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A McKay bijection for projectors

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Abstract. If $\mathfrak{F}$ is a saturated formation of groups, we define a canonical subset $\mathrm{Irr}_{\mathfrak{F}'}(G)$ of the irreducible complex characters of a finite solvable group G. If H is an $\mathfrak{F}$-projector of G, we show that $|\mathrm{Irr}_{\mathfrak{F}'}(G)| = |\mathrm{Irr}(\mathbf{N}_G(H)/H')|$, where $H' = [H, H]$ is the derived subgroup of H. In particular, if $\mathfrak{F}$ is the class of p-groups, this reproves the solvable case of the celebrated McKay conjecture.

1. Introduction

One of the main problems in the representation theory of finite groups is the McKay conjecture. This asserts that if G is a finite group, $\mathrm{Irr}(G)$ is the set of the irreducible complex characters of G, p is a prime, and P is a Sylow p-subgroup of G, then

$$|\mathrm{Irr}_{p'}(G)| = |\mathrm{Irr}_{p'}(\mathbf{N}_G(P))|,$$

where $\mathrm{Irr}_{p'}(G)$ is the set of $\chi \in \mathrm{Irr}(G)$ whose degree $\chi(1)$ is not divisible by p. In other words, important global information of a finite group can be calculated locally.

The main contribution of this paper is to provide a conceptual framework which allows for a vast generalization of this conjecture for solvable groups. (In this paper, all finite groups are solvable.)

In the 1960's, after the discovery by R. Carter of the *Carter subgroups* of any finite solvable group, the theory of formations was developed by R. Carter, K. Doerk, W. Gaschütz, B. Huppert, T. Hawkes and many others. Recall that a class of groups $\mathfrak{F}$ is a formation if it is closed under quotients, and whenever $G/N, G/M \in \mathfrak{F}$ for some finite group G, then $G/(N \cap M) \in \mathfrak{F}$. The canonical examples of formations are the classes of finite p-groups, π-groups (if π is a set of primes), or of nilpotent groups. But of course, there are many others. One of the principal ideas was to generalize Sylow and Hall subgroups and unify these with the recent discovery of Carter. Indeed, if $\mathfrak{F}$ is saturated (that is, $G \in \mathfrak{F}$ if and only if $G/\Phi(G) \in \mathfrak{F}$, where $\Phi(G)$ is the Frattini subgroup of G), then it was proved that every finite solvable group G possesses a unique conjugacy class of the so called $\mathfrak{F}$-*projectors*. If $\mathfrak{F}$ is the class of p-groups, these are the Sylow subgroups of G; if $\mathfrak{F}$ is the class of π-groups, these are the Hall π-subgroups of G; if $\mathfrak{F}$ is the class of

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nilpotent groups, these are the Carter subgroups of G. Other interesting examples of saturated formations (and therefore with $\mathfrak{F}$-projectors) are the class of supersolvable groups, or the class of nilpotent groups of bounded nilpotency class. The $\mathfrak{F}$-projectors of a finite solvable group G are easy to define: these are the subgroups H of G such that whenever $N \triangleleft G$, then HN/N is a maximal $\mathfrak{F}$-subgroup of G/N.

The principal objective of this paper is to define, for a given saturated formation $\mathfrak{F}$, a canonical subset $\mathrm{Irr}_{\mathfrak{F}'}(G) \subseteq \mathrm{Irr}(G)$ of the irreducible complex characters of every finite solvable group G. If $\mathfrak{F}$ is the class of all finite groups, we will show that $\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Lin}(G)$, the set of linear characters of G; if p is a prime and $\mathfrak{F}$ is the class of p-groups, then $\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}_{p'}(G)$, the set of the irreducible characters of G of degree not divisible by p (hence, our notation $\mathfrak{F}'$); if π is a set of primes and $\mathfrak{F}$ is the class of π-groups, then $\mathrm{Irr}_{\mathfrak{F}'}(G)$ is the set of irreducible characters χ whose degree is a π'-number (that is, no prime p dividing $\chi(1)$ is in π); and finally, if $\mathfrak{F}$ is the class of nilpotent groups, then we will show that $\mathrm{Irr}_{\mathfrak{F}'}(G)$ is the set of *head characters* of G, recently introduced by I. M. Isaacs in [5], and that have inspired the results in this paper.

Our main theorem is the following. (Recall that the derived subgroup of H, usually denoted by H' or $[H, H]$, is the smallest normal subgroup of H with abelian quotient.)

Theorem A. *If $\mathfrak{F}$ is a saturated formation, G is a solvable group, and H is an $\mathfrak{F}$-projector of G, then*

$$|\mathrm{Irr}_{\mathfrak{F}'}(G)| = |\mathbf{N}_G(H)/H'|\,.$$

In particular, if $k(G)$ is the number of conjugacy classes of G, then $k(\mathbf{N}_G(H)/H') \leq k(G)$.

Of course, if $\mathfrak{F}$ is the class of p-groups, then Theorem A gives another proof of the McKay conjecture for solvable groups. (We shall comment on this proof at the end of the paper.) If $\mathfrak{F}$ is the class of nilpotent groups, then $H = \mathbf{N}_G(H)$ is a Carter subgroup of G, and Theorem A reproves Isaacs result ([5]) that the number of head characters of G is $|H/H'|$. As we shall show in Theorem B below, we also have that $\mathrm{Irr}_{\mathfrak{F}'}(\mathbf{N}_G(H)) = \mathrm{Irr}(\mathbf{N}_G(H)/H')$, if H is an $\mathfrak{F}$-projector of G.

Example: if $\mathfrak{F}$ is the class of supersolvable groups and $G = \mathrm{GL}_2(3)$, we shall show below that $H = \mathsf{D}_{12}$ is an $\mathfrak{F}$-projector of G, $H = \mathbf{N}_G(H)$, $H/H' = \mathsf{C}_2 \times \mathsf{C}_2$, and the $\mathfrak{F}'$-characters of G are the two linear characters of G together with the two faithful irreducible characters of G of degree 2. These two latter characters have field of values $\mathbb{Q}(i\sqrt{2})$, so we cannot expect, for this $\mathfrak{F}$, a canonical bijection $\mathrm{Irr}_{\mathfrak{F}'}(G) \to \mathrm{Irr}(\mathbf{N}_G(H)/H')$ (since a canonical bijection would respect field of values of corresponding characters). If $\mathfrak{F}$ is the class of 2-groups, on the other hand, M. Isaacs did construct a canonical bijection in [2].

As we shall explain in Section 4, our strategy to define $\mathrm{Irr}_{\mathfrak{F}'}(G)$ is the following: given an $\mathfrak{F}$-projector H of G, we shall construct a chain of subgroups

$$G = U_0 > U_1 > \cdots > U_m = \mathbf{N}_G(H)\,,$$

which is uniquely determined by H. We shall define $\mathrm{Irr}_{\mathfrak{F}'}(U_m) = \mathrm{Irr}(\mathbf{N}_G(H)/H')$, and then, recursively, use $\mathrm{Irr}_{\mathfrak{F}'}(U_i)$ to define $\mathrm{Irr}_{\mathfrak{F}'}(U_{i-1})$. In each step, we will check that $|\mathrm{Irr}_{\mathfrak{F}'}(U_i)| = |\mathrm{Irr}_{\mathfrak{F}'}(U_{i-1})|$. We shall prove that $\mathrm{Irr}_{\mathfrak{F}'}(G)$ does not depend on H, and this will prove our main result.

Our $\mathfrak{F}'$-characters behave as expected. For instance, the following is the corresponding Itô's theorem.

Theorem B. *Suppose that $\mathfrak{F}$ is a saturated formation, G is a solvable group, and H is an $\mathfrak{F}$-projector of G. Then*

$$\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}(G)$$

if and only if H is a normal abelian subgroup of G.

Further properties of $\mathfrak{F}'$-characters will be explored in another paper. It is possible that some of our results can be extended to some other classes of groups, perhaps even outside solvable groups (as it happens if $\mathfrak{F}$ is the class of p-groups, because of the McKay conjecture). We shall not make this attempt here.

2. Reviewing formations

Recall that a class of groups $\mathfrak{F}$ is a formation if it is closed under quotients, and whenever $G/N, G/M \in \mathfrak{F}$ for some finite group G, then $G/(N \cap M) \in \mathfrak{F}$. A formation is *saturated* if and only if $G/\Phi(G) \in \mathfrak{F}$ implies that $G \in \mathfrak{F}$, where $\Phi(G)$ is the Frattini subgroup of G. Given a finite group G, we denote by $G^{\mathfrak{F}}$ the smallest normal subgroup of G such that $G/G^{\mathfrak{F}} \in \mathfrak{F}$. A *maximal $\mathfrak{F}$-subgroup* H of G is a subgroup H in $\mathfrak{F}$ such that if $H < M < G$, then $M \notin \mathfrak{F}$.

In this paper, we are only interested in saturated formations of solvable groups. For the reader's convenience, we have decided to collect all the facts that we need in this paper in the following theorem, which we state in the way that is convenient for us. All these results can be found in [1].

Theorem 2.1. *Let $\mathfrak{F}$ be a saturated formation, and let G be a solvable group.*

(a) *There exists a unique G-conjugacy class of subgroups H of G such that HN/N is $\mathfrak{F}$-maximal for every $N \triangleleft G$. (These subgroups are called the $\mathfrak{F}$-projectors of G.) In particular, if $G/N \in \mathfrak{F}$, then $G = HN$.*

(b) *If H is an $\mathfrak{F}$-projector of G and $N \triangleleft G$, then HN/N is an $\mathfrak{F}$-projector of G/N.*

(c) *If H is an $\mathfrak{F}$-projector of G and $H \le U \le G$, then H is an $\mathfrak{F}$-projector of U. In particular, if $N \triangleleft G$, then $\mathbf{N}_G(HN) = N\mathbf{N}_G(H)$, and if $H \triangleleft \triangleleft G$, then $H \triangleleft G$.*

(d) *Suppose that $N \triangleleft G$, U/N is an $\mathfrak{F}$-projector of G/N and H is an $\mathfrak{F}$-projector of U. Then H is an $\mathfrak{F}$-projector of G.*

(e) *Suppose that $K = G^{\mathfrak{F}}$ is abelian. If H is an $\mathfrak{F}$-projector of G, then $G = KH$ and $K \cap H = 1$.*

(f) *If $G \in \mathfrak{F}$ and p divides $|G|$, then every p-group is contained in $\mathfrak{F}$.*

(g) *Let N be a nilpotent normal subgroup of G. Let H be $\mathfrak{F}$-maximal such that $G = HN$. Then H is an $\mathfrak{F}$-projector of G.*

(h) *If $K, L \triangleleft G$ and H is an $\mathfrak{F}$-projector of G, then $KH \cap LH = (K \cap L)H$.*

Proof. Parts (a) to (d), and an introduction to the subject, can be found in Section 9.5 of [7]. The remaining parts lie deeper in the theory. Part (e) is Theorem IV.5.18 of [1].

Next, we prove part (f). By Lemma IV.4.2 of [1], the cyclic group C_p of order p is in $\mathfrak{F}$. Therefore so it is $C_p \times \cdots \times C_p$. If P is a p-group, then $P/\Phi(P) \in \mathfrak{F}$, and therefore $P \in \mathfrak{F}$. Part (g) is III.3.14 of [1]. Part (h) is Theorem IV.5.4. of [1]. ■

Corollary 2.2. *Suppose that G is solvable, $K \lhd G$ is nilpotent, H is an $\mathfrak{F}$-projector of G, $KH = G$, and $K \cap H = 1$. If $U \leq G$ is such that $KU = G$ and $K \cap U = 1$, then U is G-conjugate to H.*

Proof. We have that $U \cong G/K \in \mathfrak{F}$. Let $U \subseteq X \leq G$ be $\mathfrak{F}$-maximal in G. Then $XK = G$. Since K is nilpotent, we have that X is an $\mathfrak{F}$-projector of G by Theorem 2.1(g). Then $|X| = |H|$ and necessarily $U = X$, and U and H are G-conjugate. ■

3. Characters and formations

For the rest of this paper, $\mathfrak{F}$ stands for a saturated formation. For characters, we use the notation in [4] (and [6]). Hence, if $N \lhd G$ and $\theta \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G|\theta)$ is the set of the irreducible characters of G that lie over θ (that is, $\chi \in \mathrm{Irr}(G|\theta)$ if and only if θ is a constituent of the restriction χ_N). By Frobenius reciprocity, $\mathrm{Irr}(G|\theta)$ is the set of the irreducible constituents of the induced character θ^G. If $X \subseteq \mathrm{Irr}(N)$, then we shall write

$$\mathrm{Irr}(G|X) = \bigcup_{\theta \in X} \mathrm{Irr}(G|\theta)\,.$$

Finally, if A acts by automorphisms on G, we denote by $\mathrm{Irr}_A(G)$ the set of irreducible A-invariant characters of G.

The Isaacs restriction lemma is elementary but quite useful.

Lemma 3.1 (Isaacs). *Suppose that G is a finite group, K and L are normal subgroups of G, and $M \leq G$ such that $G = KM$ and $K \cap M = L$. If $\theta \in \mathrm{Irr}(K)$ is G-invariant and $\theta_L \in \mathrm{Irr}(L)$, then restriction defines a bijection $\mathrm{Irr}(G|\theta) \to \mathrm{Irr}(M|\theta_L)$.*

Proof. See, for instance, Lemma 2.7 of [3]. ■

The following is a key lemma in this paper. If G is a finite group, we denote by $\mathrm{Lin}(G)$ the group of its linear characters.

Lemma 3.2. *Suppose that G is solvable and let H be an $\mathfrak{F}$-projector of G. Let $K, L \lhd G$ such that $K \subseteq L$ and KH and LH are normal subgroups of G. Suppose that $\chi \in \mathrm{Irr}(LH)$ lies over some $\lambda \in \mathrm{Irr}(KH)$ such that $\lambda_K \in \mathrm{Irr}(K)$. Then $\chi_L \in \mathrm{Irr}(L)$.*

Proof. By Theorem 2.1(c), we have that H is an $\mathfrak{F}$-projector of LH. Thus we may assume that $LH = G$. Let $N = L \cap KH \lhd G$. By hypothesis, $\lambda_N \in \mathrm{Irr}(N)$. Let $I = I_G(\lambda_N)$ be the stabilizer of λ_N in G, which contains KH. Now if $x \in I$, then $\lambda^x = \mu_x \lambda$, for some unique linear $\mu_x \in \mathrm{Irr}(KH/N)$, by Gallagher's Corollary 6.17 of [4]. Since $[KH, L] \subseteq N$, we have that $\mu\colon I \to \mathrm{Lin}(KH/N)$ is a group homomorphism with kernel V containing KH. Now, $I/V \cong \mathrm{Im}(\mu)$. Also, $\mathrm{Im}(\mu) = \mathrm{Lin}(KH/W)$, for some $N \leq W \lhd KH$, by using Problem 2.7 of [4], applied to the abelian group $KH/(KH)'N$. Then $\mathrm{Im}(\mu) \cong KH/W \cong H/(W \cap H) \in \mathfrak{F}$, and therefore $I/V \in \mathfrak{F}$. Thus $I = HV$, because H is an $\mathfrak{F}$-projector of I (again by Theorem 2.1(c)). Since $H \subseteq V$, it follows that $V = I$. Hence, $I \subseteq I_G(\lambda) \subseteq I_G(\lambda_N) = I$. In particular, $L \cap I = I_L(\lambda_N)$. Now, let $\tau \in \mathrm{Irr}(I|\lambda)$ be the Clifford correspondent of χ over λ. By Isaacs' restriction lemma, we have that $\tau_{I\cap L} \in \mathrm{Irr}(I \cap L)$. Now, using the Clifford correspondence, we have that $\chi_L = (\tau^G)_L = (\tau_{L\cap I})^L \in \mathrm{Irr}(L)$, as desired. ■

Corollary 3.3. *Suppose that G is solvable, has a normal $\mathfrak{F}$-projector H, and H normalizes $L \leq G$. Let $\theta \in \mathrm{Irr}(HL)$. If θ lies over some linear λ character of H, then $\theta_L \in \mathrm{Irr}(L)$.*

Proof. We may assume that $G = LH$. Then, $\lambda_{H\cap L}$ is irreducible, and the corollary follows from Lemma 3.2. ■

We shall need the deep theory of fully ramified sections of solvable groups. Recall that if $K \lhd G$, $\varphi \in \mathrm{Irr}(K)$ is G-invariant, and $\varphi^G = e\theta$ for some $\theta \in \mathrm{Irr}(G)$, then it is said that φ is *fully ramified* in G. For properties of character triple isomorphisms, we refer the reader to Chapter 11 of [4] (or Chapter 5 of [6]).

Theorem 3.4. *Suppose that G is solvable, $K, L \lhd G$ such that K/L is a non-central chief factor of G. Suppose that $\varphi \in \mathrm{Irr}(L)$ is G-invariant and fully ramified in K. Write $\varphi^K = e\theta$, for some $\theta \in \mathrm{Irr}(K)$. Then there exists a complement U of K/L in G such that (G, K, θ) and (U, L, φ) are isomorphic character triples* (*with the natural isomorphism $G/K \to U/L$ as the group isomorphism associated*).

Proof. By using character triples, it is no loss to assume that $L \subseteq \mathbf{Z}(G)$. Now, Theorem A of [3] asserts that there is a bijection $\pi\colon \mathrm{Irr}(U|\varphi) \to \mathrm{Irr}(G|\theta)$ such that $\xi^\pi(1) = e\xi(1)$. All the work to prove Theorem A in [3] (and its extension Theorem B) reduces to proving that θ extends to $G_0 = K \rtimes (U/L)$, the semidirect product. This is the content of Theorem 6.1 of [3], applied to the group G_0. Once this is proved, using Theorem 5.3 of [3], it is easy to check that there is a bijection $\pi\colon \mathrm{Irr}(U|\varphi) \to \mathrm{Irr}(G|\theta)$ such that $\xi^\pi(1) = e\xi(1)$. Now, the construction of π in Theorem 5.3 of [3] in fact yields a character triple isomorphism (using Lemmas 11.26 and 11.27 of [4]). ■

Theorem 3.5. *Suppose that G is a finite solvable group. Assume that K/L is abelian, where K, L are normal in G. Let H/L be an $\mathfrak{F}$-projector of G/L, $U = \mathbf{N}_G(H)$, and assume that $KH \lhd G$ and $K \cap U = L$.*

(a) *If $\theta \in \mathrm{Irr}(K)$ is H-invariant, then there is a unique H-invariant $\varphi \in \mathrm{Irr}(L)$ under θ.*

(b) *If $\varphi \in \mathrm{Irr}(L)$ is H-invariant, then there is a unique H-invariant $\theta \in \mathrm{Irr}(K)$ over φ. From now on, fix φ and θ, as before.*

(c) *Let T be the stabilizer of θ in G, and let I be the stabilizer of φ in U. Then $T \cap U = I$ and (T, K, θ) and (I, L, φ) are isomorphic character triples* (*with the natural group isomorphism associated*). *Also, all the G-conjugate characters of θ and all the U-conjugate characters of φ are H-invariant.*

(d) *We have that θ extends to KH if and only if φ extends to H.*

(e) *We have that* $|\mathrm{Irr}(G|\theta)| = |\mathrm{Irr}(U|\varphi)|$.

(f) *If* Δ *is the set of extensions of* θ *to* KH *and* Ξ *is the set of extensions of* φ *to* H*, then*

$$|\mathrm{Irr}(G|\Delta)| = |\mathrm{Irr}(U|\Xi)|\,.$$

(g) *If we write* $\varphi = \theta'$*, then we have that the map* $\theta \mapsto \theta'$ *is a bijection* $\mathrm{Irr}_H(K) \to \mathrm{Irr}_H(L)$.

Proof. We first prove that $G = KU$ and $G/L = (K/L) \rtimes (U/L)$. Moreover, U/L is the unique complement of K/L up to G-conjugacy.

Since HK/K is an $\mathfrak{F}$-projector of G/K (Theorem 2.1 (b)), we have that $G = \mathbf{N}_G(KH) = K\mathbf{N}_G(H)$ (using Theorem 2.1 (c)). So $KU = G$. Since G/K is isomorphic to U/L, notice that $(KH \cap U)/L$ is a (normal) $\mathfrak{F}$-projector of U/L. Since H/L is an $\mathfrak{F}$-projector of U/L by Theorem 2.1 (c), we have that $H = KH \cap U$. Observe too that $\mathbf{C}_{K/L}(H) = 1$. Otherwise, if $L \leq W \leq K$, $[W/L, H] \subseteq L$, then W/L normalizes H/L, but then it is contained in $\mathbf{N}_{G/L}(H/L) = U/L$. Since $K \cap U = L$, then $W = L$.

Now we prove that U/L is the only complement of K/L up to G-conjugacy. First recall that H/L is an $\mathfrak{F}$-projector of KH/L by Theorem 2.1 (c). Let V/L be another complement of K/L in G. Then $(V \cap KH)/L$ is a complement of K/L in KH/L. By Corollary 2.2, we have that $V \cap KH/L$ is an $\mathfrak{F}$-projector of KH/L. Thus, there is $k \in K$ such that $V \cap KH/L = H^k/L$. Replacing V by $V^{k^{-1}}$, we may assume that $KH \cap V = H$. Then $H = KH \cap V \lhd V$ and $V \subseteq U$. Then $V = U$ because both are complements to K/L.

Next we show that certain 5-tuples of groups satisfy the hypothesis on (G, K, L, H, U). These are $(T, K, L, H, T \cap U)$ (whenever $KH \leq T \leq G$), and (G, K, M, MH, MU) and (MU, M, L, H, U), whenever $L \subseteq M \subseteq K$ and $M \lhd G$. This is straightforward to prove. To check that $MU = \mathbf{N}_G(MH)$, use that $\mathbf{N}_{G/L}((M/L)(H/L)) = (M/L)\mathbf{N}_{G/L}(H/L)$, applying Theorem 2.1 (c).

Finally, we claim that $\mathrm{Irr}_H(K/L) = 1$. This is equivalent to proving that $K = [K, H]L$. Otherwise, suppose that $M = [K, H]L < K$. Then $M \lhd G$, $\mathbf{C}_{K/M}(H) = K/M$. This contradicts the fact that (G, K, M, MH, MU) satisfies the hypothesis implies that $\mathbf{C}_{K/M}(H) = 1$ (as we showed in the second paragraph of this proof).

(a) By working in the stabilizer $I_G(\theta)$ of θ in G (that contains KH), we may assume that θ is G-invariant. Suppose that φ_1, φ_2 are H-invariant under θ. By Clifford's theorem, $\varphi_1 = (\varphi_2)^k$ for some $k \in K$. Then $H \subseteq I_{KH}(\varphi_1) = I_{KH}(\varphi_2)^k$. Hence $H, H^{k^{-1}} \subseteq I_{KH}(\varphi_2)$. Now, H and $H^{k^{-1}}$ are $\mathfrak{F}$-projectors of $I_{KH}(\varphi_2)$, and therefore, there is $t \in I_{KH}(\varphi_2)$ such that $H^{k^{-1}} = H^t$. Since $t \in KH$, we can write $t = hk_1$ for some $h \in H$ and $k_1 \in I_K(\varphi_2)$. Thus $H^{k_1 k} = H$ and $k_1 k \in \mathbf{N}_K(H)$. Then $k_1 k \in L$, and $\varphi_2 = (\varphi_2)^{k_1 k} = (\varphi_2)^k = \varphi_1$.

To prove that θ_L has an H-invariant irreducible constituent, by working by induction on $|K : L|$, we may assume that K/L is a chief factor of G. Since θ is G-invariant, by the going down theorem (Theorem 6.18 of [4]), we may assume that θ is induced from some character of L. Let $\varphi \in \mathrm{Irr}(L)$ be any constituent of θ_L. Let T be the stabilizer of φ in G. Since θ is G-invariant, we have that $TK = G$, by Clifford's theorem. Since $\varphi^K = \theta$, we have that $T \cap K = L$ (by Problem 6.1 of [4]). By the third paragraph in this proof, we have that $U = T^g$ for some $g \in G$. Therefore H stabilizes φ^g.

(b) By working in $I_U(\varphi)K$, we may assume that φ is U-invariant. Then $U \subseteq I_G(\varphi) = T$. Since induction defines a canonical bijection $\mathrm{Irr}(T \cap K|\varphi) \to \mathrm{Irr}(K|\varphi)$, we may assume that φ is G-invariant. Now, by Lemma 2.2 of [8], there is a unique subgroup $L \subseteq W \subseteq K$ such that every irreducible constituent γ of φ^W extends φ and is K-invariant. In fact, every such $\gamma \in \mathrm{Irr}(W)$ is fully ramified with respect to K/W. By uniqueness, notice that W is U-invariant, and therefore $W \triangleleft G$. If $\gamma \in \mathrm{Irr}(W|\varphi)$, then $\gamma^K = e\rho$ and $\rho_W = e\gamma$, for a unique $\rho \in \mathrm{Irr}(K)$. Therefore γ is H-invariant if and only if ρ is H-invariant. Working by induction on $|K : L|$, we may assume that $W = K$, that is, φ extends to K. Now, by using character triples, we may assume that $L \subseteq \mathbf{Z}(G)$ and φ is faithful. It is enough to show that φ^N has some H-invariant extension ϵ. Indeed, if $\epsilon_1 \in \mathrm{Irr}(K)$ is another H-invariant extension, then $\epsilon_1 = \lambda\epsilon$ for a unique (and therefore H-invariant) character of K/L. But we know that $\mathrm{Irr}_H(K/L) = 1$. Again, by induction on $|K : L|$, we may assume that K/L is a chief factor of G, so that K/L is an abelian p-group for some prime p.

Now, since K/L is abelian, φ extends to K and φ is linear and faithful, we have that K is abelian. Since K/L is a p-group, then by using that $\varphi_{p'}$ has a canonical extension to K (using Corollary 6.27 of [4]), we may assume that $\varphi_{p'} = 1$. (Here, we are writing $\varphi = \varphi_p\varphi_{p'}$, where φ_p has order a power of p, and $\varphi_{p'}$ has order not divisible by p.) Thus, we are assuming that L is a cyclic p-group and that K is an abelian p-group. Let H_1 be an $\mathfrak{F}$-projector of H. Then H_1 is an $\mathfrak{F}$-projector group of G, by Theorem 2.1 (d). Now, by Theorem 2.1 (e), the $\mathfrak{F}$-residual U_0 of KH, complements H_1. Since $KH \triangleleft G$, we have that $U_0 \triangleleft G$. Thus $KH = U_0H_1$. Notice that $H = LH_1$. Now, $L = (L \cap U_0) \times (L \cap H_1)$, and since L is a cyclic p-group, then either $L \cap U_0 = 1$ or $L \cap H_1 = 1$. In the first case, $K = U_0 \times L$, and $1_{U_0} \times \varphi$ is the H-invariant extension that we are looking for. Hence we may assume that $L \cap H_1 = 1$. Then $H = L \times H_1$. Suppose first that $\mathfrak{F}$ contains the p-groups. Then $H/H_1 \in \mathfrak{F}$, and $H/L \in \mathfrak{F}$, so $H \in \mathfrak{F}$. Since H_1 is $\mathfrak{F}$-maximal in KH, we have that $H = H_1$. Thus $L = 1$, and we are done, since we choose θ to be the trivial character of K, in this case.

Finally, suppose that $\mathfrak{F}$ does not contain the p-groups. By Theorem 2.1 (f), we have that H_1 is a p'-group. Then $L = \mathbf{C}_K(H_1)$ (using that $\mathbf{C}_{K/L}(H) = 1$). By Fitting's lemma, we have that $K = [K, H_1] \times L$. Now, $[K, H_1] \triangleleft G$ (because $H_1 \triangleleft U$) and $\theta = 1_{[K,H_1]} \times \varphi$ is H_1-invariant, and thus H-invariant.

(c) If $u \in U$ and $h \in H$, then $uhu^{-1} \in H$, and therefore $\theta^{uhu^{-1}} = \theta$, and thus we deduce that θ^u is H-invariant. In the same way, φ^u is H-invariant. Therefore, if $u \in I_U(\theta)$, then $u \in I_U(\varphi)$ by uniqueness, and conversely. Since $T = KI_U(\theta)$, we conclude that $T \cap U = I$. To prove (c), we may assume that θ is G-invariant and φ is U-invariant. By the transitivity of character triple isomorphisms (Corollary 11.25 of [4]), we may assume that K/L is a chief factor of G. By the going down theorem, we have that either $\theta_L = \varphi$, $\varphi^K = \theta$, or that $\theta_K = e\theta$ with $e^2 = |K : L|$. In the latter case, we apply Theorem 3.4. In the first case, we have that if $K \le X \le G$, then restriction defines a bijection

$$\mathrm{Irr}(X|\theta) \to \mathrm{Irr}(X \cap U|\varphi)\,.$$

It is clear that this affords a character triple isomorphism between (G, K, θ) and (U, L, φ). Finally, if $\varphi^K = \theta$, then $U = I_G(\varphi)$, and if $K \le X \le G$, then, by the Clifford correspondence, we have that induction defines a bijection

$$\mathrm{Irr}(X \cap U|\varphi) \to \mathrm{Irr}(X|\theta)\,.$$

This also affords a character triple isomorphism between (U, L, φ) and (G, K, θ).

Now, parts (d) and (e) are consequences of (c) using the Clifford correspondence maps $\mathrm{Irr}(T|\theta) \to \mathrm{Irr}(G|\theta)$ and $\mathrm{Irr}(I|\varphi) \to \mathrm{Irr}(U|\varphi)$.

(f) Let $\chi \in \mathrm{Irr}(G|\theta)$ and let $\psi \in \mathrm{Irr}(T|\theta)$ be the Clifford correspondent of χ. Suppose that χ lies over some extension of θ. Since $KH \lhd G$, by Clifford's theorem, we have that every irreducible constituent of χ_{KL} restricts irreducibly to L. Now, let $\mu \in \mathrm{Irr}(KL)$ be under ψ. Hence, μ lies under χ, and therefore restricts irreducibly to K. Since $\mu_K = e\theta$, we have that $\mu_K = \theta$. We see that the Clifford correspondence defines a bijection

$$\mathrm{Irr}(T|\Delta) \to \mathrm{Irr}(G|\Delta).$$

The same happens with

$$\mathrm{Irr}(I|\Xi) \to \mathrm{Irr}(U|\Xi).$$

Now, the character triple isomorphism in (c) gives us a bijection

$$\mathrm{Irr}(I|\Xi) \to \mathrm{Irr}(T|\Delta)\ .$$

(g) It easily follows using parts (a) and (b). ■

As we have done in the proof of Theorem 3.5, we shall write that (G, K, L, H, U) satisfies the group theoretical hypothesis of Theorem 3.5.

4. $\mathfrak{F}'$-characters

We assume again in this section that $\mathfrak{F}$ is a saturated formation. Let G be a finite solvable group, and let fix H an $\mathfrak{F}$-projector of G. We let K be the unique smallest normal subgroup of G such that $KH \lhd G$. Notice that K exists by Theorem 2.1 (h). We also use the notation $K = G^{\mathfrak{F}_n}$. We notice that K is uniquely determined by G. (Indeed, if H_1 is any other $\mathfrak{F}$-projector of G, then $H = H_1^g$ for some $g \in G$, and $KH \lhd G$ if and only if $KH_1 \lhd G$. In fact, in this case, $KH = KH_1$.) In particular, K is characteristic in G. Note that $K = 1$ if and only if $H \lhd G$. Also note that $G = K\mathbf{N}_G(H)$, since $KH \lhd G$ (by Theorem 2.1 (c)).

Our first goal is to define a uniquely defined 5-tuple (G, K, L, LH, U) satisfying Theorem 3.5, where $U = L\mathbf{N}_G(H)$. We shall prove that $U < G$ if and only if H is not normal in G. (All this is done in Lemma 4.1 below.) Since H is an $\mathfrak{F}$-projector of U and $\mathbf{N}_G(H) \subseteq U$, we can repeat this process in U, and eventually produce a chain of subgroups

$$G = U_0 > U = U_1 > \cdots > U_{m-1} > U_m = \mathbf{N}_G(H),$$

which are uniquely determined by H. We will define

$$\mathrm{Irr}_{\mathfrak{F}'}(U_m) = \mathrm{Irr}(\mathbf{N}_G(H)/H')$$

as the set of the irreducible characters χ of $\mathbf{N}_G(H)$ that contain the derived subgroup H' in their kernel. Afterwards, we will recursively define $\mathrm{Irr}_{\mathfrak{F}'}(U_{i-1})$, using the definition of $\mathrm{Irr}_{\mathfrak{F}'}(U_i)$. In particular, this defines

$$\mathrm{Irr}_{\mathfrak{F}'}(G) .$$

In each step, we will show that

$$|\mathrm{Irr}_{\mathfrak{F}'}(U_i)| = |\mathrm{Irr}_{\mathfrak{F}'}(U_{i-1})| .$$

Hence $\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}_{\mathfrak{F}'}(U_0)$ will be defined.

If instead of H we choose H^g, for some $g \in G$, it will become clear that the corresponding series would be

$$G = U_0 > (U_1)^g > \cdots > (U_{m-1})^g > (U_m)^g = \mathbf{N}_G(H^g),$$

with

$$\mathrm{Irr}_{\mathfrak{F}'}((U_i)^g) = \mathrm{Irr}_{\mathfrak{F}'}(U_i)^g .$$

(Here, if $\alpha \in \mathrm{Irr}(X)$ and $g \in G$, then α^g is the unique irreducible character of X^g such that $\alpha^g = (x^g) = \alpha(x)$ for $x \in X$.) The proof of Theorem A will be then established.

We use the following.

Lemma 4.1. *Let G be a finite solvable group, let H be an $\mathfrak{F}$-projector of G, and let $K = G^{\mathfrak{F}_n}$. Let $L = K'$ be the derived subgroup of L. If $U = L\mathbf{N}_G(H)$, then (G, K, L, HL, U) satisfies the hypotheses of Theorem 3.5. Furthermore, $U = G$ if and only if $H \lhd G$, which happens if and only if $K = 1$. Also, we have that $KH \cap U = HL$ and $\mathbf{C}_{K/L}(H) = 1$.*

Proof. By definition of the derived subgroup, we have that K/L is abelian. Since $K = L'$, we have that $L \lhd G$. By Theorem 2.1 (b), we have that HL/L is an $\mathfrak{F}$-projector of G/L. By Theorem 2.1 (c), we have that $U = \mathbf{N}_G(HL)$. By the definition of K, we have that $KH \lhd G$. It only remains to show that $K \cap U = L$. To do that, we are going to use Theorem 2.1 (e). Hence, it suffices to show that $K/L = (KH/L)^{\mathfrak{F}}$. Write $W/L = (KH/L)^{\mathfrak{F}}$. Notice that $W \lhd G$, since $KH \lhd G$ and W/L is characteristic in KH/L. Since $KH/K \cong H/H \cap K \in \mathfrak{F}$, we have that $W/L \subseteq K/L$. Now we have that $KH/W \in \mathfrak{F}$, by definition. Since H is an $\mathfrak{F}$-projector of KW (by Theorem 2.1 (c)), using Theorem 2.1 (a) we have that $WH = KH \lhd G$. Since K is the smallest normal subgroup of G such that $KH \lhd G$, then we conclude that $W = K$, as desired. Finally, notice that $G = U$ if and only if $K = L$, which happens if and only if $K = 1$, which happens if and only if H is normal in G. For the final part, since G/K is isomorphic to U/L, we have that U/L has a unique normal $\mathfrak{F}$-projector H/L. The image of the natural isomorphism $G/K \to U/L$ sends the $\mathfrak{F}$-projector of G/K onto the $\mathfrak{F}$-projector of U/L. Thus $KH \cap U = HL$. Finally, if $W/L = \mathbf{C}_{K/L}(H)$, then

$$W/L \subseteq \mathbf{N}_{G/L}(HL/L) \cap K/L = U/L \cap K/L = 1 ,$$

as desired. ∎

The following is Theorem A of the introduction.

Theorem 4.2. *Let G be a solvable group, and let H be an $\mathfrak{F}$-projector of G, where $\mathfrak{F}$ is a saturated formation. Then*

$$|\mathrm{Irr}_{\mathfrak{F}'}(G)| = \mathrm{Irr}(N_G(H)/H')| .$$

Proof. We construct a family of subgroups U_i, K_i and L_i of G (uniquely determined by H) in the following way. By Lemma 4.1 (and using its notation), we let

$$(U_0, K_0, L_0, L_0H, U_1) = (G, K, L, LH, U) .$$

We have that $U_1 = L_0\mathbf{N}_G(H)$. Thus H is an $\mathfrak{F}$-projector of U_1 (using Theorem 2.1 (c)). Also, $U_1 = U_0$ if and only if $U_0 = \mathbf{N}_G(H)$. If $U_1 < U_0$ (that is, if $\mathbf{N}_G(H) < U_0$), then we repeat the process, and apply Lemma 4.1 to U_1 and H, to produce

$$(U_1, K_1, L_1, L_1H, U_2) .$$

Therefore $K_1 = (U_1)^{\mathfrak{F}_n}$, $L_1 = (K_1)'$, and $U_2 = L_1\mathbf{N}_{U_1}(H) = L_1\mathbf{N}_G(H) \leq U_1$. Again, $U_1 = U_2$ if and only $K_1 = 1$, which happens if and only if $U_1 = \mathbf{N}_G(H)$. By Lemma 4.1, we have that $(U_1, K_1, L_1, L_1H, U_2)$ satisfies the hypothesis of Theorem 3.5, and H is an $\mathfrak{F}$-projector of U_1. If $U_2 < U_1$ (that is, if $\mathbf{N}_G(H) < U_1$), then we continue this process and construct $(U_2, K_2, L_2, L_2H, U_3)$, where $U_3 = L_2\mathbf{N}_G(H)$. We repeat this process until we arrive to the smallest m such that $U_m = \mathbf{N}_G(H)$. That is, using Lemma 4.1, when we arrive to the smallest m such that $K_m = 1$.

Notice that in each step, the 5-tuple $(U_i, K_i, L_i, L_iH, U_{i+1})$ satisfies the hypothesis of Theorem 3.5, where $U_{i+1} = L_i\mathbf{N}_G(H)$, and each of its components is uniquely determined by H. In particular, every U_i, K_i and L_i are $\mathbf{N}_G(H)$-invariant. Also, $L_i = (K_i)'$, $U_{i+1} < U_i$ if and only if $\mathbf{N}_G(H) < U_i$, and this happens for $i = 0, \ldots, m-1$. Again, by Lemma 4.1, notice that $K_iH \cap U_{i+1} = L_iH$. Since, $U_{i+1} = L_i\mathbf{N}_G(H)$, then U_{i+1}/L_i has a normal $\mathfrak{F}$-projector L_iH/L_i. Therefore, $(U_{i+1})^{\mathfrak{F}_n} = K_{i+1} \subseteq L_i$.

Once we have dealt with the part on groups, we finally arrive to the part on characters. We define

$$\mathrm{Irr}_{\mathfrak{F}'}(U_m) = \mathrm{Irr}_{\mathfrak{F}'}(\mathbf{N}_G(H)) = \mathrm{Irr}(\mathbf{N}_G(H)/H') .$$

We claim that $\mathrm{Irr}_{\mathfrak{F}'}(U_m)$ satisfies the conditions (a), (b) and (c) below:

(a) *If $\gamma \in \mathrm{Irr}(K_mH)$ lies under some $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_m)$, then γ_{K_m} is irreducible.*

We have that $K_m = 1$, and, of course, (a) is true because γ is linear.

(b) *Let Δ_{m-1} be the set of the irreducible constituents of $\chi_{L_{m-1}}$, where $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_m)$. Let X_{m-1} be the set of irreducible characters γ of $L_{m-1}H$ such that $\gamma_{L_{m-1}} = \rho \in \Delta_{m-1}$. Then every $\tau \in \Delta_{m-1}$ is H-invariant and extends to $L_{m-1}H$, Δ_{m-1} and X_{m-1} are U_m-invariant, and*

$$\mathrm{Irr}_{\mathfrak{F}'}(U_m) = \mathrm{Irr}(U_m|X_{m-1}) .$$

Let $\delta \in \mathrm{Irr}_{\mathfrak{F}'}(U_m)$, let $\mu \in \mathrm{Irr}(L_{m-1}H)$ under δ, and let $\tau \in \mathrm{Irr}(L_{m-1})$ be under μ. Notice that μ lies over some linear character of H (because δ does). By Corollary 3.3, we have that $\mu_{L_{m-1}} = \tau$. Hence, $\mu \in X_{m-1}$, and therefore

$$\mathrm{Irr}_{\mathfrak{F}'}(U_m) \subseteq \mathrm{Irr}(U_m|X_{m-1}) .$$

Now let $\psi \in \mathrm{Irr}(U_m|X_{m-1})$. Hence ψ lies over some $\gamma \in \mathrm{Irr}(L_{m-1}H)$ such that $\gamma_{L_{m-1}} = \rho \in \Delta_{m-1}$. Now ρ is an irreducible constituent of $\chi_{L_{m-1}}$, for some $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_m)$. Let $\mu \in \mathrm{Irr}(L_{m-1}H)$ be over ρ under χ. So we know that μ lies over some linear $\lambda \in \mathrm{Irr}(H)$. Thus $\mu_{L_{m-1}} = \rho$ by Corollary 3.3. By Gallagher, we have that $\gamma = \epsilon\mu$, where $\epsilon \in \mathrm{Irr}(L_{m-1}H/L_{m-1})$ is a linear character. Now, γ lies over $\epsilon_H\lambda$, which is linear. Thus $\psi \in \mathrm{Irr}(\mathbf{N}_G(H)/H') = \mathrm{Irr}_{\mathfrak{F}'}(U_m)$.

Since Δ_{m-1} is clearly U_m-invariant, it follows that X_{m-1} is U_m-invariant. This concludes the proof of (b).

By the definition, we also have that:

(c) $|\mathrm{Irr}_{\mathfrak{F}'}(U_m)| = |\mathrm{Irr}(N_G(H)/H')|$.

In this paragraph, we construct $\mathrm{Irr}_{\mathfrak{F}'}(U_{m-1})$ from $\mathrm{Irr}_{\mathfrak{F}'}(U_m)$, using Theorem 3.5. Recall that

$$(U_{m-1}, K_{m-1}, L_{m-1}, L_{m-1}H, U_m)$$

satisfies the hypothesis of Theorem 3.5. By Theorem 3.5 , there is a natural bijection

$$' : \mathrm{Irr}_H(K_{m-1}) \to \mathrm{Irr}_H(L_{m-1})$$

such that θ extends to $K_{m-1}H$ if and only if θ' extends to $L_{m-1}H$. Therefore, for each $\varphi \in \Delta_{m-1}$, there is a unique H-invariant $\theta \in \mathrm{Irr}(K_{m-1})$ lying over γ, that extends to $K_{m-1}H$. Let

$$\Delta'_{m-1} = \{\gamma \in \mathrm{Irr}_H(K_{m-1}) \mid \gamma' \in \Delta_{m-1}\}\,.$$

Let $X'_{m-1} = \{\tau \in \mathrm{Irr}(K_{m-1}H)$ such that $\tau_{K_{m-1}} \in \Delta'_{m-1}\}$. Both Δ'_{m-1} and X'_{m-1} are $\mathbf{N}_G(H)$-invariant. We let

$$\mathrm{Irr}_{\mathfrak{F}'}(U_{m-1}) = \mathrm{Irr}(U_{m-1} \mid X'_{m-1})\,.$$

Notice that if $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_{m-1})$ and $\tau \in \mathrm{Irr}(K_{m-1}H)$ lies under χ, then $\tau_{K_{m-1}}$ is irreducible (and H-invariant), by construction. (So it satisfies condition (a) with respect to $m-1$.) Notice that Δ'_{m-1} is simply the set of irreducible constituents of $\chi_{K_{m-1}}$ such that $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_{m-1})$. (Recall that if $\theta \in \mathrm{Irr}(K_{m-1})$ is H-invariant, then all $\mathbf{N}_G(H)$-conjugates are H-invariant too.)

Suppose that $A = \{\varphi_1, \ldots, \varphi_k\}$ is a complete set of representatives of the $\mathbf{N}_G(H)$-action on Δ_{m-1}. Then $A' = \{\varphi'_1, \ldots, \varphi'_k\}$ is a complete set of representatives of the $\mathbf{N}_G(H)$-action on Δ'_{m-1}. If

$$B_i = \{\tau \in \mathrm{Irr}(L_{m-1}H) \mid \tau_{L_{m-1}} = \varphi_i\} \quad \text{and} \quad B'_i = \{\tau \in \mathrm{Irr}(K_{m-1}H) \mid \tau_{K_{m-1}} = \varphi'_i\},$$

it follows that

$$\mathrm{Irr}_{\mathfrak{F}'}(U_{m-1}) = \bigcup_i \mathrm{Irr}(U_{m-1}|B'_i) \quad \text{and} \quad \mathrm{Irr}_{\mathfrak{F}'}(U_m) = \bigcup_i \mathrm{Irr}(U_m|B_i)$$

are disjoint unions. By Theorem 3.5, we have that

$$|\mathrm{Irr}_{\mathfrak{F}'}(U_{m-1})| = |\mathrm{Irr}_{\mathfrak{F}'}(U_m)|\,.$$

Suppose now that we have constructed $\mathrm{Irr}_{\mathfrak{F}'}(U_k)$, satisfying the following conditions (a), (b) and (c), below. We canonically construct $\mathrm{Irr}_{\mathfrak{F}'}(U_{k-1})$, satisfying the corresponding conditions (for the index $k-1$).

(a) *Let* $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_k)$. *If* $\gamma \in \mathrm{Irr}(HK_k)$ *lies under* χ, *then* γ_{K_k} *is irreducible.*

Notice then that since $K_k \leq L_{k-1} \triangleleft U_k$, $HK_k \triangleleft U_k$, and $L_{k-1}H \triangleleft U_k$, if $\tau \in \mathrm{Irr}(L_{k-1}H)$ lies under χ, we have that $\tau_{L_{k-1}}$ is irreducible by Lemma 3.2.

Let Δ_{k-1} be the set of the irreducible constituents $\chi_{L_{k-1}}$, for $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_k)$. If $\gamma \in \Delta_{k-1}$, then we know that γ is H-invariant and γ extends to $L_{k-1}H$. Let $X_{k-1} = \{\tau \in \mathrm{Irr}(L_{k-1}H) \mid \tau_{L_{k-1}} \in \Delta_{k-1}\}$.

(b) *We have that*

$$\mathrm{Irr}_{\mathfrak{F}'}(U_k) = \mathrm{Irr}(U_k \mid X_{k-1})\,.$$

(c) *We have that*

$$|\mathrm{Irr}_{\mathfrak{F}'}(U_k)| = |\mathrm{Irr}(\mathbf{N}_G(H)/H')|\,.$$

Now, let $\Delta'_{k-1} = \{\gamma \in \mathrm{Irr}(K_{k-1}) \mid H\text{-invariant such that } \gamma' \in \Delta_{k-1}\}$, we let $X'_{k-1} = \{\tau \in \mathrm{Irr}(K_{k-1}H)$ such that $\tau_{K_{k-1}} \in \Delta'_{k-1}\}$. Finally, we define

$$\mathrm{Irr}_{\mathfrak{F}'}(U_{k-1}) = \mathrm{Irr}(U_{k-1} \mid X'_{k-1})\,.$$

Notice that Δ'_{k-1} is the set of all the irreducible constituents of $\chi_{K_{k-1}}$ such that $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_{k-1})$.

We claim that $\mathrm{Irr}_{\mathfrak{F}'}(U_{k-1})$ satisfies our set of conditions:

(a) *If* $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_{k-1})$ *and* $\gamma \in \mathrm{Irr}(HK_{k-1})$ *lies under* χ, *then* $\gamma_{K_{k-1}}$ *is irreducible.*

Recall that K_{k-1} is the smallest normal subgroup of U_{k-1} such that $K_{k-1}H \triangleleft U_{k-1}$). Now, $K_{k-1} \subseteq L_{k-2}$, $K_{k-1}H \triangleleft U_{k-1}$ and it follows by Lemma 3.2, that if $\epsilon \in \mathrm{Irr}(L_{k-2}H)$ lies under $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_{k-1})$, then $\epsilon_{L_{k-2}}$ is irreducible.

Let Δ_{k-2} be the set of irreducible constituents of $\chi_{L_{k-2}}$, where $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_{k-1})$, and let X_{k-2} be the set of all extensions of the characters of Δ_{k-2} to $L_{k-2}H$. We claim that the following holds:

(b) $\mathrm{Irr}_{\mathfrak{F}'}(U_{k-1}) = \mathrm{Irr}(U_{k-1} \mid X_{k-2})\,.$

Indeed, if $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_{k-1})$ and $\epsilon \in \mathrm{Irr}(L_{k-2}H)$ lies under χ, we have shown in the previous paragraph that $\epsilon_{L_{k-2}}$ is irreducible. Thus $\epsilon \in X_{k-2}$, and $\chi \in \mathrm{Irr}(U_{k-1} \mid X_{k-2})$. Conversely, let $\psi \in \mathrm{Irr}(U_{k-1} \mid X_{k-2})$ and let μ be an irreducible constituent of $\psi_{L_{k-2}H}$. Then $\mu \in X_{k-2}$ and $\mu_{L_{k-2}} = \theta \in \Delta_{k-2}$. Therefore, there is $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_{k-1})$ such that θ is below χ. Let $\tau \in \mathrm{Irr}(L_{k-2}H)$ below χ and over θ. Then we know that $\tau_{L_{k-2}} = \theta$ (by the paragraph proving (a)). Hence, we have that $\mu = \lambda\tau$ for some linear $\lambda \in \mathrm{Irr}(L_{k-2}H/L_{k-2})$, by Gallagher. Also, τ lies over $\rho \in X'_{k-1}$, because $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(U_{k-1}) = \mathrm{Irr}(U_{k-1} \mid X'_{k-1})$. Therefore μ lies over $\lambda_{K_{k-1}H}\rho \in X'_{k-1}$ because K_{k-1} is in the kernel of λ. Hence $\psi \in \mathrm{Irr}(U_{k-1}|X'_{k-1}) = \mathrm{Irr}_{\mathfrak{F}'}(U_{k-1})$.

Finally, using Theorem 3.5 and (b), we also have that:

(c) $|\mathrm{Irr}_{\mathfrak{F}'}(U_{k-1})| = |\mathrm{Irr}_{\mathfrak{F}'}(U_k)|\,.$

This finishes the proof of the theorem. ∎

Let us extract, from the first and the last step in our construction of the set $\mathrm{Irr}_{\mathfrak{F}'}(G)$, the following statement for further use.

Theorem 4.3. *Let G be a solvable group, let $\mathfrak{F}$ be a saturated formation, and let H be an $\mathfrak{F}$-projector. Let K be the smallest normal subgroup of G such that $KH \lhd G$, let $L = K'$, and let $U = L\mathbf{N}_G(H)$. Denote by $'\colon \mathrm{Irr}_H(K) \to \mathrm{Irr}_H(L)$ the natural bijection established in Theorem 3.5 (g).*

(a) *If $H \lhd G$, then $\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}(G/H')$.*

(b) *We have that $\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}(G|Y)$, where Y is the set of the irreducible characters τ of KH such that $\tau_K = \gamma$ is irreducible, and $\gamma' \in \mathrm{Irr}(L)$ lies under some $\psi \in \mathrm{Irr}_{\mathfrak{F}'}(U)$. In particular, if $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(G)$, then all the irreducible constituents of χ_K are H-invariant.*

(c) *Suppose that $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(G)$ and let $N \lhd G$ with $K \subseteq N$. Then every irreducible constituent of χ_{NH} restricts irreducibly to N.*

Proof. We use the notation in Theorem 4.2.

(a) If $H \lhd G$, then $m = 0$, $U_m = G$, and, in this case, we have defined

$$\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}_{\mathfrak{F}'}(U_m) = \mathrm{Irr}(\mathbf{N}_G(H)/H') = \mathrm{Irr}(G/H')\,.$$

(b) The first part is exactly the content of our construction in the case $k = 1$, that is, from $U = U_1$, to $G = U_0$. If $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(G)$, then some irreducible constituent $\eta \in \mathrm{Irr}(K)$ of χ_K is H-invariant, by the first part. Since $G = K\mathbf{N}_G(H)$, all the G-conjugates of η are in fact $\mathbf{N}_G(H)$-conjugate. Now, if $x \in \mathbf{N}_G(H)$ and $h \in H$, then $xhx^{-1} \in H$ and therefore $(\eta^x)^h = \eta^x$.

(c) Let $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(G)$, let $\mu \in \mathrm{Irr}(NH)$ be under χ, and let $\theta \in \mathrm{Irr}(KH)$ be under μ. By part (a), we know that $\theta_K \in \mathrm{Irr}(K)$. Now, apply Lemma 3.2 in the group NH. ∎

The following is Theorem B.

Theorem 4.4. *Let G be a finite solvable group, let $\mathfrak{F}$ be a saturated formation, and let H be an $\mathfrak{F}$-projector. Then $\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}(G)$ if and only if H is normal and abelian.*

Proof. If H is normal and abelian, then $\mathbf{N}_G(H)/H' = G$, and the theorem follows from Theorem 4.3 (a).

Assume now that $\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}(G)$. Let K be the smallest normal subgroup such that $KH \lhd G$. Assume that $K > 1$, and let $L = K'$. We know that $\mathbf{C}_{K/L}(H) = 1$ by Lemma 4.1. Let $\theta \in \mathrm{Irr}(K)$, and let $\chi \in \mathrm{Irr}(G)$ be over θ. By hypothesis, we have that $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(G)$. By Theorem 4.3 (b), we know that all the irreducible constituents of χ_K are H-invariant. In particular, all the irreducible characters of K/L are H-invariant. Thus $\mathbf{C}_{K/L}(H) = K/L$, contrary to the assumption. Therefore, $K = 1$, $H \lhd G$, and $\mathrm{Irr}(G) = \mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}(G/H')$, by Theorem 4.3 (a). We conclude that $H' = 1$, that is, that H is abelian. ∎

Theorem 4.5. *If $\mathfrak{F}$ is the class of p-groups, then $\mathrm{Irr}_{\mathfrak{F}'}(G)$ is exactly the set of the irreducible characters of G of degree not divisible by p.*

Proof. We argue by induction on $|G|$. Let $P \in \mathrm{Syl}_p(G)$. The smallest normal subgroup K of G such that KP is normal in G is $\mathbf{O}^{p'p}(G)$. Now, let $L = K'$. Notice that K/L is a p'-group. Let $U = \mathbf{N}_G(P)L$.

Suppose that $\chi \in \mathrm{Irr}(G)$ has p'-degree. By degrees, we have that χ_U has some p'-degree irreducible constituent γ. By induction, $\gamma \in \mathrm{Irr}_{\mathfrak{F}'}(U)$. Let $\tau \in \mathrm{Irr}(LP)$ be under γ. Since $LP \lhd U$, we have that τ has p'-degree. Thus $\tau_L = \varphi \in \mathrm{Irr}(L)$, by Corollary 11.29 of [4]. Now, χ_{KP} has some irreducible constituent $\xi \in \mathrm{Irr}(KP)$ over φ. Since ξ_K is irreducible, it follows that $\xi = \varphi'$ is the unique P-invariant character over φ. We have that $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(G)$ by Theorem 4.3(b). Conversely, suppose that $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(G)$. By Theorem 4.3(b), χ lies over some $\tau \in \mathrm{Irr}(KP)$ such that $\tau_{KP} = \theta \in \mathrm{Irr}(K)$, and $\varphi = \theta'$ lies under some $\psi \in \mathrm{Irr}_{\mathfrak{F}'}(U)$. By induction, ψ has p'-degree. Then φ has p'-degree by Clifford's theorem, and so does θ, τ and χ, by two applications of Corollary 11.29 of [4]. ■

Notice that we cannot exactly modify the proof in Theorem 4.5 if we consider the class $\mathfrak{F}$ of π-groups, instead of p-groups: if U is a subgroup of a solvable group G and $\chi \in \mathrm{Irr}(G)$ has π'-degree, it is false that χ_U has a π'-degree irreducible constituent. (For instance, take $\pi = \{5\}'$ and $G = E : \mathrm{SL}_2(3)$, where E is an extra-special 5-group of order 5^3 and exponent 5.) The argument in the second part of the proof of Theorem 4.5 does show that $\mathrm{Irr}_{\mathfrak{F}'}(G) \subseteq \mathrm{Irr}_{\pi'}(G)$, where $\mathrm{Irr}_{\pi'}(G)$ is the set of the irreducible characters such that $\chi(1)$ is a π'-number. We have that $|\mathrm{Irr}_{\pi'}(G)| = |\mathrm{Irr}(\mathbf{N}_G(H)/H')|$, by the main theorem of [10]. Since $|\mathrm{Irr}(\mathbf{N}_G(H)/H')| = |\mathrm{Irr}_{\mathfrak{F}'}(G)|$ by Theorem A, we do conclude that $\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}_{\pi'}(G)$.

Theorem 4.6. *If $\mathfrak{F}$ is the class of nilpotent groups and G is solvable, then* $\mathrm{Irr}_{\mathfrak{F}'}(G)$ *is the set of Isaacs head characters.*

Proof. In the language of [5], our construction provides a Carter chain $G = U_0 > \cdots > U_m = C$, such that $\mathrm{Irr}_{\mathfrak{F}'}(G)$ is the image of the Isaacs associated injection $f : \mathrm{Lin}(C) \to \mathrm{Irr}(G)$. Now, $f(\mathrm{Lin}(C)) = \mathrm{Irr}_{\mathfrak{F}'}(G)$ does not depend on the Carter chain, and this is the set of Isaacs head characters (using Theorem 5.3 of [5]). ■

Of course, if $\mathfrak{F}$ is the class of all finite groups and G is solvable, then G is itself an $\mathfrak{F}$-projector and $\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}(G/G')$ is the set of linear characters of G, by Theorem 4.3(a).

Let us now review the $G = \mathrm{GL}_2(3)$ example that we pointed out in the introduction. Let $\mathfrak{F}$ be the class of supersolvable groups. Then the unique conjugacy class of subgroups of G isomorphic to D_{12} is the conjugacy class of $\mathfrak{F}$-projectors of G. Let $H \leq G$ be one of those. Now, the proper normal subgroups of G are $\{1, \mathbf{Z}(G), \mathsf{Q}_8, \mathrm{SL}_2(3)\}$. Hence, the smallest normal subgroup K of G such that G/K has a normal $\mathfrak{F}$-projector is $K = \mathsf{Q}_8$. In fact, $G/K = \mathsf{S}_3$ is supersolvable, so $G = KH$. Also, $K' = L = \mathbf{Z}(G)$, and $K \cap H = L$. Now, $H = \mathbf{N}_G(H)$, and $H' \cap L = 1$, since H' is a Sylow 3-subgroup of G. Thus, if $\mathrm{Irr}(H/H') = \{1, \lambda_2, \lambda_3, \lambda_4\}$, then, say the two first, restrict trivially to Z, and the remaining two, λ_3 and λ_4, extend the unique non-trivial irreducible character $\lambda \in \mathrm{Irr}(Z)$. Since $\lambda^K = 2\theta$, $\lambda' = \theta$, and then $\mathrm{Irr}_{\mathfrak{F}'}(G|\theta)$ are the two extensions of θ to G. In the other case, $(1_Z)' = 1_K$, and $\mathrm{Irr}_{\mathfrak{F}'}(G|1_K)$ are two linear characters of G.

In order to investigate further properties of $\mathfrak{F}'$-characters, it seems convenient to find ways to relax the rather rigid algorithm that we have used to define them, as for instance, Isaacs does with its head characters in [5] using H-composition series of G, if H is a

Carter subgroup of G. We believe that our $\mathfrak{F}'$-characters can be described in terms of $\mathbf{N}_G(H)$-composition series, but we do not attempt this here.

Also, there are some questions on head characters and zeros of characters in [5] (raised by this author) which might have some interest. With the generality in this paper, it is not true that if $\chi \in \mathrm{Irr}_{\mathfrak{F}'}(G)$ and $h \in H$, where H is an $\mathfrak{F}$-projector, then $\chi(h) \neq 0$. (As it does happen if $\mathfrak{F}$ is the class of p-groups: see Corollary 4.20 of [6].) For instance, if $\mathfrak{F}$ is the class of $\{2,3\}$ groups and $G = \mathsf{C}_6 \operatorname{wr} \mathsf{C}_5$, then $\mathrm{Irr}_{\mathfrak{F}'}(G) = \mathrm{Irr}(G)$ and G has an irreducible character of degree 5 that vanishes on an element of order 6.

As we have mentioned, Theorem 4.5 together with Theorem A gives a proof of the McKay conjecture for solvable groups. The first proof of the McKay conjecture for solvable groups was given by T. R. Wolf in [9]. This proof, however, relied on very deep results of Dade. The standard proof of the p-solvable case nowadays is due to T. Okuyama and M. Wajima, and can be found in [6]. The proof that we present here can be obtained as a combination of the proofs of [3] with [2], if we let $\mathfrak{F}$ be the class of p-groups.

Finally, since all finite groups in this paper are solvable (in order to consider many classes of groups at the same time), our proof of Theorem A does not take care of the p-solvable case of the McKay conjecture, or of the π-separable case of the McKay conjecture for Hall π-subgroups (proven in [10]). Hence, it is possible that our Theorem A can be extended to other classes of groups.

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