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Non-symplectic automorphisms of K3 surfaces with one-dimensional moduli space

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Abstract. The moduli space of K3 surfaces X with a purely non-symplectic automorphism σ of order $n \geq 2$ is one dimensional exactly when $\varphi(n) = 8$ or 10. In this paper we classify and give explicit equations for the very general members (X, σ) of the irreducible components of maximal dimension of such moduli spaces. In particular, we show that there is a unique one-dimensional component for $n = 20, 22, 24$, three irreducible components for $n = 15$ and two components in the remaining cases.

1. Introduction

An automorphism σ of finite order $n \geq 2$ of a complex K3 surface X is purely non-symplectic if $\sigma^*(\omega_X) = \zeta_n \omega_X$, where ω_X is a nowhere vanishing holomorphic 2-form of X and ζ_n is a primitive nth root of unity. By [18], Main Theorem 3, there exists one such pair (X, σ) if and only if n belongs to the set $\mathrm{TV}_{\mathrm{K3}} = \{n \in \mathbb{N} - \{60\} \,|\, \varphi(n) \leq 20\}$.

The structure of the moduli space of such K3 surfaces can be described by means of the global Torelli theorem and the surjectivity theorem for periods of K3 surfaces (see §11 in [13]). In particular, it is known that an irreducible component of the moduli space of pairs (X, σ) for $n \geq 3$ is an arithmetic quotient of a Zariski open subset of a complex ball of dimension $\dim(V^\sigma) - 1$, where V^σ is the ζ_n-eigenspace of σ^* in $H^2(X, \mathbb{C})$.

In this paper we consider the orders n such that the moduli space of K3 surfaces carrying a purely non-symplectic automorphism of order n is one dimensional. We show that the orders n with such property, as expected, are exactly those $n \in \mathrm{TV}_{\mathrm{K3}}$ with $\varphi(n) = 8$ or 10, i.e., 11, 15, 16, 20, 22, 24 and 30 (see [18]). For all these values of n, we classify pairs (X, σ) such that $\dim(V^\sigma) = 2$, i.e., we identify the fixed locus of σ and of its powers, determine the dimensions of the eigenspaces of σ^* in $H^2(X, \mathbb{C})$, and compute the Néron–Severi lattice of a very general pair. The orders $n = 11$ and $n = 16$ had been previously studied in [6, 22] and [2] respectively. We collect these results in the following theorem.

Theorem 1.1. *Let X be a complex K3 surface with a purely non-symplectic automorphism σ of order $n \geq 2$ such that $\varphi(n) = 8$ or 10 and $\dim(V^\sigma) = 2$. Then Table 1 provides all possible values for the vector d describing the dimensions of the eigenspaces of σ^**

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in $H^2(X,\mathbb{C})$, the topological invariants describing the fixed locus of powers of σ (see Section 2 for the notation) and the Néron–Severi lattice of a very general K3 surface in each case. Moreover, all cases in the table exist.

	n	d	i	g_i	k_i	N_i	NS
11a	11	$(2,2)$	11	1	0	2	U
11b	11	$(2,2)$	11	-	-	2	$U(11)$
22	22	$(2,0,0,2)$	22	-	-	6	U
			11	1	0	2	
			2	10	1	0	
15a	15	$(2,1,0,2)$	15	-	-	5	$U(3)\oplus A_2\oplus A_2$
			5	2	0	1	
			3	2	0	2	
15b	15	$(2,0,1,4)$	15	-	-	7	$H_5\oplus A_4$
			5	1	0	4	
			3	4	1	1	
15c	15	$(2,0,2,2)$	15	-	-	4	$H_5\oplus A_4$
			5	1	0	4	
			3	4	0	0	
30a	30	$(2,0,1,0,0,0,1,1)$	30	-	-	1	$U(3)\oplus A_2\oplus A_2$
			15	-	-	5	
			5	2	0	1	
			3	2	0	2	
			2	10	0	0	
30b	30	$(2,0,0,1,0,0,1,3)$	30	-	-	3	$H_5\oplus A_4$
			15	-	-	7	
			5	1	0	4	
			3	4	1	1	
			2	9	1	0	
16a	16	$(2,0,0,0,6)$	16	0	0	6	$U\oplus D_4$
			8	0	0	6	
			4	0	0	6	
			2	7	2	0	
16b	16	$(2,0,0,2,4)$	16	-	-	4	$U(2)\oplus D_4$
			8	0	0	6	
			4	0	0	6	
			2	6	1	0	
20	20	$(2,0,1,0,0,2)$	20	-	-	3	$U(2)\oplus D_4$
			10	-	-	7	
			5	2	0	1	
			4	0	0	6	
			2	6	1	0	
24	24	$(2,0,0,0,0,1,0,4)$	24	-	-	5	$U\oplus D_4$
			12	-	-	5	
			6	0	0	11	
			3	4	1	1	
			2	7	2	0	

Table 1. Non-symplectic automorphisms with $\varphi(n)=8,10$.

This classification allows us to prove the following result, which provides explicit birational models for a very general pair (X,σ) under the previous conditions (see Remark 2.3 about the generality assumption in the statement).

Theorem 1.2. *Let X be a very general complex K3 surface with a purely non-symplectic automorphism σ of order $n \geq 2$ such that $\varphi(n) = 8$ or 10 and $\dim(V^\sigma) = 2$, Then up to a birational isomorphism, (X, σ) belongs to the families described in Table 2, where $a \in \mathbb{C}$ is a parameter, ζ_n denotes a primitive nth root of unity and $(*)$ means: minimal resolution of a degree 11 covering of a principal homogeneous space of order 11 of the rational elliptic surface $y^2 = x^3 + x + t$ (see Example 3.3).*

n	X	σ
11	(a) $y^2 = x^3 + ax + (t^{11} - 1)$ (b) $(*)$	$(x, y, \zeta_{11} t)$
15	(a) $y^2 = x^3 + (t^5 - 1)(t^5 - a)$ (b) $y^2 = x_0^6 + x_0 x_1^5 + x_2^6 + a x_0^3 x_2^3$ (c) $y^3 = x_0^5 x_1 + x_1^2 x_2^2 + x_1^4 x_2 + a x_1^6$	$(\zeta_3 x, y, \zeta_5 t)$ $(x_0, \zeta_5 x_1, \zeta_3 x_2, y)$ $(\zeta_5 x_0, x_1, x_2, \zeta_3 y)$
16	(a) $y^2 = x^3 + t^2 x + a t^3 (t^8 + 1)$ (b) $y^2 = x_0(x_0^4 x_2 + x_1^5 + x_1 x_2^4 + a x_1^3 x_2^2)$	$(\zeta_{16}^2 x, \zeta_{16}^3 y, \zeta_{16}^2 t)$ $(x_0, \zeta_8^7 x_1, \zeta_8^3 x_2, \zeta_{16}^3 y)$
20	$y^2 = x_0(x_1^5 + x_2^5 + x_0^2 x_2^3 + a x_0^4 x_2)$	$(-x_0, \zeta_5 x_1, x_2, iy)$
22	$y^2 = x^3 + ax + (t^{11} - 1)$	$(x, -y, \zeta_{11} t)$
24	$y^2 = x^3 + t(t^4 - 1)(t^4 - a)$	$(\zeta_{12} x, \zeta_8 y, it)$
30	(a) $y^2 = x^3 + (t^5 - 1)(t^5 - a)$ (b) $y^2 = x_0^6 + x_0 x_1^5 + x_2^6 + a x_0^3 x_2^3$	$(\zeta_3 x, -y, \zeta_5 t)$ $(x_0, \zeta_5 x_1, \zeta_3 x_2, -y)$

Table 2. One dimensional families of $K3$ surfaces with non-symplectic automorphisms.

Corollary 1.3. *The moduli space of K3 surfaces carrying a purely non-symplectic automorphism of order n has a unique one-dimensional component for $n = 20, 22, 24$, three irreducible components for $n = 15$ and two irreducible components for $n = 11, 16, 30$.*

For orders $22, 15, 30$ and 20, we actually prove a stronger version of Theorem 1.2, since we provide projective models without assuming X to be very general.

Finally, in case $n = 22$ and $n = 15$, we classify purely non-symplectic automorphisms of order n, that is, we provide the same type of information contained in Table 1, without assuming $\dim(V^\sigma) = 2$, see Theorem 4.2 and Theorem 5.1.

The structure of the paper is the following. In Section 2 we give preliminaries on non-symplectic automorphisms of K3 surfaces and we fix the corresponding notation: fixed loci, invariant lattices and eigenspaces in cohomology, moduli spaces. In Section 3, for each order $n \in \{11, 22, 15, 30, 16, 20, 24\}$, we prove Theorem 1.1 (see Theorem 3.1 and Propositions 3.4, 3.6, 3.11, 3.14, 3.19, 3.24) and Theorem 1.2. In Sections 4 and 5, we prove Theorem 4.2 and Theorem 5.1, respectively.

2. Background and preliminary results

We will work over the complex numbers and we will denote by ζ_i a primitive ith root of unity. Let X be a K3 surface over $\mathbb{C}$ and let σ be a purely non-symplectic automorphism of X of order $n \geq 3$, i.e., $\sigma^*(\omega_X) = \zeta_n \omega_X$, where ω_X is a generator of the complex vector space $H^{2,0}(X)$.

In what follows, we will denote by σ_k an element of $\langle\sigma\rangle$ whose order is k.

2.1. Fixed locus

We start describing the fixed locus of σ. The local action of σ in a neighborhood of one of its fixed points can be linearized and can be described by a matrix of the form

$$A_{i,n} = \begin{pmatrix} \zeta_n^{i+1} & 0 \\ 0 & \zeta_n^{n-i} \end{pmatrix}, \quad i = 0, 1, 2, \ldots, \left\lfloor \frac{n-1}{2} \right\rfloor,$$

see §5 in [20]. When $i = 0$, the fixed point belongs to a fixed curve, otherwise it is an isolated fixed point. This description implies that the fixed locus of σ is the union of isolated points and disjoint smooth curves. Moreover, by the Hodge index theorem, the fixed locus contains at most one curve of genus $g \geq 2$. In what follows we will use the following notation for the fixed locus of σ:

$$\mathrm{Fix}(\sigma) = C_g \sqcup R_1 \sqcup \cdots \sqcup R_k \sqcup \{p_1, \ldots, p_N\},$$

where C_g is a smooth curve of genus g, $R_1, \ldots, R_k$ are smooth rational curves, and $p_1, \ldots, p_N$ are isolated fixed points. The fixed points such that the local action is given by the matrix $A_{i,n}$ will be called points of type $A_{i,n}$, and the number of such points will be denoted by $a_{i,n}$.

We now recall the *holomorphic Lefschetz formula* [7], which relates these numbers with the action of σ^* on the cohomology groups $H^j(X, \mathcal{O}_X)$:

$$\sum_{j=0}^{2} \mathrm{tr}\left(\sigma^*_{|H^j(X,\mathcal{O}_X)}\right) = \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \frac{a_{i,n}}{(1-\zeta_n^{i+1})(1-\zeta_n^{n-i})} + \alpha \frac{1+\zeta_n}{(1-\zeta_n)^2},$$

where $\alpha := \sum_{C \subset \mathrm{Fix}(\sigma)} (1 - g(C))$. Observe that

$$H^1(X, \mathcal{O}_X) = H^3(X, \mathcal{O}_X) = \{0\}$$

since X is a K3 surface, $\sigma^* = \mathrm{id}$ on $H^0(X, \mathcal{O}_X) \cong \mathbb{C}$ and σ^* acts as multiplication by $\bar{\zeta}_n$ on $H^2(X, \mathcal{O}_X) \cong H^{0,2}(X) = \mathbb{C}\bar{\omega}_X$. Thus the left-hand side of the formula is equal to $1 + \bar{\zeta}_n$.

Finally, we recall *Hurwitz formula* for a ramified covering $f\colon X \to Y$ of degree d between smooth complex projective varieties, which will be used several times in the paper for both curves and surfaces:

$$K_X \sim f^* K_Y + \sum_i (e_i - 1) C_i,$$

where the C_i's are the irreducible components of the ramification locus and e_i is the associated ramification index (see for example Sections 16 and 17 of [8]).

2.2. Eigenspaces and invariant lattices

We now consider the action of σ^* in $H^2(X,\mathbb{Z})$ and $H^2(X,\mathbb{C})$. We will denote by $S(\sigma^i)\subset H^2(X,\mathbb{Z})$ the invariant lattice of σ^i for $i=0,\dots,n-1$. Moreover, for any divisor k of n, let

$$H^2(X,\mathbb{C})_k^\sigma := \{x\in H^2(X,\mathbb{C}) : \sigma^* x = \zeta_k x\}$$

and let d_k be its dimension. In particular, d_1 is the rank of $S(\sigma)$ and d_n is the dimension of $V^\sigma = H^2(X,\mathbb{C})_n^\sigma$. In what follows, we will denote by d the vector whose entries are the numbers d_k, as k varies in the set of divisors of n in decreasing order:

$$d=(d_n,\dots,d_k,\dots,d_1),\quad k|n.$$

Remark 2.1. Observe that, since σ is purely non symplectic, then $S(\sigma^i)$ is contained in the Néron–Severi lattice of X for any $i=0,\dots,n-1$. In fact, given $x\in S(\sigma^i)$ we have

$$(x,\omega_X)=((\sigma^i)^*x,(\sigma^i)^*\omega_X)=(x,\zeta_n^i\omega_X)=\zeta_n^i(x,\omega_X),$$

which implies $(x,\omega_X)=0$ and thus $x\in H^2(X,\mathbb{Z})\cap\omega_X^\perp=\mathrm{NS}(X)$.

We also recall the *topological Lefschetz formula* ([7], Theorem 4.6). For simplicity, we state it only for σ:

$$\chi(\mathrm{Fix}(\sigma))=\sum_{i=0}^{4}(-1)^i\,\mathrm{tr}\left(\sigma^*|_{H^i(X,\mathbb{R})}\right),$$

where the right side is equal to $2+\mathrm{tr}(\sigma^*|_{H^2(X,\mathbb{R})})$ since $H^i(X,\mathbb{R})=\{0\}$ for $i=1,3$ and $\sigma^*=\mathrm{id}$ on $H^i(X,\mathbb{R})$ for $i=0,4$.

Finally, we recall some notation for lattices which will appear in the paper: A_ℓ ($\ell\geq 1$), D_m ($m\geq 4$) and E_n ($n=6,7,8$) denote the negative definite even lattices associated to the Dynkin diagrams of the corresponding types, U and H_5 denote the lattices with the following Gram matrices:

$$U=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad H_5=\begin{pmatrix}2&1\\1&-2\end{pmatrix},$$

and $U(r)$ with $r\geq 2$ denotes the lattice whose Gram matrix is that of U multiplied by r.

2.3. Moduli spaces

Let X be a K3 surface with an order n automorphism σ such that $\sigma^*(\omega_X)=\zeta_n\omega_X$. The period line $\mathbb{C}\omega_X$ belongs to the domain

$$\mathcal{D}^\sigma=\{\mathbb{C}z\in\mathbb{P}(V^\sigma):(z,\bar{z})>0,(z,z)=0\},$$

where V^σ is the ζ_n-eigenspace of σ^* in $H^2(X,\mathbb{C})$. Observe that, for $n\geq 3$, we have

$$(z,z)=(\sigma^*z,\sigma^*z)=\zeta_n^2(z,z),$$

thus the condition $(z,z)=0$ is not necessary and $\mathcal{D}^\sigma$ can be easily proved to be isomorphic to a complex ball. On the other hand, if $n=2$, then $\mathcal{D}^\sigma$ is a type IV Hermitian symmetric space. By Theorem 11.3 in [13], an arithmetic quotient of a Zariski open subset of $\mathcal{D}^\sigma$ parametrizes isomorphism classes of (ρ,M)-polarized K3 surfaces, where $\rho\colon C_n \to \mathrm{O}(L_{K3})$ is a representation induced by the isometry σ^* of $H^2(X,\mathbb{C})$ and the choice of an isometry $H^2(X,\mathbb{Z}) \to L_{K3}$, and $M \subseteq L_{K3}$ is the invariant lattice of $\mathrm{Im}(\rho)$. In particular, such moduli space has dimension $\dim(\mathcal{D}^\sigma) = \dim(V^\sigma) - 1$ if $n \geq 3$ and $\dim(V^\sigma) - 2$ if $n = 2$.

On the other hand, if T_X is the transcendental lattice of X, it is known ([20], Section 3) that the eigenvalues of σ^* in $T_X \otimes_{\mathbb{Z}} \mathbb{C}$ are the primitive nth roots of unity, thus $\mathrm{rank}(T_X) = \dim(V^\sigma)\varphi(n)$. Since $\mathrm{rank}(T_X) \leq 21$, this implies that

$$\dim(\mathcal{D}^\sigma) \leq \gamma(n) := \left\lfloor \frac{21}{\varphi(n)} \right\rfloor - 1.$$

In particular, the dimension of $\mathcal{D}^\sigma$ is at most one if $\gamma(n) = 1$. We show that the converse also holds.

Lemma 2.2. *Let $n \neq 60$ be a positive integer with $\varphi(n) \leq 20$ and $\gamma(n) > 1$. Then there exist a K3 surface X and a purely non-symplectic automorphism σ of X of order n such that* $\dim(\mathcal{D}^\sigma) > 1$.

Proof. We will denote by $d(n)$ the dimension of the moduli space of K3 surface carrying a purely non-symplectic automorphism of order n. The orders $n \geq 2$ with $\varphi(n) \leq 20$ and $\gamma(n) > 1$ are $n = 7, 9, 14, 18$ with $\gamma(n) = 2$, $n = 5, 8, 10, 12$ with $\gamma(n) = 4$, $n = 3, 4, 6$ with $\gamma(n) = 9$, and $n = 2$.

For prime orders $n = 3, 5, 7$, it is known by [6] that $d(n) = \gamma(n)$. Moreover, the same is true for orders $n = 6, 10, 14$ by Proposition 2.4 and [6].

For order $n = 9$, it is known by [3] that $d(n) = 2$. Moreover, the general member of one of its components of maximal dimension is an elliptic K3 surface with Weierstrass equation

$$y^2 = x^3 + t(t^3 - a)(t^3 - b)(t^3 - c), \quad a, b, c \in \mathbb{C},$$

which carries the order nine automorphism $\sigma(x, y, t) = (\zeta_9^4 x, \zeta_9^6 y, \zeta_3 t)$. This surface also admits the non-symplectic involution $\tau(x, y, t) = (x, -y, t)$, which commutes with σ, so it carries the non-symplectic automorphism $\sigma\tau$ of order 18. This shows that $d(18) = 2$ as well.

When $n = 4$, Example 6.3 in [5] is a 9-dimensional family of K3 surfaces with a purely non-symplectic automorphism of order 4.

When $n = 8$, Example 4.1 in [1] is a 2-dimensional family of K3 surfaces with a purely non-symplectic automorphism of order 8.

When $n = 12$, the family of elliptic K3 surfaces defined by the Weierstrass equation

$$y^2 = x^3 + t\prod_{i=1}^{5}(t^2 - a_i), \quad a_i \in \mathbb{C},$$

is 4-dimensional and has an order 12 automorphism, $\sigma(x, y, t) = (-\zeta_3 x, iy, -t)$, which can be easily checked to be purely non-symplectic.

When $n = 2$, it is well known that $d(2) = 19$ and there is a unique component of maximal dimension whose general element is a double cover of $\mathbb{P}^2$ branched along a smooth plane sextic. ∎

Remark 2.3. Under the hypotheses of Theorem 1.2, since T_X has the structure of a $\mathbb{Z}[\zeta_n]$-module by [20], Section 3, and $\dim(V^\sigma) = 2$, we have that $\operatorname{rk}\mathrm{NS}(\mathrm{X}) \geq 22 - 2\varphi(n)$. The generality assumption in the statement of the theorem means that the Néron–Severi lattice of X has the minimal rank.

Finally, we recall a result contained in Theorems 1.4 and 1.5 of [15], and in [12].

Proposition 2.4. *Let X be a K3 surface with a non-symplectic automorphism σ of order n. If either*

(i) $n = 5, 13, 17, 19$,

(ii) *or* $n = 7, 11$ *and the fixed locus of* σ *contains a curve,*

(iii) *or* $n = 3$ *and the the fixed locus of* σ *contains at least two curves,*

(iv) *or* $n = 3$ *and the the fixed locus of* σ *contains a curve and two points,*

then X admits a non-symplectic automorphism τ of order $2n$ with $\tau^2 = \sigma$.

Moreover, if $n = 11$ and the fixed locus of σ consists of only isolated fixed points, then X does not admit a non-symplectic automorphism τ of order 22 with $\tau^2 = \sigma$.

3. Proof of Theorem 1.1 and Theorem 1.2

In this section we prove the two main theorems for each order.

3.1. Order 11

Non-symplectic automorphisms of order 11 have been classified in [22] and [6], Section 7. In particular, the proof of Theorem 1.2 for order 11 follows from the following result.

Theorem 3.1. *Let X be a K3 surface with a non-symplectic automorphism σ of order 11 such that rank $S(\sigma) = 2$ (or equivalently $\dim(V^\sigma) = 2$). Then two cases can occur:*

(a) $\mathrm{Fix}(\sigma) = C_1 \sqcup \{p_1, p_2\}$ *and* $S(\sigma) = \mathrm{NS}(X) \cong U$,

(b) $\mathrm{Fix}(\sigma) = \{p_1, p_2\}$ *and* $S(\sigma) = \mathrm{NS}(X) \cong U(11)$,

where C_1 is a smooth curve of genus one. In both cases, $d = (2, 2)$. Moreover, up to birational isomorphisms, (X, σ) belongs to the family in Example 3.2 in case (a), *and to the family in Example 3.3 in case* (b).

Example 3.2. Given $a \in \mathbb{C}$, let X_{11a} be the elliptic fibration with Weierstrass equation

$$y^2 = x^3 + ax + (t^{11} - 1).$$

For general $a \in \mathbb{C}$, the fibration has one fiber of Kodaira type II over $t = \infty$ and twenty-two fibers of type I_1. Observe that X_{11a} carries the order 11 automorphism

$$\sigma_{11a}(x, y, t) = (x, y, \zeta_{11} t),$$

which fixes the smooth fiber over $t = 0$ and two points in the fiber over $t = \infty$.

Example 3.3. Consider the extremal rational elliptic surface $\phi\colon Y \to \mathbb{P}^1$ with Weierstrass equation

$$y^2 = x^3 + x + t.$$

The fibration has a fiber of type II* over $t = \infty$ and two fibers of type I_1 over the zeroes of $\Delta = 4 + 27t^2$, thus it is extremal. Given $\alpha \in \mathbb{P}^1$ such that $\phi^{-1}(\alpha)$ is smooth, let $\phi_{\alpha,e}\colon Y_{\alpha,e} \to \mathbb{P}^1$ be the principal homogeneous space of ϕ associated to a non-trivial 11-torsion element e in $\phi^{-1}(\alpha)$. We recall that $\phi_{\alpha,e}$ has the same configuration of singular fibers as ϕ and it has a fiber $F = 11F_0$ of multiplicity 11 over α such that $(F_0)_{|F_0} = e \in \mathrm{Pic}^0(F_0)$ (see [11], §4, Chapter V, as a reference for principal homogeneous spaces of rational Jacobian elliptic fibrations). Let $\psi_{\alpha,e}\colon Z_{\alpha,e} \to \mathbb{P}^1$ be the degree 11 base change of $\phi_{\alpha,e}$ branched along $t = \infty$ and $t = \alpha$. A minimal resolution of $Z_{\alpha,e}$ is a K3 surface $X_{\alpha,e}$ carrying an elliptic fibration $\pi_{\alpha,e}$ induced by $\psi_{\alpha,e}$ which has twenty-two fibers of type I_1 over the two fibers of type I_1 of $\psi_{\alpha,e}$ and a fiber of type II over $t = \infty$. The covering automorphism of $Z_{\alpha,e} \to Y_{\alpha,e}$ induces an order 11 automorphism σ_{11b} of $X_{\alpha,e}$:

$$\begin{array}{ccccc}
X_{\alpha,e} & \longrightarrow & Z_{\alpha,e} & \longrightarrow & Y_{\alpha,e} \\
\downarrow{\scriptstyle \pi_{\alpha,e}} & & \downarrow{\scriptstyle \psi_{\alpha,e}} & & \downarrow{\scriptstyle \phi_{\alpha,e}} \\
\mathbb{P}^1 & \longrightarrow & \mathbb{P}^1 & \xrightarrow{11:1} & \mathbb{P}^1.
\end{array}$$

We will denote by (X_{11b}, σ_{11b}) the family of K3 surfaces with automorphism obtained with this construction. The automorphism σ_{11b} fixes exactly two points in the fiber of $\pi_{\alpha,e}$ of type II.

3.2. Order 22

In this section we will give the classification of purely non-symplectic automorphisms of order 22 with $\dim(V^\sigma) = 2$. The full classification, including the cases with $\dim(V^\sigma) = 1$, will be given in Section 4.

Proposition 3.4. *Let X be a K3 surface with a purely non-symplectic automorphism σ of order 22 such that $\dim(V^\sigma) = 2$. Then the fixed loci of $\sigma = \sigma_{22}$ and of its powers $\sigma_{11} = \sigma^2$ and $\sigma_2 = \sigma^{11}$ are as follows:*

$\mathrm{Fix}(\sigma_{22})$	$\mathrm{Fix}(\sigma_{11})$	$\mathrm{Fix}(\sigma_2)$
$\{p_1, \dots, p_6\}$	$C_1 \sqcup \{p_5, p_6\}$	$C_{10} \sqcup R$

where $g(C_1) = 1,\ g(C_{10}) = 10$ and $g(R) = 0$. Moreover, $d = (2,0,0,2)$ and $\mathrm{NS}(X) \cong U$ for a very general K3 surface with such property.

Proof. Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of σ^* we obtain, with the notation in Section 2,

$$\dim H^2(X, \mathbb{C}) = 22 = 10d_{22} + 10d_{11} + d_2 + d_1 = 20 + 10d_{11} + d_2 + d_1.$$

Since $d_{22} = 2$, thus $d_1 + d_2 = 2$ and $d_{11} = 0$, so either $d = (2, 0, 1, 1)$ or $(2, 0, 0, 2)$. Let $\chi_i := \chi(\mathrm{Fix}(\sigma_i))$, $i \in \{2, 11, 22\}$. By the topological Lefschetz formulas, we have

$$\begin{cases} \chi_{22} = d_{22} - d_{11} - d_2 + d_1 + 2, \\ \chi_{11} = -d_{22} - d_{11} + d_2 + d_1 + 2, \\ \chi_2 \ = -10 d_{22} + 10 d_{11} - d_2 + d_1 + 2. \end{cases} \tag{3.1}$$

This implies $\chi_{11} = 2$. By Proposition 2.4, if a K3 surface admits a non-symplectic automorphism of order 11 without fixed curves, it does not admit a non-symplectic automorphism of order 22. This result and Theorem 3.1 imply that $\mathrm{Fix}(\sigma_{11})$ is the union of a smooth genus 1 curve C and two points p, q. On the other hand, the same equations give that $\chi_{22} = 4$ if $d = (2, 0, 1, 1)$ and $\chi_{22} = 6$ if $d = (2, 0, 0, 2)$. This implies that σ_{22} is not the identity on C, thus it acts on it as an involution with four fixed points, and it either exchanges or fixes p and q.

We will now show that σ_{22} must fix p and q, i.e., that $\chi_{22} = 6$. Observe that the fixed points of σ_{22} on C are of type $A_{10,22}$ since they are contained in a fixed curve of σ_{22}^2. If these were the only fixed points of σ_{22}, an easy computation shows that the holomorphic Lefschetz formula does not hold, giving a contradiction.

Finally, $\chi_2 = -16$. By [20], this implies that the fixed locus of σ_2 is either a genus 9 curve or the union of a genus 10 curve and a rational curve. The first case is not possible since a curve of genus 9 has no order 11 automorphisms by the Riemann–Hurwitz formula.

Observe that for a very general $K3$ surface as in the statement, $\mathrm{rk}\,\mathrm{NS}(X) = 22 - 2\varphi(22) = 2$ (see Remark 2.3) and $S(\sigma_{11}) \subseteq \mathrm{NS}(X)$ by Remark 2.1, thus $\mathrm{NS}(X) = S(\sigma_{11}) \cong U$ by Theorem 3.1. ∎

Example 3.5. The elliptic K3 surface in Example 3.2,

$$y^2 = x^3 + ax + (t^{11} - 1), \quad a \in \mathbb{C},$$

admits the order 22 automorphism

$$\sigma_{22}(x, y, t) = (x, -y, \zeta_{11} t),$$

which fixes four points in the smooth fiber over $t = 0$ and two points in the fiber of type II over $t = \infty$. The involution $\sigma_2 = \sigma_{22}^{11}$ fixes the curve $y = 0$, which has genus 10, and the sections at infinity. Since σ_2 has fixed curves and since there exist no symplectic automorphism of a K3 surface of order 11 [20], then σ is purely non-symplectic.

Proof of Theorem 1.2, order 22. Let X be a K3 surface with a purely non-symplectic automorphism $\sigma = \sigma_{22}$ of order 22. By Proposition 3.4, $\mathrm{Fix}(\sigma_{11})$ contains an elliptic curve C_1 and two points. Thus, by Theorem 3.1, (X, σ_{11}) belongs to the family in Example 3.2 up to isomorphism, i.e., it carries an elliptic fibration $\pi \colon X \to \mathbb{P}^1$ with Weierstrass equation

$$y^2 = x^3 + ax + (t^{11} - 1), \quad a \in \mathbb{C},$$

and $\sigma_{11}(x, y, t) = (x, y, \zeta_{11} t)$. The lattice generated by the class of a fiber and the class of a section of π is isometric to the lattice U and is fixed by the automorphism σ_{11}^*, thus it coincides with $S(\sigma_{11})$ by Theorem 3.1. Since σ_{22}^* preserves the lattice $S(\sigma_{11})$ and this contains a unique class of elliptic fibration and a unique class of smooth rational curve, then σ_{22}^* preserves both. By Proposition 3.4, the fixed locus of the involution σ_2 is the disjoint union of a smooth curve C_{10} of genus 10 and a smooth rational curve R. The curve C_{10} is clearly transverse to the fibers of π, thus each fiber of π contains fixed points of σ_2. This implies that the action induced by σ_2 on $\mathbb{P}^1$ is the identity, i.e., each fiber of π is preserved by σ_2. Moreover, the unique section S of π must be pointwise fixed by σ_2, so that $R = S$. Since σ_2 is an involution which preserves each fiber of π and fixes S, then it is defined by $(x, y, t) \mapsto (x, -y, t)$. This shows that the action of $\sigma_{22} = \sigma_{11} \circ \sigma_2$ on π is the one described in the statement of Theorem 1.2, concluding the proof. ∎

3.3. Order 15

In this section we will give the classification of purely non-symplectic automorphisms of order 15 with $\dim(V^\sigma) = 2$. The full classification, including the cases with $\dim(V^\sigma) = 1$, will be given in Section 5.

Proposition 3.6. *Let X be a K3 surface with a purely non-symplectic automorphism σ of order* 15 *such that* $\dim(V^\sigma) = 2$. *Then the fixed loci of $\sigma = \sigma_{15}$ and its powers $\sigma_i = \sigma^{15/i}$ are as follows*:

	$\mathrm{Fix}(\sigma_{15})$	$\mathrm{Fix}(\sigma_5)$	$\mathrm{Fix}(\sigma_3)$
(a)	$\{p_1, \dots, p_5\}$	$C_2 \sqcup \{p_1\}$	$C_2' \sqcup \{p_2, p_3\}$
(b)	$\{p_1, \dots, p_7\}$	$C_1 \sqcup \{p_1, \dots, p_4\}$	$C_4 \sqcup R \sqcup \{p_1\}$
(c)	$\{p_1, \dots, p_4\}$	$C_1 \sqcup \{p_1, q_1, q_2, q_3\}$	C_4

where $g(C_1) = 1, g(C_2) = g(C_2') = 2, g(C_4) = 4$ and $g(R) = 0$. Moreover, $d = (2,1,0,2)$ in case (a)*, $d = (2,0,1,4)$ in case* (b) *and $d = (2,0,2,2)$ in case* (c)*. Finally,* $\mathrm{NS}(X) \cong U(3) \oplus A_2 \oplus A_2$ *for a very general K3 surface X in case* (a) *and* $\mathrm{NS}(X) \cong H_5 \oplus A_4$ *for a very general K3 surface X in cases* (b) *and* (c)*, where H_5 is the lattice defined in Section* 1 *of* [6].

Proof. Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of σ^* we obtain, with the notation in Section 2,

$$22 = 8d_{15} + 4d_5 + 2d_3 + d_1 = 16 + 4d_5 + 2d_3 + d_1,$$

thus $d \in \{(2,1,0,2), (2,0,2,2), (2,0,1,4), (2,0,0,6)\}$.

Let $\chi_i := \chi(\mathrm{Fix}(\sigma_i))$, $i \in \{3, 5, 15\}$. By the topological Lefschetz fixed point formulas,

$$\begin{cases} \chi_{15} = d_{15} - d_5 - d_3 + d_1 + 2, \\ \chi_5 = -2d_{15} - d_5 + 2d_3 + d_1 + 2, \\ \chi_3 = -4d_{15} + 4d_5 - d_3 + d_1 + 2. \end{cases} \tag{3.2}$$

We will show that $d = (2,1,0,2)$, $d = (2,0,1,4)$ and $d = (2,0,2,2)$ are the only possible cases.

Assume that $d = (2,1,0,2)$. Thus $(\chi_{15}, \chi_5, \chi_3) = (5, -1, 0)$. By [6], we have that $\mathrm{Fix}(\sigma_5)$ is the union of a curve C_2 of genus 2 and one point. Since $\chi_{15} = 5$, the action of σ

on C_2 has order 3 with four fixed points, by the Riemann–Hurwitz formula. In particular, $\mathrm{Fix}(\sigma)$ is the union of five points. Finally, by [4], $\mathrm{Fix}(\sigma_3)$ is either the union of a genus 2 curve and two points, or contains a curve of genus 3. The second case is not possible since there is no genus 3 curve with an order five automorphism by Table 5 in [10].

If $d \neq (2,1,0,2)$, then $\chi_5 = 4$ and $\chi_3 = -6, -3, 0$ if $d = (2,0,2,2), (2,0,1,4)$ or $(2,0,0,6)$, respectively. By [6], $\mathrm{Fix}(\sigma_5)$ is either the union of an elliptic curve C_1 and four points, or the union of four points. Observe that C_1 can not be contained in $\mathrm{Fix}(\sigma_3)$ since by [4] this would imply $\chi_3 \geq 3$. Thus, looking at the possible actions of σ on C_1 and the four points, we find that χ_{15} is either 1, 4 or 7.

If $d = (2,0,0,6)$, then $\chi_{15} = 10$ by (3.2), giving a contradiction.

If $d = (2,0,1,4)$, then $\chi_{15} = 7$ by (3.2). Thus $\mathrm{Fix}(\sigma_5)$ is the union of an elliptic curve C_1 and four points, and $\mathrm{Fix}(\sigma)$ consists of seven points, three of them on C_1. Moreover, $\chi_3 = -3$, thus by [4] $\mathrm{Fix}(\sigma_3)$ is either the union of a genus 4 curve, a rational curve and one point, or it contains a curve of genus 3. The last case is not possible by Table 5 in [10].

If $d = (2,0,2,2)$, then $\chi_{15} = 4$ and $\chi_3 = -6$ by (3.2). Observe that σ has seven types of isolated fixed points. The fixed points of type $A_{1,15}$, $A_{4,15}$, $A_{7,15}$ are isolated fixed points for σ_3 too, while points of type $A_{2,15}$, $A_{3,15}$, $A_{5,15}$, $A_{6,15}$ lie on a curve fixed by σ_3. Observe that σ_{15} acts on the set of isolated fixed points of σ_3 with orbits of length either 1 or 5. Thus we have

$$a_{1,15} + a_{4,15} + a_{7,15} \leq a_{1,3}, \quad a_{1,15} + a_{4,15} + a_{7,15} \equiv a_{1,3} \mod 5.$$

Moreover, points of type $A_{4,15}$, $A_{5,15}$ lie on a curve fixed by σ_5, while points of type $A_{1,15}$, $A_{2,15}$, $A_{3,15}$, $A_{6,15}$, $A_{7,15}$ are isolated fixed points for σ_5 too. Checking types one has

$$a_{1,15} + a_{3,15} + a_{6,15} \leq a_{1,5}, \quad a_{2,15} + a_{7,15} \leq a_{2,5}. \tag{3.3}$$

Since $\chi_3 = -6$, then $a_{1,3} = 0$ by [4]. Applying the holomorphic Lefschetz formula to σ with this condition and using the fact that $\alpha = 0$, we find that $(a_{1,15}, a_{2,15}, \ldots, a_{7,15}) = (0, 1, 0, 0, 3, 0, 0)$. Since $a_{5,15} = 3$, then we find that $\mathrm{Fix}(\sigma_5)$ contains an elliptic curve C_1 and σ fixes three points on it.

For $\mathrm{Fix}(\sigma_3)$ there are two possibilities by [4]: it is either a curve of genus 4, or the union of a genus 5 curve and a rational curve. The second case is excluded by Lemma 3.7, where C' is the elliptic curve C_1.

We now compute the Néron–Severi lattice of a very general X in each case. Observe that since $d_{15} = 2$ and $\varphi(15) = 8$, the Néron–Severi lattice of X has rank $22 - 2 \cdot 8 = 6$. In case (a), the invariant lattice $S(\sigma^5) = S(\sigma_3)$ has rank $d_1 + 4d_5 = 6$, thus $\mathrm{NS}(X) = S(\sigma_3) \cong U(3) \oplus A_2 \oplus A_2$, where the last isomorphism is by [6]. In cases b) and c), the invariant lattice $S(\sigma^3) = S(\sigma_5)$ has rank $d_1 + 2d_3 = 6$, thus we conclude as before that $\mathrm{NS}(X) = S(\sigma_5)$. In both cases, $S(\sigma_5)$ is isomorphic to $H_5 \oplus A_4$ by [6]. ■

Lemma 3.7. *Let X be a K3 surface and let τ be a non-symplectic automorphism of order 3 of X whose fixed locus is the disjoint union of a smooth curve C of genus five and a smooth rational curve R. Then X has no purely non-symplectic automorphism σ of order 15 such that $\sigma^5 = \tau$ and such that the fixed locus of σ^3 contains a curve C' distinct from C and R.*

Proof. Let $\pi: X \to Y$ be the quotient morphism by τ. Since the fixed locus of τ is a smooth curve and the automorphism is non-symplectic of order 3, then Y is a smooth rational surface. Moreover, the invariant lattice $S(\tau)$ has rank 2 by [4]. Since π^* is injective and $\pi^*\mathrm{NS}(Y)$ is a sublattice of $S(\tau)$, then $\operatorname{rank}\mathrm{NS}(Y) \leq 2$. Thus Y is isomorphic to either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_r$, $r \geq 0$. The covering π is branched along a smooth curve B whose class $[B] \in \mathrm{NS}(Y)$ satisfies $-3K_Y = 2[B]$ by the Hurwitz formula. This excludes the case $Y \cong \mathbb{P}^2$. We recall that if $Y = \mathbb{F}_r$, then $-K_Y = (r+2)f + 2e$, where $f^2 = 0, e^2 = -r$ and $f \cdot e = 1$. Thus r must be even and

$$[B] = \frac{3(r+2)}{2} f + 3e.$$

Since $R^2 < 0$, then its image in Y has the same property and is the unique curve of negative self-intersection in Y, i.e., $[\pi(R)] = e$. Moreover, since B is the disjoint union of $\pi(R)$ and $\pi(C)$, then

$$([B] - e) \cdot e = \frac{3(r+2)}{2} - 2r = \frac{6-r}{2} = 0.$$

Thus $Y \cong \mathbb{F}_6$ and the class of $\pi(C)$ is $12f + 2e$. Let $p: \mathbb{F}_6 \to \mathbb{P}^1$ be the natural fibration. Observe that the restriction of p to $\pi(C)$ is a double cover of $\mathbb{P}^1$ since $(12f + 2e) \cdot f = 2$, thus $\pi(C)$ is hyperelliptic. This implies that there are twelve fibers of p which are tangent to $\pi(C)$.

Assume now that X has an automorphism σ of order 15 with $\sigma^5 = \tau$. Then σ induces an automorphism $\bar{\sigma}$ of Y of order 5 which preserves both $\pi(R)$ and $\pi(C)$. Since σ^3 is not the identity on R, then $\bar{\sigma}$ is not the identity on $\pi(R)$, thus we can assume that it acts on the basis of the fibration p as $(x, y) \mapsto (\zeta_5 x, y)$, where ζ_5 is a primitive 5-th root of unity. In particular, there are exactly two fibers of p which are invariant for $\bar{\sigma}$.

This implies that the image of the curve C' in $\mathrm{Fix}(\sigma^3)$ is a fiber of p, which is invariant for $\bar{\sigma}$. On the other hand, $\bar{\sigma}$ preserves $\pi(C)$ and thus permutes the twelve fibers of p which are tangent to $\pi(C)$. Thus it should preserve at least two of them. The curve $\pi(C)$ can not be tangent to $\pi(C')$, since otherwise C' would be singular, thus $\bar{\sigma}$ should leave invariant three fibers of p, a contradiction. ■

Example 3.8. Let B be the plane sextic defined by

$$F_6(x_0, x_1, x_2) = a_1 x_0^6 + a_2 x_0 x_1^5 + a_3 x_2^6 + a_4 x_0^3 x_2^3,$$

with general $a_1, a_2, a_3, a_4 \in \mathbb{C}$. Let X be a double cover of $\mathbb{P}^2$ branched along B, which can be defined by $x_3^2 - F_6(x_0, x_1, x_2) = 0$ in $\mathbb{P}(1,1,1,3)$. Then X is a K3 surface carrying an order 15 automorphism

$$\sigma_{15}(x_0, x_1, x_2, x_3) = (x_0, \zeta_5 x_1, \zeta_3 x_2, x_3)$$

whose fixed locus is the union of five points, which project to the points $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ of $\mathbb{P}^2$. Observe that σ_5 fixes the genus 2 curve defined by $x_1 = 0$ and the point $(0,1,0,0)$, while σ_3 fixes the genus 2 curve $x_2 = 0$ and the points $(0,0,1,\pm 1)$. Since both σ_3 and σ_5 fix curves, then none of them is symplectic by [20], thus σ_{15} is purely non-symplectic. This is an example of case (a) in Proposition 3.6.

Example 3.9. Consider the elliptic surface with Weierstrass equation

$$y^2 = x^3 + (t^5 - 1)(t^5 - a),$$

with general $a \in \mathbb{C}$. Then X is a K3 surface with the automorphism of order 15:

$$\sigma_{15}(x, y, t) = (\zeta_3 x, y, \zeta_5 t).$$

The elliptic fibration has one fiber of type IV over $t = \infty$ and ten fibers of type II. The automorphism σ_3 fixes the genus 4 curve defined by $x = 0$, the section at infinity and the center of the fiber of type IV. The automorphism σ_5 fixes the smooth fiber over $t = 0$ and four points in the fiber of $t = \infty$. The automorphism σ_{15} fixes three points in the fiber over $t = 0$ and four points in the fiber over $t = \infty$. As in the previous example, σ_3 and σ_5 fix curves, thus they are non-symplectic and σ_{15} is purely non-symplectic. This is an example of case (b) in Proposition 3.6.

Example 3.10. Consider $P = \mathbb{P}(1, 1, 2)$ with coordinates (x_0, x_1, x_2), and let D be a curve of degree 6 in P of equation

$$G_6(x_0, x_1, x_2) = x_0^5 x_1 + a_1 x_1^2 x_2^2 + a_2 x_1^4 x_2 + a_3 x_1^6 + a_4 x_2^3 = 0,$$

where $a_1, a_2, a_3, a_4 \in \mathbb{C}$ are general. Observe that D is smooth, since it does not pass through the singular point $(0, 0, 1)$ and its partial derivatives only vanish at the origin. Let $Y \cong \mathbb{F}_2$ be the blow up of the singular point of P and let B be the preimage of D in Y. Since $2[D] \sim -3K_P$ and the resolution $Y \to P$ is crepant, then $2[B] \sim -3K_Y$. Let X be the triple cover of Y branched along B. By the Hurwitz formula, X is a K3 surface. Observe that the curve D has the order 5 automorphism

$$(x_0, x_1, x_2) \mapsto (\zeta_5 x_0, x_1, x_2),$$

which lifts to an order 5 automorphism φ of X. The composition of φ with the covering automorphism of $X \to Y$ is an order 15 automorphism σ of X. A birational model of (X, σ) in $\mathbb{P}(1, 1, 2, 2)$ is

$$x_3^3 + G_6(x_0, x_1, x_2) = 0, \quad \sigma(x_0, x_1, x_2, x_3) = (\zeta_5 x_0, x_1, x_2, \zeta_3 x_3).$$

Embedding $\mathbb{P}(1, 1, 2, 2)$ in $\mathbb{P}^4$ via the map $(x_0, x_1, x_2, x_3) \mapsto (x_0^2, x_0 x_1, x_1^2, x_2, x_3)$, we also obtain a birational model of (X, σ) as complete intersection in $\mathbb{P}^4$ with three A_1 singularities at $q_i = (0, 0, 0, \zeta_3^i, 1)$, $i = 1, 2, 3$:

$$\begin{cases} y_1^2 - y_0 y_2 = 0, \\ y_4^3 + y_0^2 y_1 + a_1 y_2 y_3^2 + a_2 y_2^2 y_3 + a_3 y_2^3 + a_4 y_3^3 = 0, \end{cases}$$

$$\sigma(y_0, y_1, y_2, y_3, y_4) = (\zeta_5^2 y_0, \zeta_5 y_1, y_2, y_3, \zeta_3 y_4).$$

The fixed locus of σ_3 is the genus 4 curve $y_4 = 0$. The fixed locus of σ_5 is the union of the curve C_1 of genus 1 defined by $y_0 = y_1 = 0$, the point $p_1 = (1, 0, 0, 0, 0)$ and three points over q_1, q_2, q_3. Finally, σ fixes p_1 and three points in $C_1 \cap C_4$. This is an example of case (c) in Proposition 3.6.

Proof of Theorem 1.2*, order* 15. Let X be a K3 surface with a purely non-symplectic automorphism σ_{15} of order 15. By Proposition 3.6, $\mathrm{Fix}(\sigma_{15})$ contains either four, five or seven isolated fixed points.

Case (a). We first assume that $\mathrm{Fix}(\sigma_{15})$ consists of five fixed points, $\mathrm{Fix}(\sigma_5)$ is the union of a curve C_2 of genus 2 and one point, and $\mathrm{Fix}(\sigma_3)$ is the union of a genus 2 curve C_2' and two points. Let $\varphi\colon X \to \mathbb{P}^2$ be the morphism associated to the linear system $|C_2'|$, which is a degree 2 morphism branched along a plane sextic B which possibly contracts the smooth rational curves disjoint from C_2' to simple singular points of B, [25]. Since $[C_2']$ is fixed by σ_{15}^*, the automorphism σ_{15} descends to an automorphism $\bar{\sigma}_{15}$ of $\mathbb{P}^2$. Let $\bar{\sigma}_3 = \bar{\sigma}_{15}^5$ and $\bar{\sigma}_5 = \bar{\sigma}_{15}^3$. Up to a projectivity, we can assume that $\bar{\sigma}_{15}$, and thus $\bar{\sigma}_3$ and $\bar{\sigma}_5$, are diagonal. Observe that both $\bar{\sigma}_3$ and $\bar{\sigma}_5$ must fix pointwise a line and a point in $\mathbb{P}^2$, since both σ_3 and σ_5 fix pointwise a curve of positive genus. Moreover, by the previous description, the two lines must be distinct. Thus we can assume that

$$\bar{\sigma}_3(x_0, x_1, x_2) = (x_0, x_1, \zeta_3 x_2) \quad \text{and} \quad \bar{\sigma}_5(x_0, x_1, x_2) = (x_0, \zeta_5 x_1, x_2).$$

The branch sextic B of φ is invariant for $\bar{\sigma}_{15}$. Observe that B can not contain a line fixed by either $\bar{\sigma}_3$ or $\bar{\sigma}_5$ since otherwise $\mathrm{Fix}(\sigma_3)$ and $\mathrm{Fix}(\sigma_5)$ would contain a smooth rational curve. This implies that B is defined by an equation of the form $F_6(x_0, x_1, x_2) = 0$ as given in Example 3.8. If either a_2 or a_3 vanishes, then B would contain a line. If $a_1 = 0$, then B would contain a singular point of type E_8, whose central component would be fixed by both σ_3 and σ_5, a contradiction. Thus $a_1 a_2 a_3 \neq 0$, in particular B is smooth. Up to rescaling the variables, an equation for X is the one given in Table 2 with $a \in \mathbb{C}$.

Case (b). We now consider the case when $\mathrm{Fix}(\sigma_{15})$ consists of seven points, $\mathrm{Fix}(\sigma_5)$ is the union of an elliptic curve C_1 and four points, and $\mathrm{Fix}(\sigma_3)$ is the union of a genus 4 curve C_4, a rational curve R and one point. Let $\pi\colon X \to \mathbb{P}^1$ be the morphism associated to the linear system $|C_1|$, which is an elliptic fibration. Since C_1 is invariant for σ_{15}, then the elliptic fibration is invariant for σ_{15}. On the other hand, since σ_3 fixes the curve C_4 of genus $g > 1$, then it induces the identity on $\mathbb{P}^1$. The automorphism σ_3 acts on C_1 and has fixed points in $C_1 \cap C_4$, thus it has exactly three fixed points by the Riemann–Hurwitz formula. This implies that $C_1 \cdot C_4 \leq 3$. Moreover, $C_1 \cdot C_4 > 1$ since otherwise the restriction of π to C_4 would be an isomorphism onto $\mathbb{P}^1$. Thus $C_1 \cdot C_4$ is either 2 or 3. We now show that the second case does not appear.

If $C_1 \cdot C_4 = 3$, then the curve R can not intersect C_1, since otherwise C_1 would contain more than three fixed points of σ_5. Thus R must be contained in a reducible fiber F of π. The fiber F is invariant for σ_3, it can only contain an isolated fixed point of σ_3 and $C_4 \cdot F = 3$. By Lemma 4.1 in [4], F should be of type I_0^*, but this contradicts the fact that the rank of the invariant lattice of σ_3 is 4 by Theorem 2.2 in [4].

Thus $C_1 \cdot C_4 = 2$. Since the general fiber of π must contain three fixed points of σ_3 by the Riemann–Hurwitz formula, then the curve R must be a section of π. Thus π is a Jacobian elliptic fibration invariant for an order 3 automorphism and with a fixed section. This implies that, up to a coordinate change, π has Weierstrass equation

$$y^2 = x^3 + p(t),$$

with $\sigma_3(x, y, t) = (\zeta_3 x, y, t)$, where $\deg(p) \leq 12$. In these coordinates, C_1 is the fiber over $t = 0$ and C_4 is the curve $x = 0$. Since σ_5 has order 5 on R, then it induces an order 5 automorphism on $\mathbb{P}^1$ which can be assumed to be $\bar{\sigma}_5(t) = \zeta_5 t$. Since σ_5 is the identity when $t = 0$, then $\sigma_5(x, y, t) = (x, y, \zeta_5 t)$. Thus, up to a coordinate change, π has Weierstrass equation of the form

$$y^2 = x^3 + (t^5 - 1)(t^5 - a),$$

with $a \in \mathbb{C}$ and $\sigma_{15}(x, y, t) = (\zeta_3 x, y, \zeta_5 t)$, as in Example 3.9.

Case (c). Finally, assume that $\mathrm{Fix}(\sigma_{15})$ consists of four points, $\mathrm{Fix}(\sigma_5)$ is the union of an elliptic curve C_1 and four points, and $\mathrm{Fix}(\sigma_3)$ is a curve of genus 4. Following the same argument in the proof of Lemma 3.7, we obtain that the quotient of X by σ_3 is a smooth rational surface Y isomorphic to a Hirzebruch surface $\mathbb{F}_r$ for some even $r \geq 0$. This quotient is a cyclic degree 3 covering branched along a smooth curve B of genus 4 whose class is $[B] = \frac{3(r+2)}{2} f + 3e$, where $f^2 = 0$, $e^2 = -r$ and $f \cdot e = 1$. Since B is smooth, then $[B] \cdot e = \frac{6-3r}{2} \geq 0$, thus $r \leq 2$.

The case when $r = 0$, i.e., $Y \cong \mathbb{P}^1 \times \mathbb{P}^1$, can be excluded as follows. In this case B is a curve of type $(3, 3)$. The automorphism σ_5 descends to an automorphism $\bar{\sigma}_5$ of $\mathbb{P}^1 \times \mathbb{P}^1$ which has a fixed curve, thus up to a coordinate change we can assume

$$\bar{\sigma}_5 : (x_0, x_1), (y_0, y_1) \mapsto (x_0, x_1), (\zeta_5 y_0, y_1).$$

However, there is no curve of type $(3, 3)$ which is invariant for this automorphism, giving a contradiction.

Thus $Y \cong \mathbb{F}_2$, $[B] = 6f + 3e$ and $[B] \cdot e = 0$. After contracting the (-2)-curve of Y, we obtain the surface $P \cong \mathbb{P}(1, 1, 2)$. Since the contraction is a crepant morphism, the image of B is a smooth curve D with $2[D] \sim -3K_P$, i.e., of degree 6. The automorphism σ_5 descends to an automorphism $\bar{\sigma}_5$ of P which preserves D and fixes the image of the curve C_1. This implies that, after a coordinate change, we can assume $\bar{\sigma}_5(x_0, x_1, x_2) = (\zeta_5 x_0, x_1, x_2)$. The equation of D must be invariant for $\bar{\sigma}_5$, thus it is of the form given in Example 3.10. If all the coefficients of G_6 are non-zero, then one obtains the equation in Table 2 up to rescaling the variables. ∎

3.4. Order 30

Proposition 3.11. *Let X be a K3 surface with a purely non-symplectic automorphism σ_{30} of order* 30 *such that* $\dim(V^\sigma) = 2$. *Then there are two possibilities for the fixed locus of σ_{30} and of its powers*:

	$\mathrm{Fix}(\sigma_{30})$	$\mathrm{Fix}(\sigma_{15})$	$\mathrm{Fix}(\sigma_5)$	$\mathrm{Fix}(\sigma_3)$	$\mathrm{Fix}(\sigma_2)$
(a)	$\{p_1\}$	$\{p_1, \ldots, p_5\}$	$C_2 \sqcup \{p_1\}$	$C_2' \sqcup \{p_2, p_3\}$	C_{10}
(b)	$\{p_1, p_2, p_5\}$	$\{p_1, \ldots, p_7\}$	$C_1 \sqcup \{p_1, \ldots, p_4\}$	$C_4 \sqcup R \sqcup \{p_1\}$	$C_9 \sqcup R$

where C_g, C_g' have genus g and $g(R) = 0$. Moreover, $d = (2, 0, 1, 0, 0, 0, 1, 1)$ *in case* (a)*, and* $d = (2, 0, 0, 1, 0, 0, 1, 3)$ *in case* (b).

Finally, $\mathrm{NS}(X) \cong U(3) \oplus A_2 \oplus A_2$ *for a very general K3 surface X in case* (a)*, and* $\mathrm{NS}(X) \cong H_5 \oplus A_4$ *for a very general K3 surface X in case* (b).

Proof. Let $\chi_i = \chi(\mathrm{Fix}(\sigma_i))$, $i = 30, 15, 5, 3, 2$. First observe that given a one-dimensional family of K3 surfaces admitting a purely non-symplectic automorphism of order 30, every element in the family admits a purely non-symplectic automorphism of order 15. Thus this corresponds to one of the three families in Proposition 3.6, and the vector $(\chi_{15}, \chi_5, \chi_3)$ is either $(5, -1, 0)$, $(7, 4, -3)$ or $(4, 4, -6)$.

Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of σ^*, we obtain

$$22 = 8d_{30} + 8d_{15} + 4d_{10} + 2d_6 + 4d_5 + 2d_3 + d_2 + d_1. \tag{3.4}$$

Assuming $d_{30} = 2$, this gives $d_{15} = 0$. Using the topological Lefschetz fixed point formulas, we compute the topological Euler characteristic of the fixed loci of powers of σ_{30} by:

$$\begin{cases} \chi_{30} = d_{10} + d_6 - d_5 - d_3 - d_2 + d_1, \\ \chi_{15} = -d_{10} - d_6 - d_5 - d_3 + d_2 + d_1 + 4, \\ \chi_5 \ = -d_{10} + 2d_6 - d_5 + 2d_3 + d_2 + d_1 - 2, \\ \chi_3 \ = 4d_{10} - d_6 + 4d_5 - d_3 + d_2 + d_1 - 6, \\ \chi_2 \ = -4d_{10} - 2d_6 + 4d_5 + 2d_3 - d_2 + d_1 - 14. \end{cases} \tag{3.5}$$

We first assume to be in case (a) of Proposition 3.6, i.e., $\mathrm{Fix}(\sigma_5)$ is the union of a smooth curve C_2 of genus 2 and a point p_1, $\mathrm{Fix}(\sigma_3)$ is the union of a smooth curve C_2' of genus 2 and two isolated points, and $\mathrm{Fix}(\sigma_{15})$ consists of five isolated points $p_1, \dots, p_5$. In particular, $(\chi_{15}, \chi_5, \chi_3) = (5, -1, 0)$. Moreover, since the fixed locus of σ_{15} only contains isolated points, the same holds for σ_{30}. Thus $\chi_{30} \geq 0$. By (3.4) and (3.5), we get the possibilities in Table 3.

d_{30}	d_{15}	d_{10}	d_6	d_5	d_3	d_2	d_1	χ_{30}	χ_{15}	χ_5	χ_3	χ_2
2	0	1	0	0	0	1	1	1	5	-1	0	-18
2	0	1	0	0	0	0	2	3	5	-1	0	-16
2	0	0	0	1	0	0	2	1	5	-1	0	-8

Table 3

In particular, χ_{30} is either 3 or 1, thus $\mathrm{Fix}(\sigma_{30})$ is either the union of p_1 and two of the p_i's with $i \geq 2$ (and the other two are exchanged), or $\mathrm{Fix}(\sigma_{30}) = \{p_1\}$ and σ_{30} has no fixed points on C_2. By the proof of Theorem 1.2 in the case $n = 15$, the linear system associated to C_2 defines a double cover $\varphi\colon X \to \mathbb{P}^2$ which can be defined in $\mathbb{P}(1, 1, 1, 3)$ by an equation of the form

$$y^2 = x_0^6 + x_0 x_1^5 + x_2^6 + a x_0^3 x_2^3,$$

where $a \in \mathbb{C}$, and in these coordinates $\sigma_{15}(x_0, x_1, x_2, y) = (x_0, \zeta_5 x_1, \zeta_3 x_2, y)$. Since σ_{30} preserves C_2, then it induces an automorphism $\bar{\sigma}_{30}$ of $\mathbb{P}^2$. The involution σ_2 either induces the identity or an involution of $\mathbb{P}^2$. The latter is not possible since the fixed locus of σ_2 would contain a curve of genus at most 2, while $\chi_2 \leq -8$ by Table 3. Thus σ_2 coincides with the (automorphism induced by) the covering involution of φ, which fixes a smooth genus 10 curve, so that $\chi_2 = -18$ and $\chi_{30} = 1$ by Table 3. Thus σ_{30} fixes a unique point. Since C_2 is invariant for σ_{30}, then $\varphi(C_2)$ is a line which contains two fixed points for $\bar{\sigma}_{30}$. Since $\chi_{30} = 1$, their preimages by φ are four points exchanged in pairs by σ_2.

Assume now to be in case (b) of Proposition 3.6, i.e., $\mathrm{Fix}(\sigma_3)$ is the union of a curve C_4 of genus 4, a rational curve R and a point, $\mathrm{Fix}(\sigma_5)$ is union of an elliptic curve C_1 and four points, and $\mathrm{Fix}(\sigma_{15})$ is the union of seven points (three on C_1). In particular, $(\chi_{15}, \chi_5, \chi_3) = (7, 4, -3)$. Moreover, $\chi_{30} \geq 0$ since $\mathrm{Fix}(\sigma_{15})$ only contains isolated points, and thus the same holds for σ_{30}. There are five possible vectors d such that $(\chi_{15}, \chi_5, \chi_3) = (7, 4, -3)$ (see Table 4).

d_{30}	d_{15}	d_{10}	d_6	d_5	d_3	d_2	d_1	χ_{30}	χ_{15}	χ_5	χ_3	χ_2
2	0	0	1	0	0	2	2	1	7	4	-3	-16
2	0	0	1	0	0	1	3	3	7	4	-3	-14
2	0	0	0	0	1	1	3	1	7	4	-3	-10
2	0	0	1	0	0	0	4	5	7	4	-3	-12
2	0	0	0	0	1	0	4	3	7	4	-3	-8

Table 4

By the proof of Theorem 1.2, case $n = 15$, X admits an elliptic fibration $\pi: X \to \mathbb{P}^1$ with Weierstrass equation

$$y^2 = x^3 + (t^5 - 1)(t^5 - a),$$

with $a \in \mathbb{C}$ and $\sigma_{15}(x, y, t) = (\zeta_3 x, y, \zeta_5 t)$. By the same argument in the proof of Theorem 1.2 in the case $n = 15$, using Lemma 5 in [5], one concludes that the elliptic fibration is invariant for σ_{30}. Since $\chi_2 \leq -8$, then σ_2 fixes a curve of genus > 1. Such curve is clearly transverse to all fibers of π, thus σ_2 induces the identity on the basis of the fibration. Moreover, σ_2 must fix the section at infinity R of the fibration, since it preserves R and each fiber of π. This implies that $\sigma_2(x, y, t) = (x, -y, t)$. In particular, σ_2 fixes R and the curve defined by $y = 0$, which has genus 9, so that $\chi_2 = -14$. Moreover, σ_{30} fixes three points: two points on R and the center of the fiber of type IV over $t = \infty$.

Assume now to be in case (c) of Proposition 3.6, i.e., $\mathrm{Fix}(\sigma_3)$ is a curve C_4 of genus 4, $\mathrm{Fix}(\sigma_5)$ is union of an elliptic curve C_1 and four points, and $\mathrm{Fix}(\sigma_{15})$ is the union of four points. In particular, $(\chi_{15}, \chi_5, \chi_3) = (4, 4, -6)$. Moreover, $\chi_{30} \geq 0$ since $\mathrm{Fix}(\sigma_{15})$ only contains isolated points, and thus the same holds for σ_{30}. By the proof of Theorem 1.2 in the case $n = 15$, X is the minimal resolution of the double cover of $P = \mathbb{P}(1, 1, 2)$ branched along a smooth curve D of degree 6 not passing through the singular point of P. The automorphism induced by σ_5 in P can be assumed to be $(x_0, x_1, x_2) \mapsto (\zeta_5 x_0, x_1, x_2)$. The automorphism σ_2, since it commutes with σ_3, induces an involution $\bar{\sigma}_2$ of P which preserves the curve D. Moreover, it can be also diagonalized. However, no diagonal involution leaves invariant the general equation as in Example 3.10, thus this case is not possible.

The Néron–Severi lattice of a very general X in cases (a) and (b) is clearly the same as in Proposition 3.6. ■

Example 3.12. The double cover X of $\mathbb{P}^2$ in Example 3.8 carries the order 30 automorphism

$$\sigma_{30}(x_0, x_1, x_2, x_3) = (x_0, \zeta_5 x_1, \zeta_3 x_2, -x_3).$$

Observe that for a general choice of the coefficients the fixed locus of σ_2 is the smooth plane sextic defined by $x_3 = 0$, which has genus 10. Moreover, the fixed locus of σ_{30} consists of the point $(0, 1, 0, 0)$. This is an example of case (a) in Proposition 3.11.

Example 3.13. The elliptic K3 surface in Example 3.9 carries the order 30 automorphism

$$\sigma_{30}(x, y, t) = (\zeta_3 x, -y, \zeta_5 t).$$

Observe that for general $a \in \mathbb{C}$ the fixed locus of σ_2 is the curve $y = 0$, which has genus 9. Moreover, as observed in the proof of Proposition 3.11, the fixed locus of σ_{30} consists of two points in the section at infinity (over $t = 0$ and $t = \infty$) and the center of the fiber of type IV over $t = \infty$. This is an example of case (b) in Proposition 3.11.

Proof of Theorem 1.2, *order* 30. Let X be a K3 surface with a purely non-symplectic automorphism σ of order 30 such that $\dim(V^\sigma) = 2$. It is straightforward from the proof of Proposition 3.11 that, up to isomorphism, (X, σ) belongs to one of the families in Examples 3.12 and 3.13. ∎

3.5. Order 16

Purely non-symplectic automorphisms of order 16 on K3 surfaces have been classified in [2]. The following result has the same statement as that of Theorem 4.1 in [2], but we provide a slightly different proof since we use the weaker hypothesis $\dim(V^\sigma) = 2$.

Proposition 3.14. *Let σ_{16} be a purely non-symplectic automorphism of order* 16 *of a K3 surface X and assume that* $\dim(V^\sigma) = 2$ (*or equivalently, $S(\sigma_2)$ has rank* 6). *Then there are two possibilities for the fixed locus of σ_{16} and of its powers*:

	$\mathrm{Fix}(\sigma_{16})$	$\mathrm{Fix}(\sigma_8)$	$\mathrm{Fix}(\sigma_4)$	$\mathrm{Fix}(\sigma_2)$
(a)	$\{p_1, \ldots, p_6\} \sqcup R$	$\{p_1, \ldots, p_6\} \sqcup R$	$\{p_1, \ldots, p_6\} \sqcup R$	$C_7 \sqcup R \sqcup R'$
(b)	$\{p_1, p_2, p_7, p_8\}$	$\{p_1, \ldots, p_6\} \sqcup R$	$\{p_1, \ldots, p_6\} \sqcup R$	$C_6 \sqcup R$

where $g(C_6) = 6$, $g(C_7) = 7$ and $g(R) = g(R') = 0$. Moreover, $d = (2, 0, 0, 0, 6)$ in case (a), *and $d = (2, 0, 0, 2, 4)$ in case* (b).

Finally, $\mathrm{NS}(X) \cong U \oplus D_4$ *for a very general X in case* (a), *and* $\mathrm{NS}(X) \cong U(2) \oplus D_4$ *for a very general X in case* (b).

Proof. Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of σ_{16}^*, we obtain

$$22 = 8d_{16} + 4d_8 + 2d_4 + d_2 + d_1. \tag{3.6}$$

Since $d_{16} = 2$, this implies that d_8 is either 0 or 1, and gives the 14 possibilities for the vector d in Table 5.

Let N_i be the number of isolated fixed points of σ_i, let $\chi_i = \chi(\mathrm{Fix}(\sigma_i))$ and write

$$\alpha_i = \sum_{C \subset \mathrm{Fix}(\sigma_i)} (1 - g(C))$$

d_{16}	d_8	d_4	d_2	d_1	χ_{16}	χ_8	χ_4	χ_2
2	1	0	1	1	2	4	0	-8
2	0	2	1	1	2	2	8	-8
2	0	1	3	1	0	5	8	-8
2	0	0	5	1	-2	8	8	-8
2	1	0	0	2	4	4	0	-8
2	0	2	0	2	4	2	8	-8
2	0	1	2	2	2	5	8	-8
2	0	0	4	2	0	8	8	-8
2	0	1	1	3	4	5	8	-8
2	0	0	3	3	2	8	8	-8
2	0	1	0	4	6	5	8	-8
2	0	0	2	4	4	8	8	-8
2	0	0	1	5	6	8	8	-8
2	0	0	0	6	8	8	8	-8

Table 5

for $i \in \{2, 4, 8, 16\}$. By the topological Lefschetz fixed point formula, we get

$$\begin{cases} \chi_{16} = -d_2 + d_1 + 2, \\ \chi_8 = -d_4 + d_2 + d_1 + 2, \\ \chi_4 = -4d_8 + 2d_4 + d_2 + d_1 + 2, \\ \chi_2 = -8d_{16} + 4d_8 + 2d_4 + d_2 + d_1 + 2. \end{cases} \tag{3.7}$$

Table 5 shows the values of $(\chi_{16}, \chi_8, \chi_4, \chi_2)$ for each possible vector d.

Observe that $\chi_2 = -8$. By [20], $N_2 = 0$ and $\mathrm{Fix}(\sigma_2)$ is the union of a curve of genus g and k rational curves with $(g, k) = (5, 0), (6, 1)$ or $(7, 2)$.

Moreover, $\chi_4 = 0$ or 8. By Proposition 1 in [5], we have that $N_4 = 2\alpha_4 + 4$. Since $\chi_4 = 2\alpha_4 + N_4$, one has

$$\chi_4 = 4\alpha_4 + 4.$$

If $\chi_4 = 0$, then $\alpha_4 = -1$, but this is not possible since $\mathrm{Fix}(\sigma_4) \subseteq \mathrm{Fix}(\sigma_2)$ and it is not compatible with the aforementioned possibilities for $\mathrm{Fix}(\sigma_2)$. Thus $\chi_4 = 8$, $\alpha_4 = 1$ and $\mathrm{Fix}(\sigma_4)$ contains a rational curve (and no more curves) and six points. This implies that the case $(g, k) = (5, 0)$ is impossible.

The cases $(g, k) = (6, 1)$ and $(7, 2)$ are treated in Lemmas 3.15 and 3.16. We conclude that the only admissible cases are the ones in Proposition 3.14. Observe that both in case (a) and (b) we have that $d_4 = d_8 = 0$ and $d_1 + d_2 = 6$. This implies that $S(\sigma_8) = S(\sigma_4) = S(\sigma_2)$ has rank 6. If X is very general, then $\operatorname{rank} \mathrm{NS}(X) = 22 - 2\varphi(20) = 6$ and thus, by Remark 2.1, $\mathrm{NS}(X) = S(\sigma_2)$. Moreover, by Theorem 4.2.2 in [21] or Figure 1 in [6], the invariant lattice of σ_2 is isometric to $U \oplus D_4$ in case (a) and $U(2) \oplus D_4$ in case (b), see [2]. ■

Lemma 3.15. *If* $\mathrm{Fix}(\sigma_2)$ *is the union of a curve of genus* 6 *and a rational curve, then the fixed loci of* σ_{16}, σ_8 *and* σ_4 *are as follows*:

$$\mathrm{Fix}(\sigma_{16}) = \{p_1, p_2, p_7, p_8\}, \quad \mathrm{Fix}(\sigma_8) = \{p_1, \dots, p_6\} \sqcup R, \quad \mathrm{Fix}(\sigma_4) = \{p_1, \dots, p_6\} \sqcup R.$$

Proof. Let C_6 (respectively, R) be the smooth curve of genus 6 (respectively, rational curve) in $\mathrm{Fix}(\sigma_2)$. By the previous analysis, we know that σ_4 fixes pointwise R and has six isolated fixed points $p_1, \dots, p_6$ on C_6.

By the Riemann–Hurwitz formula for σ_8 on C_6, we observe that either a) two of the p_i's are fixed and the other four are permuted in pairs by σ_8, or b) the points $p_1, \dots, p_6$ are fixed points for σ_8. Observe that case a) is not possible since $\chi_4 = 8$, and $\chi_8 = 4$ does not appear in Table 5.

By the Riemann–Hurwitz formula for σ_{16} on C_6, we obtain that σ_{16} fixes two of the p_i's and exchanges the other four in pairs. Thus $(\chi_{16}, \chi_8, \chi_4, \chi_2) = (4, 8, 8, -8)$.

Observe that six of the fixed points of σ_8 lie on a curve fixed pointwise by σ_2 and not by σ_4, thus the local action of σ_8 at such points is either of type $A_{2,8}$ or $A_{3,8}$. By Proposition 2.2 in [1], we have that $6 = 2 + 4\alpha_8$, thus $\alpha_8 = 1$. This implies that $N_8 = 6$ and the curve R is pointwise fixed by σ_8. On the other hand, by Proposition 2 in [2], $N_{16} \geq 2\alpha_{16} + 1$. This implies that $\alpha_{16} = 0$, i.e., R is not pointwise fixed by σ_{16}. ■

Lemma 3.16. *If* $\mathrm{Fix}(\sigma_2)$ *is the union of a curve of genus* 7 *and two rational curves, then the fixed loci of* σ_{16}*,* σ_8 *and* σ_4 *are as follows*:

$$\mathrm{Fix}(\sigma_{16}) = \{p_1, \dots, p_4, q_1, q_2\} \sqcup R, \quad \mathrm{Fix}(\sigma_8) = \{p_1, \dots, p_4, q_1, q_2\} \sqcup R,$$
$$\mathrm{Fix}(\sigma_4) = \{p_1, \dots, p_4, q_1, q_2\} \sqcup R.$$

Proof. Let C_7 (respectively, R, R') be the smooth curve of genus 7 (respectively, rational curves) in $\mathrm{Fix}(\sigma_2)$. We already know that one rational curve is fixed by σ_4, say R. Thus σ_4 fixes two points q_1, q_2 on R' and four points $p_1, \dots, p_4$ on C_7. This implies that the curves R and R' cannot be exchanged by σ_{16} nor by σ_8 and that $\chi_{16} \geq 4$ and $\chi_8 \geq 4$.

By the Riemann–Hurwitz formula for σ_8 on C_7, either the four p_i's are fixed by σ_8 or none of them is fixed by σ_8. This implies that either $\chi_8 = 4$ or $\chi_8 = 8$. Looking at Table 5, we find that we are left with the three possibilities of Table 6.

d_{16}	d_8	d_4	d_2	d_1	χ_{16}	χ_8	χ_4	χ_2
2	0	0	2	4	4	8	8	-8
2	0	0	1	5	6	8	8	-8
2	0	0	0	6	8	8	8	-8

Table 6

In particular, $\chi_8 = 8$ and $\{p_1, \dots, p_4, q_1, q_2\} \subset \mathrm{Fix}(\sigma_8)$. Moreover, by Proposition 2.2 in [1], we obtain that $2 + 4\alpha_8 = 6$, thus $\alpha_8 = 1$. This implies that σ_8 fixes pointwise the curve R.

By the Riemann–Hurwitz formula for σ_{16} on C_7, either a) σ_{16} fixes the four p_i's and thus $\chi_{16} = 8$, or b) it does not fix any of them and $\chi_{16} = 4$. By Proposition 2 and Remark 1.3 in [2], the cases $(N_{16}, \alpha_{16}) = (2, 1)$ and $(N_{16}, \alpha_{16}) = (8, 0)$ are impossible. Thus in case a), $\alpha_{16} = 1$ and $N_{16} = 4$, i.e., the fixed locus of σ_{16} contains $p_1, \dots, p_4, q_1, q_2$ and the curve R. On the other hand, in case b) we have that $\alpha_{16} = 0$, i.e., σ_{16} fixes exactly q_1, q_2 and two points on R. We now show that this case can not appear. By Remark 1.3 in [2], if $N_{16} = 4$, then $n_{3,16} = n_{7,16} = 1$ and $n_{8,16} = 2$. Observe that the points of type $A_{8,16}$ lie on a curve fixed by σ_8, thus they must be the two points on R. This implies that the points of type $A_{3,16}$ and $A_{7,16}$ are q_1, q_2. However, two isolated fixed points of σ_{16} lying on an invariant smooth rational curve can not be of these types by the proof of Lemma 4 in [5]. ■

For the following examples, see Example 4.2 in [2].

Example 3.17. Consider the elliptic fibration defined by

$$y^2 = x^3 + t^2x + at^3 + t^{11}, \quad a \in \mathbb{C},$$

with the order 16 automorphism $\sigma_{16}(x, y, t) = (\zeta_8 x, \zeta_{16}^3 y, \zeta_8 t)$. The action of σ_{16}^* on the holomorphic two form $\omega_X = (dx \wedge dt)/2y$ is the multiplication by ζ_{16}, thus σ_{16} is purely non-symplectic. The fibration has a fiber of type I_0^* over $t = 0$ and a fiber of type II over $t = \infty$. The automorphism σ_{16} fixes the central component of the fiber of type I_0^*, four points in the other components of the same fiber and two more fixed points in the fiber over $t = \infty$. This is an example of case (a) in Proposition 3.14.

Example 3.18. Consider the plane sextic B defined by

$$F_6(x_0, x_1, x_2) = x_0(x_0^4 x_2 + a_1 x_1^5 + a_2 x_1 x_2^4 + a_3 x_1^3 x_2^2) = 0$$

for general $a_1, a_2, a_3 \in \mathbb{C}$. Observe that B is the union of a smooth plane quintic C and a line L. Let Y be the double cover of $\mathbb{P}^2$ branched along B, which can be defined by the equation $x_3^2 = F_6(x_0, x_1, x_2)$ in $\mathbb{P}(1, 1, 1, 3)$. The surface Y has the order 16 automorphism

$$\sigma_{16}(x_0, x_1, x_2, y) = (x_0, \zeta_8^7 x_1, \zeta_8^3 x_2, \zeta_{16}^3 y).$$

The surface Y has five singular points of type A_1 over the intersection points of C and L. Its minimal resolution X is a K3 surface and σ_{16} lifts to an automorphism $\tilde{\sigma}_{16}$ of X. The automorphism $\tilde{\sigma}_{16}$ has four fixed points: two of them over the points $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$, and the other two in the exceptional divisor over $(0, 0, 1, 0)$ (which is a singular point of Y). Thus this is an example of case (b) in Proposition 3.14.

Proof of Theorem 1.2*, order* 16. Let X be a K3 surface with a purely non-symplectic automorphism σ of order 16 such that $\dim(V^\sigma) = 2$. By Proposition 3.14, $\mathrm{Fix}(\sigma_{16})$ is either the union of a rational curve and six points, or the union of four isolated points.

Case (a). By Proposition 3.14, $\mathrm{NS}(X) = S(\sigma_2) \cong U \oplus D_4$ for a very general K3 surface. In what follows we assume X to be very general. By Lemma 2.1 in [17], or the proof of Corollary 3 in §3 of [23], X has an elliptic fibration $\pi: X \to \mathbb{P}^1$ with a section S and a reducible fiber of type $\tilde{D}_4 = I_0^*$. The curve C_7 fixed by the involution σ_2 has to be transverse to the fibers of π, since its genus is bigger than 1. Thus σ_2 induces the identity on the basis of the fibration. Since $\mathrm{NS}(X) = S(\sigma_2)$, σ_2^* is the identity on $\mathrm{NS}(X)$, hence each smooth rational curve is invariant for σ_2. This implies that the section S and the central component of the fiber of type I_0^* are pointwise fixed by σ_2. Since a smooth fiber of π must contain four fixed points for σ_2 and one of them is on S, then C_7 intersects it in three points. Applying Lemma 5 in [5] with $x = [C_7]$, one concludes that the elliptic fibration π is invariant under σ_{16}. The section S corresponds to the curve R' (see the notation of Proposition 3.14) i.e., it is not fixed pointwise by σ_{16}, otherwise each fiber of π, including the smooth ones, would have an order 16 automorphism with a fixed point, which is impossible for an elliptic curve, see [16]. Thus σ_{16} induces an automorphism of order 8 on the basis of π. This implies that σ_8 preserves each fiber of π and acts on it as an involution with a fixed point.

Consider a Weierstrass equation for π with respect to the section S:

$$y^2 = x^3 + A(t)x + B(t), \quad t \in \mathbb{P}^1.$$

We can assume that the two invariant fibers of σ_{16} are over $t = 0$ and $t = \infty$, and that the fiber of type I_0^* is over $t = 0$. Since $\mathrm{NS}(X) \cong U \oplus D_4$, the fiber F_0 over 0 is the only reducible fiber of π; moreover, $24 - e(F_0)$ is divisible by 8. This implies that the fiber over $t = \infty$ is of type II. By Table IV.3.1 in [19], this implies that the vanishing order $v(\Delta)$ of $\Delta(t)$ at $t = 0$ is 6 and at $t = \infty$ is 2. Thus $\Delta(t) = t^6 P(t)$, with $P(0) \neq 0$ and $\deg(P(t)) = 16$. Moreover, $v(B(\infty)) = 1$, thus $B(t) = t^3 Q(t)$ with $\deg(Q(t)) = 8$. Since the action of σ_{16} on the basis of π has order 8 and the fibers over $t = 0, \infty$ are preserved by σ_{16}, then $Q(t) = t^8 + a$ with $a \in \mathbb{C}$. By Table IV.3.1 in [19], we have that $A(t) = t^2$. Moreover, $\sigma_{16}(x, y, t) = (\zeta_8 x, \zeta_{16}^3 y, \zeta_8 t)$ and X belongs to the family in Example 3.17.

Case (b). In this case, $\mathrm{Fix}(\sigma_{16})$ is the union of four isolated points, and $S(\sigma_2) \cong U(2) \oplus D_4$ by Proposition 3.14. As before, we assume X to be very general, i.e., that $\mathrm{NS}(X) = S(\sigma_2)$. It is known that the surface X has a degree two morphism $\pi\colon X \to \mathbb{P}^2$ which is the minimal resolution of a double cover ramified along the union of a line ℓ and a quintic curve C, see [2], Section 4. In particular, X has six (-2)-curves, i.e., five exceptional divisors $E_1, \dots, E_5$ over the points $\ell \cap C$ and the proper transform E of (the double cover) of ℓ. It follows from Vinberg's algorithm (see [24]) that these are the only (-2)-curves of X. This implies that the linear system of the divisor $2E + \sum_{i=1}^5 E_i$, which is the one defining the morphism π, is invariant for σ^*. Thus the automorphism σ induces an automorphism $\bar{\sigma}$ of $\mathbb{P}^2$ preserving the branch curve $\ell \cup C$.

The involution σ_2 fixes a genus 6 curve and a rational curve R. If the induced automorphism $\bar{\sigma}_2$ were an involution, it would fix a line and a point. This would imply that the maximum possible genus in $\mathrm{Fix}(\sigma_2)$ is 2, giving a contradiction. Thus $\bar{\sigma}_2$ is the identity on $\mathbb{P}^2$, σ_2 is the covering involution, and R is the transform of the line ℓ in the branch locus. Moreover, since R is contained in $\mathrm{Fix}(\sigma_4)$ and $\mathrm{Fix}(\sigma_8)$, ℓ is fixed by $\bar{\sigma}_4$ and $\bar{\sigma}_8$. In addition, $\bar{\sigma}_{16}$ has order 8 on $\mathbb{P}^2$ and it only fixes points, $\bar{\sigma}_8$ has order 4 and $\bar{\sigma}_4$ has order 2.

Assume that ℓ is defined by $x_0 = 0$, thus

$$\bar{\sigma}_8(x_0, x_1, x_2) = (i x_0, x_1, x_2), \quad \bar{\sigma}_4(x_0, x_1, x_2) = (-x_0, x_1, x_2).$$

Since $\bar{\sigma}_{16}$ only fixes points in $\mathbb{P}^2$, we have $\bar{\sigma}_{16}(x_0, x_1, x_2) = (\zeta_8 x_0, x_1, -x_2)$, and the equation of X is obtained taking invariant monomials and recalling that we need the quintic to be smooth, otherwise $\mathrm{Fix}(\sigma_2)$ would not contain a genus 6 curve. Thus the equation of X is as in Example 3.18, and

$$\sigma_{16}(x_0, x_1, x_2, y) = (\zeta_8 x_0, x_1, -x_2, \zeta_{16} y) = (x_0, \zeta_8^7 x_1, \zeta_8^3 x_2, \zeta_{16}^{11} y),$$

where ζ_{16} is a primitive 16-th root of unity with $\zeta_{16}^2 = \zeta_8$. Observe that σ_{16}^9 is the automorphism in Example 3.18. If all the coefficients of F_6 are non-zero, then one obtains the equation in Table 2 up to rescaling the variables. ∎

3.6. Order 20

Proposition 3.19. *Let X be a K3 surface with a purely non-symplectic automorphism σ_{20} of order 20 such that $\dim(V^{\sigma}) = 2$. Then the fixed loci of σ_{20} and of its powers are as follows*:

$\mathrm{Fix}(\sigma_{20})$	$\mathrm{Fix}(\sigma_{10})$	$\mathrm{Fix}(\sigma_5)$	$\mathrm{Fix}(\sigma_4)$	$\mathrm{Fix}(\sigma_2)$
$\{p_1, p_2, p_3\}$	$\{p_1, \dots, p_7\}$	$C_2 \sqcup \{p_1\}$	$\{p_1, \dots, p_6\} \sqcup R$	$C_6 \sqcup R$

where $g(C_i) = i$ for $i = 2, 6$ and $g(R) = 0$. Moreover, $d = (2, 0, 1, 0, 0, 2)$ and $\mathrm{NS}(X) = S(\sigma_2)$ for a very general such K3 surface X.

Proof. Decomposing $H^2(X, \mathbb{C})$ as the direct sum of the eigenspaces of σ_{20}^*, we obtain

$$22 = 8d_{20} + 4d_{10} + 4d_5 + 2d_4 + d_2 + d_1.$$

Since $d_{20} = 2$, then d_{10} is either 0 or 1, and this gives 16 possibilities for the vector d. Let $\chi_i = \chi(\mathrm{Fix}(\sigma_i))$, $i \in \{2, 4, 5, 10, 20\}$. By the topological Lefschetz fixed point formula, we get

$$\begin{cases} \chi_{20} = d_{10} - d_5 - d_2 + d_1 + 2, \\ \chi_{10} = 2d_{20} - d_{10} - d_5 - 2d_4 + d_2 + d_1 + 2, \\ \chi_5 \ = -2d_{20} - d_{10} - d_5 + 2d_4 + d_2 + d_1 + 2, \\ \chi_4 \ = -4d_{10} + 4d_5 - d_2 + d_1 + 2, \\ \chi_2 \ = -8d_{20} + 4d_{10} + 4d_5 - 2d_4 + d_2 + d_1 + 2. \end{cases} \tag{3.8}$$

By (3.8), we compute χ_5 for the 16 possible d's and find that it is either -1 or 4. Lemmas 3.20 and 3.21 study these two cases separately. Observe that, since $d_{20} = 2$ and $\varphi(20) = 8$, the Néron–Severi lattice of a very general X has rank $22 - 2 \cdot 8 = 6$. Moreover, $S(\sigma_2) \subseteq \mathrm{NS}(X)$ by Remark 2.1. On the other hand, since the fixed locus of σ_2 is the union of a curve of genus 6 and a rational curve, then $\mathrm{rk}\, S(\sigma_2) = 6$ by [20], thus $S(\sigma_2) = \mathrm{NS}(X)$. ∎

Lemma 3.20. *If $\chi_5 = -1$, then the fixed loci of σ_{20}, σ_{10}, σ_5, σ_4 and σ_2 are*

$$\mathrm{Fix}(\sigma_{20}) = \{p_1, p_2, p_3\}, \quad \mathrm{Fix}(\sigma_{10}) = \{p_1, \dots, p_6, p\},$$
$$\mathrm{Fix}(\sigma_5) = C_2 \sqcup \{p\}, \quad \mathrm{Fix}(\sigma_4) = \{p_1, \dots, p_6\} \sqcup R, \quad \mathrm{Fix}(\sigma_2) = C_6 \sqcup R.$$

Proof. By [6], if $\chi_5 = -1$, then $\mathrm{Fix}(\sigma_5)$ is the union of a smooth curve C of genus 2 and one point p. This corresponds to the cases in Table 7.

In all these cases, $\chi_{10} = 7$, so that $\mathrm{Fix}(\sigma_{10})$ is the union of p and six points on C. By the Riemann–Hurwitz formula, this implies that σ_{20} has two fixed points on C, so that $\mathrm{Fix}(\sigma_{20})$ consist of the union of three points. Moreover, in all these cases $\chi_2 = -8$ by Table 7, so that by [20], $\mathrm{Fix}(\sigma_2)$ is a) the union of a curve of genus 7 and two rational curves, or b) the union of a curve of genus 6 and a rational curve, or c) a genus 5 curve. In all cases, σ_{20} acts with order 10 on the curve of positive genus, since otherwise either σ_{10} or σ_4 should contain such curve in its fixed locus, contradicting the previous remarks for σ_{10} and Theorem 0.1 in [5].

d_{20}	d_{10}	d_5	d_4	d_2	d_1	χ_{20}	χ_{10}	χ_5	χ_4	χ_2
2	0	1	0	0	2	3	7	-1	8	-8
2	0	1	0	1	1	1	7	-1	6	-8
2	1	0	0	0	2	5	7	-1	0	-8
2	1	0	0	1	1	3	7	-1	-2	-8

Table 7

In case a), σ_{20} must fix exactly three points on the curve C of genus 7 and exchange the two rational curves. By the Riemann–Hurwitz formula, this implies that σ_5 fixes the same points on C and σ_4 has exactly eight fixed points on C and exchanges the two rational curves. This is not possible since, by Theorem 0.1 in [5], the number of fixed points equals $2\alpha + 4$, where $\alpha = \sum_{C \subset \mathrm{Fix}(\sigma_4)}(1 - g(C))$.

In case b), σ_{20} has exactly one fixed point on the genus 6 curve and two points on the rational curve R, while σ_5 has exactly five fixed points on the curve by the Riemann–Hurwitz formula. By the same formula, σ_4 has six fixed points on C. By Theorem 0.1 in [5], R is fixed by σ_4. This case corresponds to the statement.

Case c) is impossible since, by the Riemann–Hurwitz formula, a curve of genus 5 can not have an order 5 automorphism with more than two fixed points (and σ_5 would have this property). ∎

Lemma 3.21. *If $\chi_5 = 4$, there are no admissible cases.*

Proof. By [6], if $\chi_5 = 4$, then $\mathrm{Fix}(\sigma_5)$ contains either four isolated points or an elliptic curve and four isolated points. In both cases, $a_{1,5} = 3$ and $a_{2,5} = 1$. Observe that points of type $A_{4,20}, A_{5,20}, A_{9,20}$ lie on a curved fixed by σ_5, while points of type $A_{i,20}$ with $i \in \{1, 2, 3, 6, 7, 8\}$ are isolated fixed points for σ_5. Since the action of σ_{20} on $\mathrm{Fix}(\sigma_5)$ has order 2 or 4, in both cases the point of type $A_{2,5}$ is fixed by σ_{20} and $a_{2,20} + a_{7,20} = 1$, and $a_{1,20} + a_{3,20} + a_{6,20} + a_{8,20}$ is either 1 or 3.

If $\mathrm{Fix}(\sigma_5)$ consists of four isolated points, $a_{4,20} + a_{5,20} + a_{9,20} = 0$ since there are no curves in $\mathrm{Fix}(\sigma_5)$. A Magma computation shows that the holomorphic Lefschetz formula has no solutions satisfying these conditions.

If $\mathrm{Fix}(\sigma_5)$ consists of four isolated points and an elliptic curve E, by the Riemann–Hurwitz formula, E contains 0, 2 or 4 isolated points for σ_{20}, thus $a_{4,20} + a_{5,20} + a_{9,20} \in \{0, 2, 4\}$. The holomorphic Lefschetz formula has no solutions with these restrictions. ∎

Example 3.22. Let B be the plane sextic defined by

$$F_6(x_0, x_1, x_2) = x_0\,(x_1^5 + a_1 x_2^5 + a_2 x_0^2 x_2^3 + a_3 x_0^4 x_2) = 0,$$

where $a_1, a_2, a_3 \in \mathbb{C}$ are general. Observe that B is the union of a smooth plane quintic C and a line L. Let Y be the double cover of $\mathbb{P}^2$ branched along B, which can be defined by the equation $x_3^2 - F_6(x_0, x_1, x_2) = 0$ in $\mathbb{P}(1, 1, 1, 3)$. The surface Y has the order 20 automorphism

$$\sigma_{20}(x_0, x_1, x_2, y) = (-x_0, \zeta_5 x_1, x_2, iy).$$

The surface Y has five singular points of type A_1 over the intersection points of C and L. Its minimal resolution X is a K3 surface and σ_{20} lifts to an automorphism $\tilde{\sigma}_{20}$ of X. The automorphism $\tilde{\sigma}_{20}$ has three fixed points over $(1,0,0)$, $(0,1,0)$, $(0,0,1)$.

Proof of Theorem 1.2, order 20. Let X be a K3 surface with a purely non-symplectic automorphism σ_{20} of order 20. By Proposition 3.19, $\mathrm{Fix}(\sigma_5)$ contains a curve C_2 of genus 2 and one point. The linear system $|C_2|$ defines a morphism $\varphi\colon X \to \mathbb{P}^2$ of degree 2 which contracts all smooth rational curves orthogonal to C_2. Since σ leaves C_2 invariant, then it induces an automorphism $\bar{\sigma}$ of $\mathbb{P}^2$ which can be assumed to be diagonal.

Let $\bar{\sigma}_2 = \bar{\sigma}^{10}$ and assume it has order 2 . Thus its fixed locus is the union of a line and one point, so that $\mathrm{Fix}(\sigma_2)$ contains a fixed curve of genus at most 2, contradicting the fact that σ_2 fixes a curve of genus 6. Thus σ_2 coincides with the covering involution of φ.

Now consider the automorphism $\bar{\sigma}_4 = \bar{\sigma}^5$, whose order is equal to 2 . Its fixed locus contains a line; we can assume it to be $L = \{x_0 = 0\}$ up to projectivities. By Proposition 3.19, the line L must be a component of the branch curve B of φ.

Finally, let $\bar{\sigma}_5 = \bar{\sigma}^4$. Since σ_5 has a fixed curve, then $\bar{\sigma}_5$ must fix a line L' which is not equal to the line fixed by $\bar{\sigma}_4$, thus up to projectivities we can assume $L' = \{x_1 = 0\}$. In these coordinates,

$$\bar{\sigma}(x_0, x_1, x_2) = (-x_0, \zeta_5 x_1, x_2).$$

The branch curve B is reduced, invariant for $\bar{\sigma}$ and must contain the line L as a component. This implies that its equation is as in Example 3.22. If all the coefficients of F_6 are non-zero, then one obtains the equation in Table 2 up to rescaling the variables. ∎

Remark 3.23. It follows from the proof of Theorem 1.2, case $n = 20$, that there are five smooth rational curves $R_1, \dots, R_5$ in X, each intersecting at one point the two fixed curves C_6 and R of σ_2. The classes of the curves $R, R_1, \dots, R_5$ all belong to the invariant lattice $S(\sigma_2)$. Observe that the classes of

$$2R + R_1 + R_2 + R_3 + R_4, \quad 2R + R_1 + R_2 + R_3 + R_5, \quad R, \quad R_1, \quad R_2, \quad R_3,$$

generate a lattice S isometric to $U(2) \oplus D_4$. Since S is contained in $S(\sigma_2)$ and $\det(S(\sigma_2)) = \det(S) = -2^4$ by Theorem 0.1 in [6], then $S = S(\sigma_2)$.

3.7. Order 24

Proposition 3.24. *Let X be a K3 surface with a purely non-symplectic automorphism $\sigma = \sigma_{24}$ of order* 24 *such that* $\dim(V^\sigma) = 2$. *Then the fixed loci of σ_{24} and some of its powers are as follows*:

$\mathrm{Fix}(\sigma_{24})$	$\mathrm{Fix}(\sigma_{12})$	$\mathrm{Fix}(\sigma_6)$
$\{p_1, p_2, p_3, p_{12}, p_{13}\}$	$\{p_1, p_2, p_3, p_{12}, p_{13}\}$	$R_1 \sqcup \{p_1, \dots, p_{11}\}$

$\mathrm{Fix}(\sigma_3)$	$\mathrm{Fix}(\sigma_2)$
$C_4 \sqcup R_1 \sqcup \{p_1\}$	$C_7 \sqcup R_1 \sqcup R_2$

where $g(C_i) = i$ for $i = 4,7$ and $g(R_1) = g(R_2) = 0$. Moreover, we have that $\chi(\mathrm{Fix}(\sigma_4)) = \chi(\mathrm{Fix}(\sigma_8)) = 8$, $d = (2,0,0,0,0,1,0,4)$, *and* $\mathrm{NS}(X) = S(\sigma_2) \cong U \oplus D_4$ *for a very general such K3 surface X.*

Proof. Decomposing $H^2(X,\mathbb{C})$ as the direct sum of the eigenspaces of σ_{24}^*, we obtain

$$22 = 8d_{24} + 4d_{12} + 4d_8 + 2d_6 + 2d_4 + 2d_3 + d_2 + d_1.$$

Let $\chi_i = \chi(\mathrm{Fix}(\sigma_i)), i \in \{2,3,4,6,8,12,24\}$. By the topological Lefschetz fixed point formula, we get

$$\begin{cases} \chi_{24} = d_6 - d_3 - d_2 + d_1 + 2, \\ \chi_{12} = 2d_{12} - d_6 - 2d_4 - d_3 + d_2 + d_1 + 2, \\ \chi_8 \;= -2d_6 + 2d_3 - d_2 + d_1 + 2, \\ \chi_6 \;= 4d_{24} - 2d_{12} - 4d_8 - d_6 + 2d_4 - d_3 + d_2 + d_1 + 2, \\ \chi_4 \;= -4d_{12} + 2d_6 - 2d_4 + 2d_3 + d_2 + d_1 + 2, \\ \chi_3 \;= -4d_{24} - 2d_{12} + 4d_8 - d_6 + 2d_4 - d_3 + d_2 + d_1 + 2, \\ \chi_2 \;= -8d_{24} + 4d_{12} - 4d_8 + 2d_6 + 2d_4 + 2d_3 + d_2 + d_1 + 2. \end{cases} \tag{3.9}$$

Computing all possible values of the vector d, one can see that $\chi_3 \in \{0,-3,-6\}$.

Assume $\chi_3 = 0$. By [4], $\mathrm{Fix}(\sigma_3)$ is either the union of genus 2 curve and two isolated points, or the union of a genus 3 curve, a smooth rational curve and two isolated points. Clearly $\mathrm{Fix}(\sigma_6) \subseteq \mathrm{Fix}(\sigma_3)$, and in this case $\chi_6 = 16$ or 8. The first case is incompatible with the structure of the fixed locus of σ_3. If $\chi_6 = 8$, then the fixed locus of σ_3 must be the union of a genus 3 curve C, a smooth rational curve R and two isolated points p, q. The automorphism σ_6 fixes four points on C and p, q. Moreover, it either fixes pointwise R or it has two isolated fixed points on it. Both cases are incompatible with Theorem 4.1 in [12], since the fixed points of σ_6 contained in the fixed curve of σ_3 are those of type $A_{2,6}$ (of type $\frac{1}{6}(3,4)$ in [12]).

If $\chi_3 = -6$, we have $\chi_6 = 10$, and this can be seen to be incompatible with Theorem 4.1 in [12] with an argument similar to the previous one.

If $\chi_3 = -3$ and by [4], $\mathrm{Fix}(\sigma_3)$ is either the union of a curve of genus 3 and one point, or the union of a curve of genus 4, a smooth rational curve and one point. In these cases we have $\chi_6 = 13$, which excludes the first possibility for $\mathrm{Fix}(\sigma_3)$. Thus $\mathrm{Fix}(\sigma_3)$ is the union of a curve C of genus 4, a smooth rational curve R and one point p. Using the Riemann–Hurwitz formula for σ_6 and the fact that $\chi_6 = 13$, we obtain that σ_6 fixes p and ten points on C. Moreover, by Theorem 4.1 in [12], the curve R is pointwise fixed by σ_6. In this case one computes that χ_{12} is either 5 or 1, but the second case is not possible since σ_{12} either fixes pointwise or has two fixed points on R. Thus $\mathrm{Fix}(\sigma_{12})$ fixed p, two points on C and it either fixes pointwise or has two fixed points on R. A computation using the holomorphic Lefschetz formula shows that the first case does not occur. In this case one computes that $\chi_{24} \in \{-1,1,3,5,7\}$. The only cases compatible with the structure of $\mathrm{Fix}(\sigma_{12})$ are $\chi_{24} = 3$ or 5. The first case is impossible by the Riemann–Hurwitz formula.

Assuming $\chi_3 = -3$, $\chi_6 = 13$, $\chi_{12} = 5$ and $\chi_{24} = 5$, we find two possible vectors $d = (2,0,0,0,0,1,0,4), (2,0,0,1,0,0,1,3)$. For these cases, $\chi_2 = -8$, $\chi_4 = 8$. Moreover, $\chi_8 = 8$ in the first case and 2 in the second case.

By [20], the fixed locus of σ_2 is either the union of a curve C_7 of genus 7 and two smooth rational curves (R_1 and R_2), or the union of a curve C_6 of genus 6 and R_1. The latter is not possible by the Riemann–Hurwitz formula applied to σ_6 restricted to C_6. Since $\chi_4 = 8$, σ_4 must fix four points on C_7, two points on R_1 and it either fixes pointwisely R_2

or it has two fixed points on it. This implies that $\mathrm{Fix}(\sigma_8)$ contains isolated points and, at most, a smooth rational curve. Thus $\chi_8 \geq \chi_{24} = 5$, which excludes the case $d = (2,0,0,1,0,0,1,3)$.

Finally, by Theorem 4.2.2 in [21] or Figure 1 in [6], the invariant lattice of σ_2 is isometric to $U \oplus D_4$. For a very general K3 surface, we have $\mathrm{rk}\,\mathrm{NS}(X) = 22 - 2\varphi(24) = 6$. Moreover, $S(\sigma_2) \subseteq \mathrm{NS}(X)$ by Remark 2.1, thus $S(\sigma_2) = \mathrm{NS}(X)$. ∎

Example 3.25. Consider the elliptic surface with equation

$$y^2 = x^3 + t(t^4 - 1)(t^4 - a), \quad a \in \mathbb{C}.$$

For general $a \in \mathbb{C}$, this is a K3 surface and carries the order 24 automorphism

$$\sigma_{24}(x, y, t) = (\zeta_{12} x, \zeta_8 y, it).$$

The action of σ^* on the holomorphic two form $\omega_X = (dx \wedge dt)/2y$ is the multiplication by $\zeta_{12}\zeta_4\zeta_8^{-1}$, thus σ_{24} is purely non-symplectic. For general $a \in \mathbb{C}$, the elliptic fibration has a singular fiber F_∞ of type I_0^* over $t = \infty$ and nine fibers of type II. The automorphism σ_2 fixes the section at infinity R_1, the genus 7 curve defined by $y = 0$ and the central component R_2 of the fiber F_∞. The automorphism σ_3 fixes R_1, the curve of genus 4 defined by $x = 0$ and the intersection point p_1 between R_2 and the component of F_∞ intersecting R_1. Observe that the remaining three components of F_∞ are permuted by σ_3. The automorphism σ_6 fixes the nine singular points $p_3, \ldots, p_{11}$ of the fibers of type II, the point p_1 and the intersection point p_2 between the fiber F_∞ and the curve $x = 0$. Finally, the automorphisms σ_{12} and σ_{24} fix the singular point p_3 of the fiber F_0 of type II over $t = 0$, the intersection points of R_1 with the fibers F_0, F_∞, p_1 and p_2.

Proof of Theorem 1.2, order 24. By Proposition 3.24, the fixed locus of σ_2 is the union of a curve C_7 of genus 7 and two rational curves R_1 and R_2. Moreover, $\mathrm{NS}(X) = S(\sigma_2) \cong U \oplus D_4$ for a very general X. Following the first part of the proof of Theorem 1.2 for order 16, we find that a very general X has a Jacobian elliptic fibration $\pi\colon X \to \mathbb{P}^1$ with a fiber of type I_0^* such that R_1 can be assumed to be a section of π, R_2 is the central component of the reducible fiber, and C_7 intersects a general fiber in three points. It follows from Lemma 5 in [5] with $x = [C_7]$ that π is invariant for σ_{24}. Since the fixed locus of σ_3 contains a curve C_4 of genus > 1, then each fiber of π is invariant for σ_3. Moreover, σ_3 fixes pointwise the curve R_1. Thus, up to a coordinate change, π has Weierstrass equation of the form

$$y^2 = x^3 + p(t),$$

where $\deg(p) \leq 12$, $\sigma_2(x, y, t) = (x, -y, t)$ and $\sigma_3(x, y, t) = (\zeta_3 x, y, t)$. Observe that σ_8 preserves R_1, but $\sigma_4 = \sigma_8^2$ does not fix it pointwisely, since otherwise R_1 would be contained in the fixed locus of σ_{12}, contradicting Proposition 3.24. Thus σ_8 induces an automorphism $\bar{\sigma}_8$ of order 4 on $\mathbb{P}^1$. Up to a coordinate change, we can assume that $\bar{\sigma}_8(t) = it$. Since the reducible fiber of type I_0^* must be preserved by σ_8, then we we can assume it to be over $t = \infty$. By [19], this implies that the $\deg(p) = 9$ (so that it has a triple root at infinity). Moreover, since its zero set is invariant for $\bar{\sigma}_8$, then $p(t) = t(t^4 - a)(t^4 - b)$ for some $a, b \in \mathbb{C}$. Finally, since $\sigma_8^4 = \sigma_2$, we can assume that $\sigma_8(x, y, t) = (-ix, \zeta_8 y, it)$. This implies that, up to a coordinate change, X belongs to the family in Example 3.25. ∎

4. Classification for order 22

We now provide a classification theorem of purely non-symplectic automorphisms σ of order 22 on a K3 surface according to their fixed locus. Observe that, since $\varphi(22) = 10$, then $\dim(V^\sigma) \in \{1, 2\}$. The case when $\dim(V^\sigma) = 2$ has been studied in Section 3.

We recall that the fixed locus of any power $\sigma_i := \sigma^{22/i}$ of σ is of the form

$$C_i \sqcup R_1 \sqcup \cdots \sqcup R_{k_i} \sqcup \{p_1, \ldots, p_{N_i}\},$$

where $g(C_i) = g_i$, and $g(R_\ell) = 0$ for $\ell = 1, \ldots, k_i$.

Remark 4.1. As in Lemma 1.3 of [3], a straightforward computation using the holomorphic Lefschetz formula shows that a non-symplectic automorphism of order 22 is purely non-symplectic.

Theorem 4.2. *Let σ be a purely non-symplectic automorphism of order 22 of a complex K3 surface X. Then the invariants (g_i, k_i, N_i) of the fixed locus of $\sigma_i := \sigma^{22/i}$, the vector $d = (d_{22}, d_{11}, d_2, d_1)$ giving the dimensions of the eigenspaces of σ^* in $H^2(X, \mathbb{C})$, and the Néron–Severi lattice of a very general K3 surface carrying an automorphism with such invariants, are given by one of the rows of Table 8. Moreover, all cases in the table exist.*

	N_{22}	g_{22}	k_{22}	N_{11}	g_{11}	k_{11}	g_2	k_2	d	NS
A1	6	-	0	2	1	0	10	1	(2,0,0,2)	U
B1	11	0	0	11	0	0	5	5	(1,0,1,11)	$U \oplus A_{10}$
B2	9	0	0	11	0	0	5	4	(1,0,2,10)	$U \oplus A_{10}$
B3	5	-	0	11	0	0	5	1	(1,0,5,7)	$U \oplus A_{10}$

Table 8. Order 22.

Proof. Let σ_{11} be the square of σ_{22}. According to Table 4 in [6], the fixed locus of σ_{11} is either a) the union of a smooth elliptic curve and two points, or b) the union of a rational curve and eleven points. In the first case, $m := \frac{1}{10}(22 - \operatorname{rank} S(\sigma_{11})) = 2$, while in the second case, $m = 1$.

Recall that fixed points of type $A_{10,22}$ lie on a curve in $\mathrm{Fix}(\sigma_{11})$, while points of type $A_{i,22}, A_{10-i,22}$ correspond to isolated points for σ_{11} of type $A_{i,11}$, $i = 1, \ldots, 4$. The Lefschetz holomorphic formula with the restrictions

$$a_{5,22} \leq a_{5,11}, \quad a_{i,22} + a_{10-i,22} \leq a_{i,11}, \quad i = 1, 2, 3, 4$$

gives the solutions as in Table 9, where we compute χ_{22} and χ_2 by (3.1).

In case A1, we have $d_1 + d_2 = 2$ and $d_{11} + d_{22} = 2$ by Table 4 in [6]. Since $\chi_{22} = 6$, then $(d_{22}, d_{11}, d_2, d_1) = (2, 0, 0, 2)$ by (3.1). The description of the fixed locus of σ_2 is thus obtained as in the proof of Proposition 3.4.

We now study the possibilities for $\mathrm{Fix}(\sigma_{22})$ when $\mathrm{Fix}(\sigma_{11})$ is the union of a rational curve and eleven points. By [20], the fixed locus of the involution σ_2 is the union of a curve of genus g_2 and k_2 rational curves and $\chi_2 = 2(1 - g_2 + k_2)$. Thus in case B1 one has $(g_2, k_2) \in \{(0, 0), (1, 1), (2, 2), (3, 3), (4, 4), (5, 5)\}$. The only admissible one is

	$(a_{1,22}, a_{2,22}, \dots, a_{10,22})$	α	χ_{22}	χ_2
A1	(0, 0, 0, 1, 0, 0, 0, 0, 1, 4)	0	6	−16
B1	(3, 2, 1, 1, 1, 2, 1, 0, 0, 0)	1	13	2
B2	(3, 2, 2, 1, 1, 0, 0, 0, 0, 0)	1	11	0
B3	(0, 0, 0, 1, 1, 0, 0, 0, 1, 2)	0	5	−6

Table 9

$(g_2, k_2) = (5, 5)$, since otherwise, recalling that isolated points of σ_{22} lie on fixed curves for σ_2, one gets a contradiction with the Riemann–Hurwitz formula.

As for case B2, one has $(g_2, k_2) \in \{(1,0), (2,1), (3,2), (4,3), (5,4), (6,5)\}$. The first four cases give a contradiction to the Riemann–Hurwitz formula. The case $(g_2, k_2) = (6, 5)$ is not admissible since, by Proposition V.2.14 in [14], a curve of genus 6 does not admit an automorphism of order 11 acting on it.

Similarly, in case B3 the possibilities are $(g_2, k_2) \in \{(4,0), (5,1), (6,2)\}$, and the only admissible one is $(g_2, k_2) = (5, 1)$. The vector $d = (d_{22}, d_{11}, d_2, d_1)$ is obtained in all cases by means of (3.1).

The Néron–Severi lattice of a very general K3 surface in case A1 has been given in Section 3.2. In the remaining cases, which have $d_{22} = 1$, the rank of the Néron–Severi lattice in the very general case is $22 - \varphi(22) = 12$. Since the lattice $S(\sigma_{11})$ is a primitive sublattice of $\mathrm{NS}(X)$ by Remark 2.1 and has rank $d_1 + d_2 = 12$ in each case, then $\mathrm{NS}(X) = S(\sigma_{11})$. By [22] or [6], Section 7, the lattice $S(\sigma_{11})$ is isometric to $U \oplus A_{10}$.

An example for case A1 has been given in Section 3.2. We now provide examples for the cases B1, B2, B3. ∎

Example 4.3. (Case B1) Let X be the elliptic K3 surface whose elliptic fibration is given by

$$y^2 = x^3 + t^7 x + t^5.$$

The singular fibers of the fibration are II^* over $t = 0$, III over $t = \infty$, and eleven fibers of type I_1. The automorphism

$$\sigma_{22} : (x, y, t) \mapsto (\zeta_{22}^2 x, \zeta_{22}^3 y, \zeta_{22}^{10} t)$$

is purely non-symplectic of order 22 since its action on the two form $\frac{dx \wedge dt}{2y}$ is the multiplication by $-\zeta_{11}$. The automorphism σ_{22} preserves the fibers over $t = 0$ and $t = \infty$. In the fiber over $t = 0$, which is of type II^*, it must fix the component of multiplicity 6 and has eight isolated fixed points in the other components. In the fiber over $t = \infty$, it fixes three isolated points. The involution σ_2 preserves each fiber of the elliptic fibration, thus it must fix R, three more components of the fiber over $t = 0$, the section at infinity and the 3-section $y = 0$, which has genus 5. This corresponds to case B1.

Example 4.4. (Case B2) Let us consider the elliptic fibration

$$y^2 = x^3 + t^5 x + t^2,$$

The fibration has a fiber of type IV over $t = 0$, a fiber of type III^* over $t = \infty$, and eleven fibers of type I_1. The automorphism

$$\sigma_{22}(x, y, t) = (\zeta_{11}^8 x, -\zeta_{11} y, \zeta_{11} t)$$

is purely non-symplectic of order 22 since its action on the two form $\frac{dx \wedge dt}{2y}$ is the multiplication by $-\zeta_{11}^8$. By Example 7.4 in [6], σ_{11} has fixed locus $R \cup \{p_1, \dots, p_{11}\}$, where R is the central component of the fiber of type III*. The involution σ_2 maps (x, y, t) to $(x, -y, t)$, thus it preserves each fiber. This implies that it fixes R and two more rational components of the fiber of type III*, as well as the section at infinity and the 3-section $y = 0$, whose genus is 5. This corresponds to case B2.

Example 4.5. (Case B3) We already observed in Section 3.2 that the elliptic K3 surface defined by

$$y^2 = x^3 + ax + (t^{11} - 1), \quad a \in \mathbb{C}^*,$$

with the automorphism $\sigma_{22}\colon (x, y, t) \mapsto (x, -y, \zeta_{11} t)$, is an example of case A. If $a^3 = -27/4$, thus the fibration admits a singular fiber of type II over $t = 0$, I_{11} over $t = \infty$, and eleven fibers of type I_1. The fixed locus of the automorphism σ_{11} is contained in the fibers over $t = 0$ and $t = \infty$. Since it fixes eleven isolated points and one rational curve, then it must fix one of the components of the fibre of type I_{11}, say R, has nine fixed points in the other components of the same fibers, and two more fixed points in the fiber of type II. The involution σ_2 fixes the section at infinity and the curve $y = 0$, which has genus 5. Moreover, σ_2 can not preserve each component of the fiber of type I_{11}, by Lemma 4 in [5]. Thus σ_2 acts on the fiber of type I_{11} as a reflection, without fixed components and with a unique invariant component. This corresponds to case B3.

Remark 4.6. By Section 2.2, the moduli space of K3 surfaces having a purely non-symplectic automorphism of order 22 whose invariants are as in cases B1, B2 or B3 is 0-dimensional, since $\dim(V^\sigma) = 1$. In fact, since $\operatorname{rk} T(X) = 10 = \varphi(22)$ and f^* has order 11 on $\mathrm{NS}(X)$, then it follows from Theorem 5.9 in [9] that there is a unique K3 surface X which carries the three types of non-symplectic automorphisms of order 22. Thus the K3 surfaces given in Examples 4.3, 5.4 and 4.5 are isomorphic.

5. Classification for order 15

We now provide a classification theorem of purely non-symplectic automorphisms σ of order 15 on a K3 surface according to their fixed locus. Observe that, since $\varphi(15) = 8$, then $\dim(V^\sigma) \in \{1, 2\}$. The case when $\dim(V^\sigma) = 2$ has been studied in Section 3.

Theorem 5.1. *Let σ be a purely non-symplectic automorphism of order 15 of a complex K3 surface X. Then the invariants (g_i, k_i, N_i) of the fixed locus of $\sigma_i := \sigma^{15/i}$, the vector $d = (d_{15}, d_5, d_3, d_1)$ giving the dimensions of the eigenspaces of σ^* in $H^2(X, \mathbb{C})$ and the Néron–Severi lattice of a very general K3 surface carrying an automorphism with such invariants, are given by one of the rows of Table 10. Moreover, all cases in the table exist.*

Proof. According to [6], the fixed locus of the cube of σ_{15}, i.e., σ_5, is the union of a smooth curve of genus g_5, k_5 rational curves, and $a_{1,5} + a_{2,5}$ isolated points, with g_5, k_5, $a_{1,5}$ and $a_{2,5}$ as in one of the lines of Table 11.

Recall that $\alpha = \sum_{C \subset \mathrm{Fix}(\sigma_{15})} (1 - g(C))$. In order to find all possibilities for $\mathrm{Fix}(\sigma_{15})$, we will look for a solution $a := (a_{1,15}, a_{2,15}, \dots, a_{7,15}, \alpha)$ of the holomorphic Lefschetz formula compatible with the system of equations (3.2).

	N_{15}	g_{15}	k_{15}	N_5	g_5	k_5	N_3	g_3	k_3	d	NS
A1	5	-	0	1	2	0	2	2	0	$(2,1,0,2)$	$U(3) \oplus A_2 \oplus A_2$
B1	7	-	0	4	1	0	1	4	1	$(2,0,1,4)$	$H_5 \oplus A_4$
B2	7	-	0	4	1	0	6	0	2	$(1,2,0,6)$	$U \oplus E_6 \oplus A_2^{\oplus 3}$
B3	4	-	0	4	1	0	0	4	0	$(2,0,2,2)$	$H_5 \oplus A_4$
D1	10	0	0	7	1	1	6	0	2	$(1,1,0,10)$	$U \oplus E_6 \oplus A_2^{\oplus 3}$
F3	9	0	0	10	0	1	4	2	2	$(1,0,2,10)$	$H_5 \oplus A_4 \oplus E_8$
F7	12	0	0	10	0	1	5	2	3	$(1,0,1,12)$	$H_5 \oplus A_4 \oplus E_8$
F8	5	-	0	10	0	1	2	2	0	$(1,0,4,6)$	$H_5 \oplus A_4 \oplus E_8$

Table 10. Order 15.

	$a_{1,5}$	$a_{2,5}$	g_5	k_5	m
A	1	0	2	0	5
B	3	1	1	0	4
C	3	1	-	-	4
D	5	2	1	1	3
E	5	2	0	0	3
F	7	3	0	1	2
G	9	4	0	2	1

Table 11. Fixed locus of σ_5.

Remark 5.2. We recall that points of type $A_{4,15}, A_{5,15}$ lie on a curve fixed by σ_5 and not by σ_{15}. Thus if $a_{4,15} + a_{5,15} > 0$, there is at least a curve in $\mathrm{Fix}(\sigma_5) \setminus \mathrm{Fix}(\sigma_{15})$.

Remark 5.3. Observe that by [10], a curve of genus 3 does not admit an automorphism of order 5. Thus if $\mathrm{Fix}(\sigma_3)$ contains a curve of genus 3, such curve is also fixed by σ_{15}.

We now analyze each line of the previous table separately.

• **Case A**: corresponds to $\chi(\mathrm{Fix}(\sigma_5)) = -1$. By equations (3.3). it follows that $a_{2,15} = a_{7,15} = 0$. The only solution of the holomorphic Lefschetz formula with this property is $a = (0,0,1,2,2,0,0,0)$. In particular, $\chi(\mathrm{Fix}(\sigma_{15})) = 5$. It follows from equations (3.2) that $d = (2,1,0,2)$. The proof thus follows as in the proof of Proposition 3.6.

• **Case B**: corresponds to $\chi(\mathrm{Fix}(\sigma_5)) = 4$, i.e., $\mathrm{Fix}(\sigma_5)$ is the disjoint union of a smooth curve of genus 1 and four points. By Example 5.6 in [6], X has an elliptic fibration $\pi : X \to \mathbb{P}^1$ which can be defined by a Weierstrass equation of the form

$$y^2 = x^3 + (t^5 + \alpha)x + (t^{10} + \beta t^5 + \gamma), \quad \alpha, \beta, \gamma \in \mathbb{C},$$

where $\sigma_5(x,y,t) = (x,y,\zeta_5 t)$. The automorphism σ_5 fixes pointwise the smooth fiber F_0 over $t = 0$ and leaves invariant the fiber F_∞ over $t = \infty$, which contains four fixed points. This property and the fact that $24 - e(F_\infty)$ must be divisible by 5, imply that F_∞ is of Kodaira type IV, i.e., the union of three smooth rational curves intersecting transversally at one point. Observe that the elliptic fibration π is invariant for σ_3, since the smooth fiber over $t = 0$ is invariant for σ_3, and thus the same holds for the associated linear system. Moreover, σ_3 must preserve all fibers of π, since otherwise 15 should divide $24 - e(F_\infty) = 20$, a contradiction. The remaining singular fibers of π, considering the fact that they are preserved by σ_3 (thus $J = 0$) and that $24 - e(F_\infty) = 20$, are either five fibers of type IV, or ten fibers of type II.

By the holomorphic Lefschetz formula and equations (3.3), we find that either $a = (0, 1, 0, 0, 3, 0, 0, 0)$ or $a = (0, 0, 0, 0, 3, 3, 1, 0)$.

If $a = (0, 1, 0, 0, 3, 0, 0, 0)$, then it follows from equations (3.2) that either $d = (2, 0, 2, 2)$ or $d = (1, 2, 1, 4)$. The first case has been considered in the proof of Proposition 3.6 (case B3). In the second case, by (3.2) and Table 1 in [4], $\chi_3 = 9$ and the fixed locus of σ_3 contains at least two curves. We now exclude this case.

The automorphism σ_{15} fixes four points: three of them lie on the unique curve F_0 fixed by σ_5 and the other one is an isolated fixed point for σ_5. By the previous description, it follows that σ_3 must fix the center of the fiber F_∞ and permutes the other three fixed points of σ_5 on it (and thus the three components of the fiber F_∞). Moreover, being of types $A_{2,15}$ and $A_{5,15}$, the fixed points of σ_{15} are all contained in a curve C fixed by σ_3. Since C passes through the center of the fiber F_∞, then it is connected, and by the Riemann–Hurwitz formula, it is the unique fixed curve of σ_3 which is transversal to the fibers of π. On the other hand, σ_3 can not fix a curve R contained in a fiber of π, since the other singular fibers are either of type II, or of type IV, and in both cases R would intersect C, a contradiction. Thus σ_3 fixes at most one (connected) curve, so that the case $d = (1, 2, 1, 4)$ is not possible.

If $a = (0, 0, 0, 0, 3, 3, 1, 0)$, then it follows from equations (3.2) that either $d = (2, 0, 1, 4)$ or $d = (1, 2, 0, 6)$. If $d = (2, 0, 1, 4)$, then by (3.2) and Table 1 in [4], $\chi_3 = -3$ and the fixed locus of σ_3 consists either of the disjoint union of a genus 3 curve and one point, or the disjoint union of a curve of genus 4 , a rational curve and one point. The first case is not possible by Remark 5.3.

If $d = (1, 2, 0, 6)$, then by (3.2) and Table 1 in [4], $\chi_3 = 12$ and the fixed locus of σ_3 consists either of the union of three disjoint rational curves and six points, or the disjoint union of a curve of genus 1 , three rational curves and six points. We now exclude the second case. Observe that in this case σ_{15} fixes three points on F_0 and four isolated points in the fiber F_∞. Six of these points are contained in a curve fixed by σ_3, which will intersect each fiber of π at three points counting multiplicity. The same argument as before shows that σ_3 can not fix a curve contained in a fiber of π. Thus σ_3 fixes at most three (connected) curves.

To conclude, the only possible cases have $a = (0, 0, 0, 0, 3, 3, 1, 0)$ and either $d = (2, 0, 1, 4)$ with σ_3 fixing a genus 4 curve, a rational curve and one point (case B1), or $d = (1, 2, 0, 6)$ with σ_3 fixing three smooth rational curves and six points (case B2).

• **Case C**: in this case, σ_5 fixes exactly four points; more precisely, $a_{1,5} = 3$ and $a_{2,5} = 1$. As before, by the holomorphic Lefschetz formula one obtains that either

$$a = (0, 1, 0, 0, 3, 0, 0, 0) \quad \text{or} \quad a = (0, 0, 0, 0, 3, 3, 1, 0).$$

In both cases, $a_{4,15} + a_{5,15} > 0$, thus this case is not possible by Remark 5.2.

• **Case D**: in this case the fixed locus of σ_5 contains an elliptic curve, a smooth rational curve R and seven isolated fixed points, with $a_{1,5} = 5$ and $a_{2,5} = 2$. The holomorphic Lefschetz formula with the restrictions of (3.3) gives four solutions for the vector a:

(5.1) $(0, 0, 0, 0, 3, 3, 1, 0), (0, 0, 1, 2, 2, 0, 0, 0), (0, 1, 0, 0, 3, 0, 0, 0), (3, 2, 2, 3, 0, 0, 0, 1).$

The only one compatible with equations in (3.2) is $a=(3,2,2,3,0,0,0,1)$. By Remark 5.2, a solution with $\alpha = 1$ means that only R is fixed by σ_{15}. By (3.2), this gives $\chi_3 = 9$. According to Table 1 in [4], there are two possibilities for $\mathrm{Fix}(\sigma_3)$:

D1: disjoint union of three smooth rational curves and six points;

D2: disjoint union of an elliptic curve, three smooth rational curves and six points.

We now show that case D2 is not possible. Let

$$\mathrm{Fix}(\sigma_3) = E \cup R_1 \cup R_2 \cup R_3 \cup \{p_1, p_2, \dots, p_6\}$$

and consider the elliptic fibration $\pi: X \to \mathbb{P}^1$ defined by the linear system $|E|$. The automorphism $\bar{\sigma}_3$ induced by σ_3 on $\mathbb{P}^1$ is not the identity, since otherwise σ_3 should act on the general fiber of π either as a translation (which is impossible since σ_3 is non-symplectic), or with fixed points (impossible, since otherwise σ_3 should fix a curve which is transverse to all fibers, and thus intersecting E). Thus $\bar{\sigma}_3$ has order 3 and fixes two points in $\mathbb{P}^1$, one of them corresponding to the fiber E. The smooth rational curves and the isolated points fixed by σ_3 must be components of the other invariant fiber. This implies that such fiber is of type $\mathrm{I}_6^* = \tilde{D}_{10}$.

Since the curve E is preserved by σ_5, thus the fibration π is preserved too. The fixed locus of σ_5 contains a curve of genus one E'. The curve E' can not be transverse to the fibers of π, since otherwise the general fiber of π would have an order 5 automorphism with a fixed point, which is impossible by [16], Corollary 4.7, IV. Thus E' is one of the fibers of π. A similar reasoning to the one used for σ_3 implies that σ_5 induces an order 5 automorphism of $\mathbb{P}^1$, thus it preserves exactly two fibers of π. Observe that σ_5 must preserve both E, since it commutes with σ_3, and the fiber of type $\mathrm{I}_6^* = \tilde{D}_{10}$, since an elliptic fibration of a K3 surface can not have five fibers of this type (the Euler number of the fiber is 12). This implies that $E = E'$, thus E would be a fixed curve of σ_{15}, a contradiction.

• **Case E**: as in the previous case, $a_{1,5} = 5, a_{2,5} = 2$ and the holomorphic Lefschetz formula with the restrictions of (3.3) has the four solutions of (5.1). Since in each case $a_{4,15} + a_{5,15} > 0$, then by Remark 5.2 the only curve fixed by σ_5 is not fixed by σ_{15} and $\alpha = 0$. For each one of the three possible a's with $\alpha = 0$, the system (3.2) has no solutions. Thus there are no σ_{15} such that σ_5 has invariants as in case E.

• **Case F**: in this case, $\mathrm{Fix}(\sigma_5)$ contains two rational curves R_1, R_2 and ten points with $a_{1,5} = 7, a_{2,5} = 3$. The holomorphic Lefschetz formula with the restrictions of (3.3) gives nine solutions, all of them with $\alpha = 0$ or 1. Thus at most one of the two curves R_i is contained in $\mathrm{Fix}(\sigma_{15})$.

If $\mathrm{Fix}(\sigma_{15})$ contains a rational curve, then $\alpha = 1$, and combining the nine solutions of the Lefschetz formula with (3.2) one gets the possibilities F1–F7 of Table 12. If $\mathrm{Fix}(\sigma_{15})$ only contains points, then $\alpha = 0$ and by (3.2) we get possibilities F8 and F9.

By Remark 5.3, we exclude cases F4 and F9.

Case F1 has to be excluded for the following reason: the total number of fixed points for σ_{15} is nine, and σ_{15} fixes a rational curve. Thus, $a_{2,15} + a_{3,15} + a_{5,15} + a_{6,15} = 5$ of the isolated fixed points for σ_{15} lie on curves fixed by σ_3. However, $\mathrm{Fix}(\sigma_3)$ contains just one rational curve, which is fixed by σ_{15}, giving a contradiction.

	$a_{1,5}$	$a_{2,5}$	$a_{1,3}$	g_3	k_3	$a_{1,15}$	$a_{2,15}$	$a_{3,15}$	$a_{4,15}$	$a_{5,15}$	$a_{6,15}$	$a_{7,15}$	α
F1	7	3	4	0	0	3	3	1	1	1	0	0	1
F2	7	3	4	1	1	3	3	1	1	1	0	0	1
F3	7	3	4	2	2	3	3	1	1	1	0	0	1
F4	7	3	4	3	3	3	3	1	1	1	0	0	1
F5	7	3	5	0	1	3	2	1	1	1	3	1	1
F6	7	3	5	1	2	3	2	1	1	1	3	1	1
F7	7	3	5	2	3	3	2	1	1	1	3	1	1
F8	7	3	2	2	0	0	0	1	2	2	0	0	0
F9	7	3	2	3	1	0	0	1	2	2	0	0	0

Table 12. Case F.

Case F2 has to be excluded for the following reason: σ_{15} acts as an automorphism of order 5 on the elliptic curve in $\mathrm{Fix}(\sigma_3)$ and it contains fixed points, which is not possible by [16], Corollary 4.7, IV. Case F6 is analogous.

In case F5, the total number of fixed points for σ_{15} is twelve: five of them are isolated for σ_3, thus seven points should lie on the rational curve in $\mathrm{Fix}(\sigma_3) \setminus \mathrm{Fix}(\sigma_{15})$. This is not possible by the Riemann–Hurwitz formula.

• **Case G**: in this case, $\mathrm{Fix}(\sigma_5)$ contains three rational curves and all solutions of the holomorphic Lefschetz formula with the restrictions of (3.3) have $\alpha = 0$ or 1. Thus at most one of the three rational curves in $\mathrm{Fix}(\sigma_5)$ is contained in $\mathrm{Fix}(\sigma_{15})$. Checking (3.2) for all solutions in both cases $\alpha = 0, 1$, we find no solutions. Thus there are no possible σ_{15} such that $\mathrm{Fix}(\sigma_5)$ is as in case G.

The Néron–Severi lattice of a very general K3 surface in cases A1, B1 and B3 has been given in Section 3.3. In the remaining cases, which have $d_{15} = 1$, the rank of the Néron–Severi lattice in the very general case is $22 - \varphi(8) = 14$. In cases B2 and D1, we have that the rank of $S(\sigma_3)$ is $d_1 + 4d_5 = 14$. Since $S(\sigma_3)$ is a primitive sublattice of $\mathrm{NS}(X)$ by Remark 2.1, we conclude that $\mathrm{NS}(X) = S(\sigma_3)$. By [4], the lattice $S(\sigma_3)$ in the two cases is isometric to $U \oplus E_6 \oplus A_2^{\oplus 3}$. A similar argument in the cases F_3, F_7 and F_8 shows that for a very general X the Néron–Severi lattice is equal to $S(\sigma_5)$ and is isometric to $H_5 \oplus A_4 \oplus E_8$ by [6]. ■

We now provide examples for all cases collected in Table 10, thus completing the proof of Theorem 5.1. Examples of cases A1, B1 and B3 can be found in Section 3.3.

Example 5.4. (Case B2). The elliptic K3 surface with Weierstrass equation

$$y^2 = x^3 + (t^5 - 1)^2$$

has six fibers of type IV, over $t = \infty$ and over the zeroes of $t^5 - 1$. It carries the order 15 automorphism

$$\sigma_{15} : (x, y, t) \mapsto (\zeta_3 x, y, \zeta_5 t).$$

The fixed locus of σ_5 is contained in the union of the smooth fiber over $t = 0$ and the fiber over $t = \infty$. The fixed locus of σ_3 contains the section at infinity, the two sections defined by $x = y \pm (t^5 - 1) = 0$ and the six centers of the fibers of type IV.

Example 5.5. (Case D1) This surface appears in [9]. Let X be the elliptic K3 surface with Weierstrass equation

$$y^2 = x^3 + t^5 x + 1,$$

The fibration has one fiber of type $\mathrm{III}^* = \tilde{E}_7$ over $t = \infty$, and fifteen fibers of type I_1. It carries the order 15 automorphism

$$\sigma_{15} : (x, y, t) \mapsto (\zeta_{15}^{10} x, y, \zeta_{15} t).$$

The automorphism $\sigma_5 = \sigma_{15}^3$ fixes the smooth fiber E over $t = 0$, the smooth rational curve of multiplicity 4 of the fiber over $t = \infty$ and seven isolated points in the same reducible fiber. Thus the invariants of σ_5 are $(g_5, k_5) = (1, 1)$, which corresponds to case D. The elliptic curve E is not fixed by $\sigma_3 = \sigma_{15}^5 : (x, y, t) \mapsto (\zeta_3 x, y, \zeta_3 t)$. The automorphism σ_3 fixes three smooth rational curves and three isolated points in the fiber over $t = \infty$, and three points in the curve E.

Example 5.6. (Case F3) Let Y be the double cover of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1, 1, 1, 3)$:

$$y^2 = x_2 (x_0^2 x_1^3 + x_2^5 + x_0^5).$$

The branch sextic B is the union of a line L and a quintic curve Q. The surface Y has four rational double points: one point of type D_7 at $(0, 1, 0, 0)$ and three points of type A_1 at $(-\zeta_3^i, 1, 0, 0)$, for $i = 0, 1, 2$. The minimal resolution of Y is a K3 surface X. The surface has the order 15 automorphism

$$\sigma_{15} : (x_0, x_1, x_2, y) \mapsto (x_0, \zeta_3 x_1, \zeta_5 x_2, \zeta_5^3 y).$$

We will denote by $\tilde{\sigma}_{15}$ the lifting of σ_{15} to X. The automorphism σ_3 fixes the genus 2 curve C_2 defined by $x_1 = 0$ and the singular point $(0, 1, 0, 0)$. Thus $\tilde{\sigma}_3$ fixes the proper transform of C_2 and the union of two components and four isolated points in the exceptional divisor of type D_7. Thus we are in case F3.

Example 5.7. (Case F7) Let Y be the double cover of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1, 1, 1, 3)$:

$$y^2 = x_2 (x_2^5 + x_1^5 + x_0^3 x_1 x_2).$$

The branch sextic B is the union of a line L and a quintic curve Q. The surface Y has a rational double point of type D_{10} at $(1, 0, 0, 0)$. The minimal resolution of Y is a K3 surface X. The surface has the order 15 automorphism

$$\sigma_{15} : (x_0, x_1, x_2, y) \mapsto (\zeta_5^2 x_0, \zeta_{15}^7 x_1, \zeta_3^2 x_2, y).$$

We will denote by $\tilde{\sigma}_{15}$ the lifting of σ_{15} to X. The automorphism σ_3 fixes the genus 2 curve C_2 defined by $x_0 = 0$ and the point $(1, 0, 0, 0)$. Thus $\tilde{\sigma}_3$ fixes the proper transform of C_2 and the union of three components and five isolated points in the exceptional divisor of type D_{10}. Thus we are in case F7.

Example 5.8. (Case F8) Let Y be the double cover of $\mathbb{P}^2$ defined by the following equation in $\mathbb{P}(1, 1, 1, 3)$:

$$y^2 = x_0^5 x_1 + (x_1^3 - x_2^3)^2.$$

The surface Y has three rational double points of type A_4 at $(0,1,\zeta_3^i,0)$, with $i=0,1,2$. The minimal resolution of Y is a K3 surface X. The surface has the order 15 automorphism

$$\sigma_{15}:(x_0,x_1,x_2,y)\mapsto(\zeta_5x_0,x_1,\zeta_3x_2,y).$$

We will denote by $\tilde{\sigma}_{15}$ the lifting of σ_{15} to X. The automorphism σ_3 fixes the genus 2 curve C_2 defined by $x_2=0$ and the smooth points $(0,0,1,\pm1)$. Thus we are either in case F8 or in case A1. The automorphism σ_5 fixes the two smooth rational curves defined by $x_0=y\pm(x_1^3-x_2^3)=0$ and the point $(1,0,0,0)$. Thus its lifting $\tilde{\sigma}_5$ fixes two smooth rational curves, so we are in case F8.

Remark 5.9. By Section 2.2, the moduli space of K3 surfaces having a purely non-symplectic automorphism of order 15 whose invariants are as in cases B2, D1, F3, F7 or F8 is 0-dimensional, since $\dim(V^\sigma)=1$. In cases B2 and D1, the isometry f^* has order 5 on $\mathrm{NS}(X)$, while in cases F3, F7 and F8, it has order 3. Moreover, in all cases $\operatorname{rk}T(X)=8=\varphi(15)$. It follows from Theorem 5.9 in [9] that there is a unique K3 surface X which carries purely non-symplectic automorphisms of order 22 of types B2 and D1, and a unique K3 surface carrying automorphisms of types F3, F7, F8. Thus the K3 surfaces given in Examples 5.4 and 5.5 are isomorphic, and the same is true for the K3 surfaces given in Examples 5.6, 5.7, 5.8.

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