})} = 0,$$

contradicting (5.42). The case $t_{1,n} \to -\infty$ is eliminated in the same way.

Therefore,

$$\vec{u}_0^n = \vec{\psi}^1 + \vec{w}_n, \quad \|S_N(\cdot)\vec{w}_n\|_{L^5L^{10}} \to 0.$$

Now notice that, by the Pythagorean expansion (4.7), together with its L^6 version (4.8), $\mathcal{E}(\vec{\psi}^1) \le E_c$, and therefore

$$\mathcal{E}(\vec{\psi}^1) = E_c,$$

otherwise, by (5.40) and the definition of E_c, u_n scatters. This implies, by the Pythagorean expansion again, together with (5.2),

$$\|\vec{w}_n\|_{\mathcal{H}(B^c)} \to 0.$$

We take $\vec{u}_c$ to be this profile:

$$\vec{u}_c := \vec{\psi}^1.$$

By the conservation of energy, we have $\mathcal{E}(\vec{\mathcal{S}}_N(t)\vec{u}_c) = E_c$ for any t, and the same argument applied to

$$\vec{\mathcal{S}}_N(t_n)\vec{u}_c$$

for any sequence $(t_n)_{n\ge 1} \in \mathbb{R}^{\mathbb{N}}$ shows that the flow $\{t \in \mathbb{R}, \vec{\mathcal{S}}_N(t)\vec{u}_c\}$ has a compact closure in $\mathcal{H}(B^c)$. Indeed, this sequence satisfies the same assumptions as $\vec{u}_n^0$ at the beginning of the proof, and will therefore have a convergent subsequence in $\mathcal{H}(B^c)$ as well. Finally, observe that $\mathcal{E}(\vec{u}_c) = E_c > 0$ ensures, in particular, that $\vec{u}_c \neq \vec{0}$. ∎

6. Rigidity

In this section we prove the following.

Theorem 6.1. *Let $(u_0, u_1) \in \mathcal{H}(B^c)$ be radial, and let $u(t) = \mathcal{S}_N(t)(u_0, u_1)$ be a solution of the energy critical defocusing wave equation outside the unit ball with Neumann boundary conditions* (1.7)–(1.9). *Assume that u is global and that*

$$K = \{\vec{u}(t) : t \in \mathbb{R}\}$$

has compact closure in $\mathcal{H}(B^c)$. Then $u = 0$.

The proof follows the lines of the proof of [13].

6.1. Preliminaries

We will use the following asymptotic energy property for the wave equation on $\mathbb{R}^3$.

Proposition 6.2. *Let $R > 0$. Let $(v_0, v_1) \in \mathcal{H}(\mathbb{R}^3)$ and let $v = S_{\mathbb{R}^3}(v_0, v_1)$ be the solution of the linear wave equation on $\mathbb{R}^3$ with initial data (v_0, v_1). Then*

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_{R+|t|}^{+\infty} |\partial_{t,r}(rv(t,r))|^2 \, dx = \int_R^{+\infty} (\partial_r(rv_0))^2 + r^2 v_1^2 \, dr.$$

We omit the easy proof, which relies on the equation $(\partial_t^2 - \partial_r^2)u(t,r) = 0$. We note that by integration by parts,

$$\int_R^{+\infty} (\partial_{t,r}(ru_0))^2 \, dr + Ru_0^2(R) = \int_R^{+\infty} (\partial_{t,r}(u_0))^2 r^2 \, dr. \tag{6.1}$$

Proposition 6.3. *There exists $\mathfrak{z} > 0$ and a radial C^∞ function $Z = Z(|x|)$ on $\{x \in \mathbb{R}^3 : |x| > \mathfrak{z}\}$ such that*

$$\Delta Z = Z^5 \quad \textit{for } r > \mathfrak{z}, \tag{6.2}$$

$$\left| r Z'(r) + \frac{1}{r} \right| + \left| Z(r) - \frac{1}{r} \right| \le \frac{C}{r^3}, \tag{6.3}$$

$$\lim_{r\to\zeta^+} |Z(r)| = +\infty, \tag{6.4}$$

$$Z'(r) \neq 0 \quad \textit{for } r > \mathfrak{z}. \tag{6.5}$$

Proof. The existence of $\mathfrak{z}$ and Z satisfying (6.2), (6.3) and (6.4) is proved in Proposition 4.1 of [16] and we omit it.

To prove (6.5), we argue by contradiction. Assume that $Z'(R) = 0$ for some $R > \mathfrak{z}$. Multiplying equation (6.2) by Z, integrating by parts for $r > R$ and using the boundary condition $Z'(R) = 0$, we obtain

$$\int_{\{|x>R\}} |\nabla Z|^2 \, dx + \int_{\{|x>R\}} |Z|^6 \, dx = 0.$$

This proves that $Z(r) = 0$ for almost every $r > R$, contradicting (6.3). ∎

Remark 6.4. Let $\ell \in \mathbb{R} \setminus \{0\}$ and

$$Z_\ell = \frac{1}{\ell} Z\Big(\frac{r}{\ell^2}\Big).$$

Then (6.2), (6.4) and (6.5) hold with Z replaced by Z_ℓ and $\mathfrak{z}$ by $\ell^2 \mathfrak{z}$, and there exists a constant C_ℓ such that

$$\Big| r Z'_\ell(r) + \frac{\ell}{r}\Big| + \Big| Z_\ell(r) - \frac{\ell}{r}\Big| \leq \frac{C_\ell}{r^3}. \tag{6.6}$$

6.2. Proof of Theorem 6.1

Step 1. Let $(u_0, u_1) \in \mathcal{H}(B^c)$ be as in Theorem 6.1. Let $\varepsilon > 0$ be a small parameter to be specified. Throughout the proof we fix $R_\varepsilon > 1$ such that

$$\int_{R_\varepsilon}^{+\infty} ((\partial_r u_0)^2 + u_1^2) r^2 \, dr \leq \varepsilon. \tag{6.7}$$

In this step, we prove

$$\int_R^{+\infty} (\partial_r(r u_0))^2 + r^2 u_1^2 \, dr \leq C R^5 u_0^{10}(R) \quad \text{for all } R \geq R_\varepsilon. \tag{6.8}$$

Let $R \geq R_\varepsilon$. We define the radial functions $v_0 \in \dot{H}^1(\mathbb{R}^3)$, $v_1 \in L^2(\mathbb{R}^3)$ as follows:

$$\begin{cases} (v_0, v_1)(r) = (u_0, u_1)(r) & \text{if } r > R, \\ (v_0, v_1)(r) = (u_0(R), 0) & \text{if } r \in (0, R). \end{cases} \tag{6.9}$$

We let $v(t) = \mathcal{S}_{\mathbb{R}^3}(t)(v_0, v_1)$ be the solution to the quintic wave equation on $\mathbb{R}^3$ with initial data (v_0, v_1), and let $v_L(r) = S_{\mathbb{R}^3}(v_0, v_1)$ be the corresponding solution to the free wave equation. We note that by the finite speed of propagation

$$v(t, r) = u(t, r), \quad r > R + |t|.$$

By the small data theory, since ε is small,

$$\sup_{t \in \mathbb{R}} \| \vec{v}(t) - \vec{v}_L(t) \|_{\dot{H}^1 \times L^2} \leq C \| (v_0, v_1) \|^5_{\dot{H}^1 \times L^2}. \tag{6.10}$$

By Proposition 6.2,

$$\sum_{\pm} \lim_{t \to \pm\infty} \int_{R+|t|}^{+\infty} |\partial_{t,r}(r v_L(t, r))|^2 \, dr = \int_R^{+\infty} (\partial_r(r u_0))^2 + u_1^2 \, dr. \tag{6.11}$$

By (6.10) and the finite speed of propagation,

$$\left| \int_{R+|t|}^{+\infty} |\partial_{t,r}(r v_L(t, r)) - \partial_{t,r}(r u(t, r))|^2 \, dr \right| \leq C \Big(\int_R^{+\infty} ((\partial_r u_0)^2 + u_1^2) r^2 \, dr \Big)^5.$$

Combining with (6.11) and using that, by the compactness of the closure of K in $\mathcal{H}(B^c)$,

$$\lim_{t\to\pm\infty}\int_{R+|t|}^{+\infty}|\partial_{t,r}(ru(t,r))|^2\,dr=0,$$

we deduce

$$\int_R^{+\infty}(\partial_r(ru_0))^2+u_1^2\,dr\leq C\left(\int_R^{+\infty}(\partial_r u_0)^2+u_1^2)r^2\,dr\right)^5.$$

Combining with the integration by parts formula (6.1) and the smallness of ε, we deduce inequality (6.8).

Step 2. In this step we prove that there exists $\ell\in\mathbb{R}$ and $C>0$ such that for large R,

$$\left|u_0(r)-\frac{\ell}{r}\right|\leq\frac{C}{r^3},\qquad\int_r^{+\infty}\rho^2u_1(\rho)\,d\rho\leq\frac{C}{r^5}.\tag{6.12}$$

First fix R and R' such that $R_\varepsilon\leq R\leq R'\leq 2R$. Letting $\zeta_0(r)=ru_0(r)$, we have, using the Cauchy–Schwarz inequality and then step 1,

$$|\zeta_0(R)-\zeta_0(R')|\leq\int_R^{R'}|\partial_r\zeta_0(r)|\,dr\leq\sqrt{R}\sqrt{\int_R^{R'}(\partial_r\zeta_0)^2\,dr}\leq\frac{1}{R^2}\zeta_0^5(R).\tag{6.13}$$

Since, by the definition (6.7) of R_ε and the integration by parts formula (6.1), one has

$$\frac{1}{R}\zeta_0^2(R)\leq\varepsilon,\tag{6.14}$$

we deduce, from (6.13),

$$|\zeta_0(R)-\zeta_0(R')|\leq\varepsilon^2\zeta_0(R).\tag{6.15}$$

We apply this inequality between 2^kR and $2^{k+1}R$ for $k\in\mathbb{N}$ and a fixed $R\geq R_\varepsilon$. This yields

$$|\zeta_0(2^{k+1}R)-\zeta_0(2^kR)|\lesssim\varepsilon^2|\zeta_0(2^kR)|$$

and thus

$$|\zeta_0(2^{k+1}R)|\leq(1+C\varepsilon^2)|\zeta_0(2^kR)|.$$

We deduce, by an easy induction,

$$|\zeta_0(2^kR)|\leq(1+C\varepsilon^2)^k|\zeta_0(R)|.$$

Combining with (6.13), we obtain

$$|\zeta_0(2^kR)-\zeta_0(2^{k+1}R)|\lesssim(1+C\varepsilon^2)^{5k}|\zeta_0(R)|^5\frac{1}{2^{2k}R^2}.\tag{6.16}$$

Choosing ε small, so that $(1+C\varepsilon^2)^5<4$, we see that $\sum|\zeta_0(2^kR)-\zeta_0(2^{k+1}R)|$ converges, and thus that $\zeta_0(2^kR)$ has a limit $\ell(R)$ as $k\to\infty$. Summing (6.16) over all $k\geq k_0$, we obtain

$$|\zeta_0(2^{k_0}R)-\ell(R)|\lesssim\frac{1}{R^2}\frac{1}{(1+c_\varepsilon)^{k_0}}|\zeta_0(R)|^5,\tag{6.17}$$

for some constant $c_\varepsilon > 0$. Combining with (6.13), we see that

$$\lim_{r\to\infty} \zeta_0(r) = \ell(R),$$

and, in particular, the limit $\ell(R)$ does not depend on R. We will simply denote it by ℓ. By (6.17) at $k_0 = 1$, since ζ_0 is bounded,

$$|\zeta_0(R) - \ell| \lesssim_{\zeta_0} \frac{1}{R^2}, \tag{6.18}$$

which yields the first inequality in (6.12).Combining with step 1, we obtain the second inequality in (6.12).

Step 3. In this step, we assume $\ell = 0$ and prove that $(u_0, u_1) \equiv (0, 0)$. Indeed, by (6.15), if $R \geq R_\varepsilon$ and $k \in \mathbb{N}$,

$$|\zeta_0(2^{k+1}R)| \geq (1 - C\varepsilon^2)|\zeta_0(2^k R)|.$$

Hence, by induction on k,

$$|\zeta_0(2^k R)| \geq (1 - C\varepsilon^2)^k |\zeta_0(R)|.$$

Since, by the preceding step and the assumption $R = 0$, $|\zeta_0(2^k R)| \lesssim 1/2^k R^2$, we deduce, choosing ε small enough and letting $k \to \infty$, that $\zeta_0(R) = 0$. Combining with (6.8), we deduce

$$R \geq R_\varepsilon \implies \int_R^{+\infty} (\partial_r \zeta_0)^2 + u_1^2(r)\, dr = 0,$$

that is, $u_0(r)$ and $u_1(r)$ are 0 for almost every $r \geq R_\varepsilon$. Going back to the definition of R_ε, we see that we can choose any $R_\varepsilon > 1$, which concludes this step.

Step 4. We next assume $\ell \neq 0$. Let Z_ℓ be as in Remark 6.4. In this step we prove that $(u_0 - Z_\ell, u_1)$ has a bounded support. Let $f = u - Z_\ell$. Then

$$\begin{cases} \partial_t^2 f - \Delta f = D_\ell(f) := \displaystyle\sum_{k=1}^{5} \binom{5}{k} Z_\ell^{5-k} f^k, \\ \vec{f}_{\restriction t=0} = (f_0, f_1) := (u_0 - Z_\ell, u_1). \end{cases} \tag{6.19}$$

For $\varepsilon > 0$ small, we fix $R'_\varepsilon \gg 1$ such that

$$\int_{R'_\varepsilon}^{+\infty} (|\partial_r f_0(r)|^2 + |f_1(r)|^2) r^2\, dr \leq \varepsilon^2, \tag{6.20}$$

$$\int_{\mathbb{R}} \left(\int_{R'_\varepsilon+|t|}^{+\infty} Z_\ell^{10}(r) r^2\, dr \right)^{1/2} dt \leq \varepsilon^5. \tag{6.21}$$

Let f_L be the solution of $\partial_t^2 f_L = \Delta f_L$, with $\vec{f}_{L\restriction t=0} = (\tilde{f}_0, \tilde{f}_1)$, where $(\tilde{f}_0, \tilde{f}_1)$ coincides with (f_0, f_1) for $r > R'_\varepsilon$ and is defined as in (6.9). Using (6.19) and the assumptions (6.20) and (6.21) on R'_ε, we obtain

$$\sup_{t\in\mathbb{R}} \left\| \mathbb{1}_{\{|x|>|t|+R'_\varepsilon\}} |\nabla_{t,x}(\tilde{f}(t) - \tilde{f}_L(t))| \right\|_{L^2} \lesssim \varepsilon^4 \|(\tilde{f}_0, \tilde{f}_1)\|_{\dot{H}^1\times L^2}. \tag{6.22}$$

Let $R \geq R'_\varepsilon$. Since, by Proposition 6.2,

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_R^{+\infty} (\partial_{t,r}(r\tilde{f}_L(t,r)))^2\, dr \gtrsim \int_R^{+\infty} ((\partial_r(r\tilde{f}))^2 + r^2\tilde{f}_1^2)\, dr$$

and

$$\sum_{\pm} \lim_{t\to\pm\infty} \int_R^{+\infty} (\partial_{t,r}(r\tilde{f}(t,r)))^2\, dr = 0,$$

we deduce, from (6.22),

$$\varepsilon^8 \int_R^{+\infty} ((\partial_r f_0)^2 + f_1^2) r^2\, dr \gtrsim \int_R^{+\infty} ((\partial_r(rf_0))^2 + r^2 f_1^2)\, dr,$$

and thus

$$\varepsilon^8 R f_0^2(R) \gtrsim \int_R^{\infty} ((\partial_r(rf_0))^2 + r^2 f_1^2)\, dr. \tag{6.23}$$

Letting $g_0 = rf_0$, we deduce by the Cauchy–Schwarz inequality that for $R \geq R'_\varepsilon$, $k \in \mathbb{N}$,

$$|g_0(2^{k+1}R) - g_0(2^k R)| \lesssim \int_{2^k R}^{2^{k+1}R} |\partial_r g_0|\, dr \lesssim \varepsilon^4 |g_0(2^k R)|.$$

This yields by an easy induction $|g_0(2^k R)| \geq (1 - C\varepsilon^4)^k |g_0(R)|$, where $C > 0$ is a constant which is independent of ε. Since, by step 2,

$$\frac{C}{(2^k R)^2} \geq |g_0(2^k R)|,$$

we obtain, choosing ε small enough, that $g_0(R) = 0$ for large R. Combining with (6.23), we deduce that $(f_0(r), f_1(r)) = 0$ a.e. for large R, concluding this step.

Step 5. In this step, we still assume $\ell \neq 0$ and deduce a contradiction. We let

$$\rho = \inf\Big\{R > c : \int_R^{+\infty} ((\partial_r f_0)^2 + f_1^2) r^2\, dr = 0\Big\}$$

and we shall prove that $\rho = \max(1, \mathfrak{z}\ell^2)$, i.e., that $u_0(r) = Z_\ell(r)$ almost everywhere for $r > \max(1, \mathfrak{z}\ell^2)$. If $\mathfrak{z}\ell^2 \geq 1$, we deduce

$$\lim_{r\to\mathfrak{z}\ell^2} |u_0(r)| = +\infty,$$

a contradiction with the radial Sobolev embedding theorem. If $\mathfrak{z}\ell^2 \leq 1$, we obtain $u_0(r) = Z_\ell(r)$ for all $r > 1$. Translating the solution in time, the same proof yields that for all t in the domain of definition of u,

$$u(t,r) = Z_\ell(r), \tag{6.24}$$

a contradiction with the Neumann boundary condition, as given by Lemma 2.10. Note that by finite speed of propagation, the limit ℓ in (6.24) is independent of t.

To prove that $\rho = \max(1, \mathfrak{z}\ell^2)$, we argue by contradiction, assuming $\rho > \max(1, \mathfrak{z}\ell^2)$. By the preceding step and the finite speed of propagation, the essential support of f is included in $\{r \leq \rho + |t|\}$. Thus, f is a solution of

$$\begin{cases} \partial_t^2 f - \Delta f = \mathbb{1}_{\{|x| \leq \rho + |t|\}} D_\ell(f), \\ \vec{f}_{\restriction t=0} = (f_0, f_1) := (u_0 - Z_\ell, u_1), \end{cases}$$

Fix $R''_\varepsilon \in (1, \rho)$ such that

$$\int_{R''_\varepsilon}^{+\infty} (|\partial_r f_0(r)|^2 + |f_1(r)|^2) r^2 \, dr \leq \varepsilon^2,$$

$$\int_{\mathbb{R}} \left(\int_{R''_\varepsilon + |t|}^{\rho + |t|} Z_\ell^{10}(r) r^2 \, dr \right)^{1/2} dt \leq \varepsilon^5.$$

The same argument as in the preceding step, replacing R'_ε by R''_ε, yields that $(f_0, f_1) = 0$ for almost every $r > R''_\varepsilon$, which contradicts the definition of ρ. The proof is complete. ∎

We are now in position to conclude.

Proof of Theorem 1.1. By contradiction, assume that E_c, as defined by (5.1), is finite. Then Theorem 5.1 shows that there exists a solution $\vec{u}_c$ to (1.7)–(1.9) such that $\{\vec{u}_c(t) : t \in \mathbb{R}\}$ has a compact closure in $\mathcal{H}(B^c)$, but by Theorem 6.1, such a solution cannot exist. Thus, $E_c = +\infty$, and by Proposition 2.12, all the solutions of (1.7)–(1.9) scatter. ∎

7. Focusing case

In this section we sketch the proofs of Theorems 1.2 and 1.3. Section 7.1 is dedicated to the proof of a trapping property for solutions below the energy of the $\mathbb{R}^3$ ground state W that is important in the proof of both of these results. Section 7.2 concerns Theorem 1.2 and Section 7.3, Theorem 1.3. Finally, in Section 7.4, we comment on the assumptions of these two theorems, and prove that the exact analogue of Theorem 1.2 is not true when $\mathbb{R}^3 \setminus B(0,1)$ is replaced by a more general domain.

7.1. Trapping by the energy

Recall that

$$W(x) = \frac{1}{(1 + |x|^2/3)^{1/2}}$$

is the ground state of the focusing critical wave equation on $\mathbb{R}^3$. If $(f, g) \in \mathcal{H}(\mathbb{R}^3)$, we denote

$$\mathcal{E}_{\mathbb{R}^3}(f, g) = \frac{1}{2} \int_{\mathbb{R}^3} |\nabla f|^2 + \frac{1}{2} \int_{\mathbb{R}^3} |g|^2 - \frac{1}{6} \int_{\mathbb{R}^3} |f|^6.$$

Proposition 7.1. *Let u be a solution of* (1.11) *with Neumann boundary condition* (1.8) *and initial data* (1.9)*. Let I be its maximal interval of existence. Assume $\mathcal{E}(u_0,u_1) < \mathcal{E}_{\mathbb{R}^3}(W,0)$. Then the sign of $\int_{B^c}|\nabla u(t)|^2 - \int_{\mathbb{R}^3}|\nabla W|^2$ is independent of $t \in I$, and there exists $\delta > 0$, depending only on $\mathcal{E}(u_0,u_1)$, such that*

$$\left|\int_{B^c}|\nabla u(t,x)|^2\,dx - \int_{\mathbb{R}^3}|\nabla W(x)|^2\,dx\right| \geq \delta \quad \textit{for all } t \in I. \tag{7.1}$$

Proof. For $(f,g) \in \mathcal{H}$, we denote by $(\tilde{f},\bar{g}) := \vec{\mathcal{P}}(f,g)$, the extension of (f,g) to $\mathcal{H}(\mathbb{R}^3)$ by $(f(1),0)$, as defined in definition 2.1. Observe that $(\tilde{f},\bar{g})$ verifies

$$\int_{\mathbb{R}^3}|\nabla\tilde{f}|^2 = \int_{B^c}|\nabla f|^2 \quad \int_{\mathbb{R}^3}\tilde{f}^6 \geq \int_{B^c}f^6, \quad \int_{\mathbb{R}^3}|\bar{g}|^2 = \int_{B^c}|g|^2$$

and

$$\mathcal{E}_{\mathbb{R}^3}(\tilde{f},\bar{g}) \leq \mathcal{E}(f,g). \tag{7.2}$$

Let u satisfy the assumptions of Proposition 7.1. Then, by conservation of the energy and (7.2),

$$\mathcal{E}_{\mathbb{R}^3}(\tilde{u}(t),\overline{\partial_t u}(t)) \leq \mathcal{E}(u_0,u_1) < \mathcal{E}_{\mathbb{R}^3}(W,0) \quad \text{for all } t \in I.$$

The conclusion of the proposition then follows from the variational properties of the ground-state W on $\mathbb{R}^3$, see, e.g., Lemma 3.4 of [27]. ∎

7.2. Scattering

Note that by Proposition 7.1 and the radial Sobolev inequality (see Remark 2.3), any solution of (1.11), (1.8)–(1.9) that satisfies $\mathcal{E}(u_0,u_1) < \mathcal{E}_{\mathbb{R}^3}(W,0)$, $\int_{B^c}|\nabla u_0|^2 < \int_{\mathbb{R}^3}|\nabla W|^2$ is global.

Using Proposition 7.1, the proof of Theorem 1.2 follows exactly the same lines as the proof of Theorem 1.1.

Recall that according to [28], any solution of the quintic focusing wave equation on $\mathbb{R}^3$ with initial data $(v_0,v_1) \in (\dot{H}^1 \times L^2)(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}|\nabla v_0|^2 < \int_{\mathbb{R}^3}|\nabla W|^2 \quad \text{and} \quad \mathcal{E}_{\mathbb{R}^3}(v_0,v_1) < \mathcal{E}_{\mathbb{R}^3}(W,0)$$

scatters to a linear solution.

Arguing by contradiction and using the arguments of Sections 3, 4 and 5, we see that it is sufficient to prove the following.

Theorem 7.2. *Let $(u_0,u_1) \in \mathcal{H}(B^c)$ be radial, and let $u(t)$ be a solution of the energy critical focusing wave equation outside the unit ball with Neumann boundary conditions* (1.11)*,* (1.8)*–*(1.9)*. Assume that u is global and that*

$$K = \{\vec{u}(t) : t \in \mathbb{R}\}$$

has compact closure in $\mathcal{H}(B^c)$. Then $u \equiv 0$.

Note that it would be sufficient to prove Theorem 7.2 with the additional assumptions $\mathcal{E}(u_0, u_1) < \mathcal{E}_{\mathbb{R}^3}(W, 0)$, $\int_{B^c} |\nabla u_0|^2 < \int_{\mathbb{R}^3} |\nabla W|^2$, but these assumptions are not needed to obtain the conclusion of the theorem.

The proof of Theorem 7.2 is the same as the proof of the Theorem 6.1 in Section 6, except that in steps 4 and 5 the solution Z_ℓ of the elliptic equation $\Delta Z_\ell = Z_\ell^5$ must be replaced by the solution W_ℓ of the elliptic equation $-\Delta W_\ell = W_\ell^5$, where

$$W_\ell(x) = \frac{\sqrt{3}}{\ell} W\Big(\frac{3x}{\ell^2}\Big) = \frac{\sqrt{3}}{\ell(1 + \frac{3|x|^2}{\ell^4})^{1/2}}, \tag{7.3}$$

so that

$$\Big|W_\ell(x) - \frac{\ell}{|x|}\Big| \lesssim \frac{1}{|x|^3}, \quad |x| \gg 1.$$

Also, since $W_\ell(x)$ is defined for all $x \in \mathbb{R}^3$, whereas $Z_\ell(x)$ is only defined for $r > \mathfrak{z}\ell^2$, we must replace $\max(1, \mathfrak{z}\ell^2)$ everywhere in these two steps of the proof by 1. The key point to obtain the contradiction is that $\partial_r W_\ell(1) \neq 0$ for any $\ell \neq 0$, i.e., that W_ℓ is not a stationary solution of the focusing wave equation on B^c with Neumann boundary condition, which can be easily checked on the explicit formula (7.3).

7.3. Blow-up

Using Proposition 7.1, the proof of Theorem 1.3 is very close to the proof of its analogue on the whole space $\mathbb{R}^3$, see Theorem 3.7 and the proof of Theorem 1.1 (ii) in Section 7 of [28]. Let us mention that this argument is inspired by the work of H. A. Levine [33].

Let us first assume that $u_0 \in H^1(B^c) = \dot{H}^1(B^c) \cap L^2(B^c)$. Using the equation satisfied by u, one sees that $u(t) \in L^2(B^c)$ for all t and, denoting $y(t) = \int_{B^c} u^2(t, x) dx$, that

$$y'(t) = 2 \int_{B^c} u \partial_t u, \quad y''(t) = 2 \int_{B^c} u^6 - 2 \int_{B^c} |\nabla u|^2 + \int_{B^c} (\partial_t u)^2.$$

Note that we have used the boundary condition $\partial_n u \restriction_{\partial B^c} = 0$, which implies $\int_{B^c} u \Delta u = -\int_{B^c} |\nabla u|^2$.

Recall that $\mathcal{E}_{\mathbb{R}^3}(W, 0) = \frac{1}{3} \int_{\mathbb{R}^3} |\nabla W|^2$. As in the proof of Theorem 3.7 of [28], one can write, for t in the domain of existence of u,

$$\begin{aligned}
y''(t) &= -12\mathcal{E}(u_0, u_1) + 4 \int_{B^c} |\nabla u|^2 + 8 \int_{B^c} (\partial_t u)^2 \\
&= 8 \int_{B^c} (\partial_t u)^2 + 4 \int_{B^c} |\nabla u|^2 - 4 \int_{\mathbb{R}^3} |\nabla W|^2 + 12\mathcal{E}_{\mathbb{R}^3}(W, 0) - 12\mathcal{E}(u_0, u_1) \\
&\geq 8 \int_{B^c} (\partial_t u)^2 + \delta_0,
\end{aligned}$$

where $\delta_0 = 12\mathcal{E}_{\mathbb{R}^3}(W, 0) - 12\mathcal{E}(u_0, u_1) > 0$ and we have used, by Proposition 7.1, that $\int_{B^c} |\nabla u(t)|^2 > \int_{\mathbb{R}^3} |\nabla W|^2$ for all t.

The end of the proof that u blows up in finite time is exactly as the end of the proof of Theorem 3.7, p. 165, of [28] and we omit it.

To treat the general case $u_0 \in \dot{H}^1(B^c)$, one should use a localized version of $\int_{B^c} u^2(t)$. These bring out new terms in the preceding computation, that can be controlled using finite speed of propagation. We refer to p. 205–206 of [28] for the details.

7.4. Comments on the assumptions

Consider the nonlinear focusing wave equation (1.1) with $\iota = -1$, and Neumann boundary condition (1.2) in a general open domain Ω of $\mathbb{R}^3$. We claim that the analogue of Theorem 1.2 does not hold in general. Indeed, first consider the case of a half-plane:

$$\Omega = \{(x_1, x_2, x_3) \in \mathbb{R}^2 : x_1 > 0\}.$$

Let w be the restriction of W to Ω. Then w is a solution of $-\Delta w = w^5$. Since W is radial, w satisfies in addition the Neumann boundary condition (1.2). This yields a non-scattering solution w of (1.1), (1.2) such that

$$\int_\Omega |\nabla w|^2 = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla W|^2, \quad \mathcal{E}(\vec{w}(0)) = \frac{1}{2}\,\mathcal{E}_{\mathbb{R}^3}(W, 0),$$

which proves that one cannot generalise Theorem 1.2 in this setting. Similarly, for $\varepsilon > 0$, the solution w_ε of (1.1), (1.2) with initial data $((1+\varepsilon)w, 0)$ blows up in finite time by [28]. This solution satisfies

$$\int_\Omega |\nabla w|^2 = \frac{(1+\varepsilon)^2}{2}\int_{\mathbb{R}^3} |\nabla W|^2, \quad \mathcal{E}(\vec{w}(0)) < \frac{1}{2}\,\mathcal{E}_{\mathbb{R}^3}(W, 0),$$

which shows that the assumptions $\mathcal{E}(u_0, u_1) < \mathcal{E}_{\mathbb{R}^3}(W, 0)$, $\int_\Omega |\nabla u_0|^2 < \int_{\mathbb{R}^3} |\nabla W|^2$ is not sufficient to ensure global existence on the half-plane.

We now give a similar example when Ω is an exterior domain. Assume that $\Omega = \mathbb{R}^3 \setminus K$, where K is bounded subset of $\mathbb{R}^3$ with a smooth boundary $\partial K = \partial\Omega$ containing a portion of a plane. Without loss of generality, we can assume (translating and rescaling Ω):

$$\{0\} \times [-1, +1]^2 \subset \partial\Omega, \quad B(0,1) \cap \{x_1 > 0\} \subset \Omega.$$

According to [30], for all $\varepsilon > 0$, there exists a radial solution z of the focusing critical wave equation on $\mathbb{R}^3$, blowing-up in finite time $T > 0$ and such that

$$\limsup_{t \to T} \int_{\mathbb{R}^3} \left|\nabla\left(z(t,x) - \frac{1}{t} W\left(\frac{x}{t^2}\right)\right)\right|^2 + (\partial_t z(t,x))^2\, dx \le \varepsilon, \quad \mathcal{E}(\vec{z}(0)) \le \mathcal{E}(W, 0) + \varepsilon.$$

Using finite speed of propagation, time translating and rescaling the solution, we can assume that the support of $\vec{z}(t)$ is included in $B(0,1)$ for all $t \in [0, T)$. The restriction u of z to $x_1 > 0$ is then a solution of (1.1), (1.2), (1.3) that satisfies

$$\mathcal{E}(u_0, u_1) \le \frac{1}{2}\,\mathcal{E}_{\mathbb{R}^3}(W, 0) + \varepsilon, \quad \limsup_{t \to T} \int_\Omega |\nabla u(t)|^2 + \int_\Omega (\partial_t u(t))^2 \le \frac{1}{2}\int_{\mathbb{R}^3} |\nabla W|^2 + \varepsilon,$$

proving that a generalization of Theorem 1.2 is hopeless in this setting also.

In view of this example, we conjecture that Theorem 1.3 cannot be either generalised to other geometries, and that the radiality assumptions in Theorems 1.2 and 1.3 are also necessary. More precisely, a natural conjecture is that the energy threshold to ensure energy trapping and a blow-up scattering/dichotomy in the case of Neumann boundary conditions is exactly $\frac{1}{2}\mathcal{E}_{\mathbb{R}^3}(W,0)$. This is of course the case when Ω is a half-plane, since one can then use the result on $\mathbb{R}^3$ after extending the solution by symmetry to the whole space.

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