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Partial Gaussian sums and the Pólya–Vinogradov inequality for primitive characters

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Abstract. In this paper we obtain a new fully explicit constant for the Pólya–Vinogradov inequality for primitive characters. Given a primitive character χ modulo q, we prove the following upper bound:

$$\Big|\sum_{1\le n\le N}\chi(n)\Big| \le c\sqrt{q}\log q,$$

where $c = 3/(4\pi^2) + o_q(1)$ for even characters and $c = 3/(8\pi) + o_q(1)$ for odd characters, with explicit $o_q(1)$ terms. This improves a result of Frolenkov and Soundararajan for large q. We proceed, following Hildebrand, to obtain the explicit version of a result by Montgomery–Vaughan on partial Gaussian sums and an explicit Burgess-like result on convoluted Dirichlet characters.

1. Introduction

It is of high interest to study the upper bound of the following quantity:

$$S(N,\chi) := \Big|\sum_{n=1}^{N}\chi(n)\Big|,$$

with $N \in \mathbb{N}$ and χ a non-principal Dirichlet character modulo q. The famous Pólya–Vinogradov inequality tells us that

$$S(N,\chi) \ll \sqrt{q}\log q,$$

and aside for the implied constant, this is the best known uniform result. Granville and Soundararajan in [9] improved the result for characters of odd fixed order, and further improvements were obtained by Goldmakher in [8]. The focus is now on the implied constant of the uniform result, with a distinction between asymptotically explicit and completely explicit results. The best asymptotic constant can be found in the papers by

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Hildebrand [12] and Granville and Soundararajan [9]. The explicit results have generally worse leading terms, the exception is for primitive characters of square-free moduli for which the author and Kerr [2] proved a result that is comparable with the asymptotic one. There have been many completely explicit results. We will be focussing on primitive characters, as these results can be easily extended to all non-principal characters. All the late results have the following shape:

$$|S(N,\chi)| \le \begin{cases} \frac{1}{\pi^2}\sqrt{q}\log q + \delta_1\sqrt{q}\log\log q + \delta_2\sqrt{q} & \text{for } \chi(-1)=1, \\ \frac{1}{2\pi}\sqrt{q}\log q + \delta_3\sqrt{q}\log\log q + \delta_4\sqrt{q} & \text{for } \chi(-1)=-1, \end{cases} \tag{1.1}$$

with the second constants improving as follows:

- $\delta_1 = 2/\pi^2, \delta_2 = 3/4, \delta_3 = 1/\pi$ and $\delta_4 = 1$ by Pomerance [16],
- Frolenkov [6] proves that for certain values of δ_2 and δ_4 it is possible to take $\delta_1 = \delta_3 = 0$,
- Frolenkov and Soundararajan [7] further improve the result showing that it is possible to take $\delta_2 = 1/2$, for $q \ge 1200$, and $\delta_4 = 1$, for $q \ge 40$.

The improvements above are on the constants of the remainder terms. Our aim is to improve on the leading constant using Hildebrand's approach [11], that relies on two results: an upper bound on partial Gaussian sums due to Montgomery and Vaughan [15] and the version of the Burgess bound for all non-principal characters from [4].

Our main result is the following theorem, that follows by computations from Theorem 1.5, see Section 5 for more details.

Theorem 1.1. *With χ a primitive Dirichlet character modulo q, assuming $q \ge q_0$ and with q_0 and $h_{1,2}(q_0,\varepsilon)$ from Tables 1 and 2, we have*

$$|S(N,\chi)| \le \begin{cases} \frac{2}{\pi^2}(\frac{3}{8}+\varepsilon)\sqrt{q}\log q + h_1(q_0,\varepsilon)\sqrt{q} & \textit{if } \chi(-1)=1, \\ \frac{1}{\pi}(\frac{3}{8}+\varepsilon)\sqrt{q}\log q + h_2(q_0,\varepsilon)\sqrt{q} & \textit{if } \chi(-1)=-1. \end{cases}$$

We first choose ε small, that forces us to take q_0 large.
We then choose ε near $1/8$ to minimize q_0.

ε	$\log\log q_0$	$h_1(q_0,\varepsilon)$	$h_2(q_0,\varepsilon)$
$\frac{1}{10}$	22	1660	3315
$\frac{1}{100}$	209	2397	4789
$\frac{1}{1000}$	2081	2490	4975
$\frac{1}{10000}$	20796	2499	4994

Table 1. Small ε.

Table 1 and the first two rows of Table 2 improve [7] for the ranges of q given in the same tables, while the last two rows of Table 2 improve [7] in the ranges of q given in Table 3 of Section 5. Note that we can obtain Theorem 1.1 for different q_0 and ε, by Theorem 1.5. In Table 3 of Section 5, we also give a version of the above result for characters

ε	$\log\log q_0$	$h_1(q_0,\varepsilon)$	$h_2(q_0,\varepsilon)$
$\frac{1}{8}(1-\frac{1}{10})$	19.7	1579	3153
$\frac{1}{8}(1-\frac{1}{100})$	17.99	1510	3015
$\frac{1}{8}(1-\frac{1}{1000})$	17.84	1503	3001
$\frac{1}{8}(1-\frac{1}{10000})$	17.83	1502	3000

Table 2. Large ε.

with moduli with the number of divisors $d(q)$ fixed, an interesting case is certainly when $d(q)=2$ and the modulus is thus prime.

We start proving the following explicit version of Corollary 1 in [15]. Let $B\geq 1$ be a constant, and let F be the class of all multiplicative functions $f:\mathbb{Z}\to\mathbb{C}$ with

$$|f(n)|\leq B. \tag{1.2}$$

With $f\in F$, α real and $e(\alpha)=\exp(2\pi i\alpha)$, write

$$S(\alpha)=\sum_{n=1}^{N} f(n)e(n\alpha).$$

Corollary 1.2. *Suppose that* $|\alpha-a/q|\leq q^{-2}$, $(a,q)=1$, $E\geq 4$ *and* $e^3/E\leq R\leq q\leq N/R$. *Then*

$$S(\alpha)\leq c_1(B,E,R)\frac{N}{\log N}+c_2(B,E,R)\frac{N\log^{3/2}(ER)}{\sqrt{R}},$$

with the functions c_1 *and* c_2 *defined in Theorem 1.5.*

Note that condition (1.2) simplifies computations compared to $\sum_{n=1}^{N}|f(n)|\leq B^2N$ found in [15].

Proving an explicit version of the Burgess bound in [4] is difficult, but the following result, that is an explicit Burgess-like result on convoluted Dirichlet characters, is enough for our purposes.

Theorem 1.3. *Let* q *and* k *be integers, with* $q>\max\{(100k)^4,\exp(\exp(8))\}$. *Let* χ *be a primitive character mod* q *and let* ψ *be any character mod* k. *For any integers* M *and* $N<q$ *we have*

$$\Big|\sum_{M<n\leq M+N}\psi(n)\chi(n)\Big|\leq 5k\,d(q)^{3/2}N^{1/2}q^{3/16}(\log q\log\log q)^{1/2},$$

with $d(-)$ *the divisor counting function.*

If we restrict to q prime, we should be able to improve the above result, but as we are mainly interested in a result for any q, we will not further exploit this possibility. Related explicit results can be found in [2], [5] and [22]. Using the above result, we are able to relax the conditions on α that appear in Corollary 1.2, thus obtaining the following fundamental result.

Lemma 1.4. *Take any* $q > \exp(\exp(8))$*,* x *such that* $q^{3/8+\varepsilon} \le x \le q$ *and, fixed a real* $\gamma \ge 2$*, for any* q *such that* $100(\log q)^\gamma \le q^{1/4}$ *and* $E \ge 4$*, we have, uniformly for all primitive characters* χ *modulo* q *as above,*

$$\Big|\sum_{n\le x}\chi(n)e(\alpha n)\Big| \le c(1,q,\gamma,\varepsilon)\,\frac{x}{\log q},$$

with the function c *defined in Theorem 1.5.*

The problem is now reduced to a computational one: we need to minimize ε, q and $c(E,q,\gamma,\varepsilon)$. We now obtain the fundamental Theorem 1.5, see Section 5 for more details, which give Theorem 1.1 by computations.

Theorem 1.5. *Take any* $q \ge \exp(\exp(8))$*. With* χ *a primitive Dirichlet character of modulo* q*, fixed* γ*, if* q *is such that* $100(\log q)^\gamma < q^{1/4}$*, with* $4 \le E \le 32$ *and* C *the Euler–Mascheroni constant, we have the following result:*

$$|S(N,\chi)| \le \begin{cases} \dfrac{2}{\pi^2}\Big(\dfrac{3}{8}+\varepsilon\Big)\sqrt{q}\log q + h_1(E,q,\gamma,\varepsilon)\sqrt{q}, & \text{if } \chi(-1)=1,\\[2mm] \dfrac{1}{\pi}\Big(\dfrac{3}{8}+\varepsilon\Big)\sqrt{q}\log q + h_2(E,q,\gamma,\varepsilon)\sqrt{q}, & \text{if } \chi(-1)=-1,\end{cases}$$

with

$$h_1(E,q,\gamma,\varepsilon) = \Big(\frac{2n(q,\varepsilon)}{\pi^2}+j\Big),\quad h_2(E,q,\gamma,\varepsilon) = \Big(\frac{n(q,\varepsilon)}{\pi}+j\Big),$$

where

$$j = j(E,q,\gamma,\varepsilon) = \frac{c(\chi)}{\pi}\Big(\frac{1}{\log q}+\frac{5}{8}-\varepsilon\Big)c(E,q,\gamma,\varepsilon)+1+\frac{(e^\pi-1-\pi)}{2\pi},$$

$$c(\chi) = \begin{cases}1 & \text{if } \chi(-1)=1,\\ 2 & \text{if } \chi(-1)=-1,\end{cases}\qquad n(q,\varepsilon) = C+\log 2+\frac{3}{q^{3/8+\varepsilon}},$$

$$c(E,q,\gamma,\varepsilon) = \max\Big\{\frac{c_1(1,E,\log^\gamma q)}{3/8+\varepsilon}+\frac{c_2(1,E,\log^\gamma q,q)(\log(E\log^\gamma q))^{3/2}}{(\log q)^{\gamma/2-1}},$$
$$15(\log q)^{2\gamma+1}\,(1+4\pi(\log q)^\gamma)\,\frac{(\log q\log\log q)^{1/2}}{q^{\varepsilon/2-\frac{3\log 2}{2\log\log q}\left(1+\frac{1}{\log\log q}+\frac{4.7626}{(\log\log q)^2}\right)}}\Big\},$$

and where

$$c_1(B,E,R) = (1+2\pi)b_1(B,E)+B\,4\pi\,\frac{\log R}{R^2},$$

$$c_2(B,E,R,q) = (1+2\pi)\Big(b_2(B,E)\,\frac{\big(e^C\log\log R+\frac{2.51}{\log\log R}\big)^{1/2}}{(\log(ER))^{3/2}}+b_3(B,E,R,q)\Big),$$

with

$$b_1(B,E) = B + (10.93 + a_1(11.8\sqrt{E} + 3.46) + 19.9a_3 + a_4(12.75\sqrt{E} + 5.28))B^2,$$

$$b_2(B,E) = a_2 B^2 \frac{\sqrt{2E}+1}{\log 2},$$

$$b_3(B,E,R,q) = B^2\Big(\frac{7.63}{\sqrt{E}}\frac{\log(4ER)}{(\log(ER))^{3/2}}\Big(1+\frac{\log(64/E)+1}{\log q}\Big) + \frac{1.48a_6}{\sqrt{\log(ER)}}$$

$$+ a_5 1.48\sqrt{\frac{\log(ER/4)}{\log(ER)}} + \frac{a_5}{\log 2}\frac{\log(R)}{(\log(ER))^{3/2}} + \frac{a_6}{\log 2}\frac{1}{(\log(ER))^{3/2}}\Big),$$

$$a_1 = 1.59B^2, \quad a_2 = \frac{\pi^2}{6}B^2 z, \quad a_3 = a_5 = 0.95B^2 z,$$

$$a_4 = B^2 z, \quad a_6 = 1.31B^2 z, \quad z = 8\sqrt{2}\prod_{p>2}\sqrt{1+\frac{1}{p^3-p^2-2p}}.$$

We will refer to the above defined functions through the paper. The outline of this article is as follows. In Section 2 we prove Corollary 1.2, and in Section 3 we prove Theorem 1.3. We proceed using these two results in Section 4 to prove Lemma 1.4 and Theorem 1.5. We conclude proving Theorem 1.1 in Section 5.

2. An explicit Montgomery–Vaughan result

We aim to prove the next explicit result, following [15].

Theorem 2.1. *Suppose that* $4 \le q \le N$, $E \ge 4$ *and* $(a,q) = 1$. *Then*

$$S(a/q) \le b_1(B,E)\frac{N}{\log N} + b_2(B,E)\frac{N}{\varphi(q)^{1/2}} + b_3(B,E,N/q,q)\sqrt{Nq}\log^{3/2} EN/q,$$

uniformly for $f \in F$.

We will deduce Corollary 1.2 from Theorem 2.1.

Define $\pi(x;q,a)$ as the numbers of primes up to x that are $\equiv a \pmod q$. An essential theorem to make the Montgomery–Vaughan result explicit is the following Brun–Titchmarsh inequality (see Theorem 3.7 in [14]).

Theorem 2.2. *Let* a *and* q *be coprime integers and let* x *and* y *be real numbers with* $1 \le q < y \le x$. *Then we have*

$$\pi(x+y;q,a) - \pi(x;q,a) \le \frac{2y}{\varphi(q)\log\frac{y}{q}},$$

for all $q \le x$ *and where* $\varphi(-)$ *is Euler's totient function.*

We introduce a precise enough result on primes from [19].

Theorem 2.3. *For $x > 1$, we have*

$$\pi(x) < 1.25506\,\frac{x}{\log x}\cdot$$

We now introduce a result on the logarithmic integral.

Lemma 2.4. *For $x > 2$, we have*

$$\mathrm{Li}(x) := \int_2^x \frac{1}{\log t}\,dt \le 1.37\,\frac{x}{\log x}\cdot$$

Proof. By Lemma 5.9 in [1], we have that

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log t}\,dt \le 1.2\,\frac{x}{\log x} \quad \text{for } x \ge 1865.$$

The result then follows computing $\mathrm{Li}(x)\frac{\log x}{x}$ for $2 < x < 1865$. ∎

Note that for our applications, the above results are sharp enough. We now introduce a result by Siebert [21].

Theorem 2.5. *Let $a \neq 0$, $b \neq 0$ be integers with $(a, b) = 1$, $2 \nmid a, b$. Then we have, for $x > 1$,*

$$\sum_{p \le x,\ ap+b \in P} 1 \le 16 \prod_{p>2}\left(1 - \frac{1}{p^2}\right) \prod_{p \mid ab,\ p>2} \frac{p-1}{p-2}\,\frac{x}{\log^2 x}, \tag{2.1}$$

where P denotes the set of all prime numbers.

Note that an improvement on the leading constant in (2.1) would lead to a significant improvement in the final result.

We now introduce some elementary results. The following upper bounds are obtained by splitting the sum in two parts, estimating the first with computer aid and the second by integration. The notation $\le_{A=\cdots}$ means that the upper bound is evaluated choosing $A = \cdots$.

Lemma 2.6. *We have*

$$\sum_{1}^{\infty} \frac{1}{2^{n/2}} = \frac{1}{\sqrt{2}-1}, \qquad \sum_{0}^{\lfloor \log_2 N \rfloor} 2^{n/2} \le \frac{\sqrt{2N}-1}{\sqrt{2}-1},$$

$$\sum_{1}^{\infty} \frac{\sqrt{n}}{2^{n/2}} \le \sum_{1}^{A} \frac{\sqrt{n}}{2^{n/2}} + \frac{2(\log 2)A + 4}{2^{A/2}(\log 2)^2} \le_{A=30} 4.15,$$

$$\sum_{0}^{\infty} \frac{\sqrt{n+1}}{2^{n/2}} \le \sum_{0}^{A} \frac{\sqrt{n+1}}{2^{n/2}} + \frac{2\log(2)(A) + 4}{2^{A/2}(\log 2)^2} \le_{A=30} 5.87,$$

$$\sum_{0}^{\infty} \frac{\sqrt{n+\log_2 e}}{2^{n/2}} \le \sum_{0}^{A} \frac{\sqrt{n+\log_2 e}}{2^{n/2}} + \frac{2\log(2)(A) + 4}{2^{A/2}(\log 2)^2} \le_{A=35} 6.34,$$

$$\sum_{1}^{\infty}\frac{\log n}{n^2}\le\sum_{1}^{A}\frac{\log n}{n^2}+\frac{\log A+1}{A}\le_{A=30}0.94,$$

$$\sum_{p\ge 2}\frac{(\log p)^2}{p^2}\le\sum_{2\le p\le A}\frac{(\log p)^2}{p^2}+\frac{(\log A)^2+2\log A+2}{A}\le_{A=15000}0.75,$$

$$\sum_{p\ge 2}\frac{\log p}{(p-1)^2}\le\sum_{2\le p\le A}\frac{\log p}{(p-1)^2}+\frac{\log(A)}{A-1}-\Big(\frac{A-1}{A}\Big)\le_{A=130}1.27,$$

$$\sum_{p\ge 2}\frac{\log p}{p(p-1)}\le\sum_{2\le p\le A}\frac{\log p}{p(p-1)}+\log\frac{\log(A)}{A-1}-\log\Big(\frac{A-1}{A}\Big)\le_{A=130}0.8,$$

$$\prod_{p>2}\Big(1+\frac{1}{p^3-p^2-2p}\Big)\le e^{\sum_{3\le p\le A}\frac{1}{p^3-p^2-2p}+\log\left(\frac{A^{1/2}}{(A+1)^{1/3}(A-2)^{1/6}}\right)}\le_{A=500}e^{0.1},$$

$$\sum_{p,j\ge 2}\frac{j\log p}{p^j}\le\sum_{2\le p,j\le A}\frac{j\log p}{p^j}+\sum_{p\le A}\frac{A\log p+1}{p^A\log p}$$
$$+\sum_{2\le j\le A}\frac{j((j-1)\log A+1)}{(j-1)^2A^{j-1}}+\frac{A+1}{(A-1)A^{A-1}}\le_{A=2000}1.99.$$

Note that the above bounds in A are meant for A not ‘too small’.

2.1. Reduction to bilinear forms

Note in the following that in the applications we will take $B=1$.

Lemma 2.7. *Let f be a multiplicative function satisfying (1.2) and let g be any real valued function. Then for any integer N we have*

$$\Big|\sum_{1\le n\le N}f(n)e(g(n))\Big|\le(B+3.59B^2)\frac{N}{\log N}+\frac{1}{\log N}\Big|\sum_{1\le np\le N}f(n)f(p)(\log p)e(g(np))\Big|.$$

Proof. We first note that, from (1.2),

$$\Big|\sum_{1\le n\le N}f(n)\log(N/n)e(g(n))\Big|\le BN,$$

and hence

$$\Big|\sum_{1\le n\le N}f(n)e(g(n))\Big|\le\frac{BN}{\log N}+\frac{1}{\log N}\Big|\sum_{1\le n\le N}f(n)(\log n)e(g(n))\Big|. \tag{2.2}$$

Since $\log n=\sum_{m|n}\Lambda(m)$, we have

$$\sum_{1\le n\le N}f(n)(\log n)e(g(n))=\sum_{1\le mn\le N}f(mn)\Lambda(m)e(g(mn)). \tag{2.3}$$

Our next step is to replace $f(mn)$ with $f(m)f(n)$, and thus we bound

$$T = \sum_{nm \le N} \Lambda(m)\,|f(mn) - f(m)f(n)| \le \Sigma_1 + \Sigma_2, \tag{2.4}$$

where

$$\Sigma_1 = \sum_{p,k\ge 1} \sum_{\substack{n \le Np^{-k} \\ p|n}} (\log p)\,|f(p^k n)|,$$

and

$$\Sigma_2 = \sum_{p,k\ge 1} (\log p)\,|f(p^k)| \sum_{j\ge 1} \sum_{m \le Np^{-k-j}} |f(p^j m)|.$$

Collecting together those terms in Σ_1 such that $p^k n$ is exactly divisible by p^j and by partial summation, using (1.2) and Lemma 2.6, we obtain

$$\begin{aligned}\Sigma_1 &\le \sum_{p,j\ge 2} (\log p)\,|f(p^j)|\,(j-1) \sum_{m \le Np^{-j}} |f(m)| \\ &\le B^2 N \sum_{p,j\ge 2} jp^{-j} \log p \le 1.99\, B^2 N.\end{aligned}$$

By (1.2),

$$\Sigma_2 \le B^2 N \sum_{p,j,k\ge 1} p^{-j-k} \log p = B^2 N \sum_{p\ge 2} \log p \Big(\sum_{j\ge 1} p^{-j} \Big)^2,$$

thus

$$\Sigma_2 \le B^2 N \sum_{p\ge 2} \frac{\log p}{(p-1)^2}$$

and, using Lemma 2.6,

$$\Sigma_2 \le 0.8\, B^2 N.$$

Thus

$$T \le 2.79\, B^2 N,$$

and hence by (2.2), (2.3) and (2.4)

$$\begin{aligned}\Big| \sum_{1\le n\le N} f(n) e(g(n)) \Big| &\le (B + 2.79\, B^2) \frac{N}{\log N} \\ &\quad + \frac{1}{\log N} \Big| \sum_{1\le nm\le N} f(n) f(m) \Lambda(m) e(g(mn)) \Big|. \end{aligned} \tag{2.5}$$

Those pairs m, n in which m is of the form p^k with $k \ge 2$ contribute an amount to the sum which is bounded by

$$\sum_{p,k\ge 2} |f(p^k)|\,(\log p) \sum_{n \le Np^{-k}} |f(n)|.$$

By (1.2), this is

$$\le B^2 N \sum_{p,k\ge 2} p^{-k}\log p = B^2 N \sum_{p\ge 2}\log p \sum_{k\ge 2} p^{-k} \le B^2 N \sum_{p\ge 2} \frac{\log p}{p(p-1)}.$$

Now, using Lemma (2.6), we have that

$$\sum_{p,k\ge 2} |f(p^k)|(\log p) \sum_{n\le Np^{-k}} |f(n)| \le 0.8\, B^2 N,$$

and hence by (2.5) the proof is completed. ∎

2.2. Partition of hyperbola into rectangles

We now partition the summation over the domain $1 \le pn \le N$ occurring in Lemma 2.7 into rectangles and their complements. Assume $N \ge q$ and let

$$\mathbf{J}_i = \min\left\{i+1, \lfloor \log_2 N\rfloor - i + 1, \left\lfloor \tfrac{1}{2}\log_2(EN/q)\right\rfloor\right\}, \tag{2.6}$$

with $4 \le E \le 32$. Define

$$\mathbf{R}_i = (0, 2^i] \times (N\,2^{-i-1}, N\,2^{-i}] \quad (0 \le i \le \log_2 N).$$

In the remaining regions, we place additional rectangles $\mathbf{R}_{ijk}$, for $j = 1, 2, \ldots, \mathbf{J}_i$ and for each j, $2^{j-1} < k \le 2^j$, defined as

$$\mathbf{R}_{ijk} = \left(2^{i+j}/k, 2^{i+j+1}/(2k-1)\right] \times \left((k-1)N\,2^{-i-j}, (2k-1)N\,2^{-i-j-1}\right]. \tag{2.7}$$

We do this for $j = 1, 2, \ldots, \mathbf{J}_i$. The choice of $\mathbf{J}_i$ ensures that each $\mathbf{R}_{ijk}$ is a rectangle of the form $(P', P''] \times (N', N'']$, with

$$P'' - P' \ge \frac{1}{4}, \quad N'' - N' \ge \frac{1}{4}, \quad (P'' - P')(N'' - N') \gg q.$$

Let $\mathbf{E}$ denote the set of points (p, n) with $pn \le N$ which do not lie in any $\mathbf{R}_i$ or $\mathbf{R}_{ijk}$. Writing $\mathbf{R}_i = \mathbf{P}_i \times \mathbf{N}_i$, we define $\mathbf{H}_i = \{(p, n) \in \mathbf{E} : n \in \mathbf{N}_i\}$. We then partition the $\mathbf{H}_i$ according to the value of $\mathbf{J}_i$. It is easy to see that

$$\mathbf{E} = \mathbf{E}_1 \cup \mathbf{E}_2 \cup \mathbf{E}_3,$$

with

$$\mathbf{E}_1 := \bigcup_{i:\mathbf{J}_i = i+1} \mathbf{H}_i,$$

$$\mathbf{E}_2 := \bigcup_{i:\mathbf{J}_i = \lfloor \log_2 N\rfloor - i + 1} \mathbf{H}_i,$$

$$\mathbf{E}_3 := \bigcup_{i:\mathbf{J}_i = \lfloor \frac{1}{2}\log_2(EN/q)\rfloor} \mathbf{H}_i.$$

See the following figure for a graphic example of the partition.

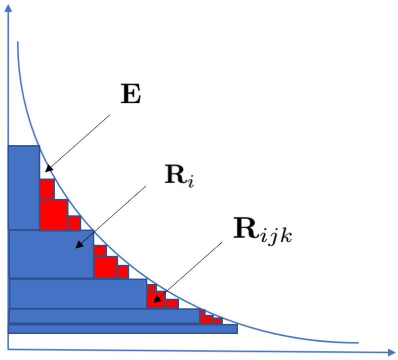

Figure 1. Example of the partition.

Lemma 2.8. *Take* $(a, q) = 1$. *The following estimate for the points* (p, n) *in* $\mathbf{E}$ *holds*:

$$\sum_{\mathbf{E}} f(p) f(n) e(pna/q) \log p$$

$$\le 7.34\, B^2 N + B^2 \frac{7.63}{\sqrt{E}} (Nq)^{1/2} (\log(4EN/q))^{1/2} (1 + \log(64/E) + \log q). \tag{2.8}$$

Proof. Consider first $\mathbf{E}_1$. For a given p, say $2^i < p \le 2^{i+1}$, the number of n for which $(p, n) \in \mathbf{E}_1$ is

$$\le \left\lceil \frac{N}{2^{i+J_i+1}} \right\rceil = \left\lceil \frac{N}{2^{2(i+1)}} \right\rceil \le \left\lceil \frac{N}{p^2} \right\rceil \le \frac{4N}{p^2},$$

for $p < 2\sqrt{N}$, where in the last step we used that by $J_i = i + 1$ we have $N > 2^{2i}$, that gives $p < 2\sqrt{N}$. Thus, for $\sqrt{N} \le p < 2\sqrt{N}$ we have $\lceil N/p^2 \rceil < 1 \le 4Np^{-2}$, and for $p \ge \sqrt{N}$ we have $\lceil N/p^2 \rceil \le 1 \le N/p^2 + 1 \le 2Np^{-2}$. Similarly, for a given n, we want to count primes p for which $(p, n) \in \mathbf{E}_1$. Taken $2^{J_i - 1} < k \le 2^{J_i}$ and using that $J_i = i + 1$, we can bound such number as follows:

$$\le \left\lceil \frac{2^{i+J_i}}{k(2k-1)} \right\rceil \le \left\lceil \frac{2^{i+1}}{(2^J_i - 1)} \right\rceil = \left\lceil \frac{2^{J_i}}{k(2k-1)} \right\rceil \le 2.$$

Hence, by Cauchy's inequality,

$$\sum_{\mathbf{E}_1} |f(p) f(n)| \log p \le \Big(\sum_{\mathbf{E}_1} |f(n)|^2 \Big)^{1/2} \Big(\sum_{\mathbf{E}_1} B^2 (\log p)^2 \Big)^{1/2}$$
$$\le B \Big(2 \sum_{n \le N} |f(n)|^2 \Big)^{1/2} \Big(\sum_{p \le 2\sqrt{N}} 4Np^{-2} (\log p)^2 \Big)^{1/2},$$

which, using Lemma 2.6, is bounded above by $2.45 B^2 N$.

Consider $\mathbf{E}_2$. For each pair $(p,n) \in \mathbf{E}_2$, we see that $n \leq (2N)^{1/2}$. This follows from the fact that, taken $2^i < p \leq 2^{i+i}$, we have

$$n \leq \frac{N}{p} \leq \frac{N}{2^i},$$

and that $J_i = \lfloor \log_2 N \rfloor - i + 1$ implies $N \leq 2^{2i+1}$. Furthermore, for a given n, the p with $(p,n) \in \mathbf{E}_2$ all lie in an interval of length $4Nn^{-2}$. This follows from the fact that, taken $2^{J_i - 1} < k \leq 2^{J_i}$ and $N/2^{i+1} < n \leq N/2^i$, such p lie in an interval of length

$$\leq \frac{2^{i+J_i}}{k(2k-1)} \leq \frac{2^{i+1}}{2^{J_i}-1} \leq \frac{2^{i+1}}{N}\Big(1 + \frac{1}{2^{J_i}-1}\Big) \leq \frac{4N}{n^2},$$

where we used that $J_i = \lfloor \log_2 N \rfloor - i + 1$. We thus have that by Theorem 2.2, there are

$$\leq 8\,\frac{N}{n^2 \log 4Nn^{-2}}$$

such p. For a given p, there is at most one n for which $(p,n) \in \mathbf{E}_2$. This follows from the fact that such n is in an interval, with one of the extremities open, of size $\leq \frac{N}{2^{i+J+1}} \leq 1$, where we used that $J_i = \lfloor \log_2 N \rfloor - i + 1$. We thus have, by partial summation,

$$\begin{aligned}
\sum_{n \leq \sqrt{2N}} |f(n)|^2 \frac{N}{n^2 \log 4Nn^{-2}} &\leq B^2\Big(\sum_{n \leq \sqrt{N}} \frac{N}{n^2 \log 4Nn^{-2}} + \frac{(\sqrt{2}-1)\sqrt{N}}{\log 2}\Big) \\
&\leq B^2 N\Big(\frac{1}{\sqrt{N}\log 4} - \int_1^{\sqrt{N}} \frac{2}{x^2 \log 4Nx^{-2}}\Big(\frac{1}{\log 4Nx^{-2}} - 1\Big)dx + \frac{(\sqrt{2}-1)}{\sqrt{N}\log 2}\Big) \\
&\leq NB^2\Big(\frac{1}{\log 4N} + \frac{\mathrm{Li}(2\sqrt{N})}{4\sqrt{N}} + \frac{(\sqrt{2}-1)}{\sqrt{N}\log 2}\Big),
\end{aligned}$$

which, by Lemma 2.4, is bounded above by $2.37\,\frac{B^2 N}{\log N}$ and

$$\sum_{p \leq N} \log^2 p \leq \pi(N)\log^2 N - 2\int_2^N \pi(x)\,\frac{2\log x}{x}\,dx \leq 1.26\,N\log N.$$

Thus

$$\begin{aligned}
\sum_{\mathbf{E}_2} |f(p)f(n)| \log p &\leq \Big(\sum_{\mathbf{E}_2} |f(n)|^2\Big)^{1/2}\Big(\sum_{\mathbf{E}_2} B^2(\log p)^2\Big)^{1/2} \\
&\leq B\Big(\sum_{n \leq \sqrt{2N}} |f(n)|^2\, 8\,\frac{N}{n^2 \log 4Nn^{-2}}\Big)^{1/2}\Big(\sum_{p \leq N} (\log p)^2\Big)^{1/2} \leq 4.89\,B^2 N.
\end{aligned}$$

Consider $\mathbf{E}_3$. Reasoning similarly to the case E_2, we have the following. For each n, the p with $(p,n) \in \mathbf{E}_3$ all lie in an interval of length $\leq \frac{8}{\sqrt{E}}(Nq)^{1/2}n^{-1}$, so that, by Theorem 2.2, there are

$$\leq \frac{16}{\sqrt{E}}\,\frac{\sqrt{Nq}}{n \log \frac{64}{E} Nqn^{-2}}$$

such p. When $(p,n) \in \mathbf{E}_3$, using that $J_i = \lfloor \frac{1}{2}\log_2(EN/q)\rfloor$ and $2^i < p \le 2^{i+1}$, we have $\frac{\sqrt{E}}{4}(N/q)^{1/2} \le p \le \frac{8}{\sqrt{E}}(Nq)^{1/2}$. From this and $N/2^{i+1} \le n \le N/p$ follows that $\frac{\sqrt{E}}{16}(N/q)^{1/2} \le n \le \frac{4}{\sqrt{E}}(Nq)^{1/2}$. Thus the following two sums on p and n will be restricted to these intervals. Using Theorem 2.3, we have

$$\sum_p \frac{\log p}{p} \le 1.26\,(1+\log(32/E)+\log q)$$

and

$$\sum_n |f(n)^2|\,\frac{1}{n} \le B^2\,(1+\log(64/E)+\log q).$$

Now, reasoning similarly to the case E_1, we have that for each p the number of n for which $(p,n) \in \mathbf{E}_3$ is $\le \frac{8}{\sqrt{E}}(Nq)^{1/2}p^{-1}$. Therefore,

$$\begin{aligned}\sum_{\mathbf{E}_3} |f(p)f(n)|\log p &\le \Big(\sum_{\mathbf{E}_3}|f(n)|^2\log N/n\Big)^{1/2}\Big(B^2\sum_{\mathbf{E}_3}\log p\Big)^{1/2}\\ &\le B\sqrt{Nq}\,\Big(\frac{16}{\sqrt{E}}\sum_n |f(n)|^2\frac{\log N/n}{n\log\frac{64}{E}Nqn^{-2}}\Big)^{1/2}\Big(\frac{8}{\sqrt{E}}\sum_p\frac{\log p}{p}\Big)^{1/2},\end{aligned}$$

and as the above ratio of the logarithms is less than $\frac{1}{4\log 2}\log(4EN/q)$,

$$\sum_{\mathbf{E}_3}|f(p)f(n)|\log p \le B^2(Nq)^{1/2}\,\frac{7.63}{\sqrt{E}}\,(\log(4EN/q))^{1/2}\,(1+\log(64/E)+\log q).$$

Combining the above estimates gives (2.8).

■

2.3. The fundamental estimate

Here we will develop a tool to bound the bilinear forms onto the rectangles defined in the previous section; in doing this, we follow [15], Section 4.

Lemma 2.9. *Let $M, X, Y \in \mathbb{R}^{\ge 1}$, $Q \in \mathbb{R}^{>1}$, $K \in \mathbb{N}$, and for $1 \le k \le K$, let*

$$\mathcal{R}(k) = \mathcal{L}(k) \times \mathcal{M}(k),$$

be a rectangle satisfying

$$\mathcal{L}(k) \subseteq (0,Q], \quad \mathcal{M}(k) \subseteq (0,M],$$

and

$$\mathcal{L}(k) = (Q'(k), Q''(k)], \quad \mathcal{M}(k) = (M'(k), M''(k)],$$

for some $Q'(k)$, $Q''(k)$ and $M'(k)$, $M''(k)$ satisfying

$$Q''(k) - Q'(k) \le X, \quad M''(k) - M'(k) \le Y, \quad M''(k) \le 2M'(k).$$

Suppose that $\mathcal{L}(k)$ and similarly $\mathcal{M}(k)$ are disjoint for each $1 \leq k \leq K$. For any function $f(n)$ satisfying (1.2), define

$$I = \sum_{k=1}^{K} \sum_{(p,n)\in\mathcal{R}(k)} f(p) f(n) e(pna/q) \log p.$$

Then, if $(a,q) = 1$ and $q \leq XY$, we have

$$I \leq B^2\Big(2.52\, MQY \log Q + 128 \prod_{p>2}\Big(1 + \frac{1}{p^3 - p^2 - 2p}\Big) MQ$$

$$\cdot\Big(\frac{\pi^4}{6^2}\frac{XY}{\varphi(q)} + 0.89X + Y\log(eX) + 0.89q\log(eXY/q) + \frac{\pi^2}{12}q\Big)\Big)^{1/2}. \tag{2.9}$$

Proof. Let $\mathcal{R} = \mathcal{L} \times \mathcal{M}$ be one of the rectangles $\mathcal{R}(k)$. By Cauchy's inequality,

$$\Big|\sum_{(p,n)\in\mathcal{R}} f(p) f(n) e(pna/q) \log p\Big|^2$$

$$\leq \Big(\sum_{n\in\mathcal{M}} |f(n)|^2\Big)\Big(\sum_{n\in\mathcal{M}}\Big|\sum_{p\in\mathcal{L}} f(p) e(pna/q) \log p\Big|^2\Big). \tag{2.10}$$

We now introduce the smoothing factor

$$w(n) = \max\{0, 2 - |2n - 2M' - Y|Y^{-1}\},$$

such that $w(n) \geq 1$ for $n \in \mathcal{M}$. Note that the above is a variation of the Fejér kernel; we choose it following Montgomery and Vaughan. Aiming to improve the result, it would surely be interesting to chose other kernels. We also introduce $g(n) = \max\{0, 1 - |n|\}$ and note that for the Fourier transform of g we have

$$|\widehat{g}(t)| = \Big(\frac{\sin(\pi t)}{\pi t}\Big)^2.$$

This allows to compute the Fourier transform of $w(n)e(\alpha n)$, with $\alpha \in \mathbb{R}$, as

$$|\widehat{w(n)e(\alpha n)}(t)| = 2Y\Big(\frac{\sin(\pi Y(\alpha - t))}{\pi Y(\alpha - t)}\Big)^2. \tag{2.11}$$

With $\|\cdot\|$ the distance to the closest integer, it is also easy to see that

$$\sum_{n\in\mathbb{Z}} \frac{1}{(\alpha - n)^2} \leq \frac{2}{\|\alpha\|^2} + \frac{\pi^2}{3}. \tag{2.12}$$

Thus the second factor on the right of (2.10) is bounded above by

$$\sum_n w(n)\Big|\sum_{p\in\mathcal{L}} f(p) e(pna/q) \log p\Big|^2$$

$$= \sum_{p,p'\in\mathcal{L}} f(p)\overline{f(p')}(\log p)(\log p') \sum_n w(n) e((p - p')na/q);$$

using the Poisson formula, (2.11), (2.12) and that $\|\alpha\| \leq 1/2$, we obtain

$$\leq B^2(\log Q)^2 \sum_{p,p' \in \mathcal{L}} \min\Big\{2Y, \frac{4/\pi^2 + 1/6}{Y\|\frac{(p-p')a}{q}\|^2}\Big\}.$$

By Cauchy's inequality, and Theorem 2.3,

$$\begin{aligned} I &\leq B(\log Q)\Big(\sum_k \sum_{n \in \mathcal{M}} |f(n)|^2\Big)^{1/2}\Big(\sum_k \sum_{p,p' \in \mathcal{L}} \min\Big\{2Y, \frac{0.58}{Y\|\frac{(p-p')a}{q}\|^2}\Big\}\Big)^{1/2} \\ &\leq B^2(\log Q)\sqrt{M}\Big(2.52Y\frac{Q}{\log Q} + \sum_{0<h\leq X} \sum_{\substack{p \leq Q \\ p+h=p'}} \min\Big\{2Y, \frac{0.58}{Y\|\frac{ha}{q}\|^2}\Big\}\Big)^{1/2}. \end{aligned}$$

Now from Theorem 2.5 we obtain

$$I \leq B^2\Big(2.52MQY\log Q + 16\prod_{p>2}\Big(1 - \frac{1}{p^2}\Big)MQV\Big)^{1/2}, \tag{2.13}$$

where

$$V = \sum_{0<h\leq X} \prod_{p|h, p>2} \frac{p-1}{p-2} \min\Big\{2Y, \frac{0.58}{Y\|\frac{ha}{q}\|^2}\Big\}.$$

Hence we need to bound V. Now we have

$$\begin{aligned} \prod_{p|h,p>2} \frac{p-1}{p-2} &= \prod_{p|h,p>2}\Big(1 + \frac{2}{p^2-p-2}\Big)\prod_{p|h,p>2}\Big(1 + \frac{1}{p}\Big) \\ &\leq \prod_{p>2}\Big(1 + \frac{2}{p^2-p-2}\Big)\sum_{m|h}\frac{1}{m}, \end{aligned}$$

so that

$$V \leq \prod_{p>2}\Big(1 + \frac{2}{p^2-p-2}\Big)\Big(\sum_{m\leq X}\frac{1}{m}\sum_{n\leq X/m}\min\Big\{2Y, \frac{0.58}{Y\|\frac{mna}{q}\|^2}\Big\}\Big).$$

The innermost sum is of the form

$$W = 0.58\frac{1}{Y}\sum_{n\leq Z}\min\Big\{(0.58)^{-1}2Y^2, \frac{1}{\|\frac{bn}{r}\|^2}\Big\}$$

with $r = q/(m,q)$ and $(b,r) = 1$. Using Lemma 14 in [13], this is seen to satisfy

$$W \leq \min\Big\{2YZ, 4\sqrt{2}(0.58)^{1/2}(Z+r)\Big(\Big(\frac{2}{0.58}\Big)^{1/2}Y + r\Big)r^{-1}\Big\}.$$

Therefore, by

$$\begin{aligned} &\sum_{\substack{m\leq X \\ (m,q)XY\leq mq}} \frac{2XY}{m^2} + \sum_{\substack{m\leq X \\ (m,q)XY>mq}} \frac{1}{m}\Big(\frac{8XY}{mq}(m,q) + 4.31\frac{X}{m} + 8Y + 4.31\frac{q}{(m,q)}\Big) \\ &\leq \sum_{r|q}\sum_{s>\frac{XY}{q}}\frac{2XY}{r^2s^2} + \sum_{r|q}\sum_{s}\frac{8XY}{rs^2q} + \frac{4.31\pi^2}{6}X + 8Y\log(eX) + \sum_{r|q}\sum_{s<\frac{XY}{q}}\frac{4.31q}{r^2s}, \end{aligned}$$

we obtain

$$V \leq 8 \prod_{p>2} \Big(1 + \frac{2}{p^2 - p - 2}\Big)\Big(\frac{\pi^4}{6^2} XY\varphi(q)^{-1} + 0.89X + Y\log(eX) + 0.54\frac{\pi^2}{6} q\log(eXY/q)) + \frac{\pi^2}{12} q\Big).$$

Thus from the above bound, Lemma 2.6 and (2.13), we obtain the desired result. ∎

Note that is easy to see that (2.9) holds also for $Q \in \mathbb{R}^{\geq 1}$.

2.4. Completion of the proof of Theorem 2.1

Note that in the following argument we will extensively use Lemma 2.6 and refer to the notation of Theorem 2.1. We will also use that given $a_i \geq 0$, we have

$$\Big(\sum_i a_i\Big)^{1/2} \leq \sum_i a_i^{1/2}. \tag{2.14}$$

We first apply (2.9) to the rectangle $\mathbf{R}_i$ and then (2.14). We take

$$K = 1, \quad X = Q = 2^i \quad \text{and} \quad Y = M = N2^{-i}.$$

Thus

$$\frac{1}{B^2}\Big|\sum_{(p,n)\in\mathbf{R}_i} f(p)f(n)e(pna/q)\log q\Big| \leq \sqrt{\log 2}a_1 N\sqrt{\frac{i}{2^i}} + a_2\frac{N}{\sqrt{\varphi(q)}} + a_3\sqrt{N2^i} + \sqrt{\log 2}\,a_4 N\sqrt{\frac{i+\log_2 e}{2^i}} + a_5\sqrt{Nq\log(N/q)} + a_6\sqrt{qN}. \tag{2.15}$$

Next, for each pair i, j with $1 \leq j \leq J_i$, we apply (2.9) to the family of 2^{j-1} rectangles $\mathbf{R}_{ijk}$ with $2^{j-1} < k \leq 2^j$. By (2.7), we may take

$$K = 2^{j-1}, \quad M = N2^{-i}, \quad Q = 2^{i+1}, \quad X = 2^{i-j+1} \quad \text{and} \quad Y = \frac{E}{2}N2^{-i-j}.$$

Thus, by (2.6), $XY \geq q$, so that the conditions for (2.9) to hold are satisfied. Hence by (2.9) and (2.14),

$$\frac{1}{B^2}\Big|\sum_{2^{j-1}<k\leq 2^j}\sum_{(p,n)\in\mathbf{R}_{ijk}} f(p)f(n)e(pna/q)\log q\Big|$$
$$\leq \sqrt{E\log 2}, \Big(a_1\sqrt{\frac{i+1}{2^{i+j}}} + a_4\sqrt{\frac{i+\log_2 e}{2^{i+j}}}\Big)N + \sqrt{2E}a_2\frac{1}{2^j}\frac{N}{\sqrt{\varphi(q)}} + a_3 2\sqrt{2^{i-j}}\sqrt{N} + a_5\sqrt{2Nq\log\Big(\frac{EN}{4q}v\Big)} + a_6\sqrt{2Nq}.$$

By (2.6), $J_i \le \frac{1}{2}\log_2(EN/q)$. Hence, summing over those j with $1 \le j \le J_i$ we obtain

$$\frac{1}{B^2}\Big|\sum_{1\le j\le J_i}\sum_{2^{j-1}<k\le 2^j}\sum_{(p,n)\in\mathbf{R}_{ijk}} f(p)f(n)e(pna/q)\log q\Big|$$
$$\le \sqrt{E\log 2}\,\frac{1}{\sqrt{2}-1}\Big(a_1\sqrt{\frac{i+1}{2^i}}+a_4\sqrt{\frac{i+\log_2 e}{2^i}}\Big)N+\sqrt{2E}a_2\frac{N}{\sqrt{\varphi(q)}}$$
$$+\frac{2}{\sqrt{2}-1}a_3\sqrt{2^i}\sqrt{N}+\frac{\sqrt{2}}{2}\Big(a_5\sqrt{Nq\log\Big(\frac{EN}{4q}\Big)}+a_6\sqrt{Nq}\Big)\log_2(EN/q).$$

Therefore, by (2.15), summing over i with $0 \le i \le \log_2 N$, we can obtain

$$\frac{1}{B^2}\Big|\sum_{\substack{pn\le N\\(p,n)\notin\mathbf{E}}} f(p)f(n)e(pna/q)\log q\Big|$$
$$\le \Big(a_1(11.8\sqrt{E}+3.46)+19.9a_3+a_4(12.75\sqrt{E}+5.28)\Big)N+a_2\frac{\sqrt{E2}+1}{\log 2}\frac{N\log N}{\varphi(q)^{1/2}}$$
$$+\Big(\frac{a_5}{\sqrt{2}\log 2}\sqrt{\frac{\log(EN/4q)}{\log(EN/q)}}+\frac{a_6}{\sqrt{2}\log 2}\frac{1}{\sqrt{\log(EN/q)}}+\frac{a_5\log(N/q)}{(\log(EN/q))^{3/2}}$$
$$+\frac{a_6}{(\log(EN/q))^{3/2}}\Big)\cdot\frac{(Nq)^{1/2}}{\log 2}(\log(EN/q))^{3/2}\log N.$$

This with (2.8) and Lemma 2.7 gives Theorem 2.1.

2.5. Proof of Corollary 1.2

Let $S(\alpha,u)=\sum_{n\le u} f(n)e(n\alpha)$. Then, with $S(\alpha) := S(\alpha,N)$, we have

$$S(\alpha)=e((\alpha-\beta)N)\,S(\beta,N)-2\pi i(\alpha-\beta)\int_1^N S(\beta,u)e((\alpha-\beta)u)\,du$$

Suppose that $\beta=b/r$ with $(b,r)=1$ and $r\le N$. Then, on using that $|S(\alpha,u)|\le Br$ when $u\le r$ and Theorem 2.1 when $u>r$, we obtain

$$|S(\alpha)|\le\Big(b_1(B,E)\frac{N}{\log N}+\frac{b_2(B,E)N}{\sqrt{\varphi(r)}}+b_3(B,E,N/r))\sqrt{rN}(\log(EN/r))^{3/2}\Big)$$
$$\cdot(1+2\pi(N-r)|\alpha-b/r|)+Br^2 2\pi|\alpha-b/r|. \tag{2.16}$$

Here we use, from Theorem 15 in [19], that for $n\ge 3$,

$$\varphi(n)>\frac{n}{e^C\log\log n+\frac{2.51}{\log\log n}},$$

with C the Euler–Mascheroni constant.

If $q > N^{1/2}$, then we take $b = a$ and $r = q$, which gives

$$S(\alpha) \le (1+2\pi)\, b_1(B,E)\,\frac{N}{\log N} + B\,2\pi + (1+2\pi)$$
$$\cdot\Big(b_2(B,E)\,\frac{(e^C \log\log R + \frac{2.51}{\log\log R})^{1/2}}{(\log(ER))^{3/2}} + b_3(B,E,R)\Big)\frac{(\log ER)^{3/2}}{\sqrt{R}}\,N.$$

If $q \le N^{1/2}$, then by Dirichlet's theorem there exist b, r such that $(b,r) = 1$, $r \le 2N/q$ and $|\ \alpha - b/r\ | \le q/(2rN)$. Thus, either $r = q$ or $1 \le |ar - bq| = rq|(\alpha - b/r) - (\alpha - a/q)| \le q^2/(2N) + r/q \le 1/2 + r/q$, thus in either case $r \ge q/2$. Therefore $|\alpha - b/r| \le N^{-1}$ and consequently, by (2.16), Corollary 1.2 follows once more.

3. Explicit Burgess bound for composite moduli

We now prove Theorem 1.3.

The proof of the following is the same as that of Lemma 2.1 in [22], which deals with the case $q = p$ prime.

Lemma 3.1. *For integers q, M, N, U satisfying*

$$N < q \quad \textit{and} \quad 28 \le U \le \frac{N}{12},$$

let $I_q(N,U)$ count the number of solutions to the congruence

$$n_1u_1 \equiv n_2u_2 \pmod q,\ \ M \le n_1, n_2 \le M+N,\ \ 1 \le u_1, u_2 \le U,\ \ (u_1u_2, q) = 1.$$

We have

$$I_q(N,U) \le 2UN\Big(\frac{NU}{q} + \log(1.85U)\Big).$$

It is useful to observe that the proof of Lemma 2.1 in [22] gives the above result as we added the condition $(u_1u_2, q) = 1$. From a simple application of Eratostene's sieve, we obtain the following result.

Lemma 3.2. *Given two integers U and q, we have*

$$\sum_{\substack{1\le u\le U\\ (u,q)=1}} 1 \ge \frac{\varphi(q)}{q}\,U - 2^{\tau(q)},$$

where $\tau(-)$ counts the number of prime divisors.

Using an idea of Burgess [3], with an improvement of Heath-Brown [10], we have the following.

Lemma 3.3. *Let q, k, V be integers with $V < q$. For any primitive $\chi \pmod q$, we have*

$$\sum_{\lambda=1}^{q}\Big|\sum_{v\le V}\chi(\lambda + kv)\Big|^4 \le 16qk^2V^2 + 4q^{1/2}k^4V^4d(q)^6,$$

where $d(-)$ is the divisor counting function.

Proof. Let

$$S = \sum_{\lambda=1}^{q} \Big| \sum_{v \le V} \chi(\lambda + kv) \Big|^4.$$

Expanding the fourth power and interchanging summation, we have

$$S \le \sum_{1 \le m_1, \dots, m_4 \le kV} \Big| \sum_{\lambda=1}^{q} \chi\Big(\frac{(\lambda + m_1)(\lambda + m_2)}{(\lambda + m_3)(\lambda + m_4)} \Big) \Big|.$$

Define

$$A_j = \prod_{i \ne j} (m_j - m_i) \quad \text{and} \quad K_j = (q, A_j).$$

Using Lemma 7 in [3], and arguing as in Burgess [3], Lemma 8, see (10), (11) and (12), we obtain

$$S \le 16qk^2V^2 + 8^{\tau(q)} q^{1/2} \sum_{j=1}^{4} \sum_{m_1,\dots,m_4}{}' K_j, \tag{3.1}$$

where $\sum'$ is the sum overall $m_1, \dots, m_4 \le kV$, which contains at least 3 distinct elements. Heath-Brown in [10], Lemma 2, proves that

$$\sum_{j=1}^{4} \sum_{m_1,\cdots,m_4 \le m}{}' K_j \le 4m^4 d(q)^3. \tag{3.2}$$

Applying (3.2) to (3.1) and using that $8^{\tau(q)} \le d(q)^3$, we thus obtain

$$S \le 16qk^2V^2 + 4q^{1/2}k^4V^4d(q)^6,$$

which completes the proof. ∎

3.1. Proof of Theorem 1.3

We begin proving the following fundamental result.

Theorem 3.4. *Let q and k be integers, let $g \ge 2$ be a real number, and let m and h be two positive real numbers. Let χ be a primitive character modulo q and ψ be any character modulo k. Assume that*

$$q > \max\Big\{ \Big(\max\Big\{29, 2^{\tau(q)+1} \frac{q}{\varphi(q)} + 1\Big\} \frac{g}{m^2 \log q \log\log q} \Big)^8, \\ \Big(\frac{12}{g}\Big)^4, (\max\{1, h\}k)^4, \Big(\frac{16 \log q}{m^2 \log\log q} \Big)^8 \Big\}. \tag{3.3}$$

Define

$$v_1(m,q) = \frac{2(1+\frac{2}{e\log q})}{m},$$

$$v_2(m,q,g) = \frac{2\,(v_1(m,q))^4}{g} + \frac{2(\log\log q)^2 \log\left(1.85\,(v_1(m,q))^2\, q^{3/8}\,\frac{\log q}{g\log\log q}\right)}{(\log q)^2},$$

$$v_3(m,q,g,h) = 2g\Big(1-\frac{1}{h}-\frac{g q^{1/4}}{m^2 q^{3/8}\log q(\log\log q)}\Big)^{-1}\Big(\frac{17\,v_2(m,q,g)}{4g^3}\Big)^{1/4} \cdot\Big(e^C+\frac{2.51}{(\log\log q)^2}\Big)+\frac{2}{\sqrt{g}}.$$

If $v_3(m,q,g,h) \le m$ holds then, for any integers M, N, we have

$$\Big|\sum_{M<n\le M+N}\psi(n)\,\chi(n)\Big| \le mk\,d(q)^{3/2}N^{1/2}q^{3/16}(\log q)^{1/2}(\log\log q)^{1/2}. \tag{3.4}$$

Proof. We proceed by induction on N. For any $K \le m^2q^{3/8}\log q(\log\log q)$, we trivially have

$$\Big|\sum_{M<n\le M+K}\psi(n)\,\chi(n)\Big| \le mk\,d(q)^{3/2}K^{1/2}q^{3/16}(\log q)^{1/2}(\log\log q)^{1/2}.$$

Also note that $\psi\chi$ is a non-principal character, with modulus $\le kq$, for otherwise $\overline{\psi}$ and χ would be induced by the same primitive character; that is impossible as χ is primitive modulo q and we have that $q > k$ from the third inequality in (3.3). We thus have, by the Pólya–Vinogradov inequality,

$$\Big|\sum_{M<n\le M+K}\psi(n)\,\chi(n)\Big| \le 2\sqrt{kq}\log kq,$$

and thus for $K > v_1(m,q)^2q^{5/8}\frac{\log q}{\log\log q}$, by comparison with (3.4) and using that $q > k$, Theorem 1.3 holds. Note that using the Pólya–Vinogradov inequality from [7] would allow to improve on m, but the above result is good enough for our purposes. This forms the basis of our induction and we assume (3.4) holds for any sum of length strictly less than N, with N such that

$$m^2q^{3/8}\log q(\log\log q) \le N \le v_1(m,q)^2q^{5/8}\frac{\log q}{\log\log q}. \tag{3.5}$$

Define

$$U = \Big\lfloor\frac{N}{gq^{1/4}}\Big\rfloor, \quad V = \Big\lfloor\frac{q^{1/4}}{k}\Big\rfloor, \tag{3.6}$$

and note that

$$UV \le \frac{N}{gk}.$$

Note also that $U \ge 1$ by the first inequality in (3.3), and that $V \ge 1$ by the third inequality in (3.3).

For any integer $y < N$, we have

$$\begin{aligned}\sum_{M<n\le M+N} \psi(n)\,\chi(n) &= \sum_{M-y<n\le M+N-y} \psi(n+y)\,\chi(n+y) \\ &= \sum_{M<n\le M+N} \psi(n+y)\,\chi(n+y) + \sum_{M-y<n\le M} \psi(n+y)\,\chi(n+y) \\ &\quad - \sum_{M+N-y<n\le M+N} \psi(n+y)\,\chi(n+y),\end{aligned}$$

and hence

$$\sum_{M<n\le M+N} \psi(n)\,\chi(n) = \sum_{M<n\le M+N} \psi(n+y)\,\chi(n+y) + 2\theta E(y),$$

with $E(y) = \max_M |\sum_{M<n\le M+y} \psi(n)\chi(n)|$ and for some $|\theta| \le 1$, different in each instance. Let $\mathcal{U}$ denote the set

$$\mathcal{U} = \{1 \le u \le U \,:\, (u,q) = 1\},$$

and average the above over integers $y = kuv$ with $u \in \mathcal{U}$ and $1 \le v \le V$ to get

$$\Big|\sum_{M<n\le M+N} \psi(n)\,\chi(n)\Big| \le \frac{1}{V|\mathcal{U}|}|W| + 2\,\frac{1}{V|\mathcal{U}|}\sum_{u,v} E(y), \tag{3.7}$$

where

$$W = \sum_{M<n\le M+N}\sum_{u\in\mathcal{U}}\sum_{1\le v\le V} \psi(n+kuv)\,\chi(n+kuv).$$

For any u, v we have $uvk \le N/g$, and thus by the induction hypothesis,

$$2\,\frac{1}{V|\mathcal{U}|}\sum_{u,v} E(y) \le \frac{2}{\sqrt{g}}\, m\,k\,d(q)^{3/2}\sqrt{N}q^{3/16}(\log q)^{1/2}(\log\log q)^{1/2}. \tag{3.8}$$

Since ψ is a character mod k, we have

$$|W| \le \sum_{M<n\le M+N}\sum_{u\in\mathcal{U}}\Big|\sum_{1\le v\le V} \chi(nu^{-1}+kv)\Big| = \sum_{\lambda=1}^{q} I(\lambda)\Big|\sum_{1\le v\le V} \chi(\lambda+kv)\Big|,$$

where $I(\lambda)$ counts the number of solutions to the congruence

$$nu^{-1} \equiv \lambda \pmod q, \quad M < n \le M+N,\ u \in \mathcal{U}.$$

By Hölder's inequality,

$$|W|^4 \le \Big(\sum_{\lambda=1}^{q} I(\lambda)\Big)^2\Big(\sum_{\lambda=1}^{q} I(\lambda)^2\Big)\Big(\sum_{\lambda=1}^{q}\Big|\sum_{1\le v\le V} \chi(\lambda+kv)\Big|^4\Big).$$

We have

$$\sum_{\lambda=1}^{q} I(\lambda) = N|\mathcal{U}| \le NU.$$

Using Lemma 3.1, since $N < q$, from (3.5) and the fourth inequality in (3.3), and since $28 \le U \le N/12$, from (3.5) and the first and second inequalities in (3.3), we obtain

$$I_q(N,U) \le 2UN\Big(\frac{NU}{q} + \log(1.85U)\Big);$$

this, using (3.5) and recalling (3.6), gives us

$$\sum_{\lambda=1}^{q} I(\lambda)^2 = I_q(N,U) \le v_2(m,q)UN\Big(\frac{\log q}{\log\log q}\Big)^2.$$

By Lemma 3.3,

$$\sum_{\lambda=1}^{q}\Big|\sum_{1\le v\le V}\chi(\lambda+kv)\Big|^4 \le 16qk^2V^2 + 4q^{1/2}k^4V^4d(q)^6.$$

Recalling (3.6), the above estimates simplify to

$$\sum_{\lambda=1}^{q} I(\lambda) \le \frac{N^2}{gq^{1/4}},$$

$$\sum_{\lambda=1}^{q} I(\lambda)^2 \le v_2(m,q)\,\frac{N^2}{gq^{1/4}}\Big(\frac{\log q}{\log\log q}\Big)^2,$$

$$\sum_{\lambda=1}^{q}\Big|\sum_{1\le v\le V}\chi(\lambda+kv)\Big|^4 \le \frac{17}{4}\,q^{3/2}d(q)^6.$$

Therefore,

$$|W|^4 \le \frac{17v_2(m,q)}{4g^3}\,N^6q^{3/4}d(q)^6\Big(\frac{\log q}{\log\log q}\Big)^2.$$

Note that using (3.5) and the third inequality in (3.3),

$$UV \ge \frac{N}{gk}\Big(1 - \frac{k}{q^{1/4}} - \frac{gq^{1/4}}{N} + \frac{gk}{N}\Big) \ge \frac{N}{gk}\Big(1 - \frac{1}{h} - \frac{gq^{1/4}}{m^2q^{3/8}\log q(\log\log q)}\Big).$$

From Lemma 3.2, (3.6), (3.5) and the first inequality in (3.3), we have

$$|\mathcal{U}| \ge \frac{\varphi(q)}{2q}\,U. \tag{3.9}$$

Thus using (3.7), (3.8), Theorem 15 in [19] and (3.9), we get

$$\Big|\sum_{M<n\le M+N}\psi(n)\,\chi(n)\Big| \le v_3(m,q,g,h)\,k\,d(q)^{3/2}N^{1/2}q^{3/16}(\log q)^{1/2}(\log\log q)^{1/2}.$$

Now, if $v_3(m,q,g,h) \le m$, we conclude the proof by induction. ∎

Theorem 1.3 follows by computationally finding g and h such that for small q we have a small m such that $v_3(m,q,g,h) \leq m$. This happens for $g = 0.45$ and $h = 100$. Note that to verify (3.3) we used Theorem 12 of [18] and Theorem 15 in [19].

By [17] and [20], p. 43, we have that for any integer $n \geq 3$,

$$\log d(n) \leq \frac{\log n}{\log\log n}\Big(\log 2 + \frac{\log 2}{\log\log n} + \frac{4.7626\log 2}{(\log\log n)^2}\Big). \tag{3.10}$$

Thus, (3.10) allows to rewrite Theorem 1.3 as follows.

Theorem 3.5. *Let q and k be as in Theorem 1.3. Let χ be a primitive character mod q and let ψ be any character mod k. For any integers M and $N < q$, we have*

$$\Big|\sum_{M<n\leq M+N}\psi(n)\,\chi(n)\Big| \leq \frac{5k\,N^{1/2}q^{3/16}(\log q\log\log q)^{1/2}}{q^{-\frac{3\log 2}{2\log\log q}\left(1+\frac{1}{\log\log q}+\frac{4.7626}{(\log\log q)^2}\right)}}. \tag{3.11}$$

4. Explicit improved Pólya–Vinogradov inequality

The aim of this section is to prove Theorem 1.5 following [11].

4.1. Two important lemmas

We use Corollary 1.2 and Theorem 1.3 to obtain the explicit version of Lemma 2 in [11] with a certain range for the modulus q.

Proof of Lemma 1.4. Let ε, χ, q, x and α be fixed and set $N = \lfloor x\rfloor$, $R = (\log q)^{\gamma}$. By $q \geq 10^5$ and $\gamma \geq 2$, we easily obtain $e^3/E \leq R \leq N$. By Dirichlet's theorem, there exist integers r and s, where $(r,s) = 1$ and $1 \leq s \leq N/R$, such that

$$\Big|\alpha - \frac{r}{s}\Big| \leq \frac{1}{sN/R}.$$

If $s \geq R$, the result follows from Corollary 1.2, since

$$c_1(1,E)\,\frac{N}{\log N} + c_2(1,E,R)\,N\,\frac{(\log R)^{3/2}}{\sqrt{R}} \leq \left(\Big(\frac{3}{8}+\varepsilon\Big)^{-1}c_1(1,E) + \frac{c_2(1,E,R)(\gamma\log\log q)^{3/2}}{(\log q)^{\gamma/2-1}}\right)\frac{x}{\log q},$$

by the definition of N and R. Now suppose $s < R$. By partial summation follows that

$$\Big|\sum_{n\leq x}\chi(n)e(N'/qn)\Big| \leq \Big(1 + 2\pi\Big|\alpha-\frac{r}{s}\Big|x\Big)\max_{u\leq x}|T(u)| \leq (1+4\pi(\log q)^{\gamma})\max_{u\leq x}|T(u)|,$$

where

$$T(u) = \sum_{n\leq u}\chi(n)\,e\Big(\frac{rn}{s}\Big).$$

By grouping the terms of the sum $T(u)$ according to the value of (n, s), we get

$$\begin{aligned} T(u) &= \sum_{dt=s} \sum_{\substack{dm\le u\\(m,t)=1}} \chi(md)\, e\left(\frac{rm}{t}\right) \\ &= \sum_{dt=s} \chi(d) \sum_{\substack{1\le a\le t\\(a,t)=1}} e\left(\frac{ra}{t}\right) \sum_{\substack{m\le u/d\\ m=a (\mathrm{mod}\ \ t)}} \chi(m) \\ &= \sum_{dt=s} \frac{\chi(d)}{\varphi(t)} \sum_{\psi \bmod t} \sum_{1\le a\le t} e\left(\frac{ra}{t}\right) \overline{\psi}(a) \sum_{m\le u/d} \chi(m)\,\psi(m). \end{aligned}$$

Applying (3.11) to the right-hand sum, we obtain

$$\left|\sum_{n\le x} \chi(n)\, e(N'/qn)\right| \le \frac{15(\log q)^{2\gamma+1}\,(1+4\pi(\log q)^{\gamma})\,(\log q \log\log q)^{1/2}}{q^{\varepsilon/2 - \frac{3\log 2}{2\log\log q}\left(1+\frac{1}{\log\log q}+\frac{4.7626}{(\log\log q)^2}\right)}}\,\frac{x}{\log q}.$$

Thus Lemma 1.4 follows. ■

We then need explicit bounds by Pomerance (see Lemmas 2 and 3 in [16]) on two trigonometric sums.

Lemma 4.1. *Uniformly for $x \ge 1$ and real α, we have*

$$\sum_{n\le x} \frac{1-\cos(\alpha n)}{n} \le \log x + C + \log 2 + \frac{3}{x}$$

and

$$\sum_{n\le x} \frac{|\sin(\alpha n)|}{n} \le \frac{2}{\pi}\log x + \frac{2}{\pi}\Big(C + \log 2 + \frac{3}{x}\Big).$$

Note that the last terms in the above upper bounds can be improved, but this would have no effect on our final result.

4.2. Proof of Theorem 1.5

We take χ primitive. We start with

$$\chi(n) = \frac{1}{d(\overline{\chi})} \sum_{a=1}^{q} \overline{\chi}(a)\, e\Big(\frac{an}{q}\Big) = \frac{1}{d(\overline{\chi})} \sum_{0<|a|<q/2} \overline{\chi}(a)\, e\Big(\frac{an}{q}\Big),$$

where $d(\overline{\chi})$ is the Gaussian sum. Summing over $1 \le n \le N$, we obtain

$$\sum_{n=1}^{N} \chi(n) = \frac{1}{d(\overline{\chi})} \sum_{0<|a|<q/2} \overline{\chi}(a) \sum_{n=1}^{N} e\Big(\frac{an}{q}\Big) = \frac{1}{d(\overline{\chi})} \sum_{0<|a|<q/2} \overline{\chi}(a)\, \frac{e\big(\frac{aN}{q}\big)-1}{1-e\big(\frac{-a}{q}\big)}.$$

Using $0 < |a| < q/2$, it is easy to see that

$$\frac{1}{1-e\left(\frac{-a}{q}\right)} = \frac{q}{2\pi i a} - \frac{\sum_{j=2}^{\infty}\frac{(-\frac{2\pi i a}{q})^{j-2}}{j!}}{-\frac{q}{2\pi i a}\left(e\left(\frac{-a}{q}\right)-1\right)},$$

and

$$\left|\sum_{j=2}^{\infty}\frac{\left(-\frac{2\pi i a}{q}\right)^{j-2}}{j!}\right| \le \sum_{j=2}^{\infty}\frac{\pi^{j-2}}{j!} = \frac{e^{\pi}-1-\pi}{\pi^2}.$$

Furthermore, with $x = 2\pi a/q$, we observe that

$$\left|-\frac{q}{2\pi i a}\left(e\left(\frac{-a}{q}\right)-1\right)\right| = \frac{1}{|x|}\sqrt{(\cos x - 1)^2 + (\sin x)^2},$$

and considering that the derivative of the right hand side is negative for $|x| \le \pi$, we obtain

$$\left|-\frac{q}{2\pi i a}\left(e\left(\frac{-a}{q}\right)-1\right)\right| > \frac{2}{\pi}.$$

Also, using that for primitive characters $|d(\overline{\chi})| = \sqrt{q}$, it follows that

$$\sum_{n=1}^{N}\chi(n) \le \frac{\sqrt{q}}{2\pi}\left|\sum_{0<|a|<q/2}\frac{\overline{\chi(a)}\left(e(\frac{aN}{q})-1\right)}{a}\right| + \frac{(e^{\pi}-1-\pi)}{2\pi}\sqrt{q}.$$

Now we split the inner sum in two parts: Σ_1 with $0 < |a| \le q_1 = q^{3/8+\varepsilon}$ and Σ_2 with $q_1 < |a| < q/2$.

By partial summation and Lemma 1.4, we have

$$|\Sigma_2| \le 2c(\chi)\left(\frac{1}{\log q} + \frac{5}{8} - \varepsilon\right)c(E,q,\gamma,\varepsilon) + 1.$$

$$\Sigma_1 = \begin{cases} 2i\displaystyle\sum_{1\le a\le q_1}\frac{\overline{\chi(a)}\sin(\frac{2\pi a N}{q})}{a} & \text{if } \chi(-1) = 1,\\ -2\displaystyle\sum_{1\le a\le q_1}\frac{\overline{\chi(a)}\left(1-\cos(\frac{2\pi a N}{q})\right)}{a} & \text{if } \chi(-1) = -1,\end{cases}$$

and from Lemma 4.1,

$$|\Sigma_1| \le \begin{cases} 2\left(\dfrac{2}{\pi}\log q_1 + \dfrac{2}{\pi}\left(C + \log 2 + \dfrac{3}{q_1}\right)\right) & \text{if } \chi(-1) = 1,\\ 2\left(\log q_1 + C + \log 2 + \dfrac{3}{q_1}\right) & \text{if } \chi(-1) = -1.\end{cases}$$

And thus we obtain the desired result.

5. Proof of Theorem 1.1

The aim of this section is to obtain a completely explicit and concise version of Theorem 1.5, thus to prove Theorem 1.1 and Tables 1 and 2, and to prove a version of Table 2 for all q such that $d(q) = U$, with U a fixed constant.

To prove Theorem 1.1, we need to optimize Theorem 1.5 in the variables ε, q, E and γ, and in doing so we aim to minimize ε and q, and at the same time $n(q, \varepsilon)$ and $m(E, q, \gamma, \varepsilon)$. We will now start introducing some bounds on these variables and make some useful comments:

- Choosing γ and a lower bound on q we must ensure that $p > (100(\log q)^{\gamma})^4$.
- To minimize the second term of $c(E, q, \gamma, \varepsilon)$, we need to choose γ in such a way that $(\log q)^2(\log\log q)^3 \leq (\log q)^{\gamma}$
- Confronting Theorem 1.5 with equation (1.1), we will assume $\varepsilon < 1/8$.
- The above point and the definition of $c(E, q, \gamma, \varepsilon)$ implies that

$$\frac{1}{16} > \frac{3\log 2}{2\log\log q}\Big(1 + \frac{1}{\log\log q} + \frac{4.7626}{(\log\log q)^2}\Big),$$

  which implies $q \geq e^{e^{17.82}}$.
- It is interesting to note that for any $\gamma > 2$ we have, for $q \to \infty$, that

$$c(E, q, \gamma, \varepsilon) \longrightarrow \frac{8}{3} c_1(1, E, (\log q)^{\gamma}).$$

- The above function quickly stabilises on the limit.
- Increasing γ reduces the left-hand term of $c(E, q, \gamma, \varepsilon)$ and increases the right-hand term.
- Choosing a small E appears to be optimal.

From Theorem 1.5 and the above observations, Tables 1 and 2 follow by computation. The optimization problem results, in this case, in a simple solution as we are forced to take q big to have $h_{1,2}(E, q, \gamma, \varepsilon)$ small enough, over this range of q the optimal γ is constant. We obtain that $\gamma = E = 4$ are optimal.

We will now prove a version of Table 2 for all q such that $d(q) = U$, with U a fixed constant. It is easy to see, by Theorem 1.3 and the proof of Lemma 1.4, that in this case Theorem 1.5 holds but with $d(q) = U$ instead of the general upper bound due to Robin. We will focus on the case where q is a prime, thus $U = 2$, and choose a small ε with the aim of minimizing q, while keeping the constant limited. The optimization problem is harder in this case as q can be taken relatively small, thus, after choosing a lower bound for q, we would have to optimize γ for each medium sized q. This means that for each medium sized q we would need to find the γ that minimizes the result and then take the maximum between all of them. To ease this problem, we can balance q and $h_{1,2}$ to ensure that we improve on [7] in the chosen range of q; this leads to two different ranges of q for h_1 and h_2, and this will give us a q big enough to make the optimization problem simpler. We thus obtain Table 3.

ε	$\log\log q_0$	$h_1(q,\varepsilon) \leq$	$\log\log q_0$	$h_2(q,\varepsilon) \leq$
$\frac{1}{8}(1-\frac{1}{10})$	13.4	1579	13.6	3153
$\frac{1}{8}(1-\frac{1}{100})$	15.6	1510	15.9	3015
$\frac{1}{8}(1-\frac{1}{1000})$	17.9	1503	18.2	3001
$\frac{1}{8}(1-\frac{1}{10000})$	20.3	1502	20.5	3000

Table 3. q prime.

It is interesting to note that in the above case, even if $h_{1,2}$ are the same as in Table 2, we have lower bounds on q that are significantly smaller compared to the case in which q is a highly composite number; it is the size of $h_{1,2}$ that forces q to be big to do better than [7]. Thus, an improvement on Corollary 1.2 would lead to an important improvement on the size of q.

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