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On stable rank of H^∞ on coverings of finite bordered Riemann surfaces

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Abstract. We prove that the Bass stable rank of the algebra of bounded holomorphic functions on an unbranched covering of a finite bordered Riemann surface is equal to one.

1. Formulation of main results

Let S' be a (not necessarily connected) unbranched covering of a finite bordered Riemann surface S. In this paper we continue the study initiated in [3] of the algebra $H^\infty(S')$ of bounded holomorphic functions on S'. (We write $H^\infty := H^\infty(\mathbb{D})$, where $\mathbb{D} \subset \mathbb{C}$ is the open unit disk.) It was shown in our previous work that algebras $H^\infty(S')$ and H^∞ share many common properties (e.g., they are Hermite, their maximal ideal spaces are two-dimensional with vanishing second Čech cohomology groups, etc., see [3–5] for the corresponding results). The purpose of this paper is to prove that these algebras have also the same Bass stable rank. The latter notion is defined as follows.

Let A be an associative ring with unit. For a natural number n let $U_n(A)$ denote the set of *unimodular* elements of A^n, i.e.,

$$U_n(A) = \{(a_1, \dots, a_n) \in A^n : Aa_1 + \cdots + Aa_n = A\}.$$

An element $(a_1, \dots, a_n) \in U_n(A)$ is called *reducible* if there exist $c_1, \dots, c_{n-1} \in A$ such that $(a_1 + c_1 a_n, \dots, a_{n-1} + c_{n-1} a_n) \in U_{n-1}(A)$. The *stable rank* $\mathrm{sr}(A)$ is the least n such that every element of $U_{n+1}(A)$ is reducible. The concept of the stable rank introduced by Bass [1] plays an important role in some stabilization problems of algebraic K-theory. Following Vaserstein [13], we call a ring of stable rank 1 a B-ring. (We refer to this paper for some examples and properties of B-rings.)

In [11], Treil proved the following result.

Theorem A. *Let $f, g \in H^\infty$ be such that $\|f\|_{H^\infty} \le 1$, $\|g\|_{H^\infty} \le 1$ and*

$$\inf_{z \in \mathbb{D}} (|f(z)| + |g(z)|) =: \delta > 0. \tag{1.1}$$

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Then there exists a function $G \in H^\infty$ such that the function $\Phi = f + gG$ is invertible in H^∞, and moreover $\|G\|_{H^\infty} \le C$ and $\|\Phi^{-1}\|_{H^\infty} \le C$, where the constant C depends only on δ.

(Here and below for a normed space B its norm is denoted by $\|\cdot\|_B$.)

By the Carleson corona theorem, condition (1.1) is satisfied if and only if $(f, g) \in U_2(H^\infty)$. Hence, Treil's theorem implies that H^∞ is a B-ring.

Theorem A was used by Tolokonnikov [10] to prove that algebras $H^\infty(U)$ are B-rings for finitely connected domains and for some Behrens domains U. Since then, no other classes of Riemann surfaces U for which $H^\infty(U)$ are B-rings were known. In the present paper, we prove the following extension of Theorem A.

Theorem 1.1. *Let S' be an unbranched covering of a finite bordered Riemann surface S. Let $f, g \in H^\infty(S')$ be such that $\|f\|_{H^\infty(S')} \le 1$, $\|g\|_{H^\infty(S')} \le 1$ and*

$$\inf_{z\in S'} (|f(z)| + |g(z)|) =: \delta > 0. \tag{1.2}$$

Then there exists a function $G \in H^\infty(S')$ such that the function $\Phi = f + gG$ is invertible in $H^\infty(S')$, and moreover $\max\{\|G\|_{H^\infty(S')}, \|\Phi^{-1}\|_{H^\infty(S')}\} \le C$, where the constant C depends only on δ and S.

By the corona theorem for $H^\infty(S')$ (see Corollary 1.6 in [3]), condition (1.2) is satisfied if and only if $(f, g) \in U_2(H^\infty(S'))$. Hence, Theorem 1.1 implies:

Theorem 1.2. *$H^\infty(S')$ is a B-ring.*

Remark 1.3. It is known that every B-ring is *Hermite* (see, e.g., Theorem 2.7 in [13]), i.e., any finitely generated stably free right module over the ring is free (equivalently, any rectangular left-invertible matrix over the ring can be extended to an invertible matrix). Let $J \subset H^\infty(S')$ be a closed ideal and $H_J^\infty := \{c + f : c \in \mathbb{C},\ f \in J\}$ be the unital closed subalgebra generated by J. Then Corollary 1.2 implies that H_J^∞ is a B-ring (see, e.g., Theorem 4 in [12]); hence, it is Hermite. This gives a generalization of Theorem 1.1 in [4] proved by a different method.

Let $M_n(H^\infty(S'))$ be the algebra of $n \times n$ matrices with entries in $H^\infty(S')$ regarded as the subspace of bounded linear operators on $(H^\infty(S'))^n$ equipped with the operator norm. We apply Theorem 1.1 to the problem of reducing a matrix with entries in $\mathrm{SL}_n(H^\infty(S')) \subset M_n(H^\infty(S'))$ (the subset of matrices with determinant 1) to the identity matrix by addition operations, that is, representing a matrix by the product of elementary matrices (i.e., those that differ from the identity matrix by at most one non-diagonal entry).

Theorem 1.4. *Every matrix in $\mathrm{SL}_n(H^\infty(S'))$ of norm $\le M$ is a product of at most $(n-1)(\frac{3n}{2}+1)$ elementary matrices whose norms are bounded from above by a constant depending only on M, n and S.*

The proof of Theorem 1.1 is based on Theorem A and some results of the author presented in [5] and [6], along with some topological results. In the next section we collect some results required for the proof of Theorem 1.1. The proof is given in Section 4.

2. Auxiliary results

2.1. Let $\mathfrak{M}(A)$ denote the maximal ideal space of a commutative complex unital Banach algebra A, i.e., the set of nonzero homomorphisms $A \to \mathbb{C}$ equipped with the *Gelfand topology*. In this part we present some facts about the maximal ideal space $\mathfrak{M}(H^\infty(S'))$, where $r\colon S' \to S$ is a (not necessarily connected but second-countable) unbranched covering of a bordered Riemann S, see Section 2 of [4] and Section 4 of [5] for details.

Recall that $H^\infty(S')$ separates points of S' and the map $\iota\colon S' \to \mathfrak{M}(H^\infty(S'))$ sending $x \in S'$ to the evaluation functional $\delta_x \in (H^\infty(S'))^*$ at x embeds S' into $\mathfrak{M}(H^\infty(S'))$ as an open dense subset – the corona theorem for $H^\infty(S')$.

The covering $r\colon S' \to S$ can be viewed as a fiber bundle over S with a discrete (at most countable) fiber F. Let $E(S,\beta F)$ be the space obtained from S' by taking the Stone–Čech compactifications of fibres under r. It is a normal Hausdorff space and r extends to a continuous map $r_E\colon E(S,\beta F) \to S$ such that $(E(S,\beta F), S, r_E, \beta F)$ is a fibre bundle on S with fibre βF and S' is an open dense subbundle of $E(S,\beta F)$. Each $f \in H^\infty(S')$ admits an extension $\hat{f} \in C(E(S,\beta F))$, and the algebra formed by such extensions separates points of $E(S,\beta F)$. Thus ι extends to a continuous injection $\hat{\iota}\colon E(S,\beta F) \to \mathfrak{M}(H^\infty(S'))$, $(\hat{\iota}(\xi))(f) := \hat{f}(\xi)$.

In what follows, we identify $E(S,\beta F)$ with its image under $\hat{\iota}$. Also, for $K \subset S$ we set $K' := r^{-1}(K)$, $K_E := r_E^{-1}(K)$ and for a subset U of a topological space we denote by $\mathring{U}$, $\bar{U}$ and ∂U its interior, closure and boundary.

It is well known that S can be regarded as a domain in a compact Riemann surface R such that $R \setminus \bar{S}$ is the finite disjoint union of open disks with analytic boundaries. Let $A(S) \subset H^\infty(S)$ be the subalgebra of functions continuous up to the boundary. We denote by $\hat{r}\colon \mathfrak{M}(H^\infty(S')) \to \bar{S}$ the continuous surjective map induced by the transpose of the homomorphism $A(S) \to H^\infty(S')$, $f \mapsto f \circ r$. Then $E(S,\beta F)$ coincides with the open set $\hat{r}^{-1}(S)$ and $\hat{r}|_{E(S,\beta F)} = r_E$.

Let $U \subset R$ be an open set such that $V := U \cap \bar{S} \neq \emptyset$. Then $\hat{r}^{-1}(V)$ is an open subset of $\mathfrak{M}(H^\infty(S'))$ and due to the corona theorem, $\mathring{V}' := r^{-1}(\mathring{V})$, where $\mathring{V} := U \cap S$, is an open dense subset of $\hat{r}^{-1}(V)$.

Proposition 2.1. *Each $f \in H^\infty(\mathring{V}')$ admits an extension $\hat{f} \in C(\hat{r}^{-1}(V))$.*

Proof. We reduce the statement to some known results.

We have to extend f continuously to each point $\xi \in \hat{r}^{-1}(V)$. The set $\hat{r}^{-1}(V)$ is the disjoint union of the open set $\mathring{V}_E = \hat{r}^{-1}(\mathring{V})$ and the set $\hat{r}^{-1}(V \cap \partial S)$. So we consider two cases.

Case 1. $\xi \in \hat{r}^{-1}(\mathring{V})$.

Let $O \subset \mathring{V}$ be an open simply connected neighbourhood of $\hat{r}(\xi)$. By the definition of the bundle $E(S,\beta F)$, the set $O_E = r_E^{-1}(O)$ is homeomorphic to $O \times \beta F$ and this homeomorphism maps $O' = r^{-1}(O)$ biholomorphically onto $O \times F$. Then Lemma 3.1 of [2] implies that $f|_{O'} \in H^\infty(O')$ admits an extension $\hat{f} \in C(O_E)$ as required (because O_E is an open neighbourhood of ξ).

Case 2. $\xi \in \hat{r}^{-1}(V \cap \partial S)$.

Let $\hat{r}(\xi)$ belong to a connected component γ of ∂S. By the definition of S, there are a relatively open neighbourhood $A_\gamma \subset \bar{S}$ of γ and a homeomorphic map $A_\gamma \to A := \{z \in \mathbb{C} : c < |z| \le 1\}, c > 0$, which maps γ onto the unit circle $\mathbb{S} \subset \mathbb{C}$ and is holomorphic on $\mathring{A}_\gamma$. Without loss of generality, we identify A_γ with A and γ with $\mathbb{S}$. Then since $V \cap A \neq \emptyset$, there is a relatively open subset $\Pi \subset V \cap A$ which is a rectangle in polar coordinates with one side of the boundary on $\mathbb{S}$ such that $\hat{r}(\xi) \in \Pi$. Repeating literally the arguments of the proof of Proposition 4.2 in [5], we obtain that each function from $H^\infty(\mathring{\Pi}')$ admits a continuous extension to $\hat{r}^{-1}(\Pi)$. Since the latter is an open neighbourhood of ξ, this gives the required extension of f to ξ. We leave the details to the readers. ■

Remark 2.2. Since $\mathring{V}'$ is dense in $\hat{r}^{-1}(V)$, the above extension preserves supremum norm. Then the transpose of the restriction homomorphism $H^\infty(S') \to H^\infty(\mathring{V}')$, $f \mapsto f|_{\mathring{V}'}$, induces a continuous map $s_V : \mathfrak{M}(H^\infty(\mathring{V}')) \to \mathfrak{M}(H^\infty(S'))$ with image $\hat{r}^{-1}(\bar{V})$ one-to-one on $s_V^{-1}(\hat{r}^{-1}(V))$.

2.2. A compact subset $K \subset \mathfrak{M}(H^\infty(S'))$ is said to be *holomorphically convex* (with respect to the algebra $H^\infty(S')$) if for every $\xi \notin K$ there is $f \in H^\infty(S')$ such that

$$\max_K |\hat{f}| < |\hat{f}(\xi)|;$$

here $\hat{f} \in C(\mathfrak{M}(H^\infty(S')))$ is the Gelfand transform of f.

A holomorphically convex subset $Z \subset \mathfrak{M}(H^\infty(S'))$ is called a *hull* if there is a proper ideal $I \subset H^\infty(S')$ such that

$$Z = \{\xi \in \mathfrak{M}(H^\infty(S')) \ : \ \hat{f}(\xi) = 0 \quad \forall f \in I\}.$$

The algebra $H^\infty(S')$ is a B-ring if and only if for every hull $Z \subset \mathfrak{M}(H^\infty(S'))$ the map $C(\mathfrak{M}(H^\infty(S')), \mathbb{C}^*) \to C(Z, \mathbb{C}^*)$, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$, induced by restriction to Z is onto, see [7].

In the next two lemmas, $S = \mathbb{D}$ and $S' = S \times F$ for some $F \subset \mathbb{N}$.

Lemma 2.3. *If $K \subset \mathfrak{M}(H^\infty(S'))$ is holomorphically convex, then for every $g \in C(K, \mathbb{C}^*)$, there exists $\tilde{g} \in C(\mathfrak{M}(H^\infty(S')), \mathbb{C}^*)$ such that $\tilde{g}|_K = g$.*

Proof. According to Lemma 5.3 in [6], the homomorphism of the Čech cohomology groups $H^1(\mathfrak{M}(H^\infty(S')), \mathbb{Z}) \to H^1(K, \mathbb{Z})$ induced by the restriction map to K is surjective. In turn, by the Arens–Royden theorem, $H^1(K, \mathbb{Z})$ and $H^1(\mathfrak{M}(H^\infty(S')), \mathbb{Z})$ are connected components of topological groups $C(K, \mathbb{C}^*)$ and $C(\mathfrak{M}(H^\infty(S')), \mathbb{C}^*)$, respectively. Hence, for each $g \in C(K, \mathbb{C}^*)$, there is $g_1 \in C(\mathfrak{M}(H^\infty(S')), \mathbb{C}^*)$ such that $g \cdot (g_1^{-1})|_K = e^h$ for some $h \in C(K)$. Let $\tilde{h} \in C(\mathfrak{M}(H^\infty(S')))$ be an extension of h (existing by the Titze–Urysohn theorem). Then $\tilde{g} = g_1 e^{\tilde{h}}$ is the required extension of g. ■

Lemma 2.4. *Suppose $K \subset \mathfrak{M}(H^\infty(S'))$ is holomorphically convex and $Z \subset \mathfrak{M}(H^\infty(S'))$ is a hull. Then $K \cup Z \subset \mathfrak{M}(H^\infty(S'))$ is holomorphically convex.*

Proof. Let $\xi \notin K \cup Z$. By the hypothesis, there exist $f, g \in H^\infty(S')$ such that $\hat{f}(\xi) = \hat{g}(\xi) = 1$ and

$$\max_K |f| =: c < 1, \quad g|_Z = 0.$$

Let $M := \max_K |g|$. We choose $n \in \mathbb{N}$ such that $c^n M < 1$. Then for $h := f^n g \in H^\infty(S')$ we have

$$\max_{K \cup Z} |\hat{h}| \le c^n M < 1 = |\hat{h}(\xi)|.$$

This shows that the set $K \cup Z$ is holomorphically convex. ∎

2.3. The proof of Theorem 1.1 relies on the following result.

Let a connected compact Hausdorff space X be such that there are a closed cover $(X_j)_{j=1}^m$ of X and continuous maps $s_j\colon \mathfrak{M}(H^\infty(\mathbb{D} \times F)) \to X$, $F \subset \mathbb{N}$, $1 \le j \le m$, satisfying

(i) $X_j \subset s_j(\mathfrak{M}(H^\infty(\mathbb{D} \times F)))$ and s_j is one-to-one on $s_j^{-1}(X_j)$, $1 \le j \le m$;

(ii) for every $J \subset \{1, \dots, m\}$ and $i \notin J$, the subset

$$s_i^{-1}((\cup_{j \in J} X_j) \cap X_i) \subset \mathfrak{M}(H^\infty(\mathbb{D} \times F))$$

is holomorphically convex.

Proposition 2.5. *Suppose $Z \subset X$ is such that for every j the set $s_j^{-1}(Z)$ is a hull. Then for every $g \in C(Z, \mathbb{C}^*)$, there exists $\tilde{g} \in C(X, \mathbb{C}^*)$ such that $\tilde{g}|_Z = g$.*

Proof. We set $Z_j := s_j^{-1}(Z)$, $1 \le j \le m$. Assume that $Z \cap X_j \ne \emptyset$. Then $s_j^* g := g \circ s_j \in C(Z_j, \mathbb{C}^*)$. Since Z_j is a hull, the Treil theorem [11] implies that there is $g_j \in C(\mathfrak{M}(H^\infty(\mathbb{D} \times F)), \mathbb{C}^*)$ which extends $s_j^* g$. Hence, due to (i), $\tilde{g}_j := g_j \circ s_j^{-1}|_{X_j} \in C(X_j, \mathbb{C}^*)$ and extends $g|_{Z \cap X_j}$. If $Z \cap X_j = \emptyset$, we define $\tilde{g}_j = 1$.

We order the sets of the cover $(X_i)_{i=1}^m$ as follows. Choose some $X_{i_1} \subset \{X_1, \dots, X_m\}$. If X_{i_p} is already chosen, we choose $X_{i_{p+1}}$ so that

$$X_{i_{p+1}} \cap (\cup_{j=1}^p X_{i_j}) \ne \emptyset.$$

This is possible because X is connected. We extend g by induction on the indices of the order.

For $j = 1$ we set $\tilde{g} = \tilde{g}_{i_1}$ on X_{i_1}. Suppose that $\tilde{g}$ is already defined on $\cup_{j=1}^p X_{i_j}$. Let us define it on $\cup_{j=1}^{p+1} X_{i_j}$. To this end let

$$g_{p,p+1} := \tilde{g} \cdot \tilde{g}_{i_{p+1}}^{-1} \quad \text{on} \quad \left(\cup_{j=1}^p X_{i_j}\right) \cap X_{i_{p+1}} =: K.$$

By (ii), $s_{i_{p+1}}^{-1}(K) \subset \mathfrak{M}(H^\infty(\mathbb{D} \times F))$ is holomorphically convex and since by the hypothesis $Z_{i_{p+1}}$ is a hull, Lemma 3.1 implies that $s_{i_{p+1}}^{-1}(K) \cup Z_{i_{p+1}}$ is a holomorphically convex subset of $\mathfrak{M}(H^\infty(\mathbb{D} \times F))$. Moreover, $s_{i_{p+1}}^*(g_{p,p+1}) \in C(s_{i_{p+1}}^{-1}(K), \mathbb{C}^*)$ and equals 1 on $s_{i_{p+1}}^{-1}(K) \cap Z_{i_{p+1}}$ if the latter is nonvoid. Then it can be extended to a function in $C(s_{i_{p+1}}^{-1}(K) \cup Z_{i_{p+1}}, \mathbb{C}^*)$ equal to 1 on $Z_{i_{p+1}}$ ($\ne \emptyset$). Due to Lemma 2.3, the extended function can be further extended to a function from $C(\mathfrak{M}(H^\infty(\mathbb{D}\times F)),\mathbb{C}^*)$. Composing this extension with $s^{-1}_{i_{p+1}}|_{X_{i_{p+1}}}$, we obtain an extension $\tilde g_{p,p+1}\in C(X_{i_{p+1}},\mathbb{C}^*)$ of $g_{p,p+1}$ equal to 1 on $Z\cap X_{i_{p+1}}$ if this set is not empty. Let us define

$$\tilde g|_{X_{i_{p+1}}} := \tilde g_{i_{p+1}}\cdot \tilde g_{p,p+1}. \tag{2.1}$$

Then $\tilde g|_{X_{i_{p+1}}}$ extends $g|_{Z\cap X_{i_{p+1}}}$ and

$$\tilde g|_{X_{i_{p+1}}}\cdot \tilde g^{-1}|_{\cup_{j=1}^{p}X_{i_j}} = \tilde g_{i_{p+1}}\tilde g_{p,p+1}\tilde g^{-1} = 1 \quad\text{on}\quad \left(\cup_{j=1}^{p}X_{i_j}\right)\cap X_{i_{p+1}},$$

i.e., (2.1) gives the required extension of $\tilde g|_{\cup_{j=1}^{p}X_{i_j}}$ to $\cup_{j=1}^{p+1}X_{i_j}$. This completes the proof of the induction step and hence of the proposition. ■

2.4. We apply Proposition 2.5 to $X=\mathfrak{M}(H^\infty(S'))$, where S' is an unbranched covering of a finite bordered Riemann surface S. To construct the required cover (X_j) of X in this case, we prove the following topological result.

Lemma 2.6. *There is a finite cover $(U_j)_{j=1}^m$ of $\bar S$ by compact subsets homeomorphic to $\bar{\mathbb{D}}$ such that each U_i is contained in an open simply connected set $V_i\subset R$ with simply connected intersection $V_i\cap S$, each U_i intersects with at most two other sets of the family, and each non-void $U_i\cap U_j$ is homeomorphic to $I:=[0,1]$.*

Proof. Since $\bar S$ is triangulable, we may regard it as a two dimensional polyhedral manifold. It follows from the Whitehead theorem (Theorem (3.5) in [14]) that there are a (finite) one-dimensional polyhedron $L\subset\bar S$ with sets of edges E_L and vertices V_L and a piecewise linear strong deformation retraction $F\colon \bar S\times I\to S$ of $\bar S$ onto L which maps ∂S onto L such that

(a) $F^{-1}(x,1)\subset\bar S$ is a connected polyhedron homeomorphic to a *star tree* with internal vertex x of degree 2 if either $x\in\mathring{e}$ for some $e\in E_L$ or $x\in V_L$ is of degree ≤ 2, and of degree >2 if $x\in V_L$ is of degree >2, and this homeomorphism maps $F^{-1}(x,1)\cap\partial S$ onto the set of external vertices of the tree.

(b) If $e\in E_L$, then $F^{-1}(\mathring{e},1)\cap\partial S$ is the disjoint union of two sets homeomorphic to I.

Let $E_L:=\{e_1,\dots,e_m\}$. We define

$$U_i:=\overline{F^{-1}(\mathring{e}_i,1)},\quad 1\le i\le m.$$

Then every U_i is a polyhedral submanifold of $\bar S$ homeomorphic to $\bar{\mathbb{D}}$ with the boundary formed by some arcs in ∂S along with some subsets of $F^{-1}(v_{i_j},1)$, $j=1,2$, homeomorphic to I; here $v_{i_1},v_{i_2}\in V_L$ are endpoints of e_i. Clearly every non-void intersection $U_i\cap U_j\subset F^{-1}(e_i\cap e_j,1)$ is homeomorphic to I. Moreover, it is readily seen that each U_i is contained in an open simply connected subset $V_i\subset R$ with simply connected intersection $V_i\cap S$ because $\bar S$ is the strong deformation retract of some of its open neighbourhoods in R (see, e.g., Theorem (3.3) in [14]). ■

3. Proof of Theorem 1.2

Proof. We retain notation of Lemma 2.6. We set

$$\Gamma_i = \overline{\partial U_i \setminus \partial S} \quad \text{and} \quad W_i := V_i \cap S, \quad 1 \le i \le m.$$

Then Γ_i consists of two connected components homeomorphic to I and W_i is an open simply connected subset of S. By the definition, $\Gamma_i \subset \overline{W}_i$.

Let $A(W_i) \subset H^\infty(W_i)$ be the subalgebra of functions continuous up to the boundary. We denote by $\hat{r}_i : \mathfrak{M}(H^\infty(W_i')) \to \overline{W}_i$ the continuous surjective map induced by the transpose of the inclusion homomorphism $r_i : A(W_i) \to H^\infty(W_i')$, $f \mapsto f \circ r_i$.

Let K be either Γ_i or its connected component. We set

$$\widetilde{K} := \hat{r}_i^{-1}(K).$$

Lemma 3.1. *The set $\widetilde{K} \subset \mathfrak{M}(H^\infty(W_i'))$ is holomorphically convex.*

Proof. By our construction, the open set $V_i \setminus K$ is connected. By the Riemann mapping theorem, there is a biholomorphic map ψ_i of V_i onto $\mathbb{D}$. Then $\mathbb{D} \setminus \psi_i(K)$ is a connected open subset of $\mathbb{D}$. Therefore the compact set $\psi_i(K) \subset \mathbb{C}$ is polynomially convex, see, e.g., [9], Chapter III, Lemma 1.3. Hence, $K \Subset V_i$ is holomorphically convex with respect to the algebra $H^\infty(V_i)$ and so it is holomorphically convex in $\overline{W}_i$ ($\Subset V_i$) with respect to the algebra $A(W_i)$. Since $\hat{r}_i$ is a surjection onto $\overline{W}_i$ and $\widetilde{K} \subset \mathfrak{M}(H^\infty(W_i'))$ is the preimage of K, it is holomorphically convex. ∎

Since $W_i' := r^{-1}(W_i)$ is biholomorphic to $\mathbb{D} \times F$, where F is the fibre of the unbranched covering $r : S' \to S$, algebras $H^\infty(W_i')$ and $H^\infty(\mathbb{D} \times F)$ are isomorphic. We denote by $s_i : \mathfrak{M}(H^\infty(\mathbb{D} \times F)) \to \mathfrak{M}(H^\infty(S'))$ the continuous map induced by the transpose of the composition of the restriction homomorphism $H^\infty(S') \to H^\infty(W_i')$ and the isomorphism $H^\infty(W_i') \to H^\infty(\mathbb{D} \times F)$. Then due to Remark 2.2, the image of s_i is $\hat{r}^{-1}(\overline{W}_i)$ and s_i is one-to-one on $s_i^{-1}(\hat{r}^{-1}(V_i \cap \bar{S}))$.

We set $X_i := \hat{r}^{-1}(U_i)$, $1 \le i \le m$. Then $(X_i)_{i=1}^m$ is a closed cover of $\mathfrak{M}(H^\infty(S'))$ satisfying condition (i) of Proposition 2.5, i.e., s_i is one-to-one on $s_i^{-1}(X_i)$ (as $U_i \subset V_i \cap \bar{S}$).

Moreover, for every $J \subset \{1, \dots, m\}$ and $i \notin J$, the set $(\cup_{j \in J} X_j) \cap X_i$ is either void or the preimage under $\hat{r}$ of Γ_i or its connected component. Hence, due to Lemma 3.1, $s_i^{-1}\big((\cup_{j \in J} X_j) \cap X_i\big) \subset \mathfrak{M}(H^\infty(\mathbb{D} \times F))$ is holomorphically convex, i.e., $(X_i)_{i=1}^m$ satisfies condition (ii) of Proposition 2.5 as well.

Finally, if $Z \subset \mathfrak{M}(H^\infty(S'))$ is a hull, then $Z_i := s_i^{-1}(Z)$ is a hull for $H^\infty(\mathbb{D} \times F)$. Hence, every function $g \in C(Z, \mathbb{C}^*)$ has an extension $\tilde{g} \in C(\mathfrak{M}(H^\infty(S')), \mathbb{C}^*)$, by Proposition 2.5. But this is equivalent to the fact that $H^\infty(S')$ is a B-ring, see Section 2.2 above.

The proof of the theorem is complete. ∎

4. Proofs of Theorems 1.1 and 1.4

Proof of Theorem 1.1. Without loss of generality, we may assume that S' is a connected unbranched covering of S.

Let $f, g \in H^\infty(S')$ be such that $\|f\|_{H^\infty(S')} \le 1$, $\|g\|_{H^\infty(S')} \le 1$ and

$$\inf_{z\in S'} (|f(z)| + |g(z)|) =: \delta > 0. \tag{4.1}$$

Due to Theorem 1.2, there exists a function $G \in H^\infty(S')$ such that the function $f + gG$ is invertible in $H^\infty(S')$. By $\mathscr{G}_{f,g,\delta,S'}$ we denote the class of such functions G. We must prove that

$$C = C(\delta, S) := \sup_{f,g,S'} \inf_{G\in\mathscr{G}_{f,g,\delta,S'}} \max\bigl\{\|G\|_{H^\infty(S')}, \|(f + gG)^{-1}\|_{H^\infty(S')}\bigr\} \tag{4.2}$$

is finite. (Here the supremum is taken over all functions f, g satisfying the above hypotheses and all connected unbranched coverings S' of S.)

Let $\{S'_i\}_{i\in\mathbb{N}}$ and $\{f_i\}_{i\in\mathbb{N}}, \{g_i\}_{i\in\mathbb{N}}$, $f_i, g_i \in H^\infty(S'_i)$, be sequences satisfying assumptions of the theorem such that

$$C = \lim_{i\to\infty} \inf_{G\in\mathscr{G}_{f_i,g_i,\delta,S'_i}} \max\bigl\{\|G\|_{H^\infty(S'_i)}, \|(f_i + g_i G)^{-1}\|_{H^\infty(S'_i)}\bigr\}. \tag{4.3}$$

The disjoint union $S' := \sqcup_{i\in\mathbb{N}} S'_i$ is clearly an unbranched covering of S and functions $f, g \in H^\infty(S')$ defined by the formulas

$$f|_{S'_i} := f_i, \quad g|_{S'_i} := g_i, \quad i \in \mathbb{N},$$

are of norms ≤ 1 and satisfy condition (4.1) on S'. Then due to Theorem 1.2 there exists a function $G \in H^\infty(S')$ such that the function $f + gG$ is invertible in $H^\infty(S')$. We set

$$G_i := G|_{S'_i}, \quad i \in \mathbb{N}.$$

Then due to (4.3),

$$\begin{aligned} C &\le \sup_{i\in\mathbb{N}} \max\bigl\{\|G_i\|_{H^\infty(S'_i)}, \|(f_i + g_i G_i)^{-1}\|_{H^\infty(S'_i)}\bigr\} \\ &= \max\bigl\{\|G\|_{H^\infty(S')}, \|(f + gG)^{-1}\|_{H^\infty(S')}\bigr\}. \end{aligned}$$

This completes the proof of the theorem. ∎

Proof of Theorem 1.4. Since $H^\infty(S')$ is a B-ring (by Theorem 1.2), Lemma 9 and Remark 10 of [8] imply that each matrix $F \in \mathrm{SL}_n(H^\infty(S'))$ can be presented as a product of at most $(n-1)(\frac{3n}{2}+1)$ elementary matrices. Let us show that if

$$\|F\|_{M_n(H^\infty(S'))} \le M, \tag{4.4}$$

then these matrices can be chosen so that their norms are bounded from above by a constant depending only on M, n and S.

As before, we may assume that S' is connected. Let $\mathcal{F}_{M,S',n}$ be the class of matrices $F \in \mathrm{SL}_n(H^\infty(S'))$ satisfying (4.4). For every $F \in \mathcal{F}_{M,S',n}$ by $\Pi_{F,M,S',n}$, we denote the set of all possible products of F by at most $(n-1)(\frac{3n}{2}+1)$ elementary matrices. By the above arguments, the set $\Pi_{F,M,S',n}$ is non-void. For each $\pi \in \Pi_{F,M,S',n}$, by $\|\pi\|$ we denote maximum of norms of elementary matrices in π. We have to prove that

$$C = C(S, M, n) := \sup_{S', F \in \mathcal{F}_{M,S',n}} \inf_{\pi \in \Pi_{F,M,S',n}} \|\pi\| < \infty; \tag{4.5}$$

here S' runs over all connected unbranched coverings of S.

Let S'_i and $F_i \in \mathcal{F}_{M,S'_i,n}$, $i \in \mathbb{N}$, be such that

$$C = \lim_{i\to\infty} \inf_{\pi \in \Pi_{F_i,M,S'_i,n}} \|\pi\|. \tag{4.6}$$

It is clear that the disjoint union $S' := \sqcup_{i\in\mathbb{N}} S'_i$ is an unbranched covering of S and the matrix $F \in H^\infty(S')$ defined by the formula

$$F|_{S'_i} := F_i, \quad i \in \mathbb{N},$$

belongs to the class $\mathcal{F}_{M,S',n}$. Then there is $\pi \in \Pi_{F,M,S',n}$. Let π_i be the product obtained by the restriction of elementary matrices in π to S'_i. Then each $\pi_i \in \Pi_{F_i,M,S'_i,n}$ and so, due to (4.6),

$$C \le \sup_{i\in\mathbb{N}} \|\pi_i\| = \|\pi\| < \infty.$$

This completes the proof of the theorem. ∎

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