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Variation of the uncentered maximal characteristic function

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Abstract. Let M be the uncentered Hardy–Littlewood maximal operator, or the dyadic maximal operator, and let $d \geq 1$. We prove that for a set $E \subset \mathbb{R}^d$ of finite perimeter, the bound $\operatorname{var} \mathrm{M}1_E \leq C_d \operatorname{var} 1_E$ holds. We also prove this for the local maximal operator.

Introduction

The uncentered Hardy–Littlewood maximal function of a non-negative locally integrable function f is given by

$$\mathrm{M}f(x) = \sup_{B \ni x} \frac{1}{\mathcal{L}(B)} \int_B f,$$

where the supremum is taken over all open balls $B \subset \mathbb{R}^d$ that contain x. Various versions of this maximal operator have been investigated. There is the (centered) Hardy–Littlewood maximal operator, where the supremum is taken only over those balls that are centered in x, or the dyadic maximal operator, which maximizes over dyadic cubes instead of balls. Those operators also have local versions, where for some open set $\Omega \subset \mathbb{R}^d$ the supremum is taken only over those balls or cubes that are contained in Ω. For example, the local dyadic maximal function with respect to Ω of $f \in L^1_{\mathrm{loc}}(\Omega)$ at $x \in \Omega$ is given by

$$\mathrm{M}f(x) = \sup_{x \in Q \subset \Omega} \frac{1}{\mathcal{L}(Q)} \int_Q f,$$

where the supremum is taken over all half open dyadic cubes $Q \subset \mathbb{R}^d$ with $x \in Q \subset \Omega$.

It is well known that many maximal operators are bounded on $L^p(\mathbb{R}^d)$ if and only if $p > 1$. The regularity of the maximal operator was first studied in [17], where Kinnunen proved for the Hardy–Littlewood maximal operator that for $p > 1$ and $f \in W^{1,p}(\mathbb{R}^d)$ also the bound

$$\|\nabla \mathrm{M}f\|_p \leq C_{d,p} \|\nabla f\|_p$$

holds, from which it follows that the Hardy–Littlewood maximal operator is bounded on $W^{1,p}(\mathbb{R}^d)$.

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The proof combines the pointwise bound $|\nabla \mathrm{M} f| \leq \mathrm{M}|\nabla f|$ with the $L^p(\mathbb{R}^d)$-bound of the maximal operator. Since the maximal operator is not bounded on $L^1(\mathbb{R}^d)$, this approach fails for $p = 1$. For $p > 1$, the gradient $L^p(\mathbb{R}^d)$-bound or some corresponding version is valid for most maximal operators. However, so far no counterexamples have been found for $p = 1$. So in 2004, Hajłasz and Onninen posed the following question in [15]: for the Hardy–Littlewood maximal operator M, is $f \mapsto |\nabla \mathrm{M} f|$ a bounded mapping $W^{1,1}(\mathbb{R}^d) \to L^1(\mathbb{R}^d)$? This question for various maximal operators has since become a well known problem and has been the subject of lots of research. In one dimension, for $L^1(\mathbb{R})$ the gradient bound has already been proven in [26] by Tanaka for the uncentered maximal function, and later in [21] by Kurka for the centered Hardy–Littlewood maximal function. The latter proof turned out to be much more complicated. In [22], Luiro has proven the gradient bound for radial functions in $L^1(\mathbb{R}^d)$ for the uncentered maximal operator. More research on this question, and also more generally on the endpoint regularity of maximal operators, can be found in [1–3, 7–9, 14, 24]. However, so far the question has been essentially unsolved in dimensions larger than one for any maximal operator.

In this paper we prove that for M being the dyadic or the uncentered Hardy–Littlewood maximal operator, and $E \subset \mathbb{R}^d$ being a set with finite perimeter, we have

$$\operatorname{var} \mathrm{M} 1_E \leq C_d \operatorname{var} 1_E .$$

This answers the question of Hajłasz and Onninen in a special case, and is the first truly higher dimensional result for $p = 1$ to the best of our knowledge. We furthermore prove a localized version, as is stated in Theorems 1.2 and 1.3. The Hardy–Littlewood uncentered maximal function and the dyadic maximal function have in common that their level sets $\{\mathrm{M} f > \lambda\}$ can be written as the union of all balls/dyadic cubes X with $\int_X f > \lambda \mathcal{L}(X)$. Our proof relies on this. Since this is not true for the centered Hardy–Littlewood maximal function, a different approach has to be found for that maximal operator.

Also related topics for various exponents $1 \leq p \leq \infty$ have been studied, such as the continuity of the maximal operator in Sobolev spaces [5] and bounds for the gradient of other maximal operators, such as fractional, convolution, discrete, local and bilinear maximal operators [6, 10, 11, 16, 19, 20, 23, 25].

1. Preliminaries and main result

We work in the setting of sets of finite perimeter, as in Evans–Gariepy [12], Section 5. For a measurable set $E \subset \mathbb{R}^d$, we denote by $\mathcal{L}(E)$ its Lebesgue measure and by $\mathcal{H}^{d-1}(E)$ its $(d-1)$-dimensional Hausdorff measure. For an open set $\Omega \subset \mathbb{R}^d$, a function $f \in L^1_{\mathrm{loc}}(\Omega)$ is said to have locally bounded variation if for each open and compactly supported $U \subset \Omega$ we have

$$\sup\Big\{\int_U f \operatorname{div} \varphi : \varphi \in C^1_{\mathrm{c}}(U; \mathbb{R}^d),\ |\varphi| \leq 1\Big\} < \infty.$$

Such a function comes with a measure μ and a function $\nu\colon \Omega \to \mathbb{R}^d$ that has $|\nu| = 1$ μ-a.e. such that, for all $\varphi \in C^1_{\mathrm{c}}(\Omega; \mathbb{R}^d)$, we have

$$\int_\Omega f \operatorname{div} \varphi = \int_\Omega \varphi\, \nu \,\mathrm{d}\mu.$$

We define the variation of f in Ω by

$$\operatorname{var}_\Omega f = \mu(\Omega).$$

For a measurable set $E \subset \mathbb{R}^d$, we define the measure theoretic boundary by

$$\partial_* E = \left\{ x : \limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \setminus E)}{r^d} > 0,\ \limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \cap E)}{r^d} > 0 \right\}.$$

The following coarea formula is our strategy to approach the variation of the maximal function.

Lemma 1.1 (Theorem 5.9 in [12]). *Let $\Omega \subset \mathbb{R}^d$ be open. Let $f \in L^1_{\mathrm{loc}}(\Omega)$. Then*

$$\operatorname{var}_\Omega f = \int_{\mathbb{R}} \mathcal{H}^{d-1}(\partial_* \{ f > \lambda \} \cap \Omega) \, \mathrm{d}\lambda.$$

We say that measurable set $E \subset \mathbb{R}^d$ has locally finite perimeter if its characteristic function 1_E has locally bounded variation. For $f = 1_E$, we call $\operatorname{var}_\Omega 1_E$ the perimeter of E and ν from above the outer normal of E. Lemma 1.1 implies

$$\operatorname{var}_\Omega 1_E = \mathcal{H}^{d-1}(\partial_* E \cap \Omega).$$

Recall the definition of the set of dyadic cubes:

$$\bigcup_{n \in \mathbb{Z}} \left\{ [x_1, x_1 + 2^n) \times \cdots \times [x_d, x_d + 2^n) : i = 1, \ldots, n,\ x_i \in 2^n \mathbb{Z} \right\}.$$

The maximal function of a characteristic function can be written as

$$\mathrm{M} 1_E(x) = \sup_{x \in X \subset \Omega} \frac{\mathcal{L}(E \cap X)}{\mathcal{L}(X)},$$

where X ranges over balls for the uncentered maximal operator, and over dyadic cubes for the dyadic maximal operator. Now we are ready to state the main results of this paper.

Theorem 1.2. *Let M be the local dyadic maximal operator with respect to an open set $\Omega \subset \mathbb{R}^d$. Let $E \subset \mathbb{R}^d$ be a set with locally finite perimeter. Then*

$$\operatorname{var}_\Omega \mathrm{M} 1_E \le C_d \, \mathcal{H}^{d-1}(\partial_* E \cap \Omega),$$

where C_d depends only on the dimension d.

Theorem 1.3. *Let M be the local uncentered maximal operator with respect to an open set $\Omega \subset \mathbb{R}^d$. Let $E \subset \mathbb{R}^d$ be a set with locally finite perimeter. Then*

$$\operatorname{var}_\Omega \mathrm{M} 1_E \le C_d \, \mathcal{H}^{d-1}(\partial_* E \cap \Omega),$$

where C_d depends only on the dimension d.

We can for example take $\Omega = \mathbb{R}^d$. Denote $\{\mathrm{M} 1_E > \lambda\} = \{x \in \Omega : \mathrm{M} 1_E(x) > \lambda\}$. We reduce Theorems 1.2 and 1.3 to the following results.

Proposition 1.4. *Let* M *be the local dyadic maximal operator with respect to some open set* $\Omega \subset \mathbb{R}^d$. *Let* $E \subset \mathbb{R}^d$ *be a set with locally finite perimeter and let* $\lambda \in (0,1)$. *Then*

$$\mathcal{H}^{d-1}(\partial_* \{\mathrm{M}1_E > \lambda\} \cap \Omega) \leq C_d \, \lambda^{-(d-1)/d} \, \mathcal{H}^{d-1}(\partial_* E \cap \Omega).$$

By Lemma 2.4, we have $\overline{E}^* \cap \Omega \subset \overline{\{\mathrm{M}1_E > \lambda\}}^*$, so that we might intersect the right-hand side with $\overline{\{\mathrm{M}1_E > \lambda\}}^*$.

Proposition 1.5. *Let* M *be the local uncentered maximal operator. Let* $E \subset \mathbb{R}^d$ *be a set with locally finite perimeter and let* $\lambda \in (0,1)$. *Then*

$$\mathcal{H}^{d-1}(\partial_* \{\mathrm{M}1_E > \lambda\} \cap \Omega) \leq C_d \, \lambda^{-(d-1)/d} (1 - \log \lambda) \, \mathcal{H}^{d-1}(\partial_* E \cap \{\mathrm{M}1_E > \lambda\}).$$

The constants C_d that appear in Theorems 1.2 and 1.3 and Propositions 1.4 and 1.5 are not equal. Since the proofs of Theorems 1.2 and 1.3 are almost the same, we do them simultaneously.

Proof of Theorems 1.2 *and* 1.3. By Lemma 1.1 and Propositions 1.4 and 1.5, we have

$$\begin{aligned} \mathrm{var}_\Omega \, \mathrm{M}1_E &= \int_0^1 \mathcal{H}^{d-1}(\partial_* \{\mathrm{M}1_E > \lambda\} \cap \Omega) \, \mathrm{d}\lambda \\ &\leq C_d \int_0^1 \lambda^{-(d-1)/d} (1 - \log \lambda) \, \mathcal{H}^{d-1}(\partial_* E \cap \Omega) \, \mathrm{d}\lambda \\ &= d(d+1) \, C_d \, \mathcal{H}^{d-1}(\partial_* E \cap \Omega). \end{aligned}$$

∎

In Sections 2 to 4 we prove Propositions 1.4 and 1.5. In Section 5 we prove Proposition 5.1, which is Proposition 1.5 without the factor $1 - \log \lambda$. The rate $\lambda^{-(d-1)/d}$ is optimal.

We introduce some notation we will use throughout the paper. By $a \lesssim b$ we mean that there exists a constant C_d that depends only on the dimension d such that $a \leq C_d b$. For a set $\mathcal{B}$ of subsets of $\mathbb{R}^d$, we write

$$\bigcup \mathcal{B} = \bigcup_{B \in \mathcal{B}} B.$$

For a ball $B = B(x,r) \subset \mathbb{R}^d$ and $c > 0$, we denote $cB = B(x, cr)$. If $\mathcal{B}$ is a set of balls, we denote

$$c\mathcal{B} = \{cB : B \in \mathcal{B}\}.$$

For a set $E \subset \mathbb{R}^d$ and a point $x \in \mathbb{R}^d$, we denote

$$\mathrm{dist}(x, E) = \inf_{y \in E} |x - y|.$$

We also need more measure theoretic quantities. We define the measure theoretic interior of E by

$$\mathrm{int}_*(E) = \Big\{ x : \limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \setminus E)}{r^d} = 0 \Big\},$$

the measure theoretic closure by

$$\overline{E}^* = \Big\{ x : \limsup_{r \to 0} \frac{\mathcal{L}(B(x,r) \cap E)}{r^d} > 0 \Big\}$$

and the measure theoretic boundary by

$$\partial_* E = \overline{E}^* \setminus \mathrm{int}_*(E).$$

Lemma 1.6. *Let $A, B \subset \mathbb{R}^d$ be measurable. Then*

$$\partial_*(A \cup B) \subset (\partial_* A \setminus \overline{B}^*) \cup (\partial_* B \setminus \overline{A}^*) \cup (\partial_* A \cap \partial_* B).$$

Proof. Let $x \in \partial_*(A \cup B)$. Then

$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r) \cap (A \cup B))}{r^d} > 0 \quad \text{and} \quad \limsup_{r\to 0} \frac{\mathcal{L}(B(x,r) \setminus (A \cup B))}{r^d} > 0.$$

By symmetry, it suffices to consider the case that

$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r) \cap A)}{r^d} > 0.$$

Then

$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r) \setminus A)}{r^d} \geq \limsup_{r\to 0} \frac{\mathcal{L}(B(x,r) \setminus (A \cup B))}{r^d} > 0,$$

which means $x \in \partial_* A$. Analogously, if

$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r) \cap B)}{r^d} > 0,$$

then $x \in \partial_* B$ so we get $x \in \partial_* A \cap \partial_* B$. Otherwise

$$\limsup_{r\to 0} \frac{\mathcal{L}(B(x,r) \cap B)}{r^d} = 0,$$

and we can conclude $x \in \partial_* A \setminus \overline{B}^*$. ∎

Let $E \subset \mathbb{R}^d$ be measurable and let μ be the measure from the definition of $\operatorname{var} 1_E$ and ν the outer normal. We define the reduced boundary $\partial^* E$ of $E \subset \mathbb{R}^d$ as the set of all points $x \in \mathbb{R}^d$ such that for all $r > 0$ we have $\mu(B(x,r)) > 0$,

$$\lim_{r\to 0} \fint_{B(x,r)} \nu \, \mathrm{d}\mu = \nu(x),$$

and $|\nu(x)| = 1$. This is Definition 5.4 in [12]. By Lemma 5.5 in [12], we have $\partial^* E \subset \partial_* E$ and $\mathcal{H}^{d-1}(\partial_* E \setminus \partial^* E) = 0$. Thus it suffices to consider only the reduced boundary when estimating the perimeter of a set. But most of the time we will formulate the results for the measure theoretic boundary. The exception is Lemma 2.4, which we could only prove for the reduced boundary because there we make use of Theorem 5.13 in [12], which states the following.

Lemma 1.7 (Theorem 5.13 in [12]). *Let $E \subset \mathbb{R}^d$ be a measurable set. Assume $0 \in \partial^* E$ with $\nu(0) = (1, 0, \ldots, 0)$. Then for $r \to 0$ we have $1_{\frac{1}{r}E} \to 1_{\{x : x_1 < 0\}}$ in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.*

A central tool used here is the relative isoperimetric inequality, see Theorem 5.11 in [12]. It states that for a ball B and any measurable set $E \subset \mathbb{R}^d$, we have

$$\min\{\mathcal{L}(E \cap B), \mathcal{L}(B \setminus E)\}^{d-1} \lesssim \mathcal{H}^{d-1}(\partial_* E \cap B)^d. \tag{1.1}$$

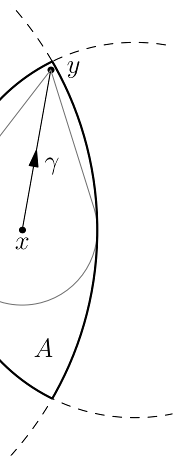

Figure 1. A John domain.

However we need the relative isoperimetric inequality also for other sets than balls. An open bounded set A is called a John domain if there is a constant K and point $x \in A$ from which every other point $y \in A$ can be reached via a path γ such that for all t we have

$$\operatorname{dist}(\gamma(t), A^{\mathrm{c}}) \geq K^{-1}|y - \gamma(t)|. \tag{1.2}$$

This is called the cone condition, see Figure 1. Theorem 107 in the lecture notes [13] by Piotr Hajłasz states that all John domains admit a relative isoperimetric inequality.

Lemma 1.8. *Let $A \subset \mathbb{R}^d$ be a John domain with constant K. Then A satisfies a relative isoperimetric inequality with constant $C_{K,d}$ only depending on K and the dimension d,*

$$\min\{\mathcal{L}(E \cap A), \mathcal{L}(A \setminus E)\}^{d-1} \leq C_{K,d}\, \mathcal{H}^{d-1}(\partial_* E \cap A)^d.$$

For example, a ball and an open cube are John domains.

Another basic tool is the Vitali covering lemma, see for example Theorem 1.24 in [12].

Lemma 1.9 (Vitali covering lemma). *Let $\mathcal{B}$ be a set of balls in $\mathbb{R}^d$ with diameter bounded by some $R \in \mathbb{R}$. Then it has a countable subset $\tilde{\mathcal{B}}$ of disjoint balls such that*

$$\bigcup \mathcal{B} \subset \bigcup 5\tilde{\mathcal{B}}.$$

Instead of considering $\{\mathrm{M}1_E > \lambda\}$, we will only consider a finite union of balls/cubes. In order to pass from there to the whole set $\{\mathrm{M}1_E > \lambda\}$, we will use an approximation result. We say that a sequence $(A_n)_n$ of sets in $\mathbb{R}^d$ converges to some set A in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$ if $(1_{A_n})_n$ converges to 1_A in $L^1_{\mathrm{loc}}(\mathbb{R}^d)$.

Lemma 1.10 (Theorem 5.2 in [12] for characteristic functions). *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $(A_n)_n$ be subsets of $\mathbb{R}^d$ with locally finite perimeter that converge to A in $L^1_{\mathrm{loc}}(\Omega)$. Then*

$$\mathcal{H}^{d-1}(\partial_* A \cap \Omega) \leq \liminf_{n\to\infty} \mathcal{H}^{d-1}(\partial_* A_n \cap \Omega).$$

Lemma 1.11. *Let $\sigma_d = \mathcal{L}(B(0,1))$ be the Lebesgue measure of the d-dimensional unit ball. Then*
$$\sqrt{\frac{2\pi}{d+1}} \le \frac{\sigma_d}{\sigma_{d-1}} \le \sqrt{\frac{2\pi}{d}}.$$

Proof. By the logarithmic convexity of the Γ-function, for all $x > 1/2$ we have

$$\frac{\Gamma(x)}{\Gamma(x+1/2)} \le \frac{\sqrt{\Gamma(x-1/2)\Gamma(x+1/2)}}{\Gamma(x+1/2)} = \sqrt{\frac{\Gamma(x-1/2)}{\Gamma(x+1/2)}} = \frac{1}{\sqrt{x-1/2}},$$
$$\frac{\Gamma(x)}{\Gamma(x+1/2)} \ge \frac{\Gamma(x)}{\sqrt{\Gamma(x)\Gamma(x+1)}} = \sqrt{\frac{\Gamma(x)}{\Gamma(x+1)}} = \frac{1}{\sqrt{x}},$$

and the result follows from $\sigma_d = \pi^{d/2}/\Gamma(d/2+1)$. ■

We will need some facts about convex sets.

Lemma 1.12. *The following properties hold for all convex and bounded sets $A, B \subset \mathbb{R}^d$.*

(i) *The set $A \cap B$ is convex.*

(ii) *If $A \subset B$, then $\mathcal{H}^{d-1}(\partial A) \le \mathcal{H}^{d-1}(\partial B)$.*

(iii) *For every $\varepsilon > 0$, we have $\mathcal{L}(\{x \in A : 0 < \operatorname{dist}(x, A^{c}) \le \varepsilon\}) \le \varepsilon \mathcal{H}^{d-1}(\partial A)$.*

Proof. (i) follows from the definition of convexity.

For every $x \in \partial B$, there is a point $z \in \partial A$ with
$$|z - x| = \min_{y \in \partial A} |y - x|.$$

A straightforward computation shows that if $z' \in \partial A$ with $|x - z'| = \min_{y\in\partial A} |x - y|$, then $|x - (z + z')/2| \le \min_{y\in\partial A} |x - y|$ and the inequality is strict if $z' \ne z$. Hence we must have $z' = z$ because $(z + z')/2 \in \overline{A}$ by convexity. We denote $p(x) = z$.

Since A is convex, in every point $z \in \partial A$ there is a hyperplane H which contains z and such that for all $y \in \partial A$ we have $\langle y - z, n\rangle \le 0$, where n is the normal of H. Because B is bounded, there is an $r \ge 0$ such that $z + rn \in \partial B$. It is easy to see that $p(z + rn) = z$. That means $p\colon \partial B \to \partial A$ is surjective.

Let $x_1, x_2 \in \partial B$. For $i = 1, 2$, denote $z_i = p(x_i)$ and let H_i be the hyperplane with normal $x_i - z_i$ which contains z_i. Then $\langle z_2 - z_1, x_1 - z_1\rangle \le 0$ because otherwise it is straightforward to find a $t > 0$ small enough with $(1 - t)z_1 + t z_2 \in \overline{A}$ which is closer to x_1 than z_1, which leads to a contradiction to $p(x_1) = z_1$. Similarly, we must have $\langle z_1 - z_2, x_2 - z_2\rangle \le 0$. We can conclude

$$\begin{aligned} |z_1 - z_2||x_1 - x_2| &\ge \langle z_1 - z_2, x_1 - x_2\rangle \\ &= \langle z_1 - z_2, x_1 - z_1\rangle + \langle z_2 - z_1, x_2 - z_2\rangle + \langle z_1 - z_2, z_1 - z_2\rangle \\ &\ge |z_1 - z_2|^2. \end{aligned}$$

This means that the map $p\colon \partial B \to \partial A$ is 1-Lipschitz, and we obtain (ii) because the Hausdorff measure does not increase under 1-Lipschitz maps by Theorem 2.8 in [12].

For every $\lambda \geq 0$, denote $A_\lambda = \{x \in A : \operatorname{dist}(x, A^c) \geq \lambda\}$. Then A_λ is convex and by Theorem 3.14 in [12] and (ii) we have

$$\mathcal{L}(\{x \in A : 0 < \operatorname{dist}(x, A^c) \leq \varepsilon\}) = \int_0^\varepsilon \mathcal{H}^{d-1}(\partial A_\lambda)\,\mathrm{d}\lambda \leq \int_0^\varepsilon \mathcal{H}^{d-1}(\partial A)\,\mathrm{d}\lambda$$
$$= \varepsilon\,\mathcal{H}^{d-1}(\partial A). \qquad \blacksquare$$

2. Tools for both maximal operators

We start with a couple of tools that are used for both maximal operators.

Lemma 2.1. *Let $X \subset \mathbb{R}^d$ be an open set with finite measure and finite perimeter which satisfies a relative isoperimetric inequality, and denote $c = \mathcal{H}^{d-1}(\partial X)^d / \mathcal{L}(X)^{d-1}$. Let $0 < \lambda \leq 1 - \varepsilon < 1$ and let E be a measurable set such that $\lambda \leq \mathcal{L}(E \cap X)/\mathcal{L}(X) \leq 1 - \varepsilon$. Then*

$$\mathcal{H}^{d-1}(\partial_* E \cap X) \gtrsim c^{-1/d} \varepsilon^{d-1} \lambda^{(d-1)/d}\, \mathcal{H}^{d-1}(\partial X).$$

Note that c is invariant under scaling of X.

Proof. We first prove

$$\mathcal{L}(E \cap X)^{d-1} \lesssim \varepsilon^{-(d-1)}\, \mathcal{H}^{d-1}(\partial_* E \cap X)^d. \tag{2.1}$$

If $\varepsilon \geq 1/2$, then (2.1) follows directly from the relative isoperimetric inequality for X. For $\varepsilon < 1/2$, we obtain (2.1) from the relative isoperimetric inequality as follows:

$$\mathcal{H}^{d-1}(\partial_* E \cap X)^d \gtrsim \mathcal{L}(X \setminus E)^{d-1} \geq \varepsilon^{d-1} \mathcal{L}(X)^{d-1} \geq \varepsilon^{d-1} \mathcal{L}(E \cap X)^{d-1}.$$

From (2.1) we conclude

$$\varepsilon^{-(d-1)} \mathcal{H}^{d-1}(\partial_* E \cap X) \gtrsim \mathcal{L}(E \cap X)^{(d-1)/d} \geq \lambda^{(d-1)/d}\, \mathcal{L}(X)^{(d-1)/d}$$
$$\geq c^{-1/d}\, \lambda^{(d-1)/d}\, \mathcal{H}^{d-1}(\partial X). \qquad \blacksquare$$

Lemma 2.2 (Boxing inequality, cf. Theorem 3.1 in Kinnunen, Korte, Shanmugalingam and Tuominen [18]). *Let $E \subset \mathbb{R}^d$ be a set with finite measure that is contained in the union of a set $\mathcal{B}$ of balls B with $\mathcal{L}(E \cap B) \leq \mathcal{L}(B)/2$. Then there is a set $\mathcal{F}$ of balls F with $\mathcal{L}(F \cap E) = \mathcal{L}(F)/2$ which covers almost all of E. Furthermore, each $F \in \mathcal{F}$ is contained in a ball $B \in \mathcal{B}$.*

Proof. It suffices to show that for every ball $B(x_1, r_1) \in \mathcal{B}$, every Lebesgue point $x \in \operatorname{int}_*(E)$ with $x \in B(x_1, r_1)$ is contained in a ball $F \subset B(x_1, r_1)$ with $\mathcal{L}(F \cap E) = \mathcal{L}(F)/2$. By assumption,

$$\mathcal{L}(E \cap B(x_1, r_1)) \leq \frac{\mathcal{L}(B(x_1, r_1))}{2},$$

and since x is a Lebesgue point, there is a ball $B(x_0, r_0)$ with $x \in B(x_0, r_0) \subset B(x_1, r_1)$ and

$$\mathcal{L}(E \cap B(x_0, r_0)) \geq \frac{\mathcal{L}(B(x_0, r_0))}{2}.$$

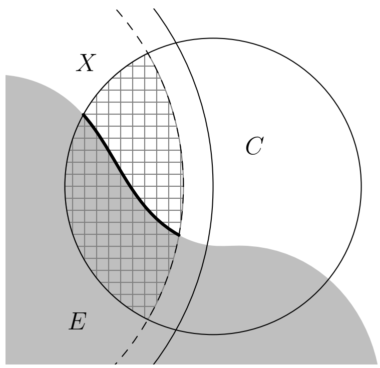

Figure 2. The regions in Lemma 2.3.

Define $x_t = (1-t)\cdot x_0 + t\cdot x_1$ and $r_t = (1-t)\cdot r_0 + t\cdot r_1$ so that $t \mapsto B(x_t, r_t)$ is a continuous transformation of balls. That means there is a t with

$$\mathcal{L}(E \cap B(x_t, r_t)) = \frac{\mathcal{L}(B(x_t, r_t))}{2}.$$

Since $x \in B(x_0, r_0) \subset B(x_t, r_t) \subset B(x_1, r_1)$, that means we have found the right ball. ∎

We will prove a more specialized version of Lemma 2.2.

Lemma 2.3. *Let X be an open cube or a ball in $\mathbb{R}^d$ and let E be a set with $\mathcal{L}(E \cap X) \geq \lambda\mathcal{L}(X)$. Then there is a cover $\mathcal{C}$ of $\partial X \setminus \overline{E}^*$ consisting of balls C with* $\operatorname{diam} C \leq 2 \operatorname{diam} X$ *and*

$$\mathcal{H}^{d-1}\Big(\partial_* E \cap \Big\{y \in C : \operatorname{dist}(y, X^{\mathrm{c}}) > \frac{\lambda \operatorname{diam} C}{4dc_d}\Big\}\Big) \gtrsim \lambda^{(d-1)/d}\, \mathcal{H}^{d-1}(\partial C), \tag{2.2}$$

where $c_d = 2^d$ if X is a ball, and $c_d = d^{d/2}\sigma_d$ if X is a cube.

The constants in Lemma 2.3 are not important and one could also impose a stronger bound on the diameter of the balls $C \in \mathcal{C}$ for λ near 1.

Proof of Lemma 2.3. It suffices to show that for each $x \in \partial X \setminus \overline{E}^*$ there is a ball C centered in x that satisfies (2.2). Let $x \in \partial X \setminus \overline{E}^*$ and for $0 < r \leq \operatorname{diam} X$ define

$$A(r) = \Big\{y \in B(x, r) : \operatorname{dist}(y, X^{\mathrm{c}}) > \frac{\lambda r}{2dc_d}\Big\}.$$

We first show that $A(r)$ is a John domain. Consider the case that X is a ball. Then there is a point $z \in X \cap B(x, r)$ such that $B(z, r/2) \subset X \cap B(x, r)$. That means

$$B\Big(z, \frac{r}{4}\Big) \subset B\Big(z, \frac{r}{2} - \frac{\lambda r}{2dc_d}\Big) \subset A(r).$$

Now let X be a cube. Then $X \cap B(x,r)$ contains a cube with diameter at least r, i.e., sidelength at least $r/\sqrt{d}$. Thus, $A(r)$ contains a cube with sidelength at least

$$\frac{r}{\sqrt{d}} - 2\frac{\lambda r}{2dc_d} \geq \frac{r}{\sqrt{d}}\Big(1 - \frac{1}{\sqrt{d}c_d}\Big) \geq \frac{r}{2\sqrt{d}},$$

which in turn contains a ball B with radius $r/(4\sqrt{d})$. The last inequality holds because $1^{(1+1)/2}\sigma_1 = 2$ and $\sqrt{d}c_d = d^{(d+1)/2}\sigma_d$ is increasing in d by Lemma 1.11. We have shown that there is a point $z \in A(r)$ such that

$$B\Big(z, \frac{r}{4\sqrt{d}}\Big) \subset A(r), \tag{2.3}$$

both if X is a cube or a ball. For any $y \in A(r)$, we have $\operatorname{dist}(y,z) \leq \operatorname{diam}(A(r)) \leq 2r$. Because $A(r)$ is convex by Lemma 1.12 (i), it contains the convex hull of $B\big(z, \frac{r}{4\sqrt{d}}\big) \cup \{y\}$. We can conclude that $A(r)$ is a John domain with $K = \frac{2r}{r/(4\sqrt{d})} = 8\sqrt{d}$.

We have

$$\begin{aligned}\mathcal{L}(B(x,r) \setminus A(r)) &\leq \mathcal{L}\Big(\Big\{y : 0 < \operatorname{dist}(y, (B(x,r) \cap X)^{\mathrm{c}}) \leq \frac{\lambda r}{2dc_d}\Big\}\Big)\\ &\leq \frac{\lambda r}{2dc_d}\,\mathcal{H}^{d-1}(\partial(B(x,r) \cap X)) \leq \frac{\lambda r}{2dc_d}\,\mathcal{H}^{d-1}(\partial B(x,r))\\ &= \frac{\lambda}{2c_d}\,\mathcal{L}(B(x,r)) \leq \frac{\lambda}{2}\,\mathcal{L}(B(x,r) \cap X),\end{aligned} \tag{2.4}$$

where the last inequality holds because as observed above, $B(x,r) \cap X$ contains a ball with radius $r/2$ if X is a ball, and a cube with sidelength $r/\sqrt{d}$ if X is a cube. Then from

$$\frac{\mathcal{L}(X \cap E)}{\mathcal{L}(X)} \geq \lambda$$

and (2.4) with $r = \operatorname{diam} X$ we get

$$\frac{\mathcal{L}(A(\operatorname{diam} X) \cap E)}{\mathcal{L}(A(\operatorname{diam} X))} \geq \frac{\mathcal{L}(A(\operatorname{diam} X) \cap E)}{\mathcal{L}(X)} \geq \lambda - \frac{\lambda}{2} = \frac{\lambda}{2}.$$

Since $x \notin \overline{E}^*$, we have $\mathcal{L}(E \cap B(x,r))/r^d \to 0$ for $r \to 0$. By (2.3) this implies that there is an r_0 with

$$\frac{\mathcal{L}(A(r_0) \cap E)}{\mathcal{L}(A(r_0))} \leq \frac{\lambda}{2}.$$

By continuity we conclude that there is an $r_0 \leq r \leq \operatorname{diam} X$ such that

$$\frac{\mathcal{L}(A(r) \cap E)}{\mathcal{L}(A(r))} = \frac{\lambda}{2}.$$

By (2.3) and Lemma 1.12 (i), we have

$$\mathcal{H}^{d-1}(\partial B(x,r)) \lesssim \mathcal{H}^{d-1}\Big(\partial B\Big(z, \frac{r}{4\sqrt{d}}\Big)\Big) \leq \mathcal{H}^{d-1}(\partial A(r)). \tag{2.5}$$

Because $A(r)$ is a John domain, it satisfies, by Lemma 1.8, a relative isoperimetric inequality, so that we can apply Lemma 2.1 with $X = A(r)$ and $\varepsilon = 1/2$ and obtain

$$\mathcal{H}^{d-1}(\partial A(r)) \lesssim \lambda^{-(d-1)/d} \mathcal{H}^{d-1}(\partial_* E \cap A(r)). \tag{2.6}$$

Combining (2.5) and (2.6) we obtain (2.2), which finishes the proof. ∎

Note that the following Lemma 2.4 addresses the reduced boundary $\partial^* E$ and not the measure theoretic boundary $\partial_* E$.

Lemma 2.4. *Let $\Omega \subset \mathbb{R}^d$ be an open set and let $E \subset \mathbb{R}^d$ be measurable. Then for every $\lambda \in [0, 1)$, and for both the dyadic and the uncentered maximal operator with domain Ω, we have $\operatorname{int}_*(E) \cap \Omega \subset \{\mathrm{M}1_E = 1\}$. For the uncentered maximal operator, we furthermore have $\partial^* E \cap \Omega \subset \{\mathrm{M}1_E = 1\}$.*

This is a slightly more precise version of $\mathrm{M}f \geq f$ almost everywhere for characteristic functions.

Proof. Let $x \in \operatorname{int}_*(E) \cap \Omega$. Then for every $\varepsilon > 0$ there is a ball $B \subset \Omega$ with center x and with $\mathcal{L}(B \setminus E) \leq \varepsilon \mathcal{L}(B)$, and a dyadic cube Q with $x \in Q \subset B$ and $\mathcal{L}(Q) \gtrsim \mathcal{L}(B)$. This means $\mathcal{L}(Q \setminus E) \leq \varepsilon \mathcal{L}(B) \lesssim \varepsilon \mathcal{L}(Q)$. We can conclude $\mathrm{M}1_E(x) = 1$.

Let $x \in \partial^* E \cap \Omega$. It suffices to consider $x = 0$ and

$$\lim_{r \to 0} \fint_{B(0,r)} \nu_E = (1, 0, \ldots, 0).$$

Then for r small enough we have $0 \in B_r = B((-r, 0, \ldots, 0), r + r^2) \subset \Omega$, and so by Lemma 1.7 we obtain

$$\lim_{r \to 0} \fint_{B_r} 1_E = \lim_{r \to 0} \frac{\mathcal{L}(\{y \in B_r : y_1 < 0\})}{\mathcal{L}(B_r)} = \lim_{r \to 0} \frac{\mathcal{L}(\{y \in B(0, r + r^2) : y_1 < r\})}{\mathcal{L}(B(0, r + r^2))} = 1.$$

∎

3. The dyadic maximal function

In this section we discuss the argument for the dyadic maximal operator. It already showcases the main idea of the proof for the uncentered maximal operator. For the superlevelset of the dyadic maximal operator we have

$$\{\mathrm{M}1_E > \lambda\} = \bigcup \big\{\text{dyadic cube } Q : \mathcal{L}(E \cap Q) > \lambda \mathcal{L}(Q)\big\}.$$

The first step in the proof of Proposition 1.4 is to consider only a finite set $\mathcal{Q}$ of cubes Q with $\mathcal{L}(E \cap Q) > \lambda \mathcal{L}(Q)$ instead of the whole set, because this allows to write

$$\mathcal{H}^{d-1}\big(\partial_* \bigcup \mathcal{Q}\big) \leq \sum_{Q \in \mathcal{Q}} \mathcal{H}^{d-1}\big(\partial Q \cap \partial_* \bigcup \mathcal{Q}\big).$$

From there we use approximation results to extend to the union of all cubes Q with $\mathcal{L}(E \cap Q) > \lambda \mathcal{L}(Q)$. The strategy for the uncentered maximal operator is similar, but with cubes replaced by balls. The main argument is Proposition 3.1, which is more or less Proposition 1.4 for the case that $\{\mathrm{M}1_E > \lambda\}$ consists of only one cube.

Proposition 3.1. *Let $0 < \lambda \le 1$, let Q be a cube and let $E \subset \mathbb{R}^d$ be a measurable set with $\mathcal{L}(E \cap Q) \ge \lambda \mathcal{L}(Q)$. Then*

$$\mathcal{H}^{d-1}(\partial Q \setminus \overline{E}^*) \lesssim \lambda^{-(d-1)/d} \mathcal{H}^{d-1}(\partial_* E \cap \mathring{Q}).$$

Proof of Proposition 3.1. We apply Lemma 2.3 to $X = \mathring{Q}$ and for the resulting cover use Lemma 1.9 to extract a disjoint subcollection $\mathcal{C}$ such that $5\mathcal{C}$ still covers $\partial Q \setminus \overline{E}^*$. Then by Lemma 1.12 (i) and (ii) and Lemma 2.3 we have

$$\begin{aligned}\mathcal{H}^{d-1}(\partial Q \setminus \overline{E}^*) &\le \sum_{C \in \mathcal{C}} \mathcal{H}^{d-1}(\partial Q \cap 5C) \le \sum_{C \in \mathcal{C}} \mathcal{H}^{d-1}(\partial 5C) \\ &\lesssim \lambda^{-(d-1)/d} \sum_{C \in \mathcal{C}} \mathcal{H}^{d-1}(\partial_* E \cap C \cap \mathring{Q}) \le \lambda^{-(d-1)/d} \mathcal{H}^{d-1}(\partial_* E \cap \mathring{Q}). \qquad \blacksquare\end{aligned}$$

Remark 3.2. For $\lambda \le 1/2$, Proposition 3.1 also follows directly from the relative isoperimetric inequality (1.1) for Q. Proposition 3.1 also holds for Q being a ball.

Proof of Proposition 1.4. For each $x \in \{\mathrm{M}1_E > \lambda\} \cap \Omega$ there is a dyadic cube $Q \subset \Omega$ with $x \in Q$ and $\mathcal{L}(E \cap Q) > \lambda \mathcal{L}(Q)$. Since there are only countably many dyadic cubes, we can enumerate them as $Q_1, Q_2, \dots$. For each n, let

$$\mathcal{Q}_n = \{Q_i : \forall j = 1, \dots, n \text{ with } j \ne i \text{ we have } Q_i \not\subset Q_j\}.$$

Then $\bigcup \mathcal{Q}_n = Q_1 \cup \dots \cup Q_n$ and thus

$$\bigcup_n Q_n = \{\mathrm{M}1_E > \lambda\}.$$

Because E and $\mathrm{int}_*(E)$ agree up to measure zero and $\mathrm{int}_*(E) \subset \{\mathrm{M}1_E > \lambda\}$ by Lemma 2.4, we have that $\bigcup \mathcal{Q}_n \cup E$ converges to $\{\mathrm{M}1_E > \lambda\}$ in $L^1_{\mathrm{loc}}(\Omega)$. Therefore, by Lemmas 1.6 and 1.10, we obtain

$$\begin{aligned}\mathcal{H}^{d-1}(\partial_*\{\mathrm{M}1_E > \lambda\} \cap \Omega) &\le \limsup_{n\to\infty} \mathcal{H}^{d-1}\big(\partial_*\big(\textstyle\bigcup \mathcal{Q}_n \cup E\big) \cap \Omega\big) \\ &\le \limsup_{n\to\infty} \mathcal{H}^{d-1}\big(\big(\partial_* \textstyle\bigcup \mathcal{Q}_n \setminus \overline{E}^*\big) \cap \Omega\big) + \mathcal{H}^{d-1}(\partial_* E \cap \Omega). \qquad (3.1)\end{aligned}$$

It is not necessary, but in the line corresponding to (3.1) in the proof for the uncentered Hardy–Littlewood maximal function, we can actually eliminate the term $\mathcal{H}^{d-1}(\partial_* E \cap \Omega)$ thanks to Lemma 2.4; see (4.1) in Section 4 and the subsequent comment. Here this is not so clear because for the dyadic maximal function, Lemma 2.4 is weaker. But in any case, it suffices to estimate the first term on the right-hand side of (3.1). We invoke Proposition 3.1 and use that the cubes in $\mathcal{Q}_n$ are disjoint and obtain

$$\begin{aligned}\mathcal{H}^{d-1}\big(\big(\partial_* \textstyle\bigcup \mathcal{Q}_n \setminus \overline{E}^*\big) \cap \Omega\big) &\le \sum_{Q \in \mathcal{Q}_n} \mathcal{H}^{d-1}((\partial_* Q \setminus \overline{E}^*) \cap \Omega) \\ &\lesssim \sum_{Q \in \mathcal{Q}_n} \lambda^{-(d-1)/d} \mathcal{H}^{d-1}(\partial_* E \cap Q) \\ &\le \lambda^{-(d-1)/d} \mathcal{H}^{d-1}(\partial_* E \cap \Omega \cap \{\mathrm{M}1_E > \lambda\}). \qquad \blacksquare\end{aligned}$$

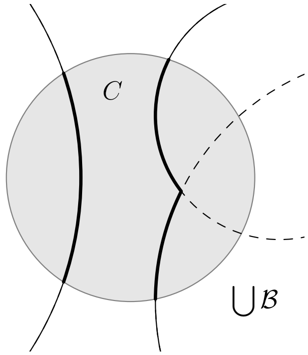

Figure 3. The objects in Lemma 4.1.

Proposition 3.1 readily implies Proposition 1.4 because $\{\mathrm{M}1_E > \lambda\}$ is a disjoint union of such cubes. Two balls however can have nontrivial intersections, which is why the proof for the uncentered Hardy–Littlewood maximal operator is much more complicated than the proof for the dyadic maximal operator.

4. The uncentered maximal function

In this section we prove Proposition 1.5. The main step is Proposition 4.3. It is Proposition 3.1 for a set $\mathcal{B}$ of finitely many balls B with $\mathcal{L}(B \cap E) > \lambda \mathcal{L}(B)$ instead of one cube. Proposition 4.3 comes with an additional but harmless factor $(1 - \log \lambda)$. We will show in Section 5 that this factor can be removed.

Lemma 4.1. *Let $K > 0$, let C be a ball and let be $\mathcal{B}$ a finite set of balls B with* $\operatorname{diam}(B) \geq K \operatorname{diam}(C)$. *Then*

$$\mathcal{H}^{d-1}\big(\partial_* \bigcup \mathcal{B} \cap C\big) \lesssim (K^{-d} + 1)\, \mathcal{H}^{d-1}(\partial C).$$

The rate K^{-d} does not play a role in the application.

Proof. By translation and scaling, it suffices to consider the case $C = B(0,1)$. Let $B(x,r)$ be a ball with $|x| \geq 4d + 1$ whose boundary intersects $B(0,1)$, which means $4d < r < 4d + 2$. For any point $y = (y_1, \dots, y_d) \in \mathbb{R}^d$, denote $\widehat{y} = (y^1, \dots, y^{d-1})$. Assume that $|x^d| = \max\{|x_1|, \dots, |x_d|\}$, so that

$$|\widehat{x}|^2 = |x|^2 - x_d^2 \leq \Big(1 - \frac{1}{d}\Big)|x|^2.$$

Then for every $y \in B(0,1)$ we have

$$\begin{aligned}|\widehat{y} - \widehat{x}| \leq |\widehat{x}| + 1 \leq \sqrt{1 - \frac{1}{d}}\,|x| + 1 \leq \Big(1 - \frac{1}{2d}\Big)(r + 1) + 1 \\ = r\Big(1 + \frac{2}{r} - \frac{1}{2d} - \frac{1}{2dr}\Big) \leq r,\end{aligned}$$

and

$$x_d - y_d \geq \sqrt{\frac{1}{d}}\,|x| - 1 \geq \frac{r - (\sqrt{d}+1)}{\sqrt{d}} > 0.$$

Therefore the function

$$y \mapsto \varphi(\hat{y}) = x_d - \sqrt{r^2 - |\widehat{y-x}|^2}$$

is well defined for $y \in B(0,1)$, we have

$$B(x,r) \cap B(0,1) = \{y \in B(0,1) : y_d > \varphi(\hat{y})\},$$

and for $y \in \partial B(x,r) \cap B(0,1)$, the gradient of φ in $y_1, \dots, y_{d-1}$ is bounded by

$$|\nabla\varphi(\hat{y})| = \frac{|\widehat{x-y}|}{\sqrt{r^2 - |\widehat{x-y}|^2}} = \frac{|\widehat{x-y}|}{|x_d - y_d|} \leq \frac{\sqrt{d}\,r}{r - (\sqrt{d}+1)} \leq \frac{4d^{3/2}}{4d - (\sqrt{d}+1)} \leq 2\sqrt{d}.$$

For the case that all balls $B \in \mathcal{B}$ have radius at least $4d$, we can conclude that the boundary of the union of all balls of the above form is a piece of the infimum of $2\sqrt{d}$-Lipschitz graphs, and thus itself a piece of a $2\sqrt{d}$-Lipschitz graph. We can conclude that

$$\begin{aligned}\mathcal{H}^{d-1}\big(\partial \textstyle\bigcup\{B(x,r) \in \mathcal{B} : x_d = \max\{|x_1|, \dots, |x_d|\}\} \cap B(0,1)\big)\\ \leq \sqrt{4d+1}\,\sigma_{d-1} = \frac{\sqrt{4d+1}\,\sigma_{d-1}}{d\sigma_d}\,\mathcal{H}^{d-1}(\partial B(0,1)).\end{aligned}$$

By rotation, we obtain the same bound for the union of those balls $B(x,r) \in \mathcal{B}$ with $\pm x_i = \max\{|x_1|, \dots, |x_d|\}$ for any $i = 1, \dots, d$ and any sign. This finishes the proof for $K \geq 4d$.

If $K < 4d$, then we cover $B(0,1)$ by $\lesssim \big(\frac{4d}{K}\big)^d$ many balls $C_1, C_2, \dots$ so that for each i we have $\operatorname{diam}(B) \geq 4d \operatorname{diam}(C_i)$. Then

$$\begin{aligned}\mathcal{H}^{d-1}\big(\partial_* \textstyle\bigcup \mathcal{B} \cap B(0,1)\big) &\leq \sum_i \mathcal{H}^{d-1}\big(\partial_* \textstyle\bigcup \mathcal{B} \cap C_i\big) \lesssim \sum_i \mathcal{H}^{d-1}(\partial C_i)\\ &\lesssim \Big(\frac{4d}{K}\Big)^d \mathcal{H}^{d-1}(\partial B(0,1)).\end{aligned}$$

■

In this section, for a set of balls $\mathcal{B}$ we denote by $\mathcal{B}_n$ the set of those $B \in \mathcal{B}$ with $\operatorname{diam}(B) \in [\frac{1}{2}, 1)2^n$. Further define $\mathcal{B}_{>n} = \bigcup_{k>n} \mathcal{B}_k$ and $\mathcal{B}_{\geq n}, \mathcal{B}_{<n}, \dots$ accordingly.

Lemma 4.2. *Let $\lambda \in (0,1)$, let $E \subset \mathbb{R}^d$ be measurable and let $\mathcal{B}$ be a finite set of balls B with $\mathcal{L}(E \cap B) > \lambda\mathcal{L}(B)$. Then there is a set of balls $\mathcal{C}$ such that for each $n \in \mathbb{Z}$ the following holds.*

(i) *The balls in $\mathcal{C}_n$ are disjoint.*

(ii) *The boundary piece $\partial_* \bigcup \mathcal{B} \cap \partial_* \bigcup \mathcal{B}_{n-1} \setminus \overline{E}^*$ is covered by $5\mathcal{C}_{\leq n}$.*

(iii) *Each $C \in \mathcal{C}_n$ has distance at most $2\operatorname{diam}(C)$ to $\partial_* \bigcup \mathcal{B} \setminus \overline{E}^*$.*

(iv) *We have*

$$\mathcal{H}^{d-1}\big(\partial_* E \cap \{x \in C : \operatorname{dist}\big(x, (\textstyle\bigcup \mathcal{B})^{\mathrm{c}}\big) \geq \lambda d^{-1} 2^{n-d-2}\}\big) \gtrsim \lambda^{(d-1)/d}\,\mathcal{H}^{d-1}(\partial C).$$

Proof. Apply Lemma 2.3 to each ball in $\mathcal{B}$ and denote by $\tilde{\mathcal{C}}$ the union of all of these balls. They cover $\partial_*\bigcup\mathcal{B}\setminus\overline{E}^*$. In particular, $\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_{n-1}\setminus\overline{E}^*$ is covered by $\tilde{\mathcal{C}}_{\leq n}$. Let $n\in\mathbb{Z}$. By Lemma 1.9, there is a subcollection $\mathcal{C}_n$ of $\tilde{\mathcal{C}}_n$ of disjoint balls with $\bigcup\tilde{\mathcal{C}}_n\subset\bigcup 5\mathcal{C}_n$. That means (i) and (ii) are satisfied. Now remove those balls C from $\mathcal{C}_n$ such that $5C$ does not touch $\partial_*\bigcup\mathcal{B}\setminus\overline{E}^*$. Then (ii) still holds and we also get (iii).

Let $C\in\mathcal{C}_n$ and let $B\in\mathcal{B}$ be the ball which gave rise to C. We use $B\subset\bigcup\mathcal{B}$ and Lemma 2.3 to obtain

$$\begin{aligned}
&\mathcal{H}^{d-1}\big(\partial_*E\cap\{x\in C:\operatorname{dist}\big(x,\textstyle\bigcup\mathcal{B}^{\mathrm{c}}\big)>\lambda d^{-1}2^{n-d-2}\}\big)\\
&\quad\geq\mathcal{H}^{d-1}\Big(\partial_*E\cap\Big\{x\in C:\operatorname{dist}\big(x,\textstyle\bigcup\mathcal{B}^{\mathrm{c}}\big)>\dfrac{\lambda\operatorname{diam}C}{2^{d+2}d}\Big\}\Big)\\
&\quad\geq\mathcal{H}^{d-1}\Big(\partial_*E\cap\Big\{x\in C:\operatorname{dist}(x,B^{\mathrm{c}})>\dfrac{\lambda\operatorname{diam}C}{2^{d+2}d}\Big\}\Big)\gtrsim\lambda^{(d-1)/d}\mathcal{H}^{d-1}(\partial C),
\end{aligned}$$

proving (iv). ∎

Proposition 4.3. *Let $\lambda\in(0,1)$. Let $E\subset\mathbb{R}^d$ be a set of locally finite perimeter and let $\mathcal{B}$ be a finite set of balls such that for each $B\in\mathcal{B}$ we have $\mathcal{L}(E\cap B)>\lambda\mathcal{L}(B)$. Then*

$$\mathcal{H}^{d-1}(\partial_*\textstyle\bigcup\mathcal{B}\setminus\overline{E}^*)\lesssim\lambda^{-(d-1)/d}(1-\log\lambda)\,\mathcal{H}^{d-1}\big(\partial_*E\cap\operatorname{int}_*\big(\bigcup\mathcal{B}\big)\big).$$

Proposition 4.3 is the key ingredient in the proof of Proposition 1.5. The idea of the proof of Proposition 4.3 is that we want to split $\partial_*\bigcup\mathcal{B}$ into pieces according to how far away a piece of $\partial_*\bigcup\mathcal{B}$ is from a significant portion of E, and then identify for each such piece of $\partial_*\bigcup\mathcal{B}$ a corresponding piece of ∂_*E with comparable size.

Proof of Proposition 4.3. We use Lemma 4.2. We first rearrange $\partial_*\bigcup\mathcal{B}\setminus\overline{E}^*$ and divide it according to the $(\mathcal{C}_n)_n$ in Lemma 4.2 and apply Lemma 4.1. We obtain

$$\begin{aligned}
\mathcal{H}^{d-1}\big(\partial_*\textstyle\bigcup\mathcal{B}\setminus\overline{E}^*\big)&=\mathcal{H}^{d-1}\big(\bigcup_k\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_k\setminus\overline{E}^*\big)\\
&=\mathcal{H}^{d-1}\big(\bigcup_k\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_k\cap\bigcup_{n\leq k+1}\bigcup5\mathcal{C}_n\big)\\
&=\mathcal{H}^{d-1}\big(\bigcup_n\bigcup_{k\geq n-1}\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_k\cap\bigcup5\mathcal{C}_n\big)\\
&=\mathcal{H}^{d-1}\big(\bigcup_n\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_{\geq n-1}\cap\bigcup5\mathcal{C}_n\big)\\
&\leq\sum_n\mathcal{H}^{d-1}\big(\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_{\geq n-1}\cap\bigcup5\mathcal{C}_n\big)\\
&\leq\sum_n\sum_{C\in\mathcal{C}_n}\mathcal{H}^{d-1}\big(\partial_*\bigcup\mathcal{B}\cap\partial_*\bigcup\mathcal{B}_{\geq n-1}\cap5C\big)\\
&\lesssim\sum_n\sum_{C\in\mathcal{C}_n}\mathcal{H}^{d-1}(\partial C).
\end{aligned}$$

In what follows we apply first (iv), then (i) and (iii). We obtain

$$\begin{aligned}
&\sum_{C\in\mathscr{C}_n} \mathscr{H}^{d-1}(\partial C)\\
&\quad\lesssim \lambda^{-(d-1)/d} \sum_{C\in\mathscr{C}_n} \mathscr{H}^{d-1}\big(\partial_* E \cap \big\{x\in C : \operatorname{dist}\big(x, \bigcup \mathscr{B}^{\mathrm{c}}\big) \geq \lambda d^{-1} 2^{n-d-2}\big\}\big)\\
&\quad= \lambda^{-(d-1)/d} \mathscr{H}^{d-1}\big(\partial_* E \cap \big\{x\in \bigcup \mathscr{C}_n : \operatorname{dist}\big(x, \bigcup \mathscr{B}^{\mathrm{c}}\big) \geq \lambda d^{-} 2^{n-d-2}\big\}\big)\\
&\quad\leq \lambda^{-(d-1)/d} \mathscr{H}^{d-1}\big(\partial_* E \cap \big\{x : \lambda d^{-1} 2^{n-d-2} \leq \operatorname{dist}\big(x, \bigcup \mathscr{B}^{\mathrm{c}}\big) \leq 2^{n+1}\big\}\big).
\end{aligned}$$

Now we sum over n. Since for a fixed number $r\in\mathbb{R}$ the condition $\lambda d^{-1}2^{n-d-2} \leq r \leq 2^{n+1}$ can only occur for $d+3+\log_2 d - \log_2\lambda$ many $n\in\mathbb{Z}$, we can bound

$$\begin{aligned}
&\mathscr{H}^{d-1}\big(\partial_* \bigcup \mathscr{B} \setminus \overline{E}^*\big)\\
&\quad\lesssim \lambda^{-(d-1)/d} \sum_n \mathscr{H}^{d-1}\big(\partial_* E \cap \big\{x : \lambda d^{-1} 2^{n-d-2} \leq \operatorname{dist}\big(x, \bigcup \mathscr{B}^{\mathrm{c}}\big) \leq 2^{n+1}\big\}\big)\\
&\quad\lesssim \lambda^{-(d-1)/d} (1-\log\lambda)\, \mathscr{H}^{d-1}\big(\partial_* E \cap \bigcup \mathscr{B}\big). \qquad \blacksquare
\end{aligned}$$

Remark 4.4. If the balls in $\bigcup_n \mathscr{C}_n$ were disjoint, then we could get rid of the factor $1-\log\lambda$ by using Remark 3.2 instead of (iv).

Now we extend Proposition 4.3 to the whole set $\{\mathrm{M}1_E > \lambda\}$.

Proof of Proposition 1.5. Note that

$$\{\mathrm{M}1_E > \lambda\} = \bigcup \{B \subset \Omega : \mathscr{L}(B\cap E) > \lambda \mathscr{L}(B)\}.$$

First we pass to a countable set of balls. By the Lindelöf property, see for example Proposition 1.5 in [4], there is a sequence of balls with

$$\{\mathrm{M}1_E > \lambda\} = B_1 \cup B_2 \cup \cdots$$

such that for each i we have $\mathscr{L}(E\cap B_i) > \lambda\mathscr{L}(B_i)$. Denote $\mathscr{B}_n = \{B_1,\dots,B_n\}$. Then $\bigcup \mathscr{B}_n$ converges to $\{\mathrm{M}1_E > \lambda\}$ in $L^1_{\mathrm{loc}}(\Omega)$. Furthermore, by Lemma 2.4 we have

$$\bigcup \mathscr{B}_n \subset \bigcup \mathscr{B}_n \cup \operatorname{int}_*(E) \subset \{\mathrm{M}1_E > \lambda\},$$

which means that also $\cup\mathscr{B}_n \cup E$ converges to $\{\mathrm{M}1_E > \lambda\}$ in $L^1_{\mathrm{loc}}(\Omega)$. Since E and $\operatorname{int}_*(E)$ agree up to a set of measure zero, we have $\overline{(\operatorname{int}_*(E))}^* = \overline{E}^*$ and $\partial_*(\operatorname{int}_*(E)) = \partial_* E$. We apply the approximation using Lemma 1.10 and then divide the boundary using Lemma 1.6 and obtain

$$\begin{aligned}
&\mathscr{H}^{d-1}(\partial_*\{\mathrm{M}1_E > \lambda\} \cap \Omega)\\
&\quad\leq \limsup_{n\to\infty} \mathscr{H}^{d-1}\big(\partial_*\big(\bigcup \mathscr{B}_n \cup \operatorname{int}_*(E)\big) \cap \Omega\big)\\
&\quad\leq \limsup_{n\to\infty} \mathscr{H}^{d-1}\big(\partial_* \bigcup \mathscr{B}_n \setminus \overline{E}^* \cap \Omega\big) + \mathscr{H}^{d-1}\big(\partial_* E \setminus \operatorname{int}_*\big(\bigcup \mathscr{B}_n\big) \cap \Omega\big).
\end{aligned} \tag{4.1}$$

By Lemma 2.4, the second summand is bounded by $\mathcal{H}^{d-1}(\partial_* E \cap \Omega \cap \{\mathrm{M}1_E > \lambda\})$. In fact, if $\mathcal{H}^{d-1}(\partial_* E \cap \Omega \cap \{\mathrm{M}1_E > \lambda\})$ is finite then the second summand in (4.1) even goes to 0 for $n \to \infty$. This is due to Lemma 2.4 for the uncentered maximal function, because

$$\mathrm{int}_*\Big(\bigcup \mathcal{B}_n\Big) \supset \bigcup \mathcal{B}_n,$$

which is an increasing sequence in n which exhausts $\{\mathrm{M}1_E > \lambda\}$. In any case, it remains to estimate the first summand in (4.1) which we do using Proposition 4.3:

$$\begin{aligned}\mathcal{H}^{d-1}\big(\partial_* \bigcup \mathcal{B}_n \setminus \overline{E}^*\big) &\lesssim \lambda^{-(d-1)/d}(1-\log\lambda)\,\mathcal{H}^{d-1}\big(\partial_* E \cap \bigcup \mathcal{B}_n\big)\\ &\leq \lambda^{-(d-1)/d}(1-\log\lambda)\,\mathcal{H}^{d-1}(\partial_* E \cap \{\mathrm{M}1_E > \lambda\}). \qquad \blacksquare\end{aligned}$$

5. The optimal rate in λ

In this section we prove the following improvement of Proposition 1.5.

Proposition 5.1. *Let* M *be the local uncentered maximal operator. Let* $E \subset \mathbb{R}^d$ *be a set with locally finite perimeter and let* $\lambda \in (0,1)$. *Then*

$$\mathcal{H}^{d-1}(\partial_*\{\mathrm{M}1_E > \lambda\} \cap \Omega) \lesssim \lambda^{-(d-1)/d}\,\mathcal{H}^{d-1}(\partial_* E \cap \{\mathrm{M}1_E > \lambda\}).$$

More important than the statement of Proposition 5.1 is maybe the proof strategy. It may be helpful when attempting to generalize Theorem 1.3 to $\operatorname{var}\mathrm{M}f \lesssim \operatorname{var} f$ for general functions f with bounded variation.

Remark 5.2. From taking $\Omega = \mathbb{R}^d$ and $E = B(0,1)$, it follows that the rate $\lambda^{-(d-1)/d}$ in Proposition 5.1 is optimal.

In order to prove Proposition 5.1, it suffices to prove the following improvement of Proposition 4.3.

Proposition 5.3. *Let* $\lambda \in [0,1/2)$, *let* $E \subset \mathbb{R}^d$ *be a set of locally finite perimeter and let* $\mathcal{B}$ *be a finite set of balls such that for each* $B \in \mathcal{B}$ *we have* $\lambda\mathcal{L}(B) < \mathcal{L}(E \cap B) \leq \frac{1}{2}\mathcal{L}(B)$. *Then*

$$\mathcal{H}^{d-1}\big(\partial_* \bigcup \mathcal{B}\big) \lesssim \lambda^{-(d-1)/d}\,\mathcal{H}^{d-1}\big(\partial_* E \cap \bigcup \mathcal{B}\big).$$

Proof of Proposition 5.1. Let $\mathcal{B}$ be a finite set of balls B with $\mathcal{L}(B \cap E) \geq \lambda\mathcal{L}(B)$. Then

$$\begin{aligned}\mathcal{H}^{d-1}\big(\partial \bigcup \mathcal{B} \setminus \overline{E}^*\big) &\leq \mathcal{H}^{d-1}(\partial\{B \in \mathcal{B} : \mathcal{L}(B \cap E) > \mathcal{L}(B)/2\} \setminus \overline{E}^*)\\ &\quad + \mathcal{H}^{d-1}(\partial\{B \in \mathcal{B} : \lambda\mathcal{L}(B) < \mathcal{L}(B \cap E) \leq \mathcal{L}(B)/2\} \setminus \overline{E}^*)\end{aligned}$$

By Proposition 4.3, the first summand in the previous display is bounded by a dimensional constant times $\mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B})$, and by Proposition 5.3, the second summand is bounded by a dimensional constant times $\lambda^{-(d-1)/d}\mathcal{H}^{d-1}(\partial_* E \cap \bigcup \mathcal{B})$. We conclude

$$\mathcal{H}^{d-1}\big(\partial \bigcup \mathcal{B} \setminus \overline{E}^*\big) \lesssim \lambda^{-(d-1)/d}\,\mathcal{H}^{d-1}\big(\partial_* E \cap \bigcup \mathcal{B}\big),$$

which is Proposition 4.3 without the factor $1 - \log\lambda$. Now we can repeat the proof of Proposition 1.5 verbatim without the factor $1 - \log\lambda$. $\blacksquare$

There is a weaker version of Proposition 5.3 which has a simpler proof, but already suffices to prove Proposition 5.1 for $\Omega = \mathbb{R}^d$.

Proposition 5.4. *There is an $\varepsilon > 0$ depending only on the dimension such that for all $\lambda \in [0, \varepsilon)$ the following holds. Let $E \subset \mathbb{R}^d$ be a set of locally finite perimeter and let $\mathcal{B}$ be a finite set of balls such that for each $B \in \mathcal{B}$ we have $\lambda\mathcal{L}(B) < \mathcal{L}(E \cap B) \le \varepsilon\mathcal{L}(B)$. Then there is a finite superset $\tilde{\mathcal{B}}$ of $\mathcal{B}$ consisting of balls B with $\mathcal{L}(E \cap B) > \lambda\mathcal{L}(B)$ that satisfies*

$$\mathcal{H}^{d-1}\big(\partial_* \bigcup \tilde{\mathcal{B}}\big) \lesssim \lambda^{-(d-1)/d}\, \mathcal{H}^{d-1}\big(\partial_* E \cap \bigcup \mathcal{B}\big).$$

Proof of Proposition 5.1 for $\Omega = \mathbb{R}^d$. Take $\varepsilon > 0$ from Proposition 5.4. For $\lambda \ge \varepsilon$, Proposition 5.1 already follows from Proposition 1.5. It suffices to consider the case that there is an $x_0 \in \mathbb{R}^d$ with $\lambda < \mathrm{M}1_E(x_0) \le \varepsilon$. Let $x \in \mathbb{R}^d$ with $\mathrm{M}1_E(x) > \lambda$. Then there is a ball $C \ni x$ with $\mathcal{L}(E \cap C) > \lambda\mathcal{L}(C)$, while $\mathcal{L}(E \cap B(x_0, |x - x_0| + 1)) \le \varepsilon\mathcal{L}(B(x_0, |x - x_0|) + 1)$. By continuously transforming C into $B(x_0, |x - x_2| + 1)$, we can conclude that $\{\mathrm{M}1_E > \lambda\}$ is a union of balls B with $\lambda\mathcal{L}(B) < \mathcal{L}(E \cap B) \le \varepsilon\mathcal{L}(B)$. Thus by the Lindelöf property there is a sequence of balls $(B_n)_n$ with $\lambda\mathcal{L}(B_n) < \mathcal{L}(E \cap B_n) \le \varepsilon\mathcal{L}(B_n)$ such that $\{\mathrm{M}1_E > \lambda\} = B_1 \cup B_2 \cup \ldots$. Let $\tilde{\mathcal{B}}_n$ be the finite superset of $\mathcal{B}_n = \{B_1, \ldots, B_n\}$ from Proposition 5.4. Then

$$\bigcup \mathcal{B}_n \subset \bigcup \tilde{\mathcal{B}}_n \subset \{\mathrm{M}1_E > \lambda\}$$

which means that $\bigcup \tilde{\mathcal{B}}_n$ converges to $\{\mathrm{M}1_E > \lambda\}$ in $L^1_{\mathrm{loc}}(\Omega)$. Thus we get as in the proof of Proposition 1.5 that

$$\mathcal{H}^{d-1}(\partial_*\{\mathrm{M}1_E > \lambda\}) \le \limsup_{n\to\infty} \mathcal{H}^{d-1}\big(\partial_* \bigcup \tilde{\mathcal{B}}_n\big).$$

By Proposition 5.4 we have

$$\begin{aligned}\mathcal{H}^{d-1}\big(\partial_* \bigcup \tilde{\mathcal{B}}_n\big) &\lesssim \lambda^{-(d-1)/d}\,\mathcal{H}^{d-1}\big(\partial_* E \cap \bigcup \mathcal{B}_n\big)\\ &\le \lambda^{-(d-1)/d}\,\mathcal{H}^{d-1}(\partial_* E \cap \{\mathrm{M}1_E > \lambda\}).\end{aligned}$$

■

5.1. The global case $\Omega = \mathbb{R}^d$

In this subsection we present a proof of Proposition 5.4. It already contains some of the ideas for the general local case Proposition 5.3.

Proof of Proposition 5.4. First, restrict $\varepsilon \le 1/2$. Let $\mathcal{F}'$ be the collection of balls from Lemma 2.2 applied to $E \cap \bigcup \mathcal{B}$ and $\mathcal{B}$. Let $\tilde{\mathcal{F}}$ be the countable disjoint subcollection from Lemma 1.9. Extract from that a finite subcollection $\mathcal{F}$ so that for every $B \in \mathcal{B}$ we have

$$\mathcal{L}\big(E \cap \bigcup 5\mathcal{F} \cap B\big) \ge \frac{\lambda}{2}\,\mathcal{L}(B). \tag{5.1}$$

This is possible since $\mathcal{B}$ is finite. Here $\mathcal{F}$ serves as a decomposition of E into pieces $F \cap E$ where each piece has a substantial amount of boundary. The overall goal is to collect for each F its contribution to $\mathcal{H}^{d-1}(\partial_*\bigcup\mathcal{B})$ and show that it is bounded by ∂F. First we enlarge $\mathcal{B}$. For every $F\in\mathcal{F}$, the ball $B=(2\lambda)^{-1/d}F$ satisfies

$$\mathcal{L}(E\cap B)\geq\mathcal{L}(E\cap F)=\frac{\mathcal{L}(F)}{2}=\lambda\mathcal{L}(B).$$

Add all those balls B to $\mathcal{B}$. Then $\mathcal{B}$ is still finite.

Restrict $\varepsilon\leq\frac{1}{2}10^{-d}$ and let $r>0$ and $F\in\mathcal{F}$ with $\operatorname{diam}F\geq 8r(2\lambda)^{1/d}$. Since we assume $\lambda\leq\varepsilon$, we obtain

$$\operatorname{diam}((2\lambda)^{-1/d}F)-\operatorname{diam}(5F)=((2\lambda)^{-1/d}-5)\operatorname{diam}F\geq(1-5(2\lambda)^{1/d})\cdot 8r\geq 4r,$$

which means that any ball $B\in\mathcal{B}$ with diameter at most r that intersects $5F$ is entirely contained in $(2\lambda)^{-1/d}F\in\mathcal{B}$. Hence we may remove B from $\mathcal{B}$ without changing $\partial_*\bigcup\mathcal{B}\setminus\overline{E}^*$. Conversely, we may assume that if $B\in\mathcal{B}$ has diameter r and $F\in\mathcal{F}$ is a ball for which $5F$ intersects B, then $\operatorname{diam}F<8r(2\lambda)^{1/d}$. We further restrict $\varepsilon\leq\frac{1}{4}20^{-d}$ and obtain

$$\frac{\mathcal{L}(5F)}{\mathcal{L}(B)}<\frac{5^d8^dr^d2\varepsilon}{2^dr^d}\leq\frac{1}{2}. \tag{5.2}$$

For each $n\in\mathbb{Z}$, denote by $\mathcal{B}_n$ the set of balls in $\mathcal{B}$ with $\operatorname{diam}B\in[\frac{1}{2},1)2^n$. Denote by $\mathcal{F}_n$ the set of those balls with $\operatorname{diam}F\in 2^n(2\lambda)^{1/d}[4,8)$. Let $B\in\mathcal{B}_n$ and let $F\in\mathcal{F}$ be such that $5F$ intersects B. Then

$$F\in\mathcal{F}_k\quad\text{for some } k\leq n. \tag{5.3}$$

By (5.2), any $F\in\mathcal{F}$ such that $5F$ intersects B is contained in $3B$. Thus we get from (5.1) and (5.3) that

$$\frac{\lambda}{2}\mathcal{L}(B)\leq\sum_{F\in\mathcal{F}_{\leq n},\ F\subset 3B}\mathcal{L}(5F\cap B).$$

We rewrite the previous display as

$$\begin{aligned}\mathcal{H}^{d-1}(\partial B)&\leq 2\sum_{k\leq n}\sum_{F\in\mathcal{F}_k,F\subset 3B}\frac{\mathcal{L}(5F\cap B)}{\lambda\mathcal{L}(B)}\mathcal{H}^{d-1}(\partial B)\\&\lesssim\sum_{k\leq n}\sum_{F\in\mathcal{F}_k,F\subset 3B}\Big(\frac{\mathcal{L}(F)}{\lambda\mathcal{L}(B)}\Big)^{1/d}\lambda^{-(d-1)/d}\Big(\frac{\mathcal{L}(F)}{\mathcal{L}(B)}\Big)^{(d-1)/d}\mathcal{H}^{d-1}(\partial B)\\&\sim\sum_{k\leq n}\sum_{F\in\mathcal{F}_k,F\subset 3B}2^{k-n}\lambda^{-(d-1)/d}\mathcal{H}^{d-1}(\partial F).\end{aligned}\tag{5.4}$$

This estimate can be seen as a way to distribute $\mathcal{H}^{d-1}(\partial B)$ over the balls F that it contains. The next step will be to turn this dependence around, and see, for a fixed F, for how much variation of $\mathcal{H}^{d-1}(\partial_*\bigcup\mathcal{B})$ it is responsible.

Since $\mathcal{B}_n$ is finite, we have

$$\mathcal{H}^{d-1}\big(\partial_*\bigcup\mathcal{B}_n\big)=\sum_{B\in\mathcal{B}_n}\mathcal{H}^{d-1}\big(\partial B\cap\partial_*\bigcup\mathcal{B}_n\big).$$

We again multiply each summand by a number bounded from below according to (5.4):

$$\begin{aligned}\sum_{B\in\mathcal{B}_n}&\mathcal{H}^{d-1}\big(\partial B\cap\partial_*\bigcup\mathcal{B}_n\big)\\&\lesssim\sum_{B\in\mathcal{B}_n}\frac{\mathcal{H}^{d-1}(\partial B\cap\partial_*\bigcup\mathcal{B}_n)}{\mathcal{H}^{d-1}(\partial B)}\sum_{k\le n}\sum_{F\in\mathcal{F}_k,F\subset 3B}2^{k-n}\lambda^{-(d-1)/d}\mathcal{H}^{d-1}(\partial F)\\&=\lambda^{-(d-1)/d}\sum_{k\le n}2^{k-n}\sum_{F\in\mathcal{F}_k}\mathcal{H}^{d-1}(\partial F)\sum_{B\in\mathcal{B}_n,3B\supset F}\frac{\mathcal{H}^{d-1}(\partial B\cap\partial_*\bigcup\mathcal{B}_n)}{\mathcal{H}^{d-1}(\partial B)}.\end{aligned}$$

Now we have reorganized $\partial_*\bigcup\mathcal{B}_n$ according to the balls in $\mathcal{F}$. We want to bound the contribution of each ball $F\in\mathcal{F}$ uniformly. For each $F\in\mathcal{F}_k$ for which there is a ball $B\in\mathcal{B}_n$ with $F\subset 3B$, denote by B_F a largest such ball B. Then for each $B\in\mathcal{B}_n$ with $F\subset 3B$, we have $B\subset 9B_F$. Thus

$$\sum_{B\in\mathcal{B}_n,3B\supset F}\frac{\mathcal{H}^{d-1}(\partial B\cap\partial_*\bigcup\mathcal{B}_n)}{\mathcal{H}^{d-1}(\partial B)}\lesssim\sum_{B\in\mathcal{B}_n,B\subset 9B_F}\frac{\mathcal{H}^{d-1}(\partial B\cap\partial_*\bigcup\mathcal{B}_n)}{\mathcal{H}^{d-1}(\partial B_F)},$$

which is uniformly bounded according to Lemma 4.1. Therefore we can conclude

$$\mathcal{H}^{d-1}\big(\partial_*\bigcup\mathcal{B}_n\big)\lesssim\lambda^{-(d-1)/d}\sum_{k\le n}2^{k-n}\sum_{F\in\mathcal{F}_k}\mathcal{H}^{d-1}(\partial F).$$

So the interaction between the scales is small enough so that we can just sum over all scales and obtain

$$\begin{aligned}\mathcal{H}^{d-1}\big(\partial_*\bigcup\mathcal{B}\big)\le\sum_n\mathcal{H}^{d-1}\big(\partial_*\bigcup\mathcal{B}_n\big)&\lesssim\lambda^{-(d-1)/d}\sum_k\sum_{n\ge k}2^{k-n}\sum_{F\in\mathcal{F}_k}\mathcal{H}^{d-1}(\partial F)\\&\lesssim\lambda^{-(d-1)/d}\sum_k\sum_{F\in\mathcal{F}_k}\mathcal{H}^{d-1}(\partial F)=\lambda^{-(d-1)/d}\sum_{F\in\mathcal{F}}\mathcal{H}^{d-1}(\partial F).\end{aligned}$$

Now we get back from $\mathcal{F}$ to E. Recall that for each $F\in\mathcal{F}$ we have $\mathcal{L}(F\cap E)=\mathcal{L}(F)/2$, so that by Lemma 2.1 we have $\mathcal{H}^{d-1}(\partial F)\lesssim\mathcal{H}^{d-1}(\partial_*E\cap F)$. Because the balls in $\mathcal{F}$ are disjoint, we can then conclude

$$\mathcal{H}^{d-1}\big(\partial_*\bigcup\mathcal{B}\big)\lesssim\lambda^{-(d-1)/d}\sum_{F\in\mathcal{F}}\mathcal{H}^{d-1}(\partial_*E\cap F)\le\lambda^{-(d-1)/d}\mathcal{H}^{d-1}\big(\partial_*E\cap\bigcup\mathcal{B}\big).\ \blacksquare$$

5.2. The general local case $\Omega\subset\mathbb{R}^d$

In this subsection we present a proof of Proposition 5.3. It requires a few more steps than the proof of Proposition 5.4.

Lemma 5.5. *Let* $0\le\lambda\le 2^{-(d+1)/2}(d+1)^{-1/2}$ *and let* B,C *be balls with* $\operatorname{diam}C\ge\operatorname{diam}B$ *and* $\mathcal{L}(B\cap C)\le\lambda\mathcal{L}(B)$. *Then* $(1-2(d+1)^{\frac{1}{d+1}}\lambda^{\frac{2}{d+1}})B$ *and* C *are disjoint.*

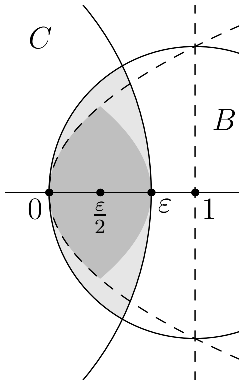

Figure 4. The lower bound for $\mathcal{L}(B \cap C)$ in the proof of Lemma 5.5.

For the application we only need that for λ small enough, B and $(3/4)^{1/d} C$ are disjoint. Since $\operatorname{diam} C \geq \operatorname{diam} B$, this follows if $(3/4)^{1/d} B$ and C are disjoint. The rate in λ plays no role.

Proof. After rescaling, rotation and translation, it suffices to consider the case that there are $r \geq 1$ and $0 < \varepsilon \leq 2$ such that $B = B(e_1, 1)$ and $C = B((\varepsilon - r)e_1, r)$. We bound $\mathcal{L}(B \cap C)$ from below by the marked area in Figure 4. For $x \in \mathbb{R}^d$, denote $\bar{x}_1 = (x_2, \ldots, x_d)$. The two spheres ∂B and ∂C intersect in a plane orthogonal to e_1 that is between $\frac{\varepsilon}{2} e_1$ and εe_1. Thus

$$\left\{x : \bar{x}_1^2 < x_1 < \frac{\varepsilon}{2}\right\} \subset \left\{x \in B : x_1 < \frac{\varepsilon}{2}\right\} \subset B \cap C,$$

and by symmetry and $r \geq 1$ also the image of the first set mirrored at $x_1 = \varepsilon/2$ is contained in $B \cap C$, so that

$$\mathcal{L}(B \cap C) > 2\mathcal{L}\left(\left\{x : \bar{x}_1^2 < x_1 < \frac{\varepsilon}{2}\right\}\right) = 2 \int_0^{\varepsilon/2} \sigma_{d-1} h^{\frac{d-1}{2}} \, \mathrm{d}h = 2^{-\frac{d-3}{2}} \frac{\sigma_{d-1}}{d+1} \varepsilon^{\frac{d+1}{2}}.$$

Therefore, since $\mathcal{L}(B \cap C) \leq \lambda \mathcal{L}(B) = \lambda \sigma_d$, we can conclude the following upper bound for ε using Lemma 1.11:

$$\varepsilon^{\frac{d+1}{2}} \leq \frac{\lambda(d+1)\sigma_d}{\sigma_{d-1}} 2^{\frac{d-3}{2}} \leq 2^{\frac{d-2}{2}} \frac{d+1}{\sqrt{d}} \sqrt{\pi} \lambda \leq 2^{\frac{d+1}{2}} \frac{d+1}{\sqrt{2d}} \lambda \leq 2^{\frac{d+1}{2}} (d+1)^{1/2} \lambda,$$
$$\varepsilon \leq 2(d+1)^{\frac{1}{d+1}} \lambda^{\frac{2}{d+1}}.$$

This finishes the proof because $(1 - \varepsilon)B$ and C are disjoint. ∎

Lemma 5.6. *Let $0 < \lambda \leq 1$, let B be a ball and let $\mathcal{F}$ be a set of balls with $\mathcal{L}(\bigcup \mathcal{F} \cap B) \geq \lambda \mathcal{L}(B)$. Then there is a ball $F \in \mathcal{F}$ which intersects $(1 - \lambda/d)B$.*

Proof. Since

$$\mathcal{L}(B \setminus (1 - \lambda/d)B) = d\mathcal{L}(B) \int_{1-\lambda/d}^{1} r^{d-1} \, \mathrm{d}r < \lambda \mathcal{L}(B),$$

the union $\bigcup \mathcal{F}$ cannot lie outside of $(1 - \lambda/d)B$. ∎

Proof of Proposition 5.3. According to Lemma 2.2, for every $B \in \mathcal{B}$, almost every point in $E \cap B$ is contained in a ball $F \subset B$ with

$$\mathcal{L}(F \cap E) = \frac{1}{2}\, \mathcal{L}(F).$$

Denote by $\mathcal{G}$ the set of all such balls F. By scaling, it suffices to consider the case that all balls in $\mathcal{G}$ and $\mathcal{B}$ have diameter at most 1. We inductively build sequences $(\mathcal{F}_n)_{n=0}^{-\infty}$ and $(\mathcal{G}_n)_{n=0}^{-\infty}$ of subsets of $\mathcal{G}$. We denote $\mathcal{F}_{>n} = \bigcup_{n<k\le 0} \mathcal{F}_n$, and $\mathcal{G}_{>n}$ and $\mathcal{F}_{n<\cdot\le k}$ accordingly. Assume we are at scale $n \le 0$. Denote by $\mathcal{B}_n$ the set of balls in $\mathcal{B}$ with $\operatorname{diam} B \in [\frac{1}{2}, 1)2^n$. Decompose $\mathcal{B}_n$ into

$$\mathcal{B}_n^0 = \Big\{B \in \mathcal{B}_n : \mathcal{L}\big(\bigcup 5\mathcal{F}_{>n} \cap B\big) \le \frac{\lambda}{2}\, \mathcal{L}(B)\Big\},$$
$$\mathcal{B}_n^1 = \Big\{B \in \mathcal{B}_n : \mathcal{L}\big(\bigcup 5\mathcal{F}_{>n} \cap B\big) > \frac{\lambda}{2}\, \mathcal{L}(B)\Big\}$$

and decompose $\mathcal{B}_n^1$ into

$$\mathcal{B}_n^{1,0} = \Big\{B \in \mathcal{B}_n^1 : \mathcal{L}\big(\bigcup \mathcal{F}_{>n} \cap B\big) \le \frac{1}{8^{(d+1)/2}(d+1)^{(d+2)/2}}\, \mathcal{L}(B)\Big\},$$
$$\mathcal{B}_n^{1,1} = \Big\{B \in \mathcal{B}_n^1 : \mathcal{L}\big(\bigcup \mathcal{F}_{>n} \cap B\big) > \frac{1}{8^{(d+1)/2}(d+1)^{(d+2)/2}}\, \mathcal{L}(B)\Big\}.$$

Denote by $\mathcal{G}_n$ the set of balls $G \in \mathcal{G}$ with $\operatorname{diam} G \in [\frac{1}{2}, 1)2^n$ which intersect $E \setminus \bigcup 5\mathcal{F}_{>n}$ or are for some $k \ge n$ and some $B \in \mathcal{B}_k^{1,0}$ contained in $B \setminus \bigcup \mathcal{F}_{n<\cdot\le k}$. Set $\mathcal{F}_n$ to be a maximal disjoint subcollection of $\mathcal{G}_n$.

Denote $\mathcal{F} = \bigcup_n \mathcal{F}_n$, $\mathcal{B}_0 = \bigcup_n \mathcal{B}_n^0$, and $\mathcal{B}^{1,0}$ and $\mathcal{B}^{1,1}$ accordingly. Here are a few properties of these ball collections.

(i) The collection $5\mathcal{F}_n$ is a cover of $\bigcup \mathcal{G}_n$.

(ii) The collection $5\mathcal{F}$ covers almost all of E.

(iii) The balls in $(3/4)^{1/d}\mathcal{F}$ are disjoint.

(iv) If $B \in \mathcal{B}_n^0$, then $5\mathcal{F}_{\le n}$ covers at least $\lambda/2$ of B.

(v) If $B \in \mathcal{B}_n^{1,0}$, then $5\mathcal{F}_{\le n}$ covers at least λ of B.

Proof. (i) By the maximality of $\mathcal{F}_n$, every $G \in \mathcal{G}_n$ intersects an $F \in \mathcal{F}_n$. Since $\operatorname{diam} G \le 2 \operatorname{diam} F$, this means $G \subset 5F$.

(ii) Let $G \in \mathcal{G}$ be a ball and let $n \in \mathbb{Z}$ be the integer with $\operatorname{diam} G \in [\frac{1}{2}, 1)2^n$. Then G intersects E, so that by definition of $\mathcal{G}_n$ we have $G \cap E \subset \bigcup 5\mathcal{F}_{>n}$ or $G \in \mathcal{G}_n$. By (i) we can conclude $G \cap E \subset \bigcup 5\mathcal{F}_{\ge n}$ in either case. Since $\mathcal{G}$ covers almost all of E, this means so does $5\mathcal{F}$.

(iii) For each n, the balls in $\mathcal{F}_n$ are disjoint. It remains to show that they are disjoint from the balls in $(3/4)^{1/d}\mathcal{F}_{>n}$. So assume $F \in \mathcal{F}_n$. If F was chosen because it intersects $E \setminus \bigcup 5\mathcal{F}_{>n}$, then it does not intersect $\mathcal{F}_{>n}$. It remains to consider the case that there is a $k \geq n$ and a $B \in \mathcal{B}_k^{1,0}$ such that $F \subset B$ and F does not intersect any $G \in \mathcal{F}_{n<\cdot\leq k}$. Since $B \in \mathcal{B}_k^{1,0}$, for every $G \in \mathcal{F}_{>k}$ we have

$$\mathcal{L}(B \cap G) \leq \frac{1}{8^{(d+1)/2}(d+1)^{(d+2)/2}} \mathcal{L}(B),$$

so that by Lemma 5.5 the balls $\big(1 - \frac{1}{4(d+1)}\big)B$ and G are disjoint. Since $(3/4)^{1/d} \leq 1 - \frac{1}{4(d+1)}$ and $\operatorname{diam} G \geq \operatorname{diam} B$, this means that $(3/4)^{1/d} G$ and B are disjoint, too. Hence also F and $\bigcup (3/4)^{1/d} \mathcal{F}_{>k}$ are disjoint.

(iv) For every $B \in \mathcal{B}_n^0$, we have $\mathcal{L}(B \cap E) \geq \lambda \mathcal{L}(B)$. Thus since $5\mathcal{F}$ covers almost all of E and

$$\mathcal{L}\big(\bigcup 5\mathcal{F}_{>n} \cap B\big) \leq \frac{\lambda}{2} \mathcal{L}(B),$$

we must have

$$\mathcal{L}\big(\bigcup 5\mathcal{F}_{\leq n} \cap B\big) \geq \frac{\lambda}{2} \mathcal{L}(B).$$

(v) Let $B \in \mathcal{B}_n^{1,0}$. It suffices to show that $5\mathcal{F}_{\leq n}$ covers $E \cap B$. By the construction of $\mathcal{G}$, using Lemma 2.2 almost all of $B \cap E$ is covered by the union of all $G \in \mathcal{G}$ with $G \subset B$ and $\operatorname{diam} G < 2^n$. Thus it suffices to show for each such G that $G \cap E$ is contained in $5\mathcal{F}_{\leq n}$. Take $k \leq n$ with $\operatorname{diam} G \in [\frac{1}{2}, 1)2^k$. If $G \cap E$ is not contained in $\bigcup 5\mathcal{F}_{k<\cdot\leq n}$, then $G \in \mathcal{G}_k$ and thus by (i) we have $G \subset \bigcup 5\mathcal{F}_k$. ∎

Denote $\tilde{\mathcal{B}} = \mathcal{B}^0 \cup \mathcal{B}^{1,0}$, so that $\mathcal{B} = \tilde{\mathcal{B}} \cup \mathcal{B}^{1,1}$. Then by Lemma 1.6 we have

$$\partial_* \bigcup \mathcal{B} \subset \Big(\partial_* \bigcup \tilde{\mathcal{B}}\Big) \cup \Big(\partial_* \bigcup \mathcal{B}^{1,1} \setminus \overline{\bigcup \tilde{\mathcal{B}}^*}\Big).$$

Note that for a finite union of balls, the topological and measure theoretical notions agree up to a set of $d-1$ dimensional measure zero. By Lemma 5.6, for every $B \in \mathcal{B}^{1,1}$ there is an $F \in \mathcal{F}$ with $\operatorname{diam} F > \operatorname{diam} B$ that intersects $(1 - 8^{-(d+1)/2} d^{-(d+4)/2})B$. Because F came about using Lemma 2.2, it is further contained in a ball $B_F \in \mathcal{B}$. Since $\operatorname{diam} B < \operatorname{diam} B_F$, we have $B \neq B_F$. For each $F \in \mathcal{F}$, denote by $\mathcal{B}(F)$ the set of $B \in \mathcal{B}$ with $\operatorname{diam} B < \operatorname{diam} F$ such that F intersects $(1 - 8^{-(d+1)/2} d^{-(d+4)/2})B$. Then

$$\begin{aligned}
\partial \bigcup \mathcal{B}^{1,1} \setminus \overline{\bigcup \tilde{\mathcal{B}}} \subset \partial \bigcup \mathcal{B}^{1,1} \setminus \bigcup \mathcal{B} &\subset \bigcup_{F \in \mathcal{F}} \partial \bigcup (\mathcal{B}^{1,1} \cap \mathcal{B}(F)) \setminus \bigcup \mathcal{B} \\
&\subset \bigcup_{F \in \mathcal{F}} \partial \bigcup (\mathcal{B}^{1,1} \cap \mathcal{B}(F)) \setminus \Big(\bigcup \mathcal{B}(F) \cup B_F\Big) \\
&\subset \bigcup_{F \in \mathcal{F}} \partial \bigcup \mathcal{B}(F) \setminus \Big(\bigcup \mathcal{B}(F) \cup B_F\Big) \\
&= \bigcup_{F \in \mathcal{F}} \partial \bigcup \mathcal{B}(F) \setminus B_F \subset \bigcup_{F \in \mathcal{F}} \partial \Big(F \cup \bigcup \mathcal{B}(F)\Big).
\end{aligned}$$

Thus Proposition 4.3 implies

$$\mathcal{H}^{d-1}\big(\partial \bigcup \mathcal{B}^{1,1} \setminus \overline{\bigcup \tilde{\mathcal{B}}}\big) \lesssim \sum_{F \in \mathcal{F}} \mathcal{H}^{d-1}(\partial F)$$

Recall that we made $(3/4)^{1/d}\mathcal{F}$ disjoint and that by Lemma 2.2 for each $F \in \mathcal{F}$ we have $F \subset \bigcup \mathcal{B}$ and $\mathcal{L}(F \cap E) = \mathcal{L}(F)/2$. Thus $\mathcal{L}((\frac{3}{4})^{1/d} F \cap E) \in [\frac{1}{4}, \frac{3}{4}]\mathcal{L}(F)$, and so by Lemma 2.1 we can conclude

$$\sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial F) = \left(\frac{4}{3}\right)^{(d-1)/d} \sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial (3/4)^{1/d} F)$$

$$\lesssim \sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial_* E \cap (3/4)^{1/d} F) \le \mathcal{H}^{d-1}\big(\partial_* E \cap \bigcup \mathcal{B}\big). \tag{5.5}$$

It remains to prove

$$\mathcal{H}^{d-1}\big(\partial \bigcup \tilde{\mathcal{B}}\big) \lesssim \lambda^{-(d-1)/d} \sum_{F\in\mathcal{F}} \mathcal{H}^{d-1}(\partial F). \tag{5.6}$$

For $n \in \mathbb{Z}$, denote by $\tilde{\mathcal{B}}_n$ the set of balls $B \in \tilde{\mathcal{B}}$ with $\operatorname{diam} B \in [\frac{1}{2}, 1)2^n$. Let $B \in \tilde{\mathcal{B}}_n$ and let $F \in \mathcal{F}_{\le n}$ be a ball such that $5F$ intersects B. Then $F \subset 21B$. By (iv) and (v), this means

$$\frac{\lambda}{2}\mathcal{L}(B) \le \mathcal{L}\big(B \cap \bigcup 5\mathcal{F}_{\le n}\big) \le \sum_{F\in\mathcal{F}_{\le n}, F\subset 21B} \mathcal{L}(5F \cap B). \tag{5.7}$$

For each $k \in \mathbb{Z}$, denote by $\tilde{\mathcal{F}}_k$ the set of balls $F \in \mathcal{F}$ with $\operatorname{diam} F \in [\frac{1}{2}, 1)2^k\lambda^{1/d}$. We make a case distinction. If there is a $k \ge n$ and a ball $F \in \tilde{\mathcal{F}}_k$ with $F \subset 21B$, we have

$$\mathcal{H}^{d-1}(\partial B) = \frac{\mathcal{H}^{d-1}(\partial B)}{\mathcal{H}^{d-1}(\partial F)} \mathcal{H}^{d-1}(\partial F) \le 2^{2(d-1)} \frac{2^{n(d-1)}}{\lambda^{(d-1)/d}\, 2^{k(d-1)}} \mathcal{H}^{d-1}(\partial F)$$

$$\sim 2^{(n-k)(d-1)} \lambda^{-(d-1)/d} \mathcal{H}^{d-1}(\partial F), \tag{5.8}$$

and we are done with this case for the moment. Now assume all balls $F \in \mathcal{F}$ with $F \subset 21B$ are contained in $\tilde{\mathcal{F}}_{<n}$. Then by (5.7) we have

$$\mathcal{H}^{d-1}(\partial B) \le 2 \sum_{k<n} \sum_{F\in\tilde{\mathcal{F}}_k, F\subset 21B} \frac{\mathcal{L}(5F \cap B)}{\lambda\mathcal{L}(B)} \mathcal{H}^{d-1}(\partial B)$$

$$\lesssim \sum_{k<n} \sum_{F\in\tilde{\mathcal{F}}_k, F\subset 21B} \left(\frac{\mathcal{L}(F)}{\lambda\mathcal{L}(B)}\right)^{1/d} \lambda^{-(d-1)/d} \left(\frac{\mathcal{L}(F)}{\mathcal{L}(B)}\right)^{(d-1)/d} \mathcal{H}^{d-1}(\partial B)$$

$$\sim \sum_{k<n} \sum_{F\in\tilde{\mathcal{F}}_k, F\subset 21B} 2^{k-n} \lambda^{-(d-1)/d} \mathcal{H}^{d-1}(\partial F). \tag{5.9}$$

If $d = 1$, then Proposition 5.3 is straightforward to prove directly, so it suffices to consider $d \ge 2$. There we can combine (5.8) and (5.9) into

$$\mathcal{H}^{d-1}(\partial B) \lesssim \lambda^{-(d-1)/d} \sum_{k} 2^{-|k-n|} \sum_{F\in\tilde{\mathcal{F}}_k, F\subset 21B} \mathcal{H}^{d-1}(\partial F)$$

for simplicity. This estimate can be seen as a way to distribute $\mathcal{H}^{d-1}(\partial B)$ over the balls F that it contains. The next step will be to turn the dependence around, and see, for a fixed ball $F \in \mathcal{F}$, for how much of $\mathcal{H}^{d-1}(\partial_* \bigcup \tilde{\mathcal{B}})$ it is responsible. Since $\tilde{\mathcal{B}}_n$ is finite, we have

$$\mathcal{H}^{d-1}\big(\partial_* \bigcup \tilde{\mathcal{B}}_n\big) = \sum_{B \in \tilde{\mathcal{B}}_n} \mathcal{H}^{d-1}\big(\partial B \cap \partial_* \bigcup \tilde{\mathcal{B}}_n\big),$$

and we multiply each summand by a number bounded from below according to (5.9):

$$\begin{aligned}
&\mathcal{H}^{d-1}\big(\partial_* \bigcup \tilde{\mathcal{B}}_n\big)\\
&\quad \lesssim \sum_{B \in \tilde{\mathcal{B}}_n} \frac{\mathcal{H}^{d-1}(\partial B \cap \partial_* \bigcup \tilde{\mathcal{B}}_n)}{\mathcal{H}^{d-1}(\partial B)} \sum_k \sum_{F \in \tilde{\mathcal{F}}_k, F \subset 21B} 2^{-|k-n|} \lambda^{-(d-1)/d} \mathcal{H}^{d-1}(\partial F)\\
&\quad = \lambda^{-(d-1)/d} \sum_k 2^{-|k-n|} \sum_{F \in \tilde{\mathcal{F}}_k} \mathcal{H}^{d-1}(\partial F) \sum_{B \in \tilde{\mathcal{B}}_n, 21B \supset F} \frac{\mathcal{H}^{d-1}(\partial B \cap \partial_* \bigcup \tilde{\mathcal{B}}_n)}{\mathcal{H}^{d-1}(\partial B)}.
\end{aligned}$$

We have reorganized $\partial_* \bigcup \tilde{\mathcal{B}}_n$ according to the balls in $\mathcal{F}$. We want to bound the contribution of each ball $F \in \mathcal{F}$ uniformly. For each $F \in \tilde{\mathcal{F}}_k$ for which there is a ball $B \in \tilde{\mathcal{B}}_n$ with $F \subset 21B$, denote by B_F a largest such B. Then for all $B \in \tilde{\mathcal{B}}_n$ with $F \subset 21B$, we have $B \subset 3B_F$. Thus,

$$\sum_{B \in \tilde{\mathcal{B}}_n, 21B \supset F} \frac{\mathcal{H}^{d-1}(\partial B \cap \partial_* \bigcup \tilde{\mathcal{B}}_n)}{\mathcal{H}^{d-1}(\partial B)} \lesssim \sum_{B \in \tilde{\mathcal{B}}_n, B \subset 63B_F} \frac{\mathcal{H}^{d-1}(\partial B \cap \partial_* \bigcup \tilde{\mathcal{B}}_n)}{\mathcal{H}^{d-1}(\partial B_F)},$$

which is uniformly bounded according to Lemma 4.1. Therefore we can conclude

$$\mathcal{H}^{d-1}\big(\partial_* \bigcup \tilde{\mathcal{B}}_n\big) \lesssim \lambda^{-(d-1)/d} \sum_k 2^{-|k-n|} \sum_{F \in \tilde{\mathcal{F}}_k} \mathcal{H}^{d-1}(\partial F).$$

So the interaction between the scales is small enough that we can just sum over all scales and obtain

$$\begin{aligned}
\mathcal{H}^{d-1}\big(\partial_* \bigcup \tilde{\mathcal{B}}\big) &\le \sum_n \mathcal{H}^{d-1}\big(\partial_* \bigcup \tilde{\mathcal{B}}_n\big) \lesssim \lambda^{-(d-1)/d} \sum_k \sum_n 2^{-|k-n|} \sum_{F \in \tilde{\mathcal{F}}_k} \mathcal{H}^{d-1}(\partial F)\\
&\lesssim \lambda^{-(d-1)/d} \sum_k \sum_{F \in \tilde{\mathcal{F}}_k} \mathcal{H}^{d-1}(\partial F) = \lambda^{-(d-1)/d} \sum_{F \in \mathcal{F}} \mathcal{H}^{d-1}(\partial F),
\end{aligned}$$

and we have proven (5.6), which was all that remained to finish the proof of Proposition 5.3. ∎

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