



Mean field equations and domains of first kind

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Abstract. In this paper, we are interested in understanding the structure of domains of *first and second kind*, a concept motivated by problems in statistical mechanics and mean field equations. We prove some openness property for domains of first kind with respect to a suitable topology, as well as some sufficient condition, in terms of the Fourier coefficients of the Riemann map, for a simply connected domain to be of first kind. Finally, we show that the set of simply connected domains of first kind is contractible.

1. Introduction

We are concerned with the mean field equation,

$$\begin{cases} -\Delta u = \lambda \frac{e^u}{\int_{\Omega} e^u} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (\mathbf{P}_-)$$

where $\lambda \in \mathbb{R}$ and $\Omega \subset \mathbb{R}^2$ is either any open and bounded domain of class C^1 or a bounded simply connected domain, *regular* according to the following definition ([10]).

Definition 1.1. Let Ω be an open and bounded domain, $\Omega \subset \mathbb{R}^2$. We say that Ω is *regular* if its boundary $\partial\Omega$ is of class C^2 but for a finite number of points $\{Q_1, \dots, Q_{N_0}\} \subset \partial\Omega$, for each of which the following conditions hold:

- (i) The inner angle θ_j of $\partial\Omega$ at Q_j satisfies $0 < \theta_j \neq \pi < 2\pi$;
- (ii) At each Q_j , there is a univalent conformal map from $B_\delta(Q_j) \cap \overline{\Omega}$ to the complex plane \mathbb{C} such that $\partial\Omega \cap B_\delta(Q_j)$ is mapped to a C^2 curve.

Clearly, any non-degenerate polygon is regular according to this definition.

Next we have the following.

Definition 1.2. Let $\Omega \subset \mathbb{R}^2$ be either an open and bounded domain of class C^1 or a regular simply connected domain. We say that Ω is of *first kind* if (\mathbf{P}_-) has no solution for $\lambda = 8\pi$. Otherwise, Ω is said to be of *second kind*. The set of domains of first/second kind will be denoted by $\mathcal{A}_I/\mathcal{A}_{II}$, respectively.

This definition is motivated by physical and analytical problems, see [8, 9], [6, 10] and more recently [1, 3, 4, 7]. In the framework of the vortex model of an Euler incompressible flow confined in Ω , the Robin function γ_Ω is essentially the renormalized free energy of a single vortex. On domains of first kind, the full range of admissible energies $E \in (0, +\infty)$ corresponds to minus the inverse statistical temperature $\lambda \in (-\infty, 8\pi)$, and as $E \rightarrow +\infty$ we have $\lambda \rightarrow (8\pi)^-$ and the vorticity of the flow concentrates to a Dirac delta, $\delta_{x=q}$. It turns out that, in this situation, q is the unique maximum point of the free energy γ_Ω . In particular, the equivalence of statistical ensembles holds and the entropy is concave for $E \in (0, +\infty)$. On domains of second kind, the states in the range $\lambda \in (-\infty, 8\pi)$ describe only a portion of the allowed energy values, say $E \in (0, E_{8\pi})$, and the peculiar phenomenon of non equivalence of statistical ensembles holds for $E \in (E_{8\pi}, +\infty)$, where in particular we have $\lambda > 8\pi$. As $E \rightarrow +\infty$, we have $\lambda \rightarrow (8\pi)^+$ and the vorticity of the flow concentrates to a Dirac delta, $\delta_{x=q}$, where q is a maximum point of the free energy γ_Ω . A full region of convexity of the entropy (negative specific heat) is found for E large enough.

On the other side, solutions of (\mathbf{P}_-) can be found as critical points of the functional $J_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 - \lambda \log(\int_\Omega e^u)$ on $H_0^1(\Omega)$, which is coercive and bounded from below if $\lambda < 8\pi$, and bounded from below but not coercive if $\lambda = 8\pi$. Whether or not the minimizing sequences for $\lambda = 8\pi$ stay compact depends in a subtle way on the geometry of the domain. Among other things, it has been shown in [6, 10] that Ω is of second kind if and only if $J_{8\pi}$ admits a minimizer.

Actually, based on the concentration/compactness behavior of minimizers of $J_{8\pi}$, the concept of domains of first/second kind was first introduced in [8, 9].

It is well known that any disk, say $B_R = B_R(0)$, is of first kind and that, in this particular case, (\mathbf{P}_-) admits a solution if and only if $\lambda < 8\pi$. Regular polygons are also of first kind ([10]). Symmetric annuli are known to be of second kind, since a radial solution of (\mathbf{P}_-) exists for any $\lambda \in \mathbb{R}$ in this case, see for example [7, 20], while $\Omega = B_R \setminus B_r(x_0)$, with $x_0 \neq 0$, is of first kind if r is small enough ([6]). Actually, this is also an example of a domain of first kind where (\mathbf{P}_-) admits solutions for $\lambda > 8\pi$ as well. Indeed, for domains with non-trivial topology, it is well known that for any $N \geq 2$ there are solutions concentrating at N distinct points as $\lambda \rightarrow 8\pi N$ [11, 14, 17], as well as solutions for any $\lambda \neq 8\pi N$ or for any λ sufficiently large [12, 18], see also [2].

It has been proven in [3] that there exists a universal constant $I_c > 4\pi$ such that any convex domain whose isoperimetric ratio $I(\Omega)$ satisfies $I(\Omega) > I_c$ is of second kind. Also, if $Q_{a,b}$ is a rectangle whose sides are $1 \leq a \leq b < +\infty$, then there exists $\eta_c \in (0, 1)$ such that $Q_{a,b}$ is of second kind if and only if $a/b < \eta_c$, see [10].

In particular, domains of first kind need not be symmetric. Let us consider a dumbbell domain $\Omega_{0,d}$ which is the union of two disks B_1 and B_2 of radii $0 < r_1 \leq r_2$, connected by any smooth thin tube of width $d > 0$. It has been shown in [10] that if $r_1 < r_2$, then for d small enough $\Omega_{0,d} \in \mathcal{A}_I$, while if $\Omega_{0,d}$ is symmetric with respect to the y axis and $r_1 = r_2$, then $\Omega_{0,d} \in \mathcal{A}_{II}$.

Remark 1.3. By the above discussion, the condition of a domain to be of first or second kind is not conformally invariant. However, we recall that (\mathbf{P}_-) is scale invariant, that is, $u(x)$ is a solution of (\mathbf{P}_-) in Ω if and only if $u(\delta x)$ is a solution of (\mathbf{P}_-) in $\frac{1}{\delta}\Omega$. Therefore $\Omega \in \mathcal{A}_I/\mathcal{A}_{II}$ if and only if $\frac{1}{\delta}\Omega \in \mathcal{A}_I/\mathcal{A}_{II}$ for some $\delta > 0$.

We are interested here in a better understanding of the structure of the set of domains of first/second kind. Indeed, besides the above-mentioned application [9], this is relevant also for other problems where, for domains of first kind, one can describe the qualitative behavior of global branches of solutions, see [5, 7] for recent results in this direction.

Let us recall that, as shown in [10] and [6], domains of first (second) kind are closed (open) in the C^1 -domain topology. These results rely on an equivalent characterization based on the geometric quantity A_Ω , see (1.4) and Theorem A below. Let $G_\Omega(x, p)$ denote the Green function of $-\Delta$ with Dirichlet boundary conditions, uniquely defined by

$$\begin{cases} -\Delta G_\Omega(x, p) = \delta_p & \text{in } \Omega, \\ G_\Omega(x, p) = 0 & \text{on } \partial\Omega, \end{cases}$$

and set

$$(1.1) \quad \begin{cases} R_\Omega(x, p) = G_\Omega(x, p) + \frac{1}{2\pi} \log |x - p|, \\ \gamma_\Omega(p) = R_\Omega(p, p). \end{cases}$$

Hence γ_Ω denotes the Robin function relative to Ω , that satisfies

$$(1.2) \quad \lim_{p \rightarrow \partial\Omega} \gamma_\Omega(p) = -\infty.$$

In view of (1.2), we see that γ_Ω admits at least one critical point, which is its maximum point. Clearly, q is a critical point of γ_Ω if and only if q is a critical point of $R(x, q)$ with respect to the x variable, i.e.,

$$(1.3) \quad \nabla_x \gamma_\Omega(x)|_{x=q} = 2 \nabla_x R_\Omega(x, q)|_{x=q} = 0.$$

Let us define

$$(1.4) \quad \pi A_\Omega(q) = \lim_{\varepsilon \rightarrow 0} \int_{\Omega \setminus B_\varepsilon(q)} \frac{e^{8\pi(R_\Omega(x,q) - \gamma_\Omega(q))} - 1}{|x - q|^4} - \int_{\Omega^c} \frac{1}{|x - q|^4}.$$

Here $B_\varepsilon(q)$ denotes the ball of center q and radius ε : also, as we will always convene later on, the standard integration measure has been omitted. Note that, in a neighborhood of q , we have

$$(1.5) \quad e^{8\pi(R_\Omega(x,q) - \gamma_\Omega(q))} - 1 = \sum_{i,j}^{1,2} a_{ij} (x_i - q_i)(x_j - q_j) + O(|x - q|^3),$$

where, since $R_\Omega(x, q)$ is harmonic in Ω , $a_{11} + a_{22} = 0$. In particular, because of (1.5), the limit in (1.4) always exists and A_Ω is finite.

According to some results in [6, 10], we have the following.

Theorem A. *A domain Ω of class C^1 is of first kind if and only if γ_Ω admits a unique maximum point q and $A_\Omega(q) \leq 0$.*

A simply connected and regular domain Ω is of first kind if and only if γ_Ω admits a unique maximum point q and $A_\Omega(q) \leq 0$.

In particular, in both cases, if γ_Ω admits a critical point q such that $A_\Omega(q) \leq 0$, then q is the unique maximum point, it is a non-degenerate critical point of γ_Ω and Ω is of first kind.

Although not stated in this form in the cited references, still Theorem A is a trivial consequence of the results obtained therein. We remark that the proof of Theorem A crucially relies on the uniqueness and non-degeneracy of solutions of (\mathbf{P}_-) for $\lambda = 8\pi$, see [10] and [6]. If we miss the regularity assumptions in the claim about Ω , then we do not know much about this point and in particular about the validity of Theorem A.

Next we focus on the case where Ω is simply connected and use complex notation. For a fixed $q \in \Omega$, let us denote by $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and by $g_q: \Omega \mapsto \mathbb{D}$ the Riemann map satisfying $g_q(q) = 0$, $g'_q(q) > 0$. Let $f_q: \mathbb{D} \mapsto \Omega$ be the inverse map, which satisfies $f_q(0) = q$ and $g_q = f_q^{-1}: \Omega \mapsto \mathbb{D}$. Next, setting $w = f_q(z)$, we find that

$$R_\Omega(w, q) = G_\Omega(w, q) + \frac{1}{2\pi} \log |w - q| = -\frac{1}{2\pi} \log \frac{|g_q(w)|}{|w - q|} = \frac{1}{2\pi} \log \frac{|f_q(z) - f_q(0)|}{|z|},$$

and it is well known that the Robin function takes the form

$$(1.6) \quad \gamma_\Omega(w) = R_\Omega(w, w) = \frac{1}{2\pi} \log \frac{(1 - |g_q(q)|^2)}{|g'_q(w)|} = \frac{1}{2\pi} \log (1 - |z|^2) |f'_q(z)|.$$

Next, let us consider the power series relative to f_q ,

$$(1.7) \quad f_q(z) = q + a_1 z + \sum_{n=2}^{+\infty} a_n z^n, \quad |z| < 1,$$

where we used that $a_1 = f'_q(0) > 0$, since by assumption $g'_q(q) > 0$. Therefore we see that

$$\gamma_\Omega(q) = \frac{1}{2\pi} \log(|a_1|),$$

and

$$(1.8) \quad \left. \frac{\partial}{\partial z} R_\Omega(z, q) \right|_{z=q} = \frac{a_2}{4\pi a_1^2} = \frac{a_2}{4\pi},$$

whence (1.3) is equivalent to $a_2 = 0$. At this point, one can prove (see [10]) that

$$(1.9) \quad D_\Omega(q) := |a_1| A_\Omega(q) = -|a_1|^2 + \sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_n|^2,$$

which is well defined and convergent since $a_2 = 0$ and $|\Omega| = \pi \sum_{n=1}^{+\infty} n |a_n|^2$ by the area theorem. We will use when needed the fact that D_Ω and A_Ω share the same sign without further comments. In particular, any bounded and simply connected domain admits a Riemann map whose series expansion takes the form

$$(1.10) \quad f_q(z) = q + a_1 z + \sum_{n=3}^{+\infty} a_n z^n, \quad |z| < 1, \quad a_1 > 0,$$

where q is a critical point of γ_Ω .

For a pair Ω, Ω_0 of class $C^1(C^{0,1})$, we will denote by $d_1(\Omega, \Omega_0)$ ($d_{0,1}(\Omega, \Omega_0)$) the distance in the $C^1(C^{0,1})$ -domain topology, see Section 2 for more details.

We then have the following result.

Theorem 1.4. (i) Let Ω be a domain of first kind of class C^1 with $A_\Omega(q) < 0$. Then there exists $\varepsilon_0 > 0$ such that if Ω_0 satisfies $d_1(\Omega, \Omega_0) < \varepsilon_0$, then $\Omega_0 \in \mathcal{A}_1$ and $A_{\Omega_0}(q_0) < 0$.

(ii) Let Ω be a simply connected and regular domain of first kind with $A_\Omega(q) < 0$. Then there exists $\varepsilon_0 > 0$ such that if Ω_0 is a simply connected and regular domain and $d_{0,1}(\Omega, \Omega_0) < \varepsilon_0$, then $\Omega_0 \in \mathcal{A}_1$ and $A_{\Omega_0}(q_0) < 0$.

It is interesting that $\mathcal{A}_1 \cap \{A_\Omega(q) < 0\}$ is open in the C^1 -topology. Obviously, the problem is more subtle for the $C^{0,1}$ -domain topology. Indeed, our proof of Theorem 1.4 crucially relies on Theorem A, that is, on a characterization of domains of first kind which unfortunately is not known for a general domain of class $C^{0,1}$.

Our next result is a coefficient-based sufficient condition that defines an open region of starlike domains in \mathcal{A}_1 which contains all disks. Let Ω be a simply connected domain, let q be a critical point of γ_Ω , and let f_q be the Riemann map defined in (1.10). Let us denote by \mathcal{S}_I the subset of those Ω such that $\Omega = f_q(\mathbb{D})$ with f_q as in (1.10) and

$$(1.11) \quad \sum_{n=3}^{+\infty} n |a_n| < |a_1|,$$

and by $\partial\mathcal{S}_I$ the subset of those Ω such that

$$(1.12) \quad \sum_{n=3}^{+\infty} n |a_n| = |a_1|.$$

Then we have the following.

Theorem 1.5. If $\Omega \in \mathcal{S}_I$, then Ω is starlike, of class C^1 and of first kind with $D_\Omega(q) < 0$. In particular, if $\Omega \in \mathcal{S}_I$ and $\{a_n\}_{n \in \mathbb{N}}$ are the coefficients of (1.10), then for any continuous (with respect to the ℓ_∞ -topology) map $\underline{a}(t) = (a_1, 0, a_3(t), a_4(t), \dots)$, $t \in [0, 1]$, satisfying $a_n(0) = 0$, $a_n(1) = a_n$ and $|a_n(t)| \leq |a_n|$ for any $n \geq 3$, then

$$f(z, t) = q + a_1 z + \sum_{n=3}^{+\infty} a_n(t) z^n, \quad |z| \leq 1, \quad t \in [0, 1],$$

is a jointly continuous family of univalent and starlike maps $f(z, t)$, $z \in \overline{\mathbb{D}}$, $t \in [0, 1]$ such that $\overline{\Omega}_t = f(\overline{\mathbb{D}}, t)$ satisfies $\Omega_t \in \mathcal{S}_I$ for any t , $\Omega_0 = q + a_1 \mathbb{D}$, $\Omega_1 = \Omega$. Therefore, in particular Ω_t is of first kind and $D_{\Omega_t}(q) < 0$ for any $t \in [0, 1]$.

Moreover, the same conclusion holds for Ω_t with $t \in [0, 1)$, whenever $\Omega \in \partial\mathcal{S}_I$ is regular and if at least one $|a_n(t)|$ is strictly increasing in a left neighborhood of $t = 1$.

Please observe that since (1.11) implies that Ω is of class C^1 , it also follows from Theorem 1.4 that \mathcal{S}_I is open. However, the interest of Theorem 1.5 relies in the fact that, as already mentioned above, it is not true that any starlike or either convex domain is of first kind.

Actually, since by the area formula the formal series built with $a_n(t)$ as in Theorem 1.5, that is,

$$D_\Omega(q) = -|a_1(t)|^2 + \sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_n(t)|^2,$$

is convergent, then it is tempting to try to adopt the same argument to prove the path-connectedness of the full set of either C^1 or regular domains of first kind. Unfortunately, this argument fails in general. Indeed, on one side if we miss (1.11) then it is not anymore guaranteed that $f(\mathbb{D})$ is either C^1 or even just regular in the sense of Definition 1.1. For example, as observed in [10], $f_3(z) = z + \frac{1}{3}z^3$ is univalent in \mathbb{D} and satisfies $D_\Omega(0) = \sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_n|^2 - |a_1|^2 = 9|a_3|^2 - |a_1|^2 = 0$, $\sum_{n=3}^{+\infty} n|a_n| = 3|a_3| = 1 = |a_1|$, but at the same time $f_3'(\pm i) = 0$ and the domain $f_3(\mathbb{D}) \in \partial\mathcal{S}_I$ has cusps (inner angle 2π) at its boundary points $\pm \frac{2}{3}i \in \partial\Omega$. In particular, $f_3(\mathbb{D})$ is not regular and then we cannot apply Theorem A above, whence the full argument breaks down. Actually, we see in this way that (1.11) is sharp as far as we are concerned with the regularity of the domain. However, (1.11) is not necessary for a domain to be of first kind, as we illustrate with an explicit example in Appendix III.

On the other side, it is neither true that if

$$D_\Omega(q) = -|a_1(t)|^2 + \sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_n(t)|^2$$

is negative then f as in (1.10) is univalent. For example, the sequence

$$\underline{a} = (1, 0, t/3, t/4, 0, 0, \dots)$$

corresponds to the holomorphic function $f(z) = z + \frac{t}{3}z^3 + \frac{t}{4}z^4$, which for $t \in [7/10, 4/5]$ is readily seen to be not univalent in \mathbb{D} , although $D_\Omega(0)$ is convergent and strictly negative.

However, we can show that a particularly simple choice of the $a_n(t)$'s in Theorem 1.5 does the job for any simply connected domain of first kind. We recall that a domain Ω is said to be *analytic* if $\partial\Omega = f(\partial\mathbb{D})$ for some f univalent in a full open neighborhood of \mathbb{D} . Then we have the following.

Theorem 1.6. *Let Ω be a simply connected domain of first kind, either regular or of class C^1 , and let f_q be the Riemann map normalized as in (1.10). Then*

$$(1.13) \quad f(z, t) = tq + \frac{f_q(tz) - q}{t}, \quad |z| \leq 1, \quad t \in [-1, 1],$$

is jointly continuous in $\overline{\mathbb{D}} \times [-1, 1]$, jointly analytic in $(z, t) \in \mathbb{D} \times t \in (-1, 1)$, $\overline{\Omega}_t = f(\overline{\mathbb{D}}, t)$ is an analytic domain for any $t \in [0, 1)$ and satisfies $\Omega_t \in \mathcal{A}_1$ for any $t \in [0, 1]$, $\Omega_1 = \Omega$ and $\Omega_0 = \mathbb{D}$.

In particular, the set of simply connected C^1 domains of first kind is contractible, while the set of simply connected regular domains of first kind is simply connected with respect to the $C^{0,1}$ -topology.

What the proof shows is that $f(z, t)$ defines a deformation retract of the identity in the subspace of C^1 domains of first kind. As remarked right after Theorem 1.4, the situation for regular domains is more delicate, which is why we come up with a weaker result in this case. In conclusion, as a consequence of Theorems 1.4 and 1.6, we have the following,

Corollary 1.7. *The set of simply connected domains of first kind of class C^1 is a contractible set with non empty interior with respect to the C^1 -topology.*

The set of regular and simply connected domains of first kind is a simply connected set with respect to the $C^{0,1}$ -topology.

It is an interesting open problem to understand how the topology of \mathcal{A}_1 is affected by the topology of the underlying domains.

This paper is organized as follows. In Section 2 we define the distances and topologies used in the introduction and list some known results which will be needed in the sequel. In Section 3 we prove Theorem 1.4. In Section 4 we prove Theorems 1.5 and 1.6. Some technical results and an example are discussed in the Appendices.

2. Preliminaries

Let us now introduce some useful definitions and distances between domains.

Definition 2.1. A domain Ω is of class $C^k(C^{0,1})$, $k \geq 2$, if for each $x_0 \in \partial\Omega$ there exist a ball $B = B_r(x_0)$ and a one-to-one map $\Phi: B \mapsto U \subset \mathbb{R}^2$ such that $\Phi \in C^k(B)(C^{0,1}(B))$, $\Phi^{-1} \in C^k(U)(C^{0,1}(U))$ and the following holds:

$$\Phi(\Omega \cap B) \subset \mathbb{R}_+^2 \quad \text{and} \quad \Phi(\partial\Omega \cap B) \subset \partial\mathbb{R}_+^2.$$

It is well known (see for example [15]) that this is equivalent to the existence of $r > 0$ and $M > 0$ such that, given any ball $B = B_r(x_0)$, $x_0 \in \mathbb{R}^2$ then, after suitable rotation and translations, it holds

$$\Omega \cap B = \{(x_1, x_2) : x_2 < \phi(x_1)\} \cap B \quad \text{and} \quad \partial\Omega \cap B = \{(x_1, x_2) : x_2 = \phi(x_1)\} \cap B,$$

for some $\phi \in C^k(\mathbb{R})(C^{0,1}(\mathbb{R}))$ whose norm is not larger than M .

We will also need some classical results about extensions up to the boundary of Riemann maps.

Remark 2.2. If $f: \mathbb{D} \rightarrow \Omega$ is univalent and if $\partial\Omega$ is the support of any rectifiable Jordan curve, then f admits a continuous and univalent extension on $\overline{\mathbb{D}}$, see for example Theorem 9.1 in [23]. Moreover, if Ω is of class C^1 , then, in view of Definition 2.1, it is not difficult to see that $\partial\Omega$ admits a C^1 -parametrization $w(t)$, $t \in [0, 2\pi]$, such that $w'(t) \neq 0$, $t \in [0, 2\pi]$. As a consequence, see Theorem 3.5 in [22], f' admits a continuous extension on $\overline{\mathbb{D}}$, with $f'(z) \neq 0$ in $\overline{\mathbb{D}}$.

Next we will use the following definition of distance in the set $\mathcal{M}_1(\Omega)$ of bounded C^1 domains which are C^1 -diffeomorphic to a given bounded domain Ω . This is a particular case of a more general definition first introduced in [19], see also [15]. For $\Omega_1, \Omega_2 \in \mathcal{M}_1(\Omega)$, we define

$$(2.1) \quad d_1(\Omega_1, \Omega_2) = \inf \left\{ \sum_{k=1}^N (\|h_k - I\|_{C^1} + \|h_k^{-1} - I\|_{C^1}) \mid h_1 \circ h_2 \circ \dots \circ h_N(\Omega_1) = \Omega_2 \right\},$$

where $I: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the identity and the infimum is taken over $N \in \mathbb{N}$ and all C^1 -diffeomorphisms $h_k: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that $D^j(h_k(x) - x) \rightarrow 0$ as $|x| \rightarrow +\infty$, $j = 0, 1$.

Equipped with this metric, $\mathcal{M}_1(\Omega)$ is complete and separable. A neighborhood of $\Omega_1 \in \mathcal{M}_1(\Omega)$ in the induced topology contains a neighborhood of the form

$$\{H(\Omega_1) : H \in C^1(\mathbb{R}^2; \mathbb{R}^2), \|H - I\|_{C^1} < \varepsilon, D^j(H(x) - x) \rightarrow 0, |x| \rightarrow +\infty, j = 0, 1\},$$

which in turn contains a ball $\{\Omega_2 : d_1(\Omega_2, \Omega_1) < \delta\}$, for some $\delta > 0$, see Appendix A.2 in [15] for proofs. This is the distance d_1 and the C^1 -domain topology which we refer to in Theorems 1.4 and 1.6, and Corollary 1.7. In particular, the set of all open and bounded domains of class C^1 splits into equivalence classes with respect to the relation $\Omega_1 \sim \Omega_2 \Leftrightarrow \Omega_2 = h(\Omega_1)$ for some C^1 -diffeomorphism h . Obviously, one of these equivalence classes is the subset of simply connected domains of class C^1 .

Concerning simply connected and regular domains (see Definition 1.1), we first observe that, in view of Remark 2.2, any such a domain Ω can be mapped one-to-one onto \mathbb{D} via a Riemann map $f: \mathbb{D} \rightarrow \Omega$, which admits a one-to-one and continuous extension on $\overline{\mathbb{D}}$. Let $Q_j \in \partial\Omega$ be any corner and assume without loss of generality that $Q_j = 0 = f(1)$. Then, by Theorem 3.9 in [22], either $\theta_j = \pi\alpha \in (0, \pi)$, and then $|f'(z)| \leq C|z - 1|^{1-\alpha}$ for $z \in \mathbb{D} \cap B_r(1)$, or $\theta_j = \pi\alpha \in (\pi, 2\pi)$, and then $|f(z)|/|z - 1| \leq C|z - 1|^{\alpha-1}$ for $z \in \mathbb{D} \cap B_r(1)$, for suitable $r > 0$. Therefore, it is not difficult to see that any regular domain is in particular of class $C^{0,1}$ in the sense defined above. As a consequence, the distance of any two regular domains, which we will denote by $d_{0,1}(\Omega_1, \Omega_2)$, can be defined just by replacing C^1 with $C^{0,1}$ in (2.1). A generalization of the notion introduced in [19], with a proof of the fact that $d_{0,1}$ is indeed a well defined metric, can be found in Chapter 3 of [13].

3. Proof of Theorem 1.4

In this section we prove Theorem 1.4, first discussing (ii). The proof of (i) when Ω is simply connected follows exactly by the same argument but it is easier, and we omit it here to avoid repetitions. Then we will be back to (i) for general connected but not simply connected domains Ω of class C^1 .

We can assume without loss of generality that $q = 0$ and in particular, by Remark 1.3, that $a_1 = f'_q(0) = 1$. By Remark 2.2, the Riemann map (1.10) can be extended to a continuous and univalent map $f_q(\overline{\mathbb{D}}) = \overline{\Omega}$ which then takes the form

$$f_q(z) = z + \sum_{n=3}^{+\infty} a_n z^n, \quad z \in \overline{\mathbb{D}}.$$

We argue by contradiction. If the claim were false, then we could find a sequence of regular domains Ω_k such that $d_{0,1}(\Omega, \Omega_k) \rightarrow 0$ in the $C^{0,1}$ -topology and $\Omega_k \in \mathcal{A}_{\text{II}}$ for any k . Let γ_{Ω_k} denote the Robin function of Ω_k . Since $d_{0,1}(\Omega, \Omega_k) \rightarrow 0$ and each Ω_k is regular, then it is not difficult to see that $\Omega_k \rightarrow \Omega$ in the sense of Caratheodory's kernel convergence [23]. Therefore, for any k large enough, since $q = 0$ is an interior point of Ω , then we have $q = 0 \in \Omega_k$ and then we can define $\hat{f}_k: \overline{\mathbb{D}} \rightarrow \overline{\Omega_k}$ to be a sequence of univalent

maps which satisfy $\widehat{f}_k(\overline{\mathbb{D}}) = \overline{\Omega}_k$ and hence

$$\widehat{f}_k(z) = \widehat{a}_{1,k}z + \sum_{n=2}^{+\infty} \widehat{a}_{n,k}z^n, \quad z \in \overline{\mathbb{D}}, \quad \widehat{a}_{1,k} > 0, \quad \forall k \in \mathbb{N}.$$

By the Caratheodory kernel theorem ([23], Theorem 1.8), we conclude that $\widehat{f}_k \rightarrow f_q$ locally uniformly and then also in $C_{\text{loc}}^3(\mathbb{D})$.

Next observe that, by Theorem A, $q = 0 = f_q(0)$ is the unique and non-degenerate maximum point of γ_Ω . As a consequence of (1.6), we see that $\gamma_{\Omega_k} \rightarrow \gamma_\Omega$ in $C_{\text{loc}}^2(D)$ and then, for k large enough, γ_{Ω_k} has a unique and non-degenerate maximum point, which we denote by q_k and satisfies $q_k \rightarrow q = 0$. At this point, we can define

$$(3.1) \quad f_{k,q_k}(z) = q_k + a_{1,k}z + \sum_{n=2}^{+\infty} a_{n,k}z^n, \quad z \in \overline{\mathbb{D}}, \quad a_{1,k} > 0, \quad \forall k,$$

to be the sequence of univalent functions which satisfies $f_{k,q_k}(\overline{\mathbb{D}}) = \overline{\Omega}_k$, $f_{k,q_k}(0) = q_k$ and $f'_{k,q_k}(0) > 0$. As a consequence of (1.8), we find that $a_{2,k} = 0$. In particular, by using once more the kernel theorem, we find that $f_{k,q_k} \rightarrow f_q$ locally uniformly in \mathbb{D} .

Next, let us recall that a sequence of compact domains U_k is said to be *uniformly locally connected* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, if $a_k, b_k \in \Omega_k$ and $|a_k - b_k| < \delta$, there exists connected compact sets $B_k \subseteq U_k$ such that $a_k, b_k \in B_k$ and $\text{diam}(B_k) < \varepsilon$. Since $d_{0,1}(\Omega, \Omega_k) \rightarrow 0$ and each Ω_k is regular, then it can be shown that:

Claim: $\overline{\Omega}_k$ is uniformly locally connected.

See Appendix I for a proof of this fact.

Therefore, by Theorem 9.11 in [23], we conclude that $f_{k,q_k} \rightarrow f_q$ uniformly in $\overline{\mathbb{D}}$. Thus, since $\|f_{k,q_k} - f_q\|_\infty \rightarrow 0$, then by Cauchy's representation formula we find that

$$\sup_{n \in \mathbb{N}} |a_{n,k} - a_n| \leq \|f_{k,q_k} - f_q\|_\infty \rightarrow 0, \quad \text{as } k \rightarrow +\infty.$$

At this point, since $D_\Omega(q) < 0$, we have

$$\lim_{N \rightarrow +\infty} \lim_{k \rightarrow +\infty} \sum_{n=3}^N \frac{n^2}{n-2} |a_{n,k}|^2 = \sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_n|^2 = |a_1|^2 - \sigma = 1 - \sigma,$$

for some $\sigma \in (0, 1)$, while on the other side, by the uniform convergence of $a_{n,k}$, we find that

$$\begin{aligned} 1 - \sigma &= \lim_{k \rightarrow +\infty} \lim_{N \rightarrow +\infty} \sum_{n=3}^N \frac{n^2}{n-2} |a_{n,k}|^2 = \lim_{k \rightarrow +\infty} \sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_{n,k}|^2 \\ &\geq \lim_{k \rightarrow +\infty} |a_{1,k}|^2 = 1, \end{aligned}$$

where the last inequality follows from Theorem A, which implies that, since $\Omega_k \in \mathcal{A}_{\text{II}}$ for any k , then

$$\sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_{n,k}|^2 > |a_{1,k}|^2 \quad \text{for any } k.$$

This contradiction shows that there exists $\varepsilon_0 > 0$ such that if Ω_0 is regular and $d_{0,1}(\Omega, \Omega_0) < \varepsilon_0$, then $\Omega_0 \in \mathcal{A}_I$.

Taking a smaller ε_0 if necessary, the same argument shows that $D_{\Omega_0}(q_0) < 0$ as well and we skip this part of the proof to avoid repetitions.

We are left with the proof of (i) in the case when Ω is not simply connected.

We will denote by $C > 0$ a uniform positive constant whose value may change from line to line. By Theorem A, Ω has a unique and non-degenerate maximum point q . If the claim were false, then we could find a sequence of C^1 domains Ω_k such that $d_1(\Omega, \Omega_k) \rightarrow 0$ and $\Omega_k \in \mathcal{A}_{II}$ for any k . Let $R_k(x, y)$ be the regular part of the Green function for Ω_k as defined in (1.1), and let γ_k be its Robin function. It can be shown, see Appendix II, that for any fixed $y \in \Omega$ we have

$$(3.2) \quad R_k(x, y) \rightarrow R_\Omega(x, y) \quad \text{in } C_{\text{loc}}^3(\Omega)$$

and

$$(3.3) \quad \gamma_k \rightarrow \gamma_\Omega \quad \text{in } C_{\text{loc}}^2(\Omega).$$

Therefore, for k large, γ_k will have a unique and non-degenerate maximum point $q_k \rightarrow q$. We can assume for the moment without loss of generality that $q_k = 0$ for any k . By assumption, for fixed k , we have,

$$A_k(0) := \lim_{\varepsilon \rightarrow 0} \int_{\Omega_k \setminus B_\varepsilon(0)} \frac{e^{8\pi(R_k(x,0) - \gamma_k(0))} - 1}{|x|^4} - \int_{\Omega^c} \frac{1}{|x|^4} > 0,$$

and then, in particular,

$$\liminf_{k \rightarrow +\infty} A_k(0) \geq 0.$$

We will obtain a contradiction by showing that there exists $\sigma > 0$ small enough such that

$$(3.4) \quad \limsup_{k \rightarrow +\infty} A_k(0) \leq -\sigma.$$

Clearly, there exists $d > 0$ small enough such that $B_d(0) \subset\subset \Omega_k$ for any k large. By (1.5) we have

$$e^{8\pi(R_k(x,0) - \gamma_k(0))} - 1 = \sum_{i,j}^{1,2} a_{k,ij} x_i x_j + c_k(x), \quad x \in \overline{B_d(0)},$$

where

$$a_{k,ij} = \partial_{x_i, x_j} R_k(x, 0) \rightarrow \partial_{x_i, x_j} R(x, 0) = a_{ij}, \quad i, j = 1, 2,$$

and by (3.2), the reminders c_k satisfy

$$|c_k(x)| \leq C|x|^3, \quad \text{for } k \text{ large.}$$

As a consequence, for any $\varepsilon \leq d/4$, we find that

$$\begin{aligned}
 \left| \int_{B_d(0) \setminus B_\varepsilon(0)} \frac{e^{8\pi(R_k(x,0) - \gamma_k(0))} - 1}{|x|^4} \right| &\leq \left| \int_{B_d(0) \setminus B_\varepsilon(0)} \frac{\sum a_{k,ij} x_i x_j + c_k(x)}{|x|^4} \right| \\
 &= \left| \int_{B_d(0) \setminus B_\varepsilon(0)} \frac{2a_{k,12} x_1 x_2 + c_k(x)}{|x|^4} \right| \\
 (3.5) \qquad \qquad \qquad &= \left| \int_{B_d(0) \setminus B_\varepsilon(0)} \frac{c_k(x)}{|x|^4} \right| \leq Cd, \quad \text{for } k \text{ large,}
 \end{aligned}$$

where we used the symmetry of the domain and the fact that, since $R_k(x, 0)$ is harmonic in Ω , $a_{k,11} + a_{k,22} = 0$ for any k . To simplify the evaluation, let us set

$$h_k(x) = \frac{e^{8\pi(R_k(x,0) - \gamma_k(0))} - 1}{|x|^4}, \quad x \in \Omega_k, \quad h(x) = \frac{e^{8\pi(R_\Omega(x,0) - \gamma_\Omega(0))} - 1}{|x|^4}, \quad x \in \Omega.$$

Since $R_k(x, q_k) \rightarrow R_\Omega(x, q)$ locally uniformly in Ω , and $\gamma_k(q_k) \rightarrow \gamma_\Omega(q)$, then, for any open and relatively compact subset $\hat{\Omega} \subset \subset \Omega$, we have

$$(3.6) \quad \hat{\Omega} \subset \subset \Omega_k, \quad \text{for } k \text{ large, and} \quad \int_{\hat{\Omega} \setminus B_d(q_k)} h_k(x) \rightarrow \int_{\hat{\Omega} \setminus B_d(q)} h(x), \quad k \rightarrow +\infty.$$

Since the symmetric difference $\Omega_k \Delta \Omega \rightarrow \emptyset$ as $k \rightarrow +\infty$, then for any $\delta > 0$ we can choose an open and relatively compact subset $\hat{\Omega}_\delta$ as in (3.6) which also satisfies

$$\Omega_k \setminus \hat{\Omega}_\delta \subset B_{2\delta}(\partial\Omega_k) \quad \text{and} \quad \Omega \setminus \hat{\Omega}_\delta \subset B_{2\delta}(\partial\Omega) \quad \text{for } k \text{ large,}$$

and then, in particular,

$$(3.7) \quad \int_{(\Omega_k \setminus B_d(q_k)) \setminus (\hat{\Omega}_\delta \setminus B_d(q_k))} |h_k(x)| \leq \int_{B_{2\delta}(\partial\Omega_k)} |h_k(x)| \leq C |\partial\Omega_k| \delta, \quad \text{for } k \text{ large,}$$

$$(3.8) \quad \int_{(\Omega \setminus B_d(q)) \setminus (\hat{\Omega}_\delta \setminus B_d(q))} |h(x)| \leq \int_{B_{2\delta}(\partial\Omega)} |h(x)| \leq C |\partial\Omega| \delta, \quad \text{for } k \text{ large,}$$

where we used the uniform bound

$$(3.9) \quad |h_k(x)| \leq C, \quad x \in B_{2\delta}(\partial\Omega_k) \cap \Omega_k,$$

see Appendix II. Thus we can estimate as follows:

$$\begin{aligned}
 &\int_{\Omega_k \setminus B_\varepsilon(q_k)} h_k(x) - \int_{\Omega \setminus B_\varepsilon(q)} h(x) \\
 &\leq \int_{B_d(q_k) \setminus B_\varepsilon(q_k)} |h_k(x)| + \int_{B_d(q) \setminus B_\varepsilon(q)} |h(x)| \\
 &\quad + \int_{(\Omega_k \setminus B_d(q_k)) \setminus (\hat{\Omega}_\delta \setminus B_d(q_k))} |h_k(x)| + \int_{(\Omega \setminus B_d(q)) \setminus (\hat{\Omega}_\delta \setminus B_d(q))} |h(x)| \\
 &\quad + \left| \int_{\hat{\Omega}_\delta \setminus B_d(q_k)} h_k(x) - \int_{\hat{\Omega}_\delta \setminus B_d(q)} h(x) \right| \\
 (3.10) \quad &\leq Cd + C\delta + \left| \int_{\hat{\Omega}_\delta \setminus B_d(q_k)} h_k(x) - \int_{\hat{\Omega}_\delta \setminus B_d(q)} h(x) \right|,
 \end{aligned}$$

where we used (3.5), (3.7) and (3.8). At this point, let us fix $\sigma > 0$ such that $A_\Omega(q) = -5\sigma$ and then choose δ and d such that $Cd + C\delta < \sigma$. For any δ and d fixed in this way, by (3.6) we have

$$\left| \int_{\widehat{\Omega}_\delta \setminus B_d(q_k)} h_k(x) - \int_{\widehat{\Omega}_\delta \setminus B_d(q)} h(x) \right| < \sigma,$$

for any k large enough. In particular, for k large we also have

$$\left| \int_{\Omega_k^c} \frac{1}{|x - q_k|^4} - \int_{\Omega^c} \frac{1}{|x - q|^4} \right| < \sigma.$$

Finally, we can choose $\varepsilon_0 > 0$ small enough to guarantee that

$$\int_{\Omega \setminus B_\varepsilon(q)} h(x) - \int_{\Omega^c} \frac{1}{|x - q|^4} \leq -4\sigma,$$

for any $\varepsilon < \varepsilon_0$. Plugging these estimates together with (3.10), we conclude that

$$\int_{\Omega_k \setminus B_\varepsilon(q_k)} h_k(x) - \int_{\Omega_k^c} \frac{1}{|x - q_k|^4} \leq \int_{\Omega \setminus B_\varepsilon(q)} h(x) - \int_{\Omega^c} \frac{1}{|x - q|^4} + 3\sigma \leq -\sigma,$$

for any k large enough and for any $\varepsilon < \varepsilon_0$. As a consequence, we conclude that (3.4) holds, which is the desired contradiction.

4. Proofs of Theorems 1.5 and 1.6

In this section we prove Theorems 1.5 and 1.6.

Proof of Theorem 1.5. By Remark 1.3, we can assume without loss of generality that $a_1 = f'_q(0) = 1$, and after a translation, we can also assume that $q = 0$. Therefore (1.10) takes the form

$$f_q(z) = z + \sum_{n=3}^{+\infty} a_n z^n, \quad |z| < 1.$$

It is well known (see for example [23], p. 44) that if $\sum_{n=2}^{+\infty} n|a_n| \leq 1$, then f'_q has positive real part and f_q is univalent and starlike in \mathbb{D} . Therefore, since $a_2 = 0$, by (1.11) f_q is in fact univalent and starlike. Setting $h(\theta) = f_q(e^{i\theta})$, $\theta \in [0, 2\pi]$, we have

$$h(\theta) = e^{i\theta} + \sum_{n=3}^{+\infty} a_n e^{in\theta},$$

and once more (1.11) shows that the real and imaginary parts of h have continuous first derivatives in $[0, 2\pi]$ satisfying $\|h'\|_\infty \leq 2$. On the other hand, we also have

$$|h'(\theta)| \geq 1 - \sum_{n=3}^{+\infty} n|a_n| > 0,$$

once more by (1.11). Therefore $h(\theta)$ is a C^1 curve and then in particular Ω is of class C^1 . Since $\sum_{n=3}^{+\infty} n|a_n| < 1$, then

$$\sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_n|^2 \leq \sum_{n=3}^{+\infty} n^2 |a_n|^2 < 1.$$

Therefore $D_\Omega(q) < 0$, and hence $\Omega \in \mathcal{A}_1$ by Theorem A.

In particular, we have shown that any $\Omega \in \mathcal{S}_I$ is a C^1 domain of first kind with $D_\Omega(q) < 0$. Now we define

$$f(z; t) = z + \sum_{n=3}^{+\infty} a_n(t) z^n, \quad |z| < 1, \quad t \in [0, 1],$$

where each $a_n(t)$ is continuous in $[0, 1]$, $a_n(0) = 0$, $a_n(1) = a_n$ and $|a_n(t)| \leq |a_n|$ for any $n \geq 3$. It is easy to see that $\Omega_t = f(\mathbb{D}, t) \in \mathcal{S}_I$ for any t . In particular, by Remark 2.2, f admits a continuous and univalent extension $f(\overline{\mathbb{D}}, 1) = \overline{\Omega}$ and $f(\overline{\mathbb{D}}, 0) = q + \overline{\mathbb{D}}$, as claimed.

Finally, if $\Omega \in \partial\mathcal{S}_I$, then the same argument shows that $\Omega_t = f(\mathbb{D}, t) \in \mathcal{S}_I$ for any $t \in [0, 1]$, whenever at least one $|a_n(t)|$ is strictly increasing for $t \simeq 1^-$, and the conclusion follows in this case as well. ■

Proof of Theorem 1.6. Let f_q be the Riemann map of Ω normalized as in (1.10). By Remark 1.3, we can assume without loss of generality that $a_1 = f'_q(0) = 1$. Obviously, $f(tz)$ is univalent in \mathbb{D} for any $t \in (0, 1]$, and so is $(f(tz) - q)/t$, which takes the form

$$(4.1) \quad \frac{f(tz) - q}{t} = z + \sum_{n=3}^{+\infty} a_n t^{n-1} z^n.$$

Since $\partial\Omega_t = f(\{|z| = t\})$, $t < 1$, then Ω_t is analytic for any $t \in (0, 1)$. Also,

$$D_{\Omega_t}(0) = \sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_n|^2 t^2 - |a_1|^2 \leq \sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_n|^2 - |a_1|^2 = D_\Omega(0) \leq 1,$$

which by Theorem A shows that $\Omega_t \in \mathcal{A}_1$ for any t . Since the domain Ω is at least regular, then, by Remark 2.2, f_q admits a continuous extension to $\overline{\mathbb{D}}$. Thus the series in (4.1) converges for $t = \pm 1$ and $z \in \partial\mathbb{D}$, and then, for fixed $t \in (-1, 1)$, it is totally convergent in $\{|z| \leq r\}$ for any $r \in (0, 1)$, and for fixed $z \in \mathbb{D}$ it is totally convergent in $\{|t| < \delta\}$, for any $\delta \in (0, 1)$. Therefore, $f(z, t)$ is a separately analytic function in $\mathbb{D} \times (-1, 1)$. By the Abel theorem, the series converges uniformly in $\overline{\mathbb{D}} \times [-1, 1]$ to a continuous function. Then $f(z, t)$ is continuous in $\overline{\mathbb{D}} \times [-1, 1]$. However, it is well known that a separately analytic and jointly continuous function is jointly analytic (see e.g. [16], Theorem 2.2.1), whence $f(z, t)$ is a jointly analytic in $\mathbb{D} \times (-1, 1)$. The first part of the claim readily follows since $f(z, 0) = z$ and $f(z, 1) = f_q(z)$.

Next, let \mathcal{A}_1 denote the topological space of C^1 simply connected domains endowed with the C^1 -topology, with $\mathcal{A}_{1,1}$ the subset of domains of first kind, and let $\mathcal{H}(\mathbb{D})$ denote the space of holomorphic functions in \mathbb{D} . Define the map $F: \mathcal{H}(\mathbb{D}) \times [0, 1] \rightarrow \mathcal{H}(\mathbb{D})$ as

$$F(f, t) = f(\cdot, t),$$

with $f(z, t)$ as in (1.13) above. The induced map $\mathcal{F}: \mathcal{A}_1 \rightarrow \mathcal{A}_1$ takes the form

$$\mathcal{F}(\Omega, t) = \Omega_t = f(\mathbb{D}, t),$$

and obviously we have

$$\mathcal{F}(\Omega, 0) = \Omega_0 = \mathbb{D} \quad \text{and} \quad \mathcal{F}(\Omega, 1) = \Omega_1 = \Omega.$$

Therefore $\mathcal{J} := \mathcal{F}(\cdot, 1)$ is the identity map in \mathcal{A}_1 and in particular, by the first part of the claim, $\mathcal{F}(\Omega, t) \in \mathcal{A}_{1,1}$ for any $t \in [0, 1]$. We claim that the restriction $\mathcal{F}: \mathcal{A}_{1,1} \times [0, 1] \rightarrow \mathcal{A}_{1,1}$ is a deformation retract of the identity in the given topology, which proves that $\mathcal{A}_{1,1}$ is contractible. By Remark 2.2, we see that f' admits a continuous extension on $\overline{\mathbb{D}}$ with $f'(z) \neq 0$ on $\overline{\mathbb{D}}$. Since obviously $d_1(\Omega_0, \mathbb{D}) = 0$ and $d_1(\Omega_1, \Omega) = 0$, then to establish the claim it will be enough to prove that $d_1(\Omega_t, \Omega)$ is continuous in $[0, 1]$. We will prove a statement which easily implies the claim, namely that

$$d_\infty(t) := \sup_{z \in \overline{\mathbb{D}}} |f(z, t) - z| + \sup_{z \in \overline{\mathbb{D}}} |f'(z, t) - 1|$$

is continuous in $[0, 1]$. We argue by contradiction and assume that there exist $t_0 \in [0, 1]$ and sequences $t_{n,i} \rightarrow t_0 \in [0, 1]$, $i = 1, 2$, such that $|d_\infty(t_{n,2}) - d_\infty(t_{n,1})| \geq \varepsilon_0$, for some $\varepsilon_0 > 0$ and any n . Clearly we can find sequences $z_{n,i}$ and $w_{n,i}$, $i = 1, 2$, which are maximizers of the corresponding absolute values, such that for any n ,

$$d_\infty(t_{n,i}) = |f(z_{n,i}, t_{n,i}) - z_{n,i}| + |f'(w_{n,i}, t_{n,i}) - 1|, \quad i = 1, 2.$$

Passing to suitable subsequences, we can assume without loss of generality that $z_{n,i} \rightarrow z_{i,0}$ and $w_{n,i} \rightarrow w_{i,0}$, $i = 1, 2$, where obviously $z_{i,0}$ and $w_{i,0}$ are maximizers of the corresponding absolute values for $t = t_0$. Consequently, as $n \rightarrow +\infty$, we would find that

$$\begin{aligned} \varepsilon_0 &\leq |d_\infty(t_{n,2}) - d_\infty(t_{n,1})| \\ &\leq \left| |f(z_{n,2}, t_{n,2}) - z_{n,2}| + |f'(w_{n,2}, t_{n,2}) - 1| - |f(z_{n,1}, t_{n,1}) - z_{n,1}| \right. \\ &\quad \left. - |f'(w_{n,1}, t_{n,1}) - 1| \right| \\ &\rightarrow \left| |f(z_2, t_0) - z_2| + |f'(w_2, t_0) - 1| - |f(z_1, t_0) - z_1| - |f'(w_1, t_0) - 1| \right| \\ &= |d_\infty(t_0) - d_\infty(t_0)| = 0, \end{aligned}$$

which yields the desired contradiction.

Finally, let $\mathcal{A}_{0,1}$ denote the set of simply connected and regular domains with metric $d_{0,1}$, with $\mathcal{A}_{0,1,1}$ the subset of domains of first kind and with $\Gamma: \mathbb{S}^1 \rightarrow \mathcal{A}_{0,1,1}$ any continuous loop of the form $\Gamma(s) = f(z; s)$, $z \in \overline{\mathbb{D}}$, $s \in \mathbb{S}^1$, where each $f(\cdot; s)$ is normalized as in (1.10) with $q_s = f(0; s)$. Then, by the first part of the statement, the map $F: \mathcal{H}(\mathbb{D}) \times [0, 1] \times \mathbb{S}^1 \mapsto \mathcal{H}(\mathbb{D})$ given by

$$F(\cdot, t, s) = tq_s + \frac{f(\cdot, t; s) - q_s}{t}, \quad (t, s) \in [0, 1] \times \mathbb{S}^1,$$

induces in $\mathcal{A}_{0,1,1}$ a continuous deformation of $\Gamma(\mathbb{S}^1) = \mathcal{F}(\mathbb{D}, 1; \mathbb{S}^1)$ to $\mathbb{D} = \mathcal{F}(\mathbb{D}, 0; \mathbb{S}^1)$. In other words, any loop $\Gamma(\mathbb{S}^1)$ in $\mathcal{A}_{0,1,1}$ can be deformed continuously to \mathbb{D} , which shows that $\mathcal{A}_{0,1,1}$ is simply connected. \blacksquare

A. Appendix I

In this appendix we prove the claim used in the proof of Theorem 1.4(ii), that is: if a sequence Ω_k of regular and simply connected domains satisfies $d_{0,1}(\Omega, \Omega_k) \rightarrow 0$, where Ω is regular and simply connected, then $\overline{\Omega}_k$ is uniformly locally connected.

We argue by contradiction and suppose that $\overline{\Omega}_k$ is not uniformly locally connected. Then $\exists \varepsilon_0 > 0$ such that $\exists a_k, b_k \in \overline{\Omega}_k$ such that $|a_k - b_k| < 1/k$ and $\{a_k, b_k\} \not\subseteq B_k$ for any compact and connected subset $B_k \subset \overline{\Omega}_k$ such that $\text{diam}(B_k) < \varepsilon_0$. Since Ω_k is uniformly bounded, then passing to a subsequence if necessary, we can assume without loss of generality that there exists $z_0 \in \mathbb{C}$ such that $a_k \rightarrow z_0$ and $b_k \rightarrow z_0$. By the kernel convergence and since $a_k, b_k \in \overline{\Omega}_k$, then $z_0 \in \overline{\Omega}$ and we are left with two possibilities: either $z_0 \in \Omega$ or $z_0 \in \partial\Omega$. We can easily exclude the first case, since then any closed disk $\overline{B}_r(z_0)$ with r small enough would contain both a_k and b_k and satisfy $\overline{B}_r(z_0) \subset \Omega_k$, for any k large enough, which is a contradiction.

If $z_0 \in \partial\Omega$, since the domain is regular, then we have two possibilities: either the boundary is locally C^2 near z_0 , or $z_0 = Q$, where Q is one of the vertex points on $\partial\Omega$. We discuss only the second case. The proof of the regular case follows exactly by the same argument but it is easier, and we omit it here to avoid repetitions. After suitable translations, we can assume that $z_0 = Q = 0 \in \Gamma_1 \cap \Gamma_2$, with θ the inner angle of Γ_1 and Γ_2 at 0, and where Γ_j are the C^2 connected components of $\partial\Omega$ near 0. Since Ω is regular, then we can find a univalent map $f: B_\delta(0) \cap \overline{\Omega} \rightarrow \mathbb{C}$ such that $B_\delta(0) \cap \partial\Omega$ is mapped to a C^2 curve.

Taking a smaller δ if necessary and composing with a suitable univalent map, we can assume without loss of generality that $f(0) = 0$, $f(B_\delta(0) \cap \partial\Omega) = \{w \in \mathbb{C} \mid w \in (-1, 1)\}$ and $f(B_\delta(0) \cap \Omega) \subset \{w \in \mathbb{C} \mid \arg(w) \in (0, \pi)\}$, where $\Omega_\delta = B_\delta(0) \cap \Omega$ is a simply connected set. Since $a_k \rightarrow 0$, then $a_k \in B_\delta(0)$ for k large enough, and since by assumption f is continuous and univalent in $B_\delta(0)$, then $f(a_k) \rightarrow f(0) = 0$. Clearly, the same holds for $b_k \rightarrow 0$ and so $f(b_k) \rightarrow 0$, and for any r there exists ν_r such that if $k > \nu_r$ then $f(a_k), f(b_k) \in \overline{B}_r(0)$. In particular, since each Ω_k is regular and converges to Ω in the kernel sense, then we can choose r_0 small enough such that $U_{k,r} = f^{-1}(\overline{B}_r(0)) \cap \overline{\Omega}_k$ is connected for any $r \leq r_0$. Since each $U_{k,r}$ is a compact subset of $\overline{\Omega}_k$ and $\text{diam}(U_{k,r}) \rightarrow 0$ as $r \rightarrow 0$, we conclude that a_k, b_k are contained in a connected compact subset of $\overline{\Omega}_k$, whose diameter is smaller than ε_0 for any k , which is the desired contradiction. ■

B. Appendix II

In this Appendix we prove (3.2), (3.3), (3.9), that is, that for any fixed $y \in \Omega$ we have

$$(B.1) \quad R_k(x, y) \rightarrow R_\Omega(x, y) \quad \text{in } C_{\text{loc}}^3(\Omega)$$

and

$$(B.2) \quad \gamma_k \rightarrow \gamma_\Omega \quad \text{in } C_{\text{loc}}^2(\Omega),$$

and that moreover h_k satisfies the uniform bound

$$(B.3) \quad |h_k(x)| \leq C, \quad x \in B_{2\delta}(\partial\Omega_k) \cap \Omega_k.$$

Proof of (B.1). We can assume without loss of generality that $y = 0$. Clearly, since by assumption $\Omega_k \rightarrow \Omega$ in the C^1 -topology, we have that $0 \in \Omega_k$ for any k large enough. Let $G_k(x, y)$ be the Green function for Ω_k and let $R_k(x, y)$ be its regular part. Since $d_1(\Omega_k, \Omega) \rightarrow 0$, then for any $\delta > 0$ we have that $\partial\Omega_k \subset B_\delta(\partial\Omega)$ for any k large, and in particular there exist $t_k \rightarrow 0^+$, and, for each k , $\varepsilon_k > 0$ and a one-to-one map

$$\Phi_k : B_{2\delta}(\overline{\Omega}) \times [0, 1] \rightarrow B_\delta(\overline{\Omega})$$

that satisfies

$$\begin{aligned} \Phi_k(x, t) &= x + tV_k(x) + o(t), \quad t \in [0, t_k + \varepsilon_k), \\ \Phi_k &\in C^1(B_{2\delta}(\overline{\Omega}) \times [0, 1]; B_\delta(\overline{\Omega})), \quad \|V_k\|_{C^1(B_{2\delta}(\overline{\Omega}); B_\delta(\overline{\Omega}))} \leq C, \\ &\lim_{t \rightarrow 0^+} \sup_{x \in B_{2\delta}(\overline{\Omega})} \frac{|o(t)|}{t} = 0, \end{aligned}$$

and

$$\Phi_k(\Omega, 0) = \Omega, \quad \Phi_k(\Omega, t_k) = \Omega_k.$$

Let $\Omega_k(t) = \Phi_k(\Omega, t)$, let $t \in [0, t_k + \varepsilon_k)$, and let $G_{\Omega_k(t)}(x, y)$ be the corresponding Green function. At this point, we can apply a result in [15] (Example 3.4) which shows that the map $t \mapsto G_{\Omega_k(t)}(x, y)$ is differentiable for $x \neq y$ and in particular that the Hadamard variational formula holds: for $t \in [0, t_k + \varepsilon_k)$,

$$\frac{\partial}{\partial t} G_{\Omega_k(t)}(x, y) = - \int_{\partial\Omega_k(t)} \frac{\partial G_{\Omega_k(t)}(x, z)}{\partial v_z}(x, z) \frac{\partial G_{\Omega_k(t)}(z, y)}{\partial v_z}(z, y) \langle V_k(z), v_z \rangle d\sigma(z),$$

for $\{x, y\} \in \Omega_k(t)$, see also [21]. Here v_z is the unit outer normal to $\Omega_k(t)$. Let $B_R = \overline{B_R}(z)$ be any relatively compact disk in Ω such that $0 \notin \overline{B_R}$. Clearly, for k large we have $\overline{B_R} \cup \{0\} \subset \Omega_k(t)$, $\forall t \in [0, t_k + \varepsilon_k)$, and then we can write

$$G_k(x, 0) = G_\Omega(x, 0) + \left(\frac{\partial}{\partial t} G_{\Omega_k(t)}(x, 0) \right)_{t=0} t_k + o_x(t_k), \quad x \in \overline{B_R},$$

where

$$\left(\frac{\partial}{\partial t} G_{\Omega_k(t)}(x, 0) \right)_{t=0} = - \int_{\partial\Omega} \frac{\partial G_\Omega}{\partial v_z}(x, z) \frac{\partial G_\Omega}{\partial v_z}(z, 0) \langle V_k(z), v_z \rangle d\sigma(z),$$

and $o_x(t_k)$ is an infinitesimal quantity which satisfies

$$(B.4) \quad \forall x \in \overline{B_R}, \quad \lim_{k \rightarrow +\infty} \frac{|o_x(t_k)|}{t_k} = 0.$$

Since Ω is of class C^1 , then

$$\sup_{z \in \partial\Omega} \left| \frac{\partial G_\Omega}{\partial v_z}(x, z) \frac{\partial G_\Omega}{\partial v_z}(z, 0) \langle V_k(z), v_z \rangle \right| \leq C_R,$$

where C_R depends only by R and Ω . As a consequence, we conclude in particular that for any $x \in \overline{B_R}$, it holds

$$|G_k(x, 0) - G_\Omega(x, 0)| \leq C_R t_k (1 + o_x(1)) \rightarrow 0, \quad \text{as } k \rightarrow 0.$$

At this point we observe that, for any smooth domain Ω_1 lying in the interior of Ω and satisfying $0 \notin \partial\Omega_1$, we have that $R_k(x, 0)$ is the unique solution of

$$\begin{cases} -\Delta R_k(x, 0) = 0 & \text{in } \Omega_1, \\ R_k(x, 0) = G_k(x, 0) + \frac{1}{2\pi} \log(|x|) & \text{on } \partial\Omega_1, \end{cases}$$

and $R_\Omega(x, 0)$ is the unique solution of

$$\begin{cases} -\Delta R_\Omega(x, 0) = 0 & \text{in } \Omega_1, \\ R_\Omega(x, 0) = G_\Omega(x, 0) + \frac{1}{2\pi} \log(|x|) & \text{on } \partial\Omega_1. \end{cases}$$

Since $\partial\Omega_1$ is compact, it can be covered with a finite number of balls $B_{R_j}(z_j)$, $z_j \in \partial\Omega$, $j = 1, \dots, N$, such that $0 \notin \overline{B_{R_j}(z_j)}$. As a consequence, $R_k(x, 0) - R_\Omega(x, 0)$ is harmonic in Ω_1 and

$$|R_k(x, 0) - R_\Omega(x, 0)| \leq C_{1,x} t_k \rightarrow 0, \quad \text{as } k \rightarrow +\infty,$$

where $C_{1,x} = \max_j \{C_{R_j}\}(1 + o_x(1))$. Therefore $R_k(x, 0) - R_\Omega(x, 0)$ converges to 0 pointwise on $\partial\Omega_1$ and then also in $C_{\text{loc}}^m(\Omega_1)$ for any $m \geq 1$. Since Ω_1 is arbitrary, the proof of (B.1) is completed.

Proof of (B.2). We first observe that

$$(B.5) \quad R_k(x, y) \rightarrow R_\Omega(x, y) \quad \text{in } C_{\text{loc}}^3(\Omega \times \Omega).$$

Indeed, from (B.1), and since $R_k(x, y) = R_k(y, x)$, we have that, for fixed $x \in \Omega$, $R_k(x, y) \rightarrow R_\Omega(x, y)$ in $C_{\text{loc}}^3(\Omega)$. Then, since $R_k(x, y)$ is harmonic, it is not difficult to check that (B.5) holds.

As a consequence, since

$$\gamma_k(x) - \gamma_\Omega(x) = R_k(x, x) - R_\Omega(x, x) = \lim_{y \rightarrow x} (R_k(x, y) - R_\Omega(x, y)),$$

then, passing to the limit as $k \rightarrow +\infty$, we see that because of (B.5) we can exchange the limits to conclude that $\gamma_k(x) \rightarrow \gamma_\Omega(x)$ pointwise and in particular locally uniformly in Ω . The same argument works for the derivatives, since for example

$$\nabla(\gamma_k(x) - \gamma_\Omega(x)) = 2\nabla(R_k(x, x) - R_\Omega(x, x)),$$

which concludes the proof of (B.2).

Proof of (B.3). Since $\inf_k \text{dist}(0, \partial\Omega_k) \geq d > 0$, by (B.2) we are reduced to prove that, for any $\delta > 0$ small enough, we have

$$R_k(x, 0) \leq C, \quad x \in B_{2\delta}(\partial\Omega_k) \cap \Omega_k.$$

But this is obvious since $R_k(x, 0)$ is harmonic in Ω_k and satisfies $2\pi R_k(x, 0) = -\log(|x|)$ for $x \in \partial\Omega_k$. Therefore

$$2\pi R_k(x, 0) \leq \sup_{x \in \partial\Omega} (-\log(|x|)) \leq -\log d$$

for any $x \in B_{2\delta}(\partial\Omega_k) \cap \Omega_k$, whenever $\delta > 0$ satisfies $3\delta < d$. ■

C. Appendix III

We discuss an example which shows that (1.11) is not necessary for a domain Ω to be of first kind. It is also shown that, increasing each $|a_n(t)|$ along certain paths, one can get well inside \mathcal{A}_{II} . Indeed, let us consider the following family of functions:

$$f(z; t) = z + t \frac{2}{3} \frac{z^3}{3} + t \frac{1}{3} \frac{z^5}{5}, \quad |z| \leq 1, \quad t \in \left[0, \frac{5}{2}\right],$$

which satisfies

$$\sum_{n=3}^{+\infty} n |a_n| = 3 |a_3(t)| + 5 |a_5(t)| = t$$

and

$$D_\Omega(0) = \sum_{n=3}^{+\infty} \frac{n^2}{n-2} |a_n(t)|^2 - |a_1(t)|^2 = \frac{13}{27} t^2 - 1.$$

Some elementary numerics shows that for $t \in [0, 5/2]$, $f(z, t)$ is univalent and maps $\overline{\mathbb{D}}$ onto a C^1 and symmetric (with respect to the x and y axis) domain $\overline{\Omega}_t = f(\overline{\mathbb{D}}, t)$ such that, putting $t_0 = 3\sqrt{3/13} \simeq 1,44$, it holds:

- if $t \leq t_0$, then $\Omega_t \in \mathcal{A}_{\text{I}}$;
- if $t \in (t_0, 3/2]$, then $\Omega_t \in \mathcal{A}_{\text{II}}$ but it is starlike and γ_{Ω_t} has a unique maximum point;
- if $t \in (3/2, 5/2]$, $\Omega_t \in \mathcal{A}_{\text{II}}$, γ_{Ω_t} has two maximum points and, for t close enough to $5/2$, Ω_t is a dumbbell shaped non-starlike domain.

Funding. A. M. is supported by the project *Geometric problems with loss of compactness* from Scuola Normale Superiore and by MIUR Bando PRIN 2015 2015KB9WPT₀₀₁. He is also member of GNAMPA as part of INdAM. D. B. is supported by S.E.E.A. project 2018 (sponsored by Univ. of Rome “Tor Vergata”), by MIUR Bando PRIN 2015 2015KB9WPT₀₀₁ and by MIUR Excellence Department Project, Department of Mathematics, Univ. of Rome Tor Vergata, CUP E83C18000100006.

References

- [1] Bartolucci, D.: Global bifurcation analysis of mean field equations and the Onsager microcanonical description of two-dimensional turbulence. *Calc. Var. Partial Differential Equations* **58** (2019), no. 1, Paper no. 18, 37 pp.
- [2] Bartolucci, D. and De Marchis, F.: On the Ambjorn–Olesen electroweak condensates. *J. Math. Phys.* **53** (2012), no. 7, 073704, 15 pp.
- [3] Bartolucci, D. and De Marchis, F.: Supercritical mean field equations on convex domains and the Onsager’s statistical description of two-dimensional turbulence. *Arch. Ration. Mech. Anal.* **217** (2015), no. 2, 525–570.
- [4] Bartolucci, D., Jevnikar, A., Lee, Y. and Yang, W.: Non-degeneracy, mean field equations and the Onsager theory of 2D turbulence. *Arch. Ration. Mech. Anal.* **230** (2018), no. 1, 397–426.

- [5] Bartolucci, D. and Jevnikar, A.: On the global bifurcation diagram of the Gel'fand problem. *Anal. PDE* **14** (2021), no. 8, 2409–2426.
- [6] Bartolucci, D. and Lin, C.-S.: Existence and uniqueness for mean field equations on multiply connected domains at the critical parameter. *Math. Ann.* **359** (2014), no. 1–2, 1–44.
- [7] Bartolucci, D. and Wolanski, G.: Maximal entropy solutions under prescribed mass and energy. *J. Differential Equations* **268** (2020), no. 11, 6646–6665.
- [8] Caglioti, E., Lions, P.-L., Marchioro, C. and Pulvirenti, M.: A special class of stationary flows for two dimensional Euler equations: a statistical mechanics description. *Comm. Math. Phys.* **143** (1992), no. 3, 501–525.
- [9] Caglioti, E., Lions, P. L., Marchioro, C. and Pulvirenti, M.: A special class of stationary flows for two dimensional Euler equations: a statistical mechanics description. II. *Comm. Math. Phys.* **174** (1995), no. 2, 229–260.
- [10] Chang, S.-Y. A., Chen, C.-C. and Lin, C.-S.: Extremal functions for a mean field equation in two dimension. In *Lecture on partial differential equations*, pp. 61–93. New Stud. Adv. Math. 2, Int. Press, Somerville, MA, 2003.
- [11] Chen, C.-C. and Lin, C.-S.: Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces. *Comm. Pure Appl. Math.* **55** (2002), no. 6, 728–771.
- [12] Chen, C. C. and Lin, C. S.: Topological degree for a mean field equation on Riemann surface. *Comm. Pure Appl. Math.* **56** (2003), no. 12, 1667–1727.
- [13] Delfour, M. C. and Zolèsio, J. P.: *Shapes and geometries. Metrics, analysis, differential calculus, and optimization*. Second edition. Advances in Design and Control 22, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.
- [14] Esposito, P., Grossi, M. and Pistoia, A.: On the existence of blowing-up solutions for a mean field equation. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **22** (2005), no. 2, 227–257.
- [15] Henry, D.: *Perturbation of the boundary in boundary-value problems of partial differential equations*. London Mathematical Society Lecture Note Series 318, Cambridge University Press, Cambridge, 2005.
- [16] Hörmander, L.: *An introduction to complex analysis in several variables*. North-Holland Mathematical Library 7, North-Holland Publishing Co., Amsterdam, 1990.
- [17] Kowalczyk, M., Musso, M. and del Pino, M.: Singular limits in Liouville-type equations. *Calc. Var. Partial Differential Equations* **24** (2005), no. 1, 47–81.
- [18] Malchiodi, A.: Morse theory and a scalar field equation on compact surfaces. *Adv. Differential Equations* **13** (2008), no. 11–12, 1109–1129.
- [19] Micheletti, A. M.: Metrica per famiglie di domini limitati e proprietà generiche degli autovalori. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (3)* **26** (1972), 683–694.
- [20] Nagasaki, K. and Suzuki, T.: Radial and nonradial solutions for the nonlinear eigenvalue problem $\Delta u + \lambda e^u = 0$ on annuli in \mathbb{R}^n . *J. Differential Equations* **87** (1990), no. 1, 144–168.
- [21] Peetre, J.: On Hadamard's variational formula. *J. Differential Equations* **36** (1980), no. 3, 335–346.
- [22] Pommerenke, Ch.: *Boundary behaviour of conformal maps*. Grundlehren der Math. Wissenschaften 299, Springer-Verlag, Berlin, 1992.
- [23] Pommerenke, Ch.: *Univalent functions*. Mathematische Lehrbücher 25, Vandenhoeck & Ruprecht, Göttingen, 1975.

Received September 14, 2020. Published online April 30, 2022.

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