A construction of equivariant bundles on the space of symmetric forms

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Abstract. We construct stable vector bundles on the space $\mathbb{P}(S^d\mathbb{C}^{n+1})$ of symmetric forms of degree d in $n+1$ variables which are equivariant for the action of $\mathrm{SL}_{n+1}(\mathbb{C})$ and admit an equivariant free resolution of length 2. For $n = 1$, we obtain new examples of stable vector bundles of rank $d-1$ on $\mathbb{P}^d$, which are moreover equivariant for $\mathrm{SL}_2(\mathbb{C})$. The presentation matrix of these bundles attains Westwick's upper bound for the dimension of vector spaces of matrices of constant rank and fixed size.

1. Introduction

It is notoriously difficult to construct rank-r non-splitting vector bundles (namely, not isomorphic to the direct sum of line bundles) on $\mathbb{P}^N$ if r is small with respect to N. A famous conjecture of Hartshorne entails that the task is in fact impossible when $r < N/3$. As for the meaning of the word *small* here, it basically refers to any value of $r \leqslant N - 1$ (say for $N \geqslant 4$). For instance, no example of non-splitting vector bundle of rank $r \leqslant N - 3$ is known, at least if we work in characteristic zero, which we tacitly assume from now on. Moreover, only two sporadic (yet quite important) constructions are available for $r = N - 2$, that are due to Horrocks–Mumford (for $N = 4$) and Horrocks (for $N = 5$). Once such a vector bundle is constructed, one can pull it back via any finite self-map of $\mathbb{P}^N$ to obtain a new bundle. Together with the method of affine pull-backs developed by Horrocks (see [14], see also [1]), this essentially exhausts the set of techniques currently available, to the best of our knowledge.

The situation improves slightly when $r = N - 1$. In this range, basically two classes of bundles are known: instanton bundles and Tango bundles (we refer to [17] for a treatment of these classical cases). These have been recently generalized by Bahtiti, see [3–5]. More examples are given by Cascini's weighted Tango bundles, see [10], and by the Sasakura bundle of rank 3 on $\mathbb{P}^4$, see [2]. That is roughly the list of all the examples known so far in this range.

From another perspective, one may try to construct vector bundles starting from their presentation matrix. Such matrix will have constant corank r when evaluated at any point of $\mathbb{P}^N$. While the opposite procedure (constructing a matrix from a bundle) is also interesting, as we tried to show in [8,9], it is not quite clear how to construct matrices of constant corank $r \leqslant N - 1$. Difficulties are even greater if we impose constraints on the matrix, for instance that it should have a given size, say $a \times b$, or that its coefficients should have a fixed degree, notably degree one. A simple calculation (see [22]) implies (say $a \leqslant b$) that such a matrix can exist a priori only if $r + 1$ divides $(a - 1)!/(b - r - 1)!$. Under this divisibility condition, some matrices of linear forms of size $a \times (a + r - 1)$ and constant corank r where given in [23], attaining the upper bound for the dimension of vector spaces of matrices of constant rank and fixed size. The construction is a bit obscure to us and in any case it says very little about the bundle itself.

The goal of this paper is to introduce a simple technique to construct non-splitting vector bundles on $\mathbb{P}^N$. For this, one has to view $\mathbb{P}^N$ as the space of homogeneous forms of degree d on $\mathbb{P}^n$ for some (d, n) and use representation theory of $G = \mathrm{SL}_{n+1}(\mathbb{C})$.

The resulting bundles satisfy a much stronger property than just being non-splitting, namely they are stable in sense of Mumford–Takemoto, or slope-stable. Also, by construction they are homogeneous for the action of G and again by definition their dual bundles are presented by a matrix of linear forms which is equivariant for G.

For $n = 1$, one has $N = d$ and our bundles have rank $d - 1$. As we will see, for $d \geqslant 4$ these bundles turn out to be different from all the bundles of rank $d - 1$ on $\mathbb{P}^d$ constructed so far (except for the single case of the classical Tango bundle).

Also, the matrix of linear forms will have size $a \times (a + r - 1)$ and constant corank $r = d - 1$, thus giving a new approach to achieve Westwick's bound.

Finally, for $n = 1$ and $d = 3$, our bundles agree with the $\mathrm{SL}_2(\mathbb{C})$-invariant instantons defined and studied in [11]. These instantons are parametrized by $\mathbb{N}$ in the sense that the second Chern class (the so-called "charge") of an $\mathrm{SL}_2(\mathbb{C})$-invariant instanton over $\mathbb{P}^3 = \mathbb{P}(V_3)$ must equal $\binom{m}{2}$ for some integer $m \geqslant 2$, and given such m, there is one and only one such instanton. This instanton is a special case of our bundles; moreover, our new approach simplifies some of the proofs given in that paper.

For higher n, our bundles have rank bigger than the dimension $N = \binom{d+n}{n} - 1$ of the ambient space. Nevertheless, they seem interesting as are they stable, homogeneous for the action of a rather big group operating on $\mathbb{P}^N$, but still of much smaller rank than most $\mathrm{SL}_{N+1}(\mathbb{C})$-homogeneous bundles. To study them, we will work on the copy of $\mathbb{P}^n$ sitting in $\mathbb{P}^N$ as the degree d Veronese map and use the theory of $\mathrm{SL}_{n+1}(\mathbb{C})$-homogeneous bundles in terms of quiver representations developed in [19].

We now state our results more precisely. For an integer $n \geqslant 1$, let V be a complex vector space of dimension $n + 1$ and let $G \simeq \mathrm{SL}_{n+1}(\mathbb{C})$ denote the special linear group of automorphisms of V. The representation theory of G is governed by the fundamental weights $\lambda_1, \ldots, \lambda_n$ of G, in the sense that an irreducible representation of G is uniquely determined by its dominant weight $\lambda = a_1\lambda_1 + \cdots + a_n\lambda_n$, where $a_i \in \mathbb{N}$ for all i. We write V_λ for this representation. By convention, the standard representation is V_{λ_1} and we often write $V = V_{\lambda_1}$. We also abbreviate V_d for $V_{d\lambda_1}$.

Now suppose $n \geqslant 2$ and take integers $d \leqslant e$. The Littlewood–Richardson rule gives

$$(1.1) \qquad V_d \otimes V_e \simeq \bigoplus_{i=0}^{d} V_{(d+e-2i)\lambda_1+i\lambda_2}.$$

From (1.1), we extract a G-invariant non-zero morphism

$$V_d \otimes V_e \to V_{(d+e-2)\lambda_1+\lambda_2}.$$

Assume that e is a multiple of d, say $e = (m-1)d$; the inclusion of the second summand $V_{(md-2)\lambda_1+\lambda_2}$ from the decomposition (1.1) into the product $V_d \otimes V_{(m-1)d}$ induces a G-equivariant morphism

$$\Phi_{m,d} : V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}V_d} \longrightarrow V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}V_d}(1),$$

which is a matrix of linear forms. We put $\mathcal{W}_{m,d} := \mathrm{Ker}(\Phi_{m,d})$.

The sheaves $\mathcal{W}_{m,d}$ constitute the main object of study of this paper. Here is our main result about them.

Theorem A. *Let $d, m, n \geqslant 2$ be integers. Then $\Phi_{m,d}$ has constant corank 1 and $\mathcal{W}_{m,d}$ is a slope-stable $\mathrm{SL}_{n+1}(\mathbb{C})$-equivariant vector bundle on $\mathbb{P}V_d$ of rank*

$$\mathrm{rk}(\mathcal{W}_{m,d}) = (md-1)\binom{md+n-1}{n-1} - \binom{(m-1)d+n}{n} + 1$$

that fits into the exact sequence
(1.2)
$$0 \to \mathcal{W}_{m,d} \to V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}V_d} \xrightarrow{\Phi_{m,d}} V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}V_d}(1) \xrightarrow{\Psi_{m,d}} \mathcal{O}_{\mathbb{P}V_d}(m) \to 0,$$

where the map $\Psi_{m,d}$ is a pull-back via a Veronese embedding of the natural surjection

$$V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}V_{(m-1)d}} \to \mathcal{O}_{\mathbb{P}V_{(m-1)d}}(1),$$

see Lemma 2.2 and Remark 2.3.

If $n = 1$ (so that $V \simeq \mathbb{C}^2$), formula (1.1) simplifies significantly and reads

$$(1.3) \qquad V_d \otimes V_{(m-1)d} = V_{md} \oplus V_{md-2} \oplus \cdots \oplus V_{(m-2)d}.$$

The second summand is just the symmetric power V_{md-2} and the space of symmetric forms $\mathbb{P}V_d \simeq \mathbb{P}^d$. The following analogue of Theorem A is our second result.

Theorem B. *Let $d, m \geqslant 2$ be integers and let $n = 1$. Then $\mathcal{W}_{m,d}$ is a slope-stable vector bundle of rank $d-1$ on $\mathbb{P}^d$, homogeneous under the action of $\mathrm{SL}_2(\mathbb{C})$, that fits into the exact sequence*

$$(1.4) \qquad 0 \to \mathcal{W}_{m,d} \to V_{md-2} \otimes \mathcal{O}_{\mathbb{P}^d} \xrightarrow{\Phi_{m,d}} V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^d}(1) \xrightarrow{\Psi_{m,d}} \mathcal{O}_{\mathbb{P}^d}(m) \to 0.$$

The bundles we construct here, as well as the presentation matrices defining them, are actually defined over $\mathbb{P}^n_{\mathbb{Q}}$ and therefore over $\mathbb{Z}$. However one cannot reduce modulo an arbitrary prime number p to obtain bundles defined in characteristic p (unless p is high enough), as the rank of the defining matrices may drop modulo p.

The presentation matrices can be defined in an algorithmic fashion simply by using the action of the Lie algebra of $SL_{n+1}(\mathbb{C})$. We provide an ancillary file containing a `Macaulay2` [13] package to do this on a computer[1].

The paper is structured as follows. In §2 we prove that the maps appearing in the sequence defining $\mathcal{W}_{m,d}$ compose to zero and that the resulting equivariant complex is exact at the sides. In §3 we prove our main result for $n = 1$, namely for the case of $SL_2(\mathbb{C})$-bundles. This is intended to guide the reader through the argument, which is a bit different (and much simpler) in this case. Also, this allows to quickly exhibit our $SL_2(\mathbb{C})$-bundles, which are the most interesting ones as far as the search of low-rank bundles on $\mathbb{P}^d$ is concerned. In §4 we treat the case of higher n, where the treatment of homogeneous bundles via representations of quivers comes into play. In §5 we show that our bundles are always new except for the case of the classical Tango bundle.

2. The equivariant complex

Recall the fundamental sequence appearing in Theorem A:

$$0 \to \mathcal{W}_{m,d} \to V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}V_d} \xrightarrow{\Phi_{m,d}} V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}V_d}(1) \xrightarrow{\Psi_{m,d}} \mathcal{O}_{\mathbb{P}V_d}(m) \to 0,$$

and its analogue for the case $n = 1$, from Theorem B:

$$0 \to \mathcal{W}_{m,d} \to V_{md-2} \otimes \mathcal{O}_{\mathbb{P}^d} \xrightarrow{\Phi_{m,d}} V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^d}(1) \xrightarrow{\Psi_{m,d}} \mathcal{O}_{\mathbb{P}^d}(m) \to 0.$$

In this section we show that these maps form equivariant complexes of vector bundles.

Lemma 2.1. *For $n \geq 2$, the space of G-invariant morphisms*

$$\mathrm{Hom}(V_{(md-2)\lambda_1+\lambda_2}, S^m V_d)^G$$

is zero. In particular, the same is true for the composition of the two maps:

$$V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}V_d} \xrightarrow{\Phi_{m,d}} V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}V_d}(1) \xrightarrow{\Psi_{m,d}} \mathcal{O}_{\mathbb{P}V_d}(m).$$

The same result holds for $n = 1$, the space $\mathrm{Hom}(V_{md-2}, S^m V_d)^G$ and the composition

$$V_{md-2} \otimes \mathcal{O}_{\mathbb{P}^d} \xrightarrow{\Phi_{m,d}} V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^d}(1) \xrightarrow{\Psi_{m,d}} \mathcal{O}_{\mathbb{P}^d}(m).$$

Proof. In virtue of Schur's lemma, it is enough to show that the irreducible representation $V_{(md-2)\lambda_1+\lambda_2}$ does not appear in the decomposition of $S^m V_d$.

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[1]The package `SLnEquivariantMatrices` is available in the current stable version of Macaulay2, and also at http://www.paololella.it/software/SLnEquivariantMatrices.m2.

We follow the standard notation from [12] and denote by H_i the diagonal matrix $E_{i,i}$ and by L_i the linear operator such that $L_i(H_j) = \delta_{ij}$, so that the fundamental weights of G are $\lambda_i = L_1 + \cdots + L_i$, for $i = 1, \ldots, n$. The Lie algebra $\mathfrak{g}$ of G is generated by $E_{i,j}, E_{j,i}$ and $H_i - H_j$, with the standard commutation relations $[H_i - H_j, E_{i,j}] = 2E_{i,j}$, $[H_i - H_j, E_{j,i}] = -2E_{j,i}$, $[E_{i,j}, E_{j,i}] = H_i - H_j$, again for $1 \leqslant i < j \leqslant n$.

Now suppose that V is generated by $\mathsf{x}_1, \ldots, \mathsf{x}_{n+1}$; then the space of symmetric d-forms V_d has a basis defined by $\mathsf{x}^{(k_1, \ldots, k_{n+1})} = \mathsf{x}_1{}^{k_1} \mathsf{x}_2{}^{k_2} \cdots \mathsf{x}_{n+1}{}^{k_{n+1}}$ and indexed by all partitions $(k_1, \ldots, k_{n+1})$ of d. We rename the basis elements of V_d as $y_1, \ldots, y_\alpha$, with $\alpha = \binom{n+d}{d}$, according to the lexicographic ordering on the monomials of degree d in the variables $\mathsf{x}_1, \ldots, \mathsf{x}_{n+1}$. We repeat the process for the symmetric power $V_{(m-1)d}$, endowing it with the basis $z_1, \ldots, z_\beta$, with $\beta = \binom{n+(m-1)d}{(m-1)d}$.

The natural action of G extends linearly to the product $V_d \otimes V_{(m-1)d}$, which in turn splits as in (1.1). The highest weight vector of the irreducible representation $V_{(md-2)\lambda_1+\lambda_2}$ of highest weight $(md - 2)\lambda_1 + \lambda_2 = (md - 1)L_1 + L_2$ is $y_1 z_2 - y_2 z_1$.

On the other hand, G also acts on the symmetric power $S^m V_d$; with our notation, $S^m V_d$ has a basis defined by monomials of type $y_1^{h_1} y_2^{h_2} \cdots y_\alpha^{h_\alpha}$, indexed by all partitions $(h_1, \ldots, h_\alpha)$ of m. The only such monomial with highest weight $(md - 1)L_1 + L_2$ is $y_1^{m-1} y_2$, and this makes it (up to a constant) the only candidate for being the highest weight vector of $V_{(md-2)\lambda_1+\lambda_2}$. Notice however that

$$E_{12}(y_1^{m-1} y_2) = E_{12}(\mathsf{x}_1{}^{(m-1)d} \mathsf{x}_1{}^{d-1} \mathsf{x}_2) = \mathsf{x}_1{}^{md} = y_1^m \neq 0,$$

hence $(md - 1)L_1 + L_2$ cannot be a highest weight in $S^m V_d$ and $V_{(md-2)\lambda_1+\lambda_2}$ cannot appear as irreducible summand in its decomposition.

For the second part of the statement, notice that

$$\Psi_{m,d} \circ \Phi_{m,d} \in \mathrm{Hom}(V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}V_d}, \mathcal{O}_{\mathbb{P}V_d}(m))^G \simeq \mathrm{Hom}(V_{(md-2)\lambda_1+\lambda_2}, S^m V_d)^G.$$

Finally, in the case of $\mathrm{SL}_2(\mathbb{C})$, recall that the Lie algebra $\mathfrak{g}$ of G is generated by X, Y and H, with $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$; the same proof as above applies with minor modifications which we omit. ∎

In the study of the sequences (1.2) and (1.4), it is useful to restrict them to the closed orbit of the G-action on $\mathbb{P}V_d$, namely the Veronese variety. For this, let $v_{n,d}$ be the Veronese map $\mathbb{P}V \to \mathbb{P}V_d$ given by the complete linear system $|\mathcal{O}_{\mathbb{P}V}(d)|$ and put

$$X_d := \mathrm{Im}(v_{n,d} : \mathbb{P}V \to \mathbb{P}V_d).$$

The idea behind this is that $\mathbb{P}V$ is a G-homogeneous space and the pull-back of $\mathcal{W}_{n,d}$ to $\mathbb{P}V$ is a G-homogeneous bundle. This remark enables to prove several results concerning this pull-back. The next observation is that many of our key statements extend to the whole ambient space $\mathbb{P}V_d$ by continuity and G-equivariance, because the G-orbit of any point of $\mathbb{P}V_d$ contains X_d in its Zariski closure.

Restricting (1.2) and (1.4) to X_d and pulling back via $\mathbb{P}V \simeq \mathbb{P}^n \to X_d$, we get the sequences
(2.1)

$$0 \to \mathcal{V}_{m,d} \to V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\phi_{m,d}} V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^n}(d) \xrightarrow{\psi_{m,d}} \mathcal{O}_{\mathbb{P}^n}(md) \to 0,$$

and

$$(2.2) \quad 0 \to \mathcal{V}_{m,d} \to V_{md-2} \otimes \mathcal{O}_{\mathbb{P}^1} \xrightarrow{\phi_{m,d}} V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^1}(d) \xrightarrow{\psi_{m,d}} \mathcal{O}_{\mathbb{P}^1}(md) \to 0,$$

respectively, where in both cases $\mathcal{V}_{m,d} \simeq \mathcal{W}_{n,d}|_{X_d}$ is a vector bundle on $\mathbb{P}^n$ which is homogeneous under G, the map $\phi_{m,d}$ is the pull-back to $\mathbb{P}^n$ of $\Phi_{m,d}$ and similarly $\psi_{m,d}$ is the pull-back of $\Psi_{m,d}$.

Studying these restricted sequences, we obtain the following.

Lemma 2.2. *The morphism* $\Psi_{m,d} \colon V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}V_d}(1) \longrightarrow \mathcal{O}_{\mathbb{P}V_d}(m)$ *is surjective.*

Proof. Let us consider a different Veronese embedding than before, namely $v_{n,(m-1)d}$, the one of degree $(m-1)d$. On the projective space $\mathbb{P}V_{(m-1)d}$ we have the Euler sequence with the natural surjection

$$V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}V_{(m-1)d}} \to \mathcal{O}_{\mathbb{P}V_{(m-1)d}}(1).$$

This surjection, pulled-back via $v_{n,(m-1)d}$ to $\mathbb{P}^n$, becomes the obvious epimorphism

$$V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}((m-1)d).$$

This map is precisely the morphism $\psi_{m,d}$, twisted by $\mathcal{O}_{\mathbb{P}V}(-d)$, so $\psi_{m,d}$ is surjective. The surjectivity of $\Psi_{m,d}$ follows, because the rank of the map can only increase with respect to the value on the closed orbit. ∎

Remark 2.3. The restricted morphism $\psi_{m,d}$ also corresponds to the natural surjection $\mathcal{P}^k(F) \twoheadrightarrow F$ of the *bundle of k-jets* (also known as *principal parts sheaf*) of a vector bundle F onto F itself. Indeed in [20] it is shown that the k-jets of line bundles on a projective space $\mathbb{P}V$ have the simple form $\mathcal{P}^k(\mathcal{O}_{\mathbb{P}V}(h)) = V_k \otimes \mathcal{O}_{\mathbb{P}V}(h-k)$. In particular, $\mathcal{P}^{(m-1)d}(\mathcal{O}_{\mathbb{P}V}(md)) = V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}V}(d)$.

Lemmas 2.1 and 2.2 entail that both sequences (1.2) and (1.4) are equivariant complexes, exact at the sides. In the next two sections we will show that exactness holds in the middle as well, and this will conclude the proof of our main results.

3. The case of binary forms

This section is devoted to the special case $G = \mathrm{SL}_2(\mathbb{C})$. We start by proving our main theorem in this case. In §5 we will draw a few more remarks about our bundles of rank $d-1$ on $\mathbb{P}^d$, which we see as the space of binary forms of degree d.

3.1. The equivariant matrix of linear forms

Recall that the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is generated by X, Y and H, with $[H, X] = 2X$, $[H, Y] = -2Y$ and $[X, Y] = H$; moreover, we consider V as being generated by x_1 and x_2, so that V_d has a basis defined by $y_k = x_1^{d-k+1} x_2^{k-1}$, for $k = 1, \ldots, d+1$. Similarly, $V_{(m-1)d}$ has a basis defined by $z_h = x_1^{(m-1)d-h+1} x_2^{h-1}$, for $h = 1, \ldots, (m-1)d + 1$.

An element of weight $md - 2j$ in $V_d \otimes V_{(m-1)d}$ is a linear combination

$$v^{(j)} = \sum_{i=1}^{j+1} c_{i,j}\, y_{j-i+2}\, z_i,$$

and acting with Y on $v^{(j-1)}$, one obtains

$$Y\big(v^{(j-1)}\big) = (d - j + 1)c_{1,j-1}\, y_{j+1} z_1$$

$$+ \sum_{i=2}^{j} \big[(d - j + i)c_{i,j-1} + ((m - 1)d - i + 2)c_{i-1,j-1}\big] y_{j-i+2}\, z_i$$

$$+ ((m - 1)d - j + 1)c_{j,j-1}\, y_1 z_{j+1}.$$

As the highest weight vector of V_{md-2} is $y_1 z_2 - y_2 z_1$, a basis of the representation V_{md-2} in $V_d \otimes V_{(m-1)d}$ is given by the set $\{v^{(1)}, \dots, v^{(md-1)}\}$ with coefficients $c_{i,j}$ defined by $c_{1,1} = -1$, $c_{2,1} = 1$ and for $2 \leqslant j \leqslant md - 1$,

$$(3.1) \qquad c_{i,j} = \begin{cases} (d - j + 1)c_{1,j-1} & i = 1, \\ (d - j + i)c_{i,j-1} + ((m - 1)d - i + 2)c_{i-1,j-1} & 2 \leqslant i \leqslant j, \\ ((m - 1)d - j + 1)c_{j,j-1} & i = j + 1. \end{cases}$$

Resolving the recurrence relation, one finds

$$c_{i,j} = \begin{cases} -\prod_{k=2}^{j}(d - k + 1), & i = 1, \\ -\Big[\prod_{k=1}^{i-2}((m - 1)d - k)\Big]\Big[\prod_{k=1}^{j-i+1}(d - k + 1)\Big]\big[\binom{j-1}{i-1}m - \binom{j}{i-1}\big], & i \geqslant 2. \end{cases}$$

These coefficients can be simplified when describing the matrix $\Phi_{m,d}$. In fact, given $j \geqslant d + 2$, the coefficient $c_{i,j}$ vanishes if $i \leqslant j - d$ or $i \geqslant j + 2$, and for $j - d + 1 \leqslant i \leqslant j + 1$ all coefficients $c_{i,j}$ are multiples of the product $\prod_{k=1}^{j-d-1}((m - 1)d - k)$ that can be removed. Moreover, a basis of the representation V_{md-2} in $V_d \otimes V_{(m-1)d}$ can also be obtained starting from the lowest weight vector $y_d z_{(m-1)d+1} - y_{d+1} z_{(m-1)d}$ and acting with X. In this way, one obtains a recurrence relation analogous to (3.1) that reveals a symmetry of the coefficients appearing in the matrix $\Phi_{m,d}$. Taking into account these remarks, the matrix representing $\Phi_{m,d}$ in the case of $G = \mathrm{SL}_2(\mathbb{C})$ is

$$\big(\Phi_{m,d}\big)_{i,j} = \begin{cases} \widetilde{c}_{i,j}\, y_{j-i+2}, & 1 \leqslant j - i + 2 \leqslant d + 1, \\ 0, & \text{otherwise}, \end{cases}$$

where, for $j \leqslant \lceil \frac{md-1}{2} \rceil$ and $i = 1$,

$$\widetilde{c}_{i,j} = \prod_{k=2}^{j}(d - k + 1);$$

for $j \leqslant \lceil \frac{md-1}{2} \rceil$ and $i = 1$,

$$\widetilde{c}_{i,j} = -\Big[\prod_{k=\max(1,j-d)}^{i-2}((m - 1)d - k)\Big]\Big[\prod_{k=1}^{j-i+1}(d - k + 1)\Big]\big[\binom{j-1}{i-1}m - \binom{j}{i-1}\big];$$

and, for $j > \lceil \frac{md-1}{2} \rceil$,

$$\widetilde{c}_{i,j} = -\widetilde{c}_{(m-1)d-i+2,md-j}.$$

3.2. Proof of Theorem B

We proceed in several steps, articulated along the next subsections, which we briefly outline here. First, we show exactness of the equivariant complex. Then, we pull-back to $\mathbb{P}^1$ via the Veronese embedding of $\mathbb{P}^1$ in $\mathbb{P}^d$ determined by the identification $\mathbb{P}^d = \mathbb{P}V_d$. We study the pull-back bundle $\mathcal{V}_{m,d}$ of $\mathcal{W}_{m,d}$ and prove that it is isomorphic to $V_{d-2} \otimes \mathcal{O}_{\mathbb{P}^1}((1-m)d)$. We finally deduce the stability of $\mathcal{W}_{m,d}$.

3.2.1. Exactness of the equivariant complex.
As mentioned above, Lemmas 2.1 and 2.2 entail that the sequence (1.4) is a complex, exact at the sides; in particular, $\Phi_{m,d}$ is a $((m-1)d+1) \times (md-1)$ matrix of linear forms in $d+1$ variables and whose rank is at most $(m-1)d$.

On the other hand, $\mathrm{rk}(\Phi_{m,d})$ is bounded below by the rank of $\Phi_{m,d}|_y$, where y is any point of the closed orbit, the rational normal curve X_d of degree d in $\mathbb{P}^d$. At the point $y = [1:0:\ldots:0] \in X_d$, the entries of the matrix $\Phi_{m,d}|_y$ are all zero except those on the subdiagonal, where the values are

$$\left(\Phi_{m,d}|_y\right)_{j+1,j} = \prod_{k=1}^{j-1}\left((m-1)d-k\right) \neq 0, \quad \forall\, j = 1,\ldots,(m-1)d.$$

Hence the rank is $(m-1)d$, which is what we wanted.

From (1.4) we compute that the rank of the vector bundle $\mathcal{W}_{m,d}$ is equal to $md-1-(m-1)d = d-1$, as required.

3.2.2. The pulled-back image bundle.
Recall the morphism of bundles $\psi_{m,d}$, defined on $\mathbb{P}^1$ as pull-back of $\Psi_{m,d}$ via the Veronese map $v_{1,d}\colon \mathbb{P}^1 \to \mathbb{P}^d$. We first study the image of $\psi_{m,d}$, so put $\mathcal{L}_{m,d} = \mathrm{Im}(\psi_{m,d})$. This is a vector bundle defined by the exact sequence

$$0 \to \mathcal{L}_{m,d} \to V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^1}(d) \xrightarrow{\psi_{m,d}} \mathcal{O}_{\mathbb{P}^1}(md) \to 0.$$

We first show that

(3.2) $$\mathcal{L}_{m,d} \simeq V_{(m-1)d-1} \otimes \mathcal{O}_{\mathbb{P}^1}(d-1).$$

To see this, note that for each $t \geqslant 0$ the map $\psi_{m,d}$ induces an equivariant surjection

$$V_{(m-1)d} \otimes \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}(t)) \to \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^1}((m-1)d+t)),$$

all the surjections for $t \geqslant 1$ being induced by the case $t = 0$, which in turn is obvious. Therefore for $t = 0$ we have $\mathrm{H}^0(\mathcal{L}_{m,d}(-d)) = 0$ while, for $t = 1$, using (1.3), we get

$$\mathrm{H}^0(\mathcal{L}_{m,d}(1-d)) \simeq V_{(m-1)d-1},$$

which easily implies (3.2). Next, we rewrite the exact sequence defining $\mathcal{V}_{m,d}$ as

$$0 \to \mathcal{V}_{m,d} \to V_{md-2} \otimes \mathcal{O}_{\mathbb{P}^1} \to V_{(m-1)d-1} \otimes \mathcal{O}_{\mathbb{P}^1}(d-1) \to 0.$$

(a) The matrix $\Phi_{4,4}$.

(b) The matrix $\Phi_{3,5}$.

Figure 1. Two examples of the matrix $\Phi_{m,d}$ in the case of $G = \mathrm{SL}_2(\mathbb{C})$.

3.2.3. The pulled-back kernel bundle. Next we want to prove

$$(3.3) \qquad \mathcal{V}_{m,d} \simeq V_{d-2} \otimes \mathcal{O}_{\mathbb{P}^1}((1-m)d).$$

To check this, note that (1.3) gives, using the same argument that we mentioned to define $\Psi_{m,d}$ (that is, a pull-back via a Veronese embedding of a natural surjection, see Lemma 2.2 and Remark 2.3), an equivariant map

$$\vartheta_{m,d} : V_{d-2} \otimes \mathcal{O}_{\mathbb{P}^1}((1-m)d) \to V_{md-2} \otimes \mathcal{O}_{\mathbb{P}^1}.$$

The highest weight vector defining the irreducible representation V_{d-2} as a direct summand of $V_{(m-1)d} \otimes V_{md-2}$ is the following:

$$\sum_{i=0}^{(m-1)d} (-1)^i \binom{(m-1)d}{i} x_1{}^i x_2{}^{(m-1)d-i} \otimes x_1{}^{md-2-i} x_2{}^i.$$

Acting k times with Y gives

$$\sum_{i=0}^{(m-1)d} (-1)^i \binom{(m-1)d}{i} x_1{}^i x_2{}^{(m-1)d-i} \otimes x_1{}^{md-2-i-k} x_2{}^{i+k}.$$

Hence, the map $\vartheta_{m,d}$ is described by the following matrix:

$$
\begin{bmatrix}
C & & & & & \\
 & C & & & & \\
 & & C & & & \\
 & & & \ddots & & \\
 & & & & C & \\
 & & & & & C
\end{bmatrix}
\qquad \text{with } C =
\begin{bmatrix}
x_2{}^{(m-1)d} \\
-\binom{(m-1)d}{1} x_1 x_2{}^{(m-1)d-1} \\
\vdots \\
(-1)^i \binom{(m-1)d}{i} x_1{}^i x_2{}^{(m-1)d-i} \\
\vdots \\
(-1)^{(m-1)d-1} \binom{(m-1)d}{(m-1)d-1} x_1{}^{(m-1)d-1} x_2 \\
(-1)^{(m-1)d} x_1{}^{(m-1)d}
\end{bmatrix}
$$

It is clear that this matrix is injective at one (and hence at any) point of $\mathbb{P}^1$.

Now, again (1.3) says that $V_{(m-1)d-1} \otimes V_{md-1} \simeq V_{2md-d-2} \oplus \cdots \oplus V_d$, so the representation V_{d-2} does not occur in this direct sum. Then, by the same argument as in end of the proof of Lemma 2.1, the composition of maps

$$V_{d-2} \otimes \mathcal{O}_{\mathbb{P}^1}((1-m)d) \to V_{md-2} \otimes \mathcal{O}_{\mathbb{P}^1} \to V_{(m-1)d-1} \otimes \mathcal{O}_{\mathbb{P}^1}(d-1)$$

is zero. Therefore, we get an injective map $V_{d-2} \otimes \mathcal{O}_{\mathbb{P}^1}((1-m)d) \to \mathcal{V}_{m,d}$. Therefore, this map is an isomorphism, because it is clear from exactness of the equivariant complex that $V_{d-2} \otimes \mathcal{O}_{\mathbb{P}^1}((1-m)d)$ and $\mathcal{V}_{m,d}$ are vector bundles on $\mathbb{P}^1$ having the same rank and first Chern class.

3.2.4. Proof of stability. The isomorphism (3.3) implies plainly that $\mathcal{W}_{m,d}$ is slope-semistable. Indeed, the pull-back of a subbundle of $\mathcal{W}_{m,d}$ having strictly higher slope than $\mathcal{W}_{m,d}$ would be a subbundle of $V_{d-2} \otimes \mathcal{O}_{\mathbb{P}^1}(d(1-m))$ again with strictly higher slope, which is absurd because this bundle is semistable on $\mathbb{P}^1$.

Now we prove that $\mathcal{W}_{m,d}$ is actually slope-stable. Indeed, assume $\mathcal{W}_{m,d}$ has a non-trivial filtration by subsheaves, whose quotients are slope-stable sheaves. We write the associated graded object in the form

$$\mathrm{gr}(\mathcal{W}_{m,d}) = \bigoplus_{i=1}^{s} \mathcal{F}_i^{\oplus r_i},$$

where r_i are positive integers and $\mathcal{F}_1, \ldots, \mathcal{F}_s$ are stable sheaves on $\mathbb{P}^d$, with $\mathcal{F}_i$ not isomorphic to $\mathcal{F}_j$ for $i \neq j$. For any $g \in G$, we get a linear automorphism of $\mathbb{P}^d$, which we still denote by g, and an isomorphism $g^*(\mathcal{W}_{m,d}) \simeq \mathcal{W}_{m,d}$ which in turn induces an automorphism $\mathrm{gr}(\mathcal{W}_{m,d}) \to \mathrm{gr}(\mathcal{W}_{m,d})$. Since $\mathcal{F}_i \not\simeq \mathcal{F}_j$ for $i \neq j$, we get isomorphisms $\mathcal{F}_i^{\oplus r_i} \to \mathcal{F}_i^{\oplus r_i}$, and since each $\mathcal{F}_i$ is stable, all such morphisms are of the form $g_i \otimes \mathrm{id}_{\mathcal{F}_i}$, for some linear isomorphism $g_i \colon \mathbb{C}^{r_i} \to \mathbb{C}^{r_i}$. In other words, there are G-representations $R_1, \ldots, R_s$ such that

$$\mathrm{gr}(\mathcal{W}_{m,d}) \simeq \bigoplus_{i=1}^{s} R_i \otimes \mathcal{F}_i.$$

Let us now look at the pull-back of the filtration to $\mathbb{P}^1$ via $v_{1,d}$. By the semi-stability of $\mathcal{V}_{m,d}$ and the homogeneity of the $\mathcal{F}_i$, each $\mathcal{F}_i$ pulls back to $T_i \otimes \mathcal{O}_{\mathbb{P}^1}(d(1-m))$, for some G-representation T_i. So the associated graded object is of the form

$$\mathrm{gr}(\mathcal{V}_{m,d}) \simeq \bigoplus_{i=1}^{s} R_i \otimes T_i \otimes \mathcal{O}_{\mathbb{P}^1}(d(1-m)).$$

Therefore, (3.3) gives $s = 1$ and $R_1 = V_{d-2}$ and $T_1 \simeq \mathbb{C}$, or $T_1 \simeq V_{d-2}$ and $R_1 \simeq \mathbb{C}$. We want to exclude the former case, the latter corresponding to the fact that $\mathcal{W}_{m,d}$ is stable. But if $R_1 = V_{d-2}$ and $T_1 \simeq \mathbb{C}$, we get that $\mathcal{F}_1$ is a line bundle on $\mathbb{P}^d$ which implies immediately $\mathcal{W}_{m,d} \simeq V_{d-2} \otimes \mathcal{O}_{\mathbb{P}^d}(1-m)$. However, a straightforward computation on the equivariant complex shows that the second Chern class of $\mathcal{W}_{m,d}(m-1)$ is non-zero, so this is impossible (see Lemma 5.2 for more details on the Chern classes of $\mathcal{W}_{m,d}$). The proof of Theorem B is thus complete.

Remark 3.1. For $d = 3$, our result agrees with the resolution-theoretic approach to the construction of $SL_2(\mathbb{C})$-equivariant instantons achieved in [11]. In fact, in that paper the classification of instantons on $\mathbb{P}^3$ which are invariant for any linear action of $SL_2(\mathbb{C})$ on $\mathbb{P}^3$ was completed. To our knowledge, no such classification is available for equivariant vector bundle on higher-dimensional projective spaces.

We postpone to §5 a more detailed study of our $SL_2(\mathbb{C})$-equivariant bundles, where we will show in Proposition 5.3 that, as soon as $d \geqslant 4$ and $m \geqslant 3$, the $\mathcal{W}_{m,d}$'s are not isomorphic to any previously known rank $d - 1$ bundle on $\mathbb{P}^d$.

4. The general case

This section is devoted to the proof of Theorem A. The steps are similar to what is done in the previous case of $G = SL_2(\mathbb{C})$; however, the situation is much more complicated this time, as we have to deal with representations whose highest weight is not just a multiple of λ_1, but a linear combination of λ_1, λ_2 and in some cases also λ_3. We exploit the theory of representations of Lie groups and of quiver representations.

We illustrate the proof in the next subsections; we start by recalling some facts about the equivalence of categories between G-homogeneous bundles and quiver representations. Then, similarly to what we did in the previous section, we work on the pulled-back bundle.

4.1. Quiver representations

The category of G-homogeneous bundles on $\mathbb{P}^n$ is naturally equivalent to the category of representations of the parabolic subgroup P of G such that $\mathbb{P}^n \simeq G/P$; this equivalence sends indecomposable bundles to irreducible representations, see for example [15], where the author attributes this result to Matsushima, though it was probably known before. An excellent reference is also [18].

The semisimple part of P, denoted by R, is a copy of $SL_n(\mathbb{C})$; an irreducible representation of R can be extended to P by letting the unipotent part act trivially. Homogenous bundles corresponding to these irreducible representation of R are called *completely reducible*. A completely reducible bundle is canonically determined by its P-dominant weight, which is of the form $\lambda = a_1 \lambda_1 + a_2 \lambda_2 \cdots + a_n \lambda_n$, where $a_1 \in \mathbb{Z}$ and $a_2, \ldots, a_n \in \mathbb{N}$ and is therefore denoted by $\mathcal{E}_\lambda \otimes \mathbb{C}^{k(\lambda)}$, where the tensor product accounts for its multiplicity. If the multiplicity is 1, we simply write $\mathcal{E}_\lambda$ for $\mathcal{E}_\lambda \otimes \mathbb{C}$.

The natural numbers a_i for $i \geqslant 2$ are the coefficients in the basis $(\lambda_2, \ldots, \lambda_n)$ of the dominant weight of the representation of R corresponding to the bundle. Notice that, given a weight λ as above, if we set $\mu_i := a_i + \cdots + a_n$ for $i = 1, \ldots, n$, we have

$$\mathcal{E}_\lambda \simeq \Gamma^\mu \Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}\left(\textstyle\sum_{i=1}^n \mu_i\right),$$

where Γ^μ is the Schur functor defined by the partition $\mu = (\mu_2 \geqslant \cdots \geqslant \mu_n)$ and $\Omega_{\mathbb{P}^n}$ is the cotangent bundle. In particular,

$$\mathcal{E}_{k\lambda_1 + \nu} \simeq \mathcal{E}_\nu(k), \quad \mathcal{E}_{\lambda_n} = T_{\mathbb{P}^n}(-1), \quad \text{and} \quad \mathcal{E}_{\lambda_2} = \Omega_{\mathbb{P}^n}(2).$$

A fundamental fact is that, when λ is also G-dominant, that is, when $a_1 \geqslant 0$ in the definition of λ above, the bundle $\mathcal{E}_\lambda$ is globally generated and satisfies

$$\mathrm{H}^0(\mathbb{P}^n, \mathcal{E}_\lambda)^G \simeq V_\lambda.$$

Any G-homogeneous bundle $\mathcal{E}$ on $\mathbb{P}^n$ admits a filtration of the form

(4.1) $$0 = \mathcal{E}^s \subset \mathcal{E}^{s-1} \subset \cdots \subset \mathcal{E}^1 \subset \mathcal{E}^0 = \mathcal{E},$$

where $\mathcal{F}^k = \mathcal{E}^{k-1}/\mathcal{E}^k$ is completely reducible for all $k = 1, \ldots, s$. We write

$$\mathrm{gr}(\mathcal{E}) = \bigoplus_{k=1}^s \mathcal{F}^k,$$

the *graded homogeneous vector bundle* associated with $\mathcal{E}$.

We can compute the graded bundle associated with $\Gamma^\sigma V \otimes \mathcal{O}_{\mathbb{P}^n}$ for any partition $\sigma = (\sigma_1 \geqslant \cdots \geqslant \sigma_n)$. The result is probably folklore, but the reader can find a detailed proof in [21], Proposition 3. We have that

(4.2) $$\mathrm{gr}(\Gamma^\sigma V \otimes \mathcal{O}_{\mathbb{P}^n}) = \bigoplus_\nu \Gamma^\nu \Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(|\sigma|),$$

the sum extended over all ν obtained from σ by removing any number of boxes from its Young diagram, with no two in any column.

There is a third category that is equivalent to the two above, namely the category of finite dimensional representations of a certain quiver $\mathcal{Q}_{\mathbb{P}^n}$ with relations, see [6, 7, 19] and the already quoted [21] among many other references. The vertices of $\mathcal{Q}_{\mathbb{P}^n}$ are given by R, the semisimple part of P, and correspond to completely reducible bundles $\mathcal{E}_\lambda$; we label them with their highest weight λ. The arrows encode the unipotent action, that is, the data of the extensions of $\mathcal{F}^k$ and $\mathcal{F}^j$ induced by the filtration (4.1), for $k, j = 1, \ldots, s$; we label them with the weights $\xi_1, \ldots, \xi_n$ of the cotangent bundle $\Omega_{\mathbb{P}^n}$. The relations on $\mathcal{Q}_{\mathbb{P}^n}$ are the commutative ones.

4.2. Exactness of the equivariant complex

As noticed before, Lemmas 2.1 and 2.2 together entail that the sequences (1.2) and (2.1) are complexes, exact at the sides; once again, we need to show that exactness holds in the middle as well. We work on the restricted sequence (2.1), then extend the result to (1.2) by semicontinuity.

We split (2.1) into two short sequences

(4.3) $$0 \to \mathcal{V}_{m,d} \to V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\phi_{m,d}} \mathcal{L}_{m,d},$$

where $\mathcal{L}_{m,d}$ is a G-homogeneous bundle on $\mathbb{P}^n$, defined as the kernel of $\psi_{m,d}$ by

(4.4) $$0 \to \mathcal{L}_{m,d} \to V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^n}(d) \xrightarrow{\psi_{m,d}} \mathcal{O}_{\mathbb{P}^n}(md) \to 0.$$

We study the bundle $\mathscr{L}_{m,d}$ more in detail; rewrite (4.4) twisted by $\mathcal{O}_{\mathbb{P}^n}(-d)$:

(4.5) $$0 \to \mathscr{L}_{m,d}(-d) \to V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}((m-1)d) \to 0.$$

Two immediate remarks are in order. First, the bundle $\mathscr{L}_{m,d}^*(d)$ is generated by an irreducible module of global sections, and this implies that $\mathscr{L}_{m,d}$ is indecomposable. Second, the map induced on global sections by the surjection $V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow \mathcal{O}_{\mathbb{P}^n}((m-1)d)$ is an isomorphism, and from this we deduce the cohomology vanishings

$$\mathrm{H}^i(\mathbb{P}^n, \mathscr{L}_{m,d}(-d)) = 0, \quad \text{for all } i \in \mathbb{N}.$$

From formula (4.2) we compute

(4.6) $$\mathrm{gr}(V_\ell \otimes \mathcal{O}_{\mathbb{P}^n}) \simeq \bigoplus_{k=0}^{\ell} \mathcal{E}_{(\ell-2k)\lambda_1 + k\lambda_2}.$$

Combining (4.5) and (4.6) and recalling that $\mathcal{O}_{\mathbb{P}^n}(m) = \mathcal{E}_{m\lambda_1}$, we compute the graded bundle associated to $\mathscr{L}_{m,d}(-d)$ and from it,

(4.7) $$\mathrm{gr}(\mathscr{L}_{m,d}) \simeq \bigoplus_{k=1}^{(m-1)d} \mathcal{E}_{(md-2k)\lambda_1 + k\lambda_2}.$$

Therefore, in the filtration (4.1) of $\mathscr{L}_{m,d}$, the index s is $s = (m-1)d$ and the completely reducible quotients $\mathcal{F}^k$ are

$$\mathcal{F}^k = \mathscr{L}_{m,d}^{k-1}/\mathscr{L}_{m,d}^k \simeq \mathcal{E}_{((m-1)d-2k)\lambda_1 + k\lambda_2}.$$

Notice that all summands in (4.7) are completely reducible homogeneous bundles and that they all appear with multiplicity 1 in the decomposition. Moreover, the first summand (that is, $k = 1$) is the only one satisfying

$$\mathrm{H}^0(\mathbb{P}^n, \mathcal{E}_{(md-2)\lambda_1 + \lambda_2}) \simeq V_{(md-2)\lambda_1 + \lambda_2}.$$

Let us now look at the associated quiver representation; we will denote by $[\mathcal{E}]$ the representation associated to a homogeneous bundle $\mathcal{E}$. Computing the action of the nilpotent part of the parabolic P, we see that the support of both quiver representations $[V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^n}]$ and $[\mathscr{L}_{m,d}]$ is connected with all arrows in the same direction, namely the one associated with the first weight ξ_1 of the cotangent bundle $\Omega_{\mathbb{P}^n}$. In other words, the support of these two quiver representations is an A_p-type quiver contained in
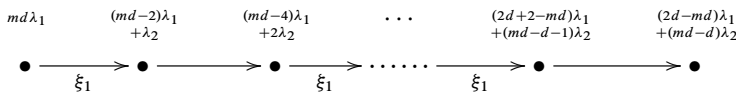
The three quiver representations $[V_{(m-1)d} \otimes \mathcal{O}_{\mathbb{P}^n}]$, $[\mathscr{L}_{m,d}]$ and $[\mathcal{O}_{\mathbb{P}^n}(md)]$ associated to the bundles in (4.4) have dimension vectors $[1\,1\,1\ldots1\,1]$, $[0\,1\,1\ldots1\,1]$ and $[1\,0\,0\ldots0\,0]$ respectively.

Now consider the two equivariant morphisms

$$V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{\phi_{m,d}} \mathcal{L}_{m,d} \xrightarrow{\pi_{m,d}} \mathcal{E}_{(md-2)\lambda_1+\lambda_2}$$

and their composition. The induced map on global sections is an equivariant map of irreducible representations $V_{(md-2)\lambda_1+\lambda_2} \to V_{(md-2)\lambda_1+\lambda_2}$, so it is either zero or an isomorphism. If it were zero, then $\phi_{m,d}$ would factor through the kernel $\mathcal{K}_{m,d}$ of $\pi_{m,d}$, whose graded object is

(4.8)
$$\mathrm{gr}(\mathcal{K}_{m,d}) \simeq \bigoplus_{k=2}^{(m-1)d} \mathcal{E}_{(md-2k)\lambda_1+k\lambda_2}.$$

Hence we would get a non-zero map from an irreducible G-module to the space of global sections of a bundle whose graded object has no summands with this particular G-module as its space of global sections, a contradiction.

We conclude that the composition $\pi_{m,d} \circ \phi_{m,d}$ induces an isomorphism on global sections. Therefore, it is a surjective morphism of sheaves because $\mathcal{E}_{(md-2)\lambda_1+\lambda_2}$ is completely reducible of multiplicity 1 and associated with a dominant weight, and is thus globally generated.

Now let us finally prove that $\phi_{m,d} \colon V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{L}_{m,d}$ is surjective and hence that the sequence (2.1) is exact. Setting $Q_{m,d} = \mathrm{Coker}\,\phi_{m,d}$, we have the following commutative exact diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}'_{m,d} & \longrightarrow & V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}^n} & \longrightarrow & \mathcal{E}_{(md-2)\lambda_1+\lambda_2} & \longrightarrow & 0 \ , \\
& & \downarrow & & \downarrow{\scriptstyle\phi_{m,d}} & & \| & & \\
0 & \longrightarrow & \mathcal{K}_{m,d} & \longrightarrow & \mathcal{L}_{m,d} & \xrightarrow{\pi_{m,d}} & \mathcal{E}_{(md-2)\lambda_1+\lambda_2} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & & & \\
& & Q'_{m,d} & \longrightarrow & Q_{m,d} & & & &
\end{array}
$$

where all maps are equivariant, $\mathcal{K}'_{m,d}$ is defined as the kernel of the surjection $\pi_{m,d} \circ \phi_{m,d}$ above, and $Q'_{m,d}$ is the cokernel of the induced morphism $\mathcal{K}'_{m,d} \to \mathcal{K}_{m,d}$. The snake lemma implies that the map $Q'_{m,d} \to Q_{m,d}$ is an isomorphism.

Since $Q'_{m,d}$ is an equivariant quotient of $\mathcal{K}_{m,d}$ and in view of (4.8), the completely reducible bundles occurring in $\mathrm{gr}(Q'_{m,d})$ are all of the form $\mathcal{E}_{(md-2k)\lambda_1+k\lambda_2}$, for some $k \in \{2, \ldots, (m-1)d\}$. In particular for some $k \geqslant 2$ we get a surjection

(4.9)
$$\mathcal{L}_{m,d} \xrightarrow{\hspace{2cm}} \mathcal{E}_{(md-2k)\lambda_1+k\lambda_2}.$$

$$[0,1,1,1,\ldots,1,1] \qquad [0,0,0,1,\ldots,0,0]$$

Set $\mathcal{G}^k := \mathcal{L}_{m,d}/\mathcal{L}^k$, with graded bundle

$$\mathrm{gr}(\mathcal{G}^k) \simeq \bigoplus_{j=1}^{k} \mathcal{F}^j.$$

We observe that the surjection (4.9) necessarily factors through

$$\mathcal{L}_{m,d} \to \mathcal{G}^k \to \mathcal{F}^k \simeq \mathcal{E}_{(md-2k)\lambda_1 + k\lambda_2};$$

indeed the morphism $\mathcal{L}_{m,d} \to \mathcal{F}^k$ restricts to zero to $\mathcal{L}_{m,d}^k$ because

$$\mathrm{gr}(\mathcal{L}_{m,d}^k) \simeq \bigoplus_{\ell=k+1}^{(m-1)d} \mathcal{F}^\ell$$

and none of these summands $\mathcal{F}^\ell$ has non-trivial maps to $\mathcal{F}^k$ for $k \neq \ell$.

Now, by definition of the filtration, we have an exact sequence

$$0 \to \mathcal{F}^k \to \mathcal{G}^k \to \mathcal{G}^{k-1} \to 0,$$

and composing the injection $\mathcal{F}^k \hookrightarrow \mathcal{G}^k$ with the surjection $\mathcal{G}^k \twoheadrightarrow \mathcal{F}^k$ we obtain an isomorphism. Indeed, if the composition were zero, then the same would be true for map $\mathcal{G}^k \to \mathcal{F}^k$, as no other summand of $\mathrm{gr}(\mathcal{G}^k)$ maps non-trivially to $\mathcal{F}^k$.

We conclude that $\mathcal{G}^k \simeq \mathcal{F}^k \oplus \mathcal{G}^{k-1}$ and this in turn entails a splitting $\mathcal{L}_{m,d} \simeq \mathcal{G}^{k-1} \oplus \mathcal{E}^{k-1}$, for some $k \geq 2$, which is a contradiction because we have already proved that $\mathcal{L}_{m,d}$ is indecomposable.

4.3. Indecomposability and rank computation

Notice that the vector bundle $\mathcal{V}_{m,d}$ is indecomposable, again because its dual $\mathcal{V}_{m,d}^*$ is generated by an irreducible module of global sections and thus $\mathcal{W}_{m,d}$ is indecomposable, as well.

Moreover, $\mathrm{rk}(\mathcal{W}_{m,d}) = \mathrm{rk}(\mathcal{V}_{m,d})$ and this rank, using the dimension formula of [12], p. 224, equals

$$\dim V_{(md-2)\lambda_1+\lambda_2} - \dim V_{(m-1)d} + 1 = (m^2 d^2 - 1) - \binom{(m-1)d+2}{2} + 1.$$

4.4. Slope-stability and quiver μ-stability

To conclude the proof of Theorem A, we need to show that $\mathcal{W}_{m,d}$ is slope-stable. Once again, we prove our result on the closed orbit and exploit the connection between slope-stability of a homogeneous bundle and μ-stability "à la King" of the associated quiver representation, see [16].

Given a homogenous vector bundle $\mathcal{E}$ and its associated quiver representation $[\mathcal{E}]$, we define

$$\mu_{[\mathcal{E}]}(-) = c_1(\mathrm{gr}(\mathcal{E}))\,\mathrm{rk}(-) - \mathrm{rk}(\mathrm{gr}(\mathcal{E}))\,c_1(-),$$

and call the representation $[\mathcal{E}]$ μ-stable if for all subrepresentations $[\mathcal{E}']$ one has that $\mu_{[\mathcal{E}]}(\mathrm{gr}(\mathcal{E}')) \geq 0$ and $\mu_{[\mathcal{E}]}(\mathrm{gr}(\mathcal{E}')) = 0$ if and only $[\mathcal{E}']$ is either $[\mathcal{E}]$ or $[0]$.

From Theorem 7.2 in [19], we learn that $[\mathcal{V}_{m,d}]$ is μ-stable if and only if $\mathcal{V}_{m,d} = W \otimes \mathcal{F}$, with $\mathcal{F}$ a slope-stable homogeneous bundle and W an irreducible G-module. Now, if we had $\mathcal{V}_{m,d} = W \otimes \mathcal{F}$, then the resolution of $\mathcal{V}_{m,d}$ (better yet, of its dual)

would be given by the resolution of $\mathcal{F}$ tensored by the irreducible representation W, a contradiction with (2.1). Therefore to prove that $\mathcal{V}_{m,d}$ is slope-stable we only need to show that the associated representation $[\mathcal{V}_{m,d}]$ is μ-stable. For this, we study $[\mathcal{V}_{m,d}]$ and its subrepresentations.

Let us first suppose that $n \geqslant 3$. From the short exact sequence (4.3) we deduce the equality

$$\mathrm{gr}(V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}^n}) = \mathrm{gr}(\mathcal{V}_{m,d}) \oplus \mathrm{gr}(\mathcal{L}_{m,d}).$$

The graded bundle $\mathrm{gr}(\mathcal{L}_{m,d})$ was already computed in (4.7). Moreover, from formula (4.2) we have
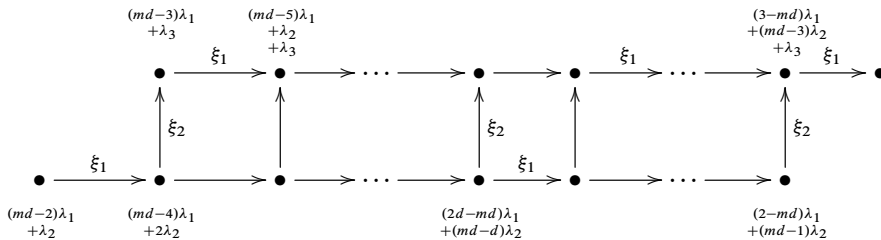
$$(4.10) \quad \mathrm{gr}(V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}^n}) \simeq \bigoplus_{k=1}^{md-1} (\mathcal{E}_{(md-1-2k)\lambda_1+(k-1)\lambda_2+\lambda_3} \oplus \mathcal{E}_{(md-2k)\lambda_1+k\lambda_2}),$$

so all in all,

$$(4.11) \quad \mathrm{gr}(\mathcal{V}_{m,d}) = \bigoplus_{k=1}^{md-1} \mathcal{E}_{(md-1-2k)\lambda_1+(k-1)\lambda_2+\lambda_3} \oplus \bigoplus_{k=md-d+1}^{md-1} \mathcal{E}_{(md-2k)\lambda_1+k\lambda_2},$$

where we remark that all summands in both (4.10) and (4.11) appear with multiplicity 1.

The subquiver of $\mathcal{Q}_{\mathbb{P}^n}$ corresponding to the support of the representations $[\mathcal{V}_{m,d}]$, $[V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}^n}]$ and $[\mathcal{L}_{m,d}]$ is contained in

The short exact sequence (4.3) is associated to the following sequence of quiver representations:

$$\mathcal{V}_{m,d} \hookrightarrow V_{(md-2)\lambda_1+\lambda_2} \otimes \mathcal{O}_{\mathbb{P}^n} \twoheadrightarrow \mathcal{L}_{m,d}$$

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 & \cdots & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{bmatrix}$$

Computing the action of the nilpotent part of the parabolic, it is easy to see that all maps of type $\mathbb{C} \to \mathbb{C}$ in the quiver representations mentioned above are non-zero. Indeed, if any of them were zero, then the support of the quiver representation would disconnect and hence the bundle would decompose, which we already know is impossible; this is apparent for the ξ_1-type arrows, whereas for the ξ_2-type arrows one needs to remember the commutativity relations holding on $\mathcal{Q}_{\mathbb{P}^n}$.

What do the subrepresentations of $[\mathcal{V}_{m,d}]$ look like? The graded bundle $\mathrm{gr}(\mathcal{V}_{m,d})$ has two types of summands; we call $\mathcal{A}_k := \mathcal{E}_{(md-1-2k)\lambda_1+(k-1)\lambda_2+\lambda_3}$ the first type, with $k =$

$1, \ldots, md - 1$ and $\mathcal{B}_k := \mathcal{E}_{(md-2k)\lambda_1 + k\lambda_2}$ the second one, $k = md - d + 1, \ldots, md - 1$. For the reader's convenience, we re-draw the support of $[\mathcal{V}_{m,d}]$:
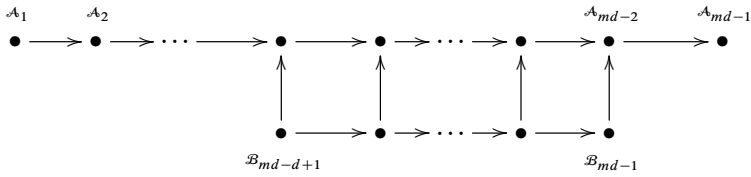
Any subrepresentation of $[\mathcal{V}_{m,d}]$ is either of the A_p-type,

$$\begin{bmatrix} 0 \ 0 & \cdots & 0 \ 0 & \cdots & 0 \ 0 \ 1 \\ & & 0 \ 0 & \cdots & 0 \ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \ 0 & \cdots & 0 \ 0 & \cdots & 0 \ 1 \ 1 \\ & & 0 \ 0 & \cdots & 0 \ 0 \end{bmatrix}, \text{ and so on until: } \begin{bmatrix} 1 \ 1 & \cdots & 1 \ 1 & \cdots & 1 \ 1 \ 1 \\ & & 0 \ 0 & \cdots & 0 \ 0 \end{bmatrix},$$

or else it is of the hook form

$$\begin{bmatrix} 0 \ 0 & \cdots & 0 \ 0 & \cdots & 0 \ 1 \ 1 \\ & & 0 \ 0 & \cdots & 0 \ 1 \end{bmatrix}, \quad \begin{bmatrix} 0 \ 0 & \cdots & 0 \ 0 & \cdots & 1 \ 1 \ 1 \\ & & 0 \ 0 & \cdots & 1 \ 1 \end{bmatrix},$$

and so on until

$$[\mathcal{V}_{m,d}] = \begin{bmatrix} 1 \ 1 & \cdots & 1 \ 1 & \cdots & 1 \ 1 \ 1 \\ & 1 \ 1 & \cdots & 1 \ 1 \end{bmatrix},$$

where again all maps of type $\mathbb{C} \to \mathbb{C}$ above are non-zero.

We have that $A_k = \Gamma^{k,1}\Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(md)$. Therefore, denoting by $\mu(\mathcal{E})$ the usual slope of sheaves, we compute that

$$1 \leqslant i < j \leqslant md - 1 \implies \mu(A_i) = md - \frac{(i+1)(n+1)}{n} > \mu(A_j).$$

Therefore any nonzero subrepresentation $[\mathcal{E}']$ of $[\mathcal{V}_{m,d}]$ of A_p-type will satisfy

$$c_1(\mathrm{gr}(\mathcal{V}_{m,d}))\,\mathrm{rk}(\mathcal{E}') - \mathrm{rk}(\mathrm{gr}(\mathcal{V}_{m,d}))\,c_1(\mathcal{E}') > 0.$$

For the hook type subrepresentations we compute, since $\mathcal{B}_k = S^k\Omega_{\mathbb{P}^n} \otimes \mathcal{O}_{\mathbb{P}^n}(md)$, that $\mu(\mathcal{B}_i) = md - \frac{i(n+1)}{n}$. In particular,

$$md - d + 1 \leqslant i < j \leqslant md - 1 \implies \mu(\mathcal{B}_i) = \mu(A_{i-1}) > \mu(A_{j-1}) = \mu(\mathcal{B}_j).$$

Therefore all these nonzero subrepresentation $[\mathcal{E}']$ of $[\mathcal{V}_{m,d}]$ satisfy $c_1(\mathrm{gr}(\mathcal{V}_{m,d}))\,\mathrm{rk}(\mathcal{E}') - \mathrm{rk}(\mathrm{gr}(\mathcal{V}_{m,d}))c_1(\mathcal{E}') \geqslant 0$. The fact that the only hook subrepresentation of $[\mathcal{V}_{m,d}]$ with $\mathrm{gr}(\mathcal{E}') = \mathrm{gr}(\mathcal{V}_{m,d})$ is $[\mathcal{V}_{m,d}]$ itself concludes the proof of stability for $n \geqslant 3$.

In the case $n = 2$, all summands of type A_k in formulas (4.10) and (4.11) must be substituted with summands of type $\mathcal{E}_{(md-1-2k)\lambda_1 + (k-1)\lambda_2}$ and the quiver representations (and subrepresentations) look the same as in the previous case. Also, $\mu(\mathcal{B}_k) = md - \frac{3}{2}k$ and $\mu(A_k) = md - \frac{3}{2}(k-1)$, therefore the same argument as above applies.

This concludes the proof of Theorem A.

5. Further remarks

Let us point out some cohomological features of our equivariant bundles. We denote by $H^p_*(\mathcal{F})$ the cohomology module $\oplus_{t \in \mathbb{Z}} H^p(\mathbb{P}^N, \mathcal{F}(t))$ of a coherent sheaf $\mathcal{F}$ over $\mathbb{P}^N$. This is an artinian module over the polynomial ring $R = \mathbb{C}[x_0, \ldots, x_N]$ if $\mathcal{F}$ is locally free and $0 < p < N$. Recall that, once the values of n and d are fixed, we have $N = \binom{d+n}{n} - 1$ and for all $m \geqslant 1$ the vector bundles $\mathcal{W}_{m,d}$ are defined on $\mathbb{P}^N$. Their cohomology modules are G-homogeneous. We assume throughout that $N \geqslant 3$ and $m \geqslant 2$.

Lemma 5.1. *The R-module $H^2_*(\mathcal{W}_{m,d})$ is cyclic, generated in degree $-m$, with $V_{(m-1)d}$ as space of minimal degree relations in degree 1. Also, $H^p_*(\mathcal{W}_{m,d}) = 0$ for $p = 3, \ldots, N-1$.*

Proof. We split the equivariant complex (1.2) into two short exact sequences and put $\mathcal{Z}_{m,d} = \mathrm{Im}(\Phi_{m,d})$. It is easy to see that $H^p_*(\mathcal{Z}_{m,d}) = 0$ for $p = 2, \ldots, N-1$. Moreover, we deduce that $H^1(\mathcal{Z}_{m,d}(t)) = 0$ for $t \geqslant -m-1$ and that $R(m) \simeq H^0_*(\mathcal{O}_{\mathbb{P}V_d}(m))$ surjects onto $H^1_*(\mathcal{Z}_{m,d})$, so that this module is cyclic, generated in degree $-m$.

Next, we have $H^0(\mathcal{Z}_{m,d}(t)) = 0$ for $t \leqslant -2$ and thus, for $t = -1$, we obtain that $H^0(\mathcal{Z}_{m,d}(-1))$ is the kernel of the map $V_{(m-1)d} \to S^{m-1}V_d$ induced by $\Psi_{m,d}$. By construction, this map is injective, so $H^0(\mathcal{Z}_{m,d}(-1)) = 0$ and therefore $H^1(\mathcal{Z}_{m,d}(-1))$ is the quotient of $S^{m-1}V_d$ by $V_{(m-1)d}$, which is to say, the module of relations of $H^1_*(\mathcal{Z}_{m,d})$ contains $V_{(m-1)d}$, sitting in degree 1 and no relations of smaller degree.

Finally, it is clear that $H^p_*(\mathcal{W}_{m,d}) \simeq H^{p-1}_*(\mathcal{Z}_{m,d})$ for all $p = 2, \ldots, N-1$, so the lemma is proved. ∎

We now focus our attention on the case of binary forms, when $n = 1$ and $N = d$; as mentioned in the introduction, few classes of indecomposable vector bundles of rank $d - 1$ on $\mathbb{P}^d$ are known, namely the classical mathematical instantons, Tango bundles, their generalizations by Cascini [10] and Bahtiti [3–5], and the Sasakura bundle of rank 3 on $\mathbb{P}^4$ [2].

In what follows we show that, as soon as $d \geqslant 4$, our equivariant bundles are new, except for a single case that we illustrate. Set $N_{m,d,k} = \binom{(m-1)d+k}{k}$.

Lemma 5.2. *The normalized bundle associated to $\mathcal{W}_{m,d}$ is $\mathcal{N}_{m,d} = \mathcal{W}_{m,d}(m-1)$ with Chern classes*

$$c_k(\mathcal{N}_{m,d}) = (-1)^k N_{m,d,k} m^k$$
$$+ \sum_{i=0}^{k-1} (-1)^i \frac{(m-1)d + (k-i)}{md - (k-i)} \binom{md-1}{k-i} N_{m,d,i} (m-1)^{k-i-1} m^{i+1}.$$

Proof. The statement follows from splitting (1.4) into two short exact sequences followed by a fairly cumbersome computation. ∎

Proposition 5.3. *For $n = 1$ and $d \geqslant 4$, the bundle $\mathcal{W}_{m,d}$ is not isomorphic to any of the following bundles, even up to dualizing and taking pull-backs by finite self-maps of $\mathbb{P}^d$:*

(1)   *mathematical instanton bundles;*

(2)   *weighted generalized Tango bundle, except if $m = 2$, in which case $\mathcal{W}^*_{2,d}(-1)$ is a Tango bundle;*

(3)   *the Sasakura bundle, for $d = 4$.*

Proof. Let us prove (1). The odd Chern classes of mathematical instanton bundles vanish; on the other hand, from Lemma 5.2 above we get that

$$c_3(\mathcal{N}_{m,d}) = -\tfrac{(d-3)}{3}m(m-1)(2m-1),$$

which does not vanish for $d \geqslant 4$ and $m \geqslant 2$.

Let us now prove (2). Given integers $\gamma > 0$ and $\alpha \geqslant \beta$, we refer to $\mathcal{F}_{\gamma,\alpha,\beta}$ as the vector bundle arising from Bahtiti's construction of generalized weighted Tango bundles; this includes Cascini's weighted Tango bundles, which are indeed homogeneous for $SL_2(\mathbb{C})$ acting on the space of binary forms. Let us recall that $\mathcal{F} = \mathcal{F}_{\gamma,\alpha,\beta}$ fits into the long exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^d}(-3\gamma) \to \bigoplus_{i=0}^{d} \mathcal{O}_{\mathbb{P}^d}(d\alpha + i(\beta - \alpha) - 2\gamma)$$

$$\to \bigoplus_{i=0}^{2d-1} \mathcal{O}_{\mathbb{P}^d}(2d\alpha + i(\beta - \alpha) - \gamma) \to \mathcal{F} \to 0.$$

The image of the middle map is usually denoted by $Q_{\gamma,\alpha,\beta}(-2\gamma)$.

We borrow the notation from the proof of Lemma 5.1 and use essentially the same argument and the fact that $d \geqslant 4$, to show that $\mathrm{H}^2_*(Q_{\gamma,\alpha,\beta}) = 0$ and consequently, that $\mathrm{H}^1_*(\mathcal{F}) = 0$. Also, by stability of $\mathcal{W}_{m,d}$ we have $\mathrm{H}^0(\mathcal{W}_{m,d}) = 0$ so that $\mathrm{H}^1(\mathcal{W}_{m,d})$ is the cokernel of an injective G-equivariant map $V_{md-2} \to \mathrm{H}^0(\mathcal{Z}_{m,d})$. By construction, we get

$$\mathrm{H}^1(\mathcal{W}_{m,d}) \simeq V_{md-4} \oplus \cdots \oplus V_{(m-2)d}.$$

We conclude that $\mathcal{F}_{\gamma,\alpha,\beta}$ is not isomorphic to $\mathcal{W}_{m,d}$.

We turn now our attention to the dual bundle $\mathcal{F}^*$ of $\mathcal{F}_{\gamma,\alpha,\beta}$. Let us assume that $\mathcal{F}^*$ is isomorphic to some twist of $\mathcal{W}_{m,d}$ and prove that this forces $\mathcal{W}_{m,d}$ to be a Tango bundle and $m = 2$, $\gamma = 1$, $\alpha = \beta = 0$. Because of Lemma 5.2 and since $c_1(\mathcal{F}) = 0$, already we should have $\mathcal{N}_{m,d} \simeq \mathcal{F}^*$.

By the long exact sequence above, using again the argument of Lemma 5.1, we see that the R-module $\mathrm{H}^2_*(\mathcal{F}^*)$ is cyclic, generated in degree -3γ. This, together with Lemma 5.1, implies $3\gamma = 2m - 1$.

Next, the statement on the relations of the module $\mathrm{H}^2_*(\mathcal{W}_{m,d})$ given in Lemma 5.1 implies that the kernel of $R(2m - 1) \to \mathrm{H}^2_*(\mathcal{N}_{m,d})$ has $(m - 1)d + 1$ generators in degree $-m$. On the other hand, the kernel of the epimorphism $R(3\gamma) \to \mathrm{H}^2_*(\mathcal{F}^*)$ has $d + 1$ generators, the ith generator being of degree $d\alpha + i(\beta - \alpha) - 2\gamma$ for $i = 0, \dots, d$. This implies that $m = 2$ (hence $\gamma = 1$) and that $d\alpha + i(\beta - \alpha) = 0$ for all i, which taken at $i = 0$ and $i = d$ says $\alpha = \beta = 0$, so $\mathcal{F}_{\gamma,\alpha,\beta}$ is a Tango bundle.

The converse implication is clear: indeed if $m = 2$ we get that $\mathcal{W}_{2,d}^*(-1)$ is a Tango bundle on $\mathbb{P}^d$ and fits in the dual of the exact sequence in [10], Section 2.

Finally, let us consider the case of finite self-maps f of $\mathbb{P}^d$. If f is defined by homogeneous polynomials of degree e, then the resolutions of $f^*\mathcal{W}_{m,d}$ and $f^*\mathcal{F}^*_{\gamma,\alpha,\beta}$ are pull-back by f of the resolutions of $\mathcal{W}_{m,d}$ and $\mathcal{F}^*_{\gamma,\alpha,\beta}$. Note that a line bundle of the

form $\mathcal{O}_{\mathbb{P}^d}(p)$ appearing in any of these resolutions is pulled-back by f to $\mathcal{O}_{\mathbb{P}^d}(ep)$. So our argument excluding that $\mathcal{W}_{m,d}$ and $\mathcal{F}^*_{\gamma,\alpha,\beta}$ are isomorphic remains valid as all factors appearing in that argument get multiplied by e. Also, given our map f we have

$$f_*(\mathcal{O}_{\mathbb{P}^d}) \simeq \bigoplus_{i=0}^{s} \mathcal{O}_{\mathbb{P}^d}(a_i),$$

for some integers s and $0 = a_1 > a_2 \geqslant \cdots \geqslant a_s$. By the projection formula, we exclude directly that $\mathcal{W}_{m,d} \simeq \mathcal{F}_{\gamma,\alpha,\beta}$, as $\mathrm{H}^1_*(f^*\mathcal{W}_{m,d}) \neq 0$ but $\mathrm{H}^1_*(f^*\mathcal{F}_{\gamma,\alpha,\beta}) = 0$.

Finally we prove (3). Let $\mathcal{S}$ denote the Sasakura bundle; from the monadic description given in [2] we compute that the cohomology module $\mathrm{H}^2_*(\mathcal{S})$ is generated in degree -4. The cohomology module $\mathrm{H}^2_*(\mathcal{S}^*)$ of the dual bundle is isomorphic to the previous one up to a shift and is generated in degree 1. Applying the same reasoning as in part (2), we see that Lemma 5.1 implies that $2m - 1 = -4$ in the case of $\mathcal{S}$ and $m = 1$ in the case of $\mathcal{S}^*$ and none of these two are possible. ∎

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