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On the completeness of dual foliations on nonnegatively curved symmetric spaces

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Abstract. We prove Wilking's conjecture about the completeness of dual leaves for the case of Riemannian foliations on nonnegatively curved symmetric spaces. Moreover, we conclude that such foliations split as a product of trivial foliations and a foliation with a single dual leaf.

1. Introduction

A *singular Riemannian foliation* $\mathcal{F}$ on M is a singular foliation, i.e., a decomposition of M into integral submanifolds, called *leaves*, of an involutive family of smooth vector fields, such that geodesics emanating perpendicularly to a leaf stays perpendicular to leaves.

Given a singular Riemannian foliation $\mathcal{F}$, *the dual leaf at* $x \in M$ is the subset

$$L^\#_x = \{q \in M \mid \exists c : [0,1] \to M,\ c(0) = x,\ c(1) = q,\ c \text{ is perpendicular to leaves}\}.$$

The set of dual leaves define the *dual foliation*. These concepts and their foundations were introduced by Wilking [13] and have been used in different situations in literature (see [2, 4, 10, 11]).

In particular, Wilking proves that the dual foliation is a singular foliation (Proposition 2.1 in [13]), which is Riemannian when dual leaves are complete and M is complete with nonnegative sectional curvature. This is the case in many interesting situations:

Theorem 1 (Wilking, Theorem 3 in [13]). *Suppose that M is a complete nonnegatively curved manifold with a singular Riemannian foliation $\mathcal{F}$. Then the dual foliation has intrinsically complete leaves if, in addition, one of the following holds*:

(1) *$\mathcal{F}$ is given by the orbit decomposition of an isometric group action*;
(2) *$\mathcal{F}$ is a non-singular foliation and M is compact*;
(3) *$\mathcal{F}$ is given by the fibers of a Sharafutdinov retraction.*

Although Theorem 1 gives many interesting conditions for completeness of dual leaves, Wilking conjectures that it should be the general case in nonnegative sectional curvature:

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Conjecture 2 (Wilking, [13]). *Suppose $\mathcal{F}$ is a singular Riemannian foliation on a complete nonnegatively curved manifold. Then $\mathcal{F}$ has complete dual leaves.*

In this note we give an affirmative answer for Wilking's conjecture ([13], Conjecture) in the case of a nonnegatively curved symmetric space.

Theorem 3. *Let $\mathcal{F}$ be a singular Riemannian foliation on M, a simply connected symmetric space with nonnegative sectional curvature. Then, the dual foliation $\mathcal{F}^\#$ has complete leaves.*

Following Lytchak [7], we actually prove a much stronger statement.

Corollary 4. *Let $\mathcal{F}$ be a singular Riemannian foliation on M, a symmetric space with nonnegative sectional curvature. Then $\mathcal{F}$ decomposes as a product $\mathcal{F}_1 \times \mathcal{F}_2$, where $\mathcal{F}_1$ has a single dual leaf and $\mathcal{F}_2$ consists of a single leaf. That is, there is a metric decomposition $M = Z \times N$, together with a singular Riemannian foliation $\mathcal{F}_1$ on Z, satisfying $L^\#_{(z,n)} = Z \times \{n\}$, for all $n \in N$, and*

$$\mathcal{F} = \{L \times N \mid L \in \mathcal{F}_1\}. \tag{1.1}$$

The result is new even for foliations on the Euclidean space (the result could be traced only for the low dimensional and regular cases, where the classification is complete: see [9] and [6, 12], respectively.) and recovers an important result on polar foliations:

Theorem 5. *Let $\mathcal{F}$ be a polar foliation on M and let $\Sigma \looparrowright M$ be a polar section. If the action of the Weyl group on Σ splits, then $\mathcal{F}$ splits.*

Theorem 5 recovers results of Ewert (Theorem 3 in [3]), Lytchak (Lemma 4.1 in [7]) and Liu–Radeschi (Proposition 3.4 in [5]). There are no analogous results in the general case, although the slice theorem in [8] might give some insights for a generalization.

The paper is divided in the following way: in Section 2 we recall a result of Lytchak [7] and present a direct application; in Section 3 we prove Theorem 3. Section 4 relates Theorem 3 to polar foliations.

2. Preliminaries

To prove Theorem 3, we use a result by Lytchak to decompose M as a metric product $M = Z \times N$, where Z is a minimal dual leaf. To this aim, we recall:

Theorem 6 (Lytchak, Proposition 3.1 in [7]). *Let M be a symmetric space with nonnegative sectional curvature. If $L^\#$ is a dual leaf, then M factors as $M = Z \times N$, where $L^\#$ is an open subset of $Z \times \{n\}$, for some $n \in N$.*

It follows that the dual leaf of minimal dimension is complete.

Proposition 7. *Let M be a symmetric space with nonnegative sectional curvature. If $L^\#$ is a dual leaf with minimal dimension, then $L^\#$ is complete. Moreover,*

$$M = L^\# \times N.$$

Proof. Suppose that $L^\#$ is not complete. Then, Theorem 6 gives us a totally geodesic submanifold with the same dimension as $L^\#$ such that $L^\# \subsetneq Z$.

By hypothesis, the topological boundary of $L^\#$ on Z is not empty, on the other hand $\mathrm{bd}(L^\#) = \bigcup F^\# \subset Z$ is a disjoint union of dual leaves (see Wilking [13], p. 1312). Moreover, since $F^\# \subseteq Z$ and $\dim L^\#$ must have minimal dimension among dual leaves,

$$\dim L^\# \leq \dim F^\# \leq \dim Z = \dim L^\#.$$

Therefore, by applying Theorem 6 again, each $F^\#$ is an open subset of Z.

We conclude that the closure of $L^\#$, $\mathrm{cl}(L^\#) = L^\# \cup \mathrm{bd}(L^\#)$, is covered by non-trivial disjoint open subsets. However, $L^\# \cup \mathrm{bd}(L^\#)$ is a closed connected subset of Z, since $L^\#$ is connected, a contradiction. ■

3. Proof of Theorem 3

We begin by constructing a very particular vertical vectors field outside a minimal dual leaf. Then use its flow lines to connect every point in a slice $N_{z'}$ to Z_n. We denote by V and H the vertical and horizontal spaces, that is, the space tangent to the leaves and the space orthogonal to V, respectively.

Let $L^\# = Z$ and $M = Z \times N$ be a fixed closed dual leaf and its respective metric decomposition given by Proposition 7. Fix $(z,n) \in Z \times N$ such that $L^\# = Z \times \{n\}$. Denote $Z \times \{n\} = Z_n$ and $\{z\} \times N = N_z$. We denote $TM = TZ + TN$, whenever it creates no ambiguity.

Let U be a tubular neighborhood of Z_n where the square of the distance function $f\colon U \to \mathbb{R}$,

$$f(z',n') = d_M((z',n'), Z_n)^2 = d_N(n',n)^2,$$

is smooth. Note that the neighborhood U can be chosen as $Z \times B_n(r)$, where $B_n(r)$ is a convex radius r open ball around $n \in N$, and that r does not depend on n, since the injectivity radius on a symmetric space does not depend on the point.

Lemma 8. *For every $(z',n') \in U - Z_n$, there is a vector $v(z',n') \in V \cap TN$ such that*

$$\langle v(z',n'), \nabla f(z',n') \rangle < 0.$$

Proof. We claim that

$$\nabla f(z',n') \notin \tilde{H}_{(z',n')} = \mathrm{pr}_{TN} H_{(z',n')},$$

where the right-hand side is the orthogonal projection of $H_{(z',n')}$ in $T_{(z',n')}N$. Recall that $-\frac{1}{2}\nabla f(z',n')$ is the velocity vector of a minimizing geodesic connecting (z',n') to (z',n). Observing that geodesics in $M = Z \times N$ are product geodesics, we conclude that no horizontal vector can be of the form $X + \nabla f$, X tangent to $Z_{n'}$, since the geodesic defined by $X + \nabla f$ connects (z',n') to the dual leaf Z_n. (Recall that the velocity of a geodesic is horizontal at one point if it is horizontal at every point. Therefore, $(z',n') \in Z_n$ if $X + \nabla f$ is horizontal, since Z_n is a dual leaf.)

Since $\nabla f \notin \tilde{H}$, ∇f must have a non-zero component on $\tilde{H}^\perp \cap TN_{z'}$. However, $\tilde{H}^\perp \cap TN_{z'} = H^\perp \cap TN_{z'} = V \cap TN_{z'}$. We conclude that any negative multiple of $\mathrm{pr}_{TN \cap V}(\nabla f(z',n'))$ has the desired properties. ■

Proof of Theorem 3. Let $Z = L^{\#} = Z_n$, $M = Z \times N$ and $U = Z \times B_n(r)$ be as above, with some fixed r independent of n. Let $\pi_Z : Z \times N \to Z$ be the projections to the Z-component. At every Z-slice we can define the foliations

$$\mathcal{F}_Z = \{L \cap Z \mid L \in \mathcal{F}\} \quad \text{and} \quad \mathcal{F}_{n'} = \{L \cap Z_{n'} \mid L \in \mathcal{F}\}$$

(we observe that the second is not necessarily Riemannian.) The splitting in equation (1.1) is equivalent to the statement:

$$L \in \mathcal{F}_{n'} \iff \pi_Z(L) \in \mathcal{F}_Z. \tag{3.1}$$

In other words, that each slice $N_{z'}$ lies in a single leaf. Since $\pi_Z|_{Z_{n'}}$ is an isometry, we also conclude that $\mathcal{F}_{n'}$ has a single dual leaf, whenever (3.1) holds.

Equation (3.1) can be proved, for instance, by showing that (z', n') and (z', n) lies in the same leaf for every $(z', n') \in M$. But the flow of v, defined in Lemma 8, gives us a vertical curve connecting (z', n') to (z', n), as long as v can be chosen to form a smooth vector field, proving the assertion for every $(z', n') \in U$. Once chosen v smooth, the existence of such curve is a standard argument and, for convenience, we recall it in the next paragraphs.

Let (z, n) and U be as in the beginning of the section and fix L. Observe that $L \cap N_{z'}$ is a submanifold for almost every $z' \in Z$: the projection into the Z coordinate, $\pi : L \to Z$, $\pi(z', n') = z'$, is obviously smooth and has $\pi^{-1}(z') = L \cap N_{z'}$. On the other hand, Sard's theorem guarantees that almost every z' is a regular value for π.

Suppose that z' is a regular value and denote by $\Phi_t(z', n')$ the flow, starting at (z', n'), defined by the vector field

$$v = -\frac{pr_{TL\cap V}(\nabla f)}{\|pr_{TL\cap V}(\nabla f)\|^2}.$$

Observe that v is a smooth vector field in $L \cap N_{z'} \cap U - \{(z', n)\}$ (Lemma 8). Moreover,

$$\frac{d}{dt} f(\Phi_t(z', n')) = -\frac{\langle \nabla f, pr_{TL\cap V}(\nabla f)\rangle}{\|pr_{TL\cap V}(\nabla f)\|^2} = -1, \tag{3.2}$$

therefore $\Phi_t(z', n')$ stays in $U = f^{-1}([0, r^2))$ and $\Phi_t(z', n')$ must be defined for $t \in [0, f(z', n'))$, since v is locally Lipschitz away from $f^{-1}(0)$. But

$$\lim_{t \to f(z', n')^-} \Phi_t(z', n') = (z', n). \tag{3.3}$$

Indeed, Φ_t fixes the first coordinate, since $v \in TN$, and $f(\Phi_t(z', n')) \to 0$ as $t \to f(z', n')^-$. Equation (3.3) then follows by equation (3.2) and $f^{-1}(0) = Z_n$. Since $v \in V$ and $\Phi_t(z', n')$ is converging, there is a sequence $\Phi_{t_k}(z', n')$, $t_k \to f(z', n')^-$, which is Cauchy on $L_{(z', n')}$. By recalling that leaves are locally immersed submanifolds, therefore intrinsically complete, we conclude that $(z', n) \in L_{(z', n')}$, as desired.

If z' is not a regular value, consider a sufficiently small neighborhood U' of (z', n) and suppose that $L \cap N_{z'} \cap U$ intersects a closed connected component $\tilde{L}$ of $L \cap U'$. Then, there is a sequence z_i of regular values of $f|_{\tilde{L} \cap U \cap N_{z'}}$ converging to z', Therefore, the argument in the last paragraph shows that $(z_i, n) \in \tilde{L}$. Since $\tilde{L}$ is closed, $(z', n) \in \tilde{L}$, as desired.

Finally, note that the arguments here show that $\mathcal{F}_{n''}$ satisfy (3.1) whenever n'' is r-close to a point n' such that $\mathcal{F}_{n'}$ satisfy (3.1). The proof is complete since we can cover N with balls of radius $r/2$. ∎

4. An application to polar foliations

A singular Riemannian foliation is called *polar* if it admits a totally geodesic horizontal section, i.e., an immersed, simply connected, connected, totally geodesic submanifold $\Sigma \looparrowright M$ such that Σ intersects every leaf perpendicularly.

One sees that the intersection of Σ with the singular strata happens in a collection of totally geodesic hypersurfaces of Σ. If M is simply connected, such reflections define a group W, called the *Weyl group* (see [1] for details). The metric quotient Σ/W is isometric to the leaf space $M/\mathcal{F}$.

Now suppose that the action of the Weyl group splits, i.e., $W = W_1 \times W_2$ as a product of groups, and $\Sigma = \Sigma_1 \times \Sigma_2$ as a metric product, such that W_i only acts on the i-th coordinate Σ_i. In this case, the foliation splits (see Ewert [3] and Liu–Radeschi [5]), i.e., M is a metric product $M = M_1 \times M_2$ and there are singular Riemannian foliations $\mathcal{F}_1, \mathcal{F}_2$ on M_1, M_2 such that

$$\mathcal{F} = \{L_1 \times L_2 \mid L_i \in \mathcal{F}_i\}.$$

This is a fundamental fact used both in [7] and [5]. Here we give a simpler proof for this fact based on Theorem 3 and arguments in Lytchak [7].

Proof of Theorem 5. With Theorem 3 at hand, Theorem 5 follows directly from the arguments in [7], see Section 2.5 and the proof of Proposition 4.2. For convenience, we briefly recall them here.

Suppose that a polar foliation $\mathcal{F}$ is given by a metric quotient $p\colon M \to \Delta$, so Δ is isometric to Σ/W. Further suppose that Σ admits a polar foliation $\mathcal{G}$ which is invariant by W (i.e., W takes $\mathcal{G}$-leaves to $\mathcal{G}$-leaves), thus W acts on $\Sigma/\mathcal{G} = \Delta'$. It follows that the fibers of $q \circ p\colon M \to \Delta'$ defines a polar foliation on M (we refer to [7], Section 2.5, for details).

This is certainly the case when the action of W splits. Indeed, denote $q_i\colon \Delta \to \Delta_i = \Sigma_i/W_i$ the metric quotients. Then $q_1 \circ p, q_2 \circ p$ define two polar foliations $\mathcal{F}_1', \mathcal{F}_2'$ on M, whose sections are Σ_1 and Σ_2, respectively. Observe that each leaf in $\mathcal{F}$ is the intersection of a leaf in $\mathcal{F}_1'$ and one in $\mathcal{F}_2'$.

By Theorem 3, M decomposes as $M = M_1 \times M_2$, where the leaves of $\mathcal{F}_1'$ are products of M_2 with the leaves of a foliation $\mathcal{F}_1$ in M_1. Since every $\mathcal{F}_1'$-horizontal curve is mapped by p into a Δ_1-factor, and hence by p_2 to a point, any dual leaf to $\mathcal{F}_1'$ is contained in a $\mathcal{F}_2'$-leaf. Thus, the M_1-factor is $\mathcal{F}_2'$-vertical. The proof is concluded by applying the arguments in the last paragraph of the proof of Theorem 3 to conclude that $\mathcal{F}_2'$ splits as a foliation $\mathcal{F}_2$ in M_2 and the one-leaf foliation on M_1. ∎

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