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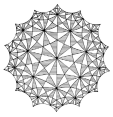

Nonsymplectic automorphisms of prime order on O'Grady's sixfolds

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Abstract. We classify nonsymplectic automorphisms of prime order on irreducible holomorphic symplectic manifolds of O'Grady's 6-dimensional deformation type. More precisely, we give a classification of the invariant and coinvariant sublattices of the second integral cohomology group.

1. Introduction

1.1. Background

Irreducible holomorphic symplectic manifolds X are simply connected compact Kähler manifolds carrying a nowhere degenerate holomorphic symplectic form σ_X which spans $H^0(X, \Omega_X^2)$.

In dimension 2, irreducible holomorphic symplectic manifolds are K3 surfaces. Fujiki [17] and Beauville [5] found examples in higher dimensions: more precisely the Hilbert scheme of n points on a K3 surface and the generalized Kummer manifold in the sense of Beauville [5] are irreducible holomorphic symplectic manifolds of dimension $2n$. Manifolds which are deformation equivalent to the Hilbert scheme and to the generalized Kummer manifold are called manifolds of $\mathrm{K3}^{[n]}$ type and of Kum_n type, respectively.

Mukai [28] discovered a symplectic form on moduli spaces of sheaves on symplectic surfaces assuming some conditions on them. However, he proved that all nonsingular irreducible holomorphic symplectic manifolds obtained in this way were a deformation of known examples.

The singular ones admit a resolution of singularities which is irreducible holomorphic symplectic only in two cases discovered by O'Grady: one in dimension 6, see [32], and one in dimension 10, see [31]. Manifolds that are deformation equivalent to O'Grady's sixfold and to O'Grady's tenfold are called manifolds of OG6 type and manifolds of OG10 type, respectively.

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1.2. Automorphisms of irreducible holomorphic symplectic manifolds

An automorphism of an irreducible holomorphic symplectic manifold X is *symplectic* if its pullback acts trivially on σ_X. An automorphism is nonsymplectic if its pullback acts nontrivially on the space $H^{2,0}(X) = \mathbb{C}\sigma_X$. A cyclic group $G \subset \mathrm{Aut}(X)$ is called *symplectic* if it is generated by a symplectic automorphism.

Automorphisms of irreducible holomorphic symplectic manifolds can be classified studying the induced action on $H^2(X, \mathbb{Z})$ which carries a lattice structure provided by the Beauville–Bogomolov–Fujiki quadratic form. A *marking* is an isometry $\eta: H^2(X, \mathbb{Z}) \to L$, where L is a lattice; the pair (X, η) is called a *marked pair*. If (X, η) is a marked pair, an isometry $\varphi \in \mathrm{O}(L)$ is *symplectic* if $\varphi \otimes \mathbb{C} \in \mathrm{O}(L \otimes \mathbb{C})$ acts trivially on $\eta(\sigma_X)$, and it is *nonsymplectic* if $\varphi \otimes \mathbb{C} \in \mathrm{O}(L \otimes \mathbb{C})$ acts nontrivially on the space $\mathbb{C}\eta(\sigma_X)$. A cyclic group $G \subset \mathrm{O}(L)$ is called *nonsymplectic* if G is generated by a nonsymplectic isometry.

We are interested in the image of the following representation map:

$$\eta_*: \mathrm{Aut}(X) \to \mathrm{O}(L), \quad f \mapsto \eta \circ f^* \circ \eta^{-1}. \tag{1.1}$$

If an isometry $\varphi \in \mathrm{O}(L)$ and there exists $g \in \mathrm{Aut}(X)$ such that $\eta_*(g) = \varphi$, then φ is *effective*. A group $G \subset \mathrm{O}(L)$ is called *effective* if its elements are effective.

The aim of this paper is to study effective nonsymplectic groups $G \subset \mathrm{O}(L)$ of prime order on manifolds of OG6 type.

The global Torelli theorem for K3 surfaces, due to Piatetski-Shapiro–Shafarevich, allows us to reconstruct automorphisms of a K3 surface S starting from Hodge isometries of $H^2(S, \mathbb{Z})$ which preserve the Kähler cone. Huybrechts [20], Markman [23] and Verbitsky [37] (see also [38]) formulated similar results of Torelli type for irreducible holomorphic symplectic manifolds.

Recently, Mongardi–Rapagnetta [26] computed the monodromy group for manifolds of OG6 type and due to the features of this group, the global Torelli theorem (Theorem 1.3 in [23]) holds in a stronger form for OG6 type manifolds, namely a necessary and sufficient condition to have a bimeromorphic map between two manifolds of OG6 type is to have a Hodge isometry of the second integral cohomology.

Classifying finite groups of automorphisms G of a certain deformation type of irreducible holomorphic symplectic manifolds can mean one of the following:

(1) classifying invariant and coinvariant sublattices of the induced action of G in $H^2(X, \mathbb{Z})$ up to isometry;

(2) classifying the connected components of the moduli space of pairs (X, G).

In general, classification (2) is finer than (1). This paper deals with level (1) of classification of nonsymplectic groups $G \subset \mathrm{Aut}(X)$ on manifolds of OG6 type.

In the case of manifolds of $\mathrm{K3}^{[2]}$ type, the symplectic automorphisms are treated by Camere [12] and Mongardi [25]; the study of nonsymplectic automorphisms was started by Beauville [6] and continued by Ohashi–Wandel [33], Boissière–Camere–Mongardi–Sarti [7], Boissière–Camere–Sarti [8], and Camere–G. Kaputska–M. Kaputska–Mongardi [15]; furthermore, Boissière–Nieper-Wißkirchen–Sarti [10] describe the fixed locus of these automorphisms. Camere–Cattaneo–Cattaneo [14] study nonsymplectic involutions of $\mathrm{K3}^{[n]}$ type manifolds and Camere–Cattaneo [13] study nonsymplectic automorphisms of $\mathrm{K3}^{[n]}$ type manifolds, where $n \geq 3$. Moreover, Joumaah [22], building on a work by

Ohashi–Wandel [33], gives a criterion to find the classification (2) in the case of involutions on manifolds of $K3^{[n]}$ type.

The study of automorphisms of generalized Kummer manifolds was started by Mongardi–Tari–Wandel [27], and continued by Boissière–Nieper-Wißkirchen–Tari [9] and by Brandhorst–Cattaneo [11].

Recently, the author together with Onorati and Veniani [19] classified symplectic birational transformations on manifolds of OG6 type in the case of finite cyclic groups, hence this paper completes the classification of automorphisms of manifolds of OG6 type. The classification of nonsymplectic automorphisms on manifolds of OG10 type was started by Brandhorst–Cattaneo [11], and recent progress by Onorati [34] about the monodromy group and the wall divisors for this deformation class constitutes a starting point for the study of the symplectic case.

1.3. Contents of the paper

In Subsection 1.2 we give a summary of basic results about irreducible holomorphic symplectic manifolds, and we introduce the main tools to approach the study of automorphisms. In Section 2 we give basic notions of lattice theory and we recall the properties of the second integral cohomology of an irreducible holomorphic symplectic manifold. If X is an irreducible holomorphic symplectic manifold, the second integral cohomology group is equipped with an integral quadratic form called *Beauville–Bogomolov–Fujiki quadratic form*. With this form, $H^2(X,\mathbb{Z})$ is a lattice of signature $(3, b_2(X)-3)$. If X is a manifold of OG6 type, then $H^2(X,\mathbb{Z})$ is isomorphic to the rank 8 and signature $(3,5)$ lattice $\mathbf{U}^{\oplus 3}\oplus[-2]^{\oplus 2}$ (see [35]), and we denote this lattice by $\mathbf{L}$ throughout the paper. If a group G acts on $\mathbf{L}$, then the invariant and coinvariant sublattices are denoted by $\mathbf{L}^G$ and $\mathbf{L}_G$, respectively.

In Section 3, in order to obtain the classification of invariant and coinvariant lattices of $\mathbf{L}$, we classify isometries of prime order of the smallest unimodular lattice in which $\mathbf{L}$ embeds, namely $\mathbf{\Lambda}=\mathbf{U}^{\oplus 5}$. We denote by $\mathbf{\Lambda}^G$ and $\mathbf{\Lambda}_G$ respectively the invariant and the coinvariant sublattices of $\mathbf{\Lambda}$ with respect to the action of a subgroup $G\subset \mathrm{O}(\mathbf{\Lambda})$. Taking into account Proposition 3.9, we determine which pairs of p-elementary lattices can represent the invariant and the coinvariant sublattices with respect to an action of a group $G\subset \mathrm{O}(\mathbf{\Lambda})$ of prime order p. We give the first crucial result of this paper.

Theorem 1.1. *Let $G\subset \mathrm{O}(\mathbf{\Lambda})$ be a subgroup of prime order p. If either $p=2$ and $\mathrm{sgn}(\mathbf{\Lambda}_G)=(2,\mathrm{rk}(\mathbf{\Lambda}_G)-2)$, or $p=2$ and $\mathrm{sgn}(\mathbf{\Lambda}_G)=(3,\mathrm{rk}(\mathbf{\Lambda}_G)-3)$, or $p\in\{3,5,7\}$ and $\mathrm{sgn}(\mathbf{\Lambda}_G)=(2,\mathrm{rk}(\mathbf{\Lambda}_G)-2)$, then the pair $(\mathbf{\Lambda}^G,\mathbf{\Lambda}_G)$ appears in Table 1.*

In Proposition 3.3, we determine the possible prime orders of nonsymplectic groups of isometries on manifolds of OG6 type and in Proposition 3.4 we find that, for manifolds of OG6 type, nonsymplectic groups $G\subset \mathrm{O}(\mathbf{L})$ of prime order are effective. In this way we obtain the classification of invariant and coinvariant sublattices of $\mathbf{L}$ with respect to effective nonsymplectic groups $G\subset \mathrm{O}(\mathbf{L})$ of prime order on a manifold of OG6 type. In Section 4 we finally come to the main result of this paper.

Theorem 1.2. *Let X be a manifold of* OG6 *type and let $G \subset \mathrm{O}(\mathbf{L})$ be a nonsymplectic group of prime order p. Then G is an effective nonsymplectic group if and only if $|G| \in \{2, 3, 5, 7\}$, and the pair $(\mathbf{L}^G, \mathbf{L}_G)$ appears in Table 5.*

Finally we remark that in [18] we determine if nonsymplectic automorphisms of manifolds of OG6 type classified in Table 5 in Theorem 1.2 are induced (see Definition 3.7 in [18]), and induced at the quotient (see Definition 4.2 in [18]).

2. Preliminaries

In Subsection 2.1 we gather the required background and we give an overview of lattice theory and of finite quadratic forms, recalling the fundamental definitions and results which we will use throughout this paper.

How to construct primitive embeddings of lattices is explained in Subsection 2.2, and in Subsection 2.3 we recall basic results about nonsymplectic groups of isometries of lattices and p-elementary lattices.

2.1. Lattices

Our main references for lattices are Nikulin's paper [30], but we also use [16] and [21], Chapter 14. A lattice L is a free $\mathbb{Z}$-module of finite rank together with a symmetric bilinear form

$$(\ ,\ ): L \times L \to \mathbb{Z},$$

which we assume to be nondegenerate. A lattice L is called *even* if $x^2 := (x, x) \in 2\mathbb{Z}$ $\forall x \in L$. For any lattice L, the *discriminant group* is the finite group associated to L defined as $L^\sharp = L^*/L$, where $L \hookrightarrow L^* := \mathrm{Hom}_{\mathbb{Z}}(L, \mathbb{Z})$, $x \longmapsto (x, \cdot)$. The discriminant group is a finite abelian group of order $|\det(L)|$ and this number is called the *discriminant* of L. A lattice is called *unimodular* if $L^\sharp = \{\mathrm{id}\}$, and *$p$-elementary* if the discriminant group $L^\sharp$ is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^{\oplus a}$ for some $a \in \mathbb{Z}_{\geq 0}$. The *length* of the discriminant group $L^\sharp$, denoted by $l(L^\sharp)$, is the minimal number of generators of the finite group $L^\sharp$. The *divisibility* $\mathrm{div}(v)$ of an element $v \in L$ is the positive generator of the ideal $(v, L) = \mathrm{div}(v)\mathbb{Z}$. The pairing $(\ ,\ )$ on L induces a $\mathbb{Q}$-valued pairing on L^* and hence a pairing $L^\sharp \times L^\sharp \to \mathbb{Q}/\mathbb{Z}$. If the lattice L is even, then the $\mathbb{Q}$-valued quadratic form on L^* yields

$$q_L: L^\sharp \to \mathbb{Q}/2\mathbb{Z}.$$

The form q_L is called the *discriminant quadratic form* on L. There exists a natural homomorphism $\mathrm{O}(L) \to \mathrm{O}(L^\sharp)$. If $G \subset \mathrm{O}(L)$ is a subgroup of isometries, then its image in $\mathrm{O}(L^\sharp)$ is denoted by $G^\sharp$. If $g \in G \subset \mathrm{O}(L)$ is an isometry, we denote by $g^\sharp$ its image in $\mathrm{O}(L^\sharp)$.

Definition 2.1. Let L be a 2-elementary even lattice, and let q_L be the discriminant quadratic form on L. We define

$$\delta(L) = \begin{cases} 0 & \text{if } q_L(x) \in \mathbb{Z}/2\mathbb{Z} \text{ for all } x \in L^\sharp, \\ 1 & \text{otherwise.} \end{cases}$$

A fundamental invariant in the theory of lattices is given by the *genus*.

Two lattices L and L' are in the same genus if $L \oplus \mathbf{U} \cong L' \oplus \mathbf{U}$, or equivalently if and only if they have the same signature and discriminant quadratic form, see Corollary 1.9.4 in [30]. Sometimes we will need to know that a lattice is unique in its genus. In this paper all the lattices that we use are unique in their genus, either because they are indefinite and p-elementary (Theorem 3.6.2 in [30]), or by an application of Theorem 1.14.2 in [30]. Finally, if L is unique in its genus also $L(n)$ satisfies this property.

We introduce two lattices that will be useful in Section 4:

$$\mathbf{H}_5 = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}, \quad \mathbf{K}_7 = \begin{pmatrix} -4 & 1 \\ 1 & -2 \end{pmatrix},$$

and we recall that $\mathbf{U}$ is the even unimodular lattice of rank 2 and $\mathbf{A}_n$, $\mathbf{D}_n$ and $\mathbf{E}_n$ denote the positive definite ADE lattices. Moreover, $[n]$ with $n \in \mathbb{Z}$ denotes the rank 1 lattice generated by a of square n; in particular, $\mathbf{A}_1 = [2]$.

2.2. Embeddings of lattices

A morphism between two lattices $S \to L$ is by definition a linear map that respects the quadratic forms. If $S \hookrightarrow L$ has finite index, then we say that L is an *overlattice* of S. An injective morphism $S \hookrightarrow L$ is called a *primitive embedding* if its cokernel is torsion free and in this setting we denote by $S^\perp \subset L$ the orthogonal complement.

Throughout the paper we refer to Propositions 1.15.1 and 1.5.1 in [30] for the classification of primitive embeddings and the computation of orthogonal complements of primitive embeddings.

By Proposition 1.15.1 in [30], a primitive embedding $S \hookrightarrow L$, where $T = S^\perp \subset L$, is given by a subgroup $H \subset L^\sharp$ which is called the *embedding subgroup*, and an isometry $\gamma\colon H \to H' \subset S^\sharp$ that we call the *embedding isometry*. If Γ is the graph of γ in $L^\sharp \oplus S(-1)^\sharp$, then

$$T^\sharp \cong \Gamma^\perp/\Gamma \quad \text{and} \quad |\det(T)| = |\det(L)| \cdot |\det(S)|/|H|^2.$$

Equivalently, by Proposition 1.5.1 in [30] we deduce that, if L is unique in its genus, a primitive embedding $S \hookrightarrow L$, where $T = S^\perp \subset L$, is given by a subgroup $K \subset S^\sharp$ which is the *gluing subgroup*, and an isometry $\gamma\colon H \to H' \subset T^\sharp$ that we call the *gluing isometry*. If Γ is the graph of γ in $S^\sharp \oplus T(-1)^\sharp$, then

$$L^\sharp \cong \Gamma^\perp/\Gamma \quad \text{and} \quad |\det(L)| = |\det(S)| \cdot |\det(T)|/|K|^2.$$

2.3. Isometries

If L is a lattice and $G \subset \mathrm{O}(L)$, then the invariant sublattice of L is

$$L^G = \{x \in L \text{ such that } g(x) = x, \forall g \in G\},$$

and the coinvariant sublattice is

$$L_G = (L^G)^\perp.$$

It holds

$$L \otimes \mathbb{Q} = (L^G \oplus L_G) \otimes \mathbb{Q}.$$

Both the invariant and the coinvariant lattices are primitive sublattices of L. In fact, they can be expressed as kernels of endomorphisms. In particular, if $G \subset \mathrm{O}(L)$ is a cyclic group of order n generated by g, then

$$L^G = \mathrm{Ker}(g - \mathrm{id}) \quad \text{and} \quad L_G = \mathrm{Ker}(\mathrm{id} + g + \cdots + g^{n-1}).$$

The Néron–Severi lattice is the $(1,1)$–part of $H^2(X,\mathbb{Z})$, i.e., $\mathrm{NS}(X) = H^2(X,\mathbb{Z}) \cap H^{1,1}(X,\mathbb{C})$. The transcendental lattice $\mathrm{T}(X)$ is the orthogonal complement of $\mathrm{NS}(X)$ in $H^2(X,\mathbb{Z})$ i.e., $\mathrm{T}(X) = \mathrm{NS}(X)^\perp$, and it is the smallest sublattice of $H^2(X,\mathbb{Z})$ such that $H^{2,0}(X) \subset \mathrm{T}(X) \otimes_{\mathbb{Z}} \mathbb{C}$. It holds

$$H^2(X,\mathbb{Z}) \otimes \mathbb{Q} = (\mathrm{NS}(X) \oplus \mathrm{T}(X)) \otimes \mathbb{Q}.$$

Proposition 2.2 (see, for instance, [29], §3). *If (X,η) is a marked irreducible holomorphic symplectic manifolds with marking $\eta\colon H^2(X,\mathbb{Z}) \to L$ and $G \subset \mathrm{O}(L)$ is a non-symplectic group, then $L^G \subset \mathrm{NS}(X)$ and $\mathrm{T}(X) \subset L_G$.*

Our convention is that the (real) spinor norm

$$\mathrm{spin}\colon \mathrm{O}(L) \to \mathbb{R}^\times/(\mathbb{R}^\times)^2 = \{\pm 1\}$$

takes the value $+1$ on a reflection τ_v when v is a such that $v^2 < 0$. Moreover, the Cartan–Dieudonné theorem guarantees that $\mathrm{O}(L \otimes \mathbb{R})$ is generated by reflections in nonisotropic elements. For more details about this choice, we refer to [24], where the opposite convention is used. We denote by $\mathrm{O}^+(L)$ the kernel of the spinor norm.

Remark 2.3. An isometry $g \in \mathrm{O}(L)$ belongs to the kernel of the spinor norm if and only if g preserves the orientation of a positive definite subspace $V \subset L \otimes \mathbb{R}$ of maximal rank.

Lemma 2.4. *Let $G \subset \mathrm{O}(L)$ be a subgroup of order 2 and let ψ be a generator of G. Then*

$$\mathrm{spin}(\psi) = (-1)^{s_+},$$

where (s_+, s_-) is the signature of L_G.

Proof. Let (t_+, t_-) be the signature of L^G. Denote by $\rho_v \in \mathrm{O}(L \otimes \mathbb{R})$ the reflection with respect to the $v \in L \otimes \mathbb{R}$. Choose an orthonormal basis $\{e_1, \ldots, e_{s_+}, f_1, \ldots, f_{s-}\}$ of $L_G \otimes \mathbb{R}$ (so $e_i^2 = 1 = -f_j^2$), and observe that if v belongs to this basis then ρ_v sends v to $-v$ and preserves both L^G and the other basis elements. Moreover, $\mathrm{spin}(\rho_{e_k}) = -1$ and $\mathrm{spin}(\rho_{f_l}) = 1$. Since ψ acts as $-\mathrm{id}$ on $L_G \otimes \mathbb{R}$ and as id on $L^G \otimes \mathbb{R}$, we have that $\psi = \rho_{e_1} \circ \cdots \circ \rho_{e_{s+}} \circ \rho_{f_1} \circ \cdots \circ \rho_{f_{s-}}$. Applying the spinor norm we have that

$$\mathrm{spin}(\psi) = \mathrm{spin}(\rho_{e_1}) \cdots \mathrm{spin}(\rho_{e_{s_+}}) \, \mathrm{spin}(\rho_{f_1}) \cdots \mathrm{spin}(\rho_{f_{s-}}) = (-1)^{s_+}. \qquad \blacksquare$$

Lemma 2.5 (see, for instance, [10], §5.3). *If Λ is a unimodular lattice and $G \subset \mathrm{O}(\Lambda)$ is a subgroup of prime order p, then Λ_G and Λ^G are p-elementary lattices and $(\Lambda_G)^\sharp \cong (\Lambda^G(-1))^\sharp$.*

Lemma 2.6 (Boissière–Nieper-Wißirchen–Sarti, [10], Lemma 5.3; Mongardi–Tari–Wandel, [27], Lemma 1.8). *Let L be a lattice and let $G \subset \mathrm{O}(L)$ be a subgroup of prime order p. Then $(p-1) \mid \mathrm{rk}(L_G)$ and*

$$\frac{L}{L^G \oplus L_G} \cong (\mathbb{Z}/p\mathbb{Z})^a.$$

There are natural embeddings of $L/(L^G \oplus L_G)$ into the discriminant groups $(L^G)^\sharp$ and $(L_G)^\sharp$. Moreover, if $m(p-1) = \mathrm{rk}(L_G)$, then $a \leq m$.

Proposition 2.7 (Boissière–Camere–Mongardi–Sarti, [7], §4). *Let L be a lattice with a nontrivial action of order p, with rank $p-1$, and discriminant d. Then d/p^{p-2} is a square in $\mathbb{Q}$.*

3. Invariant and coinvariant lattices of $\mathbf{\Lambda}$

The main goal of this paper is to classify effective nonsymplectic groups $G \subset \mathrm{O}(\mathbf{L})$ of prime order on manifolds of OG6 type. In order to pursue this goal, it is convenient to consider *the* primitive embedding $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$, where $\mathbf{\Lambda} = \mathbf{U}^{\oplus 5}$ is the smallest unimodular lattice in which $\mathbf{L}$ embeds. Such an embedding is unique up to isometry of $\mathbf{\Lambda}$ by Proposition 1.15.1 in [30]. In Subsection 3.1 we show that nonsymplectic groups $G \subset \mathrm{O}(\mathbf{L})$ of prime order are effective. In Subsection 3.2 we exhibit the possible values of the signature of p-elementary sublattices of $\mathbf{\Lambda}$. These values are related to the signature of the invariant and the coinvariant sublattices of $\mathbf{L}$ with respect to a nonsymplectic group $G \subset \mathrm{O}(\mathbf{L})$ of prime order $p \in \{2,3,5,7\}$. Afterwards, in Subsection 3.3, we give a criterion to determine if there exists a group $G \subset \mathrm{O}(\mathbf{\Lambda})$ of prime order such that the p-elementary sublattices of $\mathbf{\Lambda}$ are the invariant and the coinvariant sublattices with respect to the action of G. Finally, in Subsection 3.4 we prove Theorem 1.1.

3.1. Nonsymplectic groups of prime order are effective

Consider the primitive embedding $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$.We call $\mathbf{R} = \mathbf{L}^\perp \hookrightarrow \mathbf{\Lambda}$ the *residual lattice*. Then $\mathbf{R} \cong [2]^{\oplus 2}$, thus it is a 2-elementary lattice of signature $(2,0)$. Given an isometry $g \in \mathrm{O}(\mathbf{L})$, by Corollary 1.5.2 in [30], there exists $g' \in \mathrm{O}(\mathbf{\Lambda})$ such that g' restricts to g on $\mathbf{L}$ if and only if there exists an isometry $g'' \in \mathrm{O}(\mathbf{R})$ with the following property: $g^\sharp$ and $g''^\sharp$ (see Subsection 2.1 for the notation) coincide through an isomorphism $\mathbf{L}^\sharp \cong \mathbf{R}^\sharp$. In particular (see Lemma 2.12 in [25]) we have the following result, where we denote by $\varphi^\sharp$ the image of φ in $\mathrm{O}(\mathbf{L}^\sharp)$.

Lemma 3.1. *Let X be a manifold of* OG6 *type. If $\varphi \in \mathrm{O}(\mathbf{L})$ is an isometry such that $\varphi^\sharp = \mathrm{id}$, then there exists a primitive embedding $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$ and φ extends to an element $\widetilde{\varphi} \in \mathrm{O}(\mathbf{\Lambda})$ that acts trivially on $\mathbf{L}^\perp \subset \mathbf{\Lambda}$.*

Proof. Let $[v_1/2]$ and $[v_2/2]$ be two generators of $\mathbf{L}^\sharp$ such that $v_1^2 = -2$ and $v_2^2 = -2$. Then $\varphi^\sharp([v_1/2]) = [v_1/2]$ and $\varphi^\sharp([v_2/2]) = [v_2/2]$, i.e., $\varphi(v_1) = v_1 + 2w_1$ and $\varphi(v_2) = v_2 + 2w_2$, with $w_i \in \mathbf{L}$ for $i \in \{1,2\}$. Consider a rank 2 lattice generated by two orthogonal elements r_1 and r_2 of square 2; its discriminant group is also $(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and it is generated

by $[r_1/2]$ and $[r_2/2]$ with discriminant form given by $q(r_1/2) = 1/2$, $q(r_2/2) = 1/2$ and $(r_1, r_2) = 0$. Notice that $\mathbf{L} \oplus \mathbb{Z}r_1 \oplus \mathbb{Z}r_2$ has an overlattice isometric to $\mathbf{\Lambda}$ which is generated by $\mathbf{L}$, $\frac{r_1+v_1}{2}$ and $\frac{r_2+v_2}{2}$. We extend φ to $\mathbf{L} \oplus \mathbb{Z}r_1 \oplus \mathbb{Z}r_2$ by imposing $\varphi(r_1) = r_1$, $\varphi(r_2) = r_2$ and we obtain an extension $\widetilde{\varphi}$ of φ on $\mathbf{\Lambda}$ defined as follows:

$$\widetilde{\varphi}\Big(\frac{r_i + v_i}{2}\Big) = \frac{r_i + \varphi(v_i)}{2}.$$

■

Remark 3.2. If X is an irreducible holomorphic symplectic manifold and G is a cyclic group generated by a nonsymplectic isometry, then at a generic point of the moduli space of pairs (X, G) the invariant lattice is the Néron-Severi lattice and the coinvariant one is the transcendental lattice [29], §3.

Proposition 3.3. *If X is an irreducible holomorphic symplectic manifold of* OG6 *type and $G \subset \mathrm{O}(\mathbf{L})$ is a nonsymplectic group of prime order p, then $p \in \{2, 3, 5, 7\}$.*

Proof. The result directly follows from Proposition 6 in [4]. ■

The following result allows us to classify nonsymplectic automorphisms of prime order starting from their action on the second integral cohomology.

Proposition 3.4. *If X is an irreducible holomorphic symplectic manifold of* OG6 *type and if $G \subset \mathrm{O}(\mathbf{L})$ is a nonsymplectic group of prime order p, then G is effective.*

Proof. Since G is nonsymplectic, by Proposition 2.2, $\mathrm{T}(X) \subseteq \mathbf{L}_G$ and $\mathbf{L}^G \subseteq \mathrm{NS}(X)$. By construction, a nonsymplectic group of isometries is a group of Hodge isometries. We need to check that a Kähler class is sent to a Kähler class by the elements of G. If X admits the action of a nonsymplectic group of automorphisms then X is projective, see [4], §4. Since $\mathbf{L}^G \subseteq \mathrm{NS}(X)$, there exists an invariant ample class. More precisely, in the nonsymplectic case the signature of the invariant lattice $\mathbf{L}^G$ is $(1, \mathrm{rk}(\mathbf{L}^G) - 1)$, and the signature of the coinvariant lattice $\mathbf{L}_G$ is $(2, \mathrm{rk}(\mathbf{L}_G) - 2)$. We know by Theorem 5.4(1) in [26] that in the OG6 case $\mathrm{Mon}^2(X) = \mathrm{O}^+(\mathbf{L})$. If $p \neq 2$, then φ preserves the orientation of the positive cone since p is an odd number and $\varphi^p = \mathrm{id}$. Then we have $\varphi \in \mathrm{O}^+(\mathbf{L}) \cong \mathrm{Mon}^2(X)$. If $p = 2$, then $\mathrm{spin}(\varphi) = (-1)^2 = 1$ by Lemma 2.4, and the signature of $\mathbf{L}_G$ is $(2, \mathrm{rk}(\mathbf{L}_G) - 2)$ so $\varphi \in \mathrm{O}^+(\mathbf{L}) = \mathrm{Mon}^2(X)$. Using Theorem 1.3 in [23], we conclude the proof. ■

3.2. Admissible signature of invariant and coinvariant sublattices of $\mathbf{\Lambda}$

Proposition 3.5. *Let X be a manifold of* OG6 *type and let $G \subset \mathrm{O}(\mathbf{L})$ be a subgroup of prime order p. Consider a primitive embedding $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$, and let r_1 and r_2 be the generators of $\mathbf{R} = \mathbf{L}^\perp \subset \mathbf{\Lambda}$. Consider a group of isometries $G' \subset \mathrm{O}(\mathbf{\Lambda})$ such that G' restricts to G on $\mathbf{L}$. If $|G^\sharp| = 1$, then $\mathbf{L}_G \cong \mathbf{\Lambda}_{G'}$, hence $\mathrm{sgn}(\mathbf{L}_G) = \mathrm{sgn}(\mathbf{\Lambda}_{G'})$ and $\mathrm{sgn}(\mathbf{L}^G) = \mathrm{sgn}(\mathbf{\Lambda}^{G'}) - (2, 0)$.*

Proof. If $|G^\sharp| = 1$, then $\mathbf{L}_G \cong \mathbf{\Lambda}_{G'}$ by Lemma 3.1. ■

Proposition 3.6. *Let X be a manifold of* OG6 *type and let $G \subset \mathrm{O}(\mathbf{L})$ be a subgroup of prime order p. Consider a primitive embedding $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$, and let r_1 and r_2 be the generators of $\mathbf{R} = \mathbf{L}^{\perp} \subset \mathbf{\Lambda}$. Consider a group of isometries $G' \subset O(\mathbf{\Lambda})$ such that G' restricts to G on $\mathbf{L}$. If $|G^{\sharp}| = 2$, then $\mathbf{L}_G \cong (r_1 - r_2)^{\perp} \subset \mathbf{\Lambda}_{G'}$ and $\mathbf{L}^G \cong (r_1 + r_2)^{\perp} \subset \mathbf{\Lambda}^{G'}$, hence $\mathrm{sgn}(\mathbf{L}_G) = \mathrm{sgn}(\mathbf{\Lambda}_{G'}) - (1,0)$ and $\mathrm{sgn}(\mathbf{L}^G) = \mathrm{sgn}(\mathbf{\Lambda}^{G'}) - (1,0)$.*

Proof. We extend the action on $\mathbf{\Lambda}$ by the isometry $\psi = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ on $\mathbf{R}$. Since $\mathbf{\Lambda}$ is unimodular, the gluing subgroup of $\mathbf{L} \oplus \mathbf{R} \subset \mathbf{\Lambda}$ is $H = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. By Proposition 1.5.1 in [30], there is an isometry $\mathbf{L}^{\sharp} \to \mathbf{R}(-1)^{\sharp}$. Since $G^{\sharp}$ acts on $\mathbf{L}^{\sharp}$ exchanging the generators, $\psi^{\sharp}$ exchanges the generators of $\mathbf{R}^{\sharp}$. Since $G' = G_{|\mathbf{L}} \oplus \psi_{|\mathbf{R}}$, this gives $(r_1 + r_2) \in \Lambda^{G'}$ and $(r_1 - r_2) \in \mathbf{\Lambda}_{G'}$. ∎

3.3. Existence of the actions of prime order on $\mathbf{\Lambda}$

Now we give a classification of all the possible p-elementary sublattices S of $\mathbf{\Lambda}$, and their orthogonal complements $T = S^{\perp} \subset \mathbf{\Lambda}$, combining the constraints on the signature of Propositions 3.5 and 3.6 with the content of Remark 3.2. More precisely, for $p = 2$ we look for 2-elementary sublattices S of $\mathbf{\Lambda}$ with $\mathrm{sgn}(S) = (2, \mathrm{rk}(S) - 2)$ and $\mathrm{sgn}(S) = (3, \mathrm{rk}(S) - 3)$, and for $p \in \{3, 5, 7\}$ we look for p-elementary sublattices S of $\mathbf{\Lambda}$ with $\mathrm{sgn}(S) = (2, \mathrm{rk}(S) - 2)$. The pairs (T, S) obtained in this way are candidates to be the invariant and the coinvariant sublattices with respect to $G \subset \mathrm{O}(\mathbf{\Lambda})$ of prime order p. If $p = 2$, we know that all the 2-elementary lattices that we find correspond to an involution which acts as $-\mathrm{id}$ on $\mathbf{\Lambda}_G$. On the other hand, if p is an odd prime number, we need the following criterion to determine if the pair (T, S) corresponds to an invariant and a coinvariant sublattice respectively, with respect to $G \subset \mathrm{O}(\mathbf{\Lambda})$ of prime order p.

Lemma 3.7. *If $G \subset \mathrm{O}(\mathbf{\Lambda})$ is a group of prime order $p \in \{2, 3, 5, 7\}$ and $\mathrm{sgn}(\mathbf{\Lambda}_G) = (2, \mathrm{rk}(\mathbf{\Lambda}_G) - 2)$, then there exists a primitive embedding $\mathbf{\Lambda}_G \hookrightarrow H^2(X, \mathbb{Z})$, where X is a K3 surface.*

Proof. In our assumptions, the lattice $\mathbf{\Lambda}_G$ is a p-elementary lattice with $\mathrm{sgn}(\mathbf{\Lambda}_G) = (2, \mathrm{rk}(\mathbf{\Lambda}_G) - 2)$. Since $\mathbf{\Lambda}_G$ is a sublattice of $\mathbf{\Lambda}$ and $\mathrm{sgn}(\mathbf{\Lambda}) = (5, 5)$, we have $\mathrm{rk}(\mathbf{\Lambda}_G) \leq 7$. The signature of the lattice $H^2(X, \mathbb{Z})$ is $(3, 19)$, hence we can apply Corollary 1.12.3 in [30] to prove that there exists a primitive embedding of $\mathbf{\Lambda}_G$ in the unimodular lattice $H^2(X, \mathbb{Z})$. We have $19 - 3 = 16 \equiv 0\ (8)$, moreover $3 - 2 \geq 0$ and $19 - (\mathrm{rk}(\mathbf{\Lambda}_G) - 2) \geq 0$, since $(\mathrm{rk}(\mathbf{\Lambda}_G) - 2) \leq 5$ by assumption. The inequality $22 - \mathrm{rk}(\mathbf{\Lambda}_G) > l(\mathbf{\Lambda}_G^{\sharp})$ is satisfied since $22 - \mathrm{rk}(\mathbf{\Lambda}_G) \geq 15$ and $l(\mathbf{\Lambda}_G^{\sharp}) \leq 7$. ∎

Definition 3.8. Let X be a K3 surface. We denote by $\mathbf{M}$ the isometry class of $H^2(X, \mathbb{Z})$.

Proposition 3.9. *Let $p \in \{2, 3, 5, 7\}$ and let S be a p-elementary lattice such that there exists a primitive embedding $S \hookrightarrow \mathbf{\Lambda}$ and such that $\mathrm{sgn}(S) = (2, \mathrm{rk}(S) - 2)$. If there exists a group $G \subset \mathrm{O}(\mathbf{\Lambda})$ of prime order p such that $S \cong \mathbf{\Lambda}_G$, then there exists a K3 surface X with a marking $H^2(X, \mathbb{Z}) \to \mathbf{M}$ and a nonsymplectic group $G' \subset \mathrm{O}(\mathbf{M})$ of prime order p such that $S \cong \mathbf{M}_{G'}$.*

Proof. Let $G \subset \mathrm{O}(\mathbf{\Lambda})$ be a group of prime order p and let $\mathbf{\Lambda}_G$ be the coinvariant lattice, a p-elementary lattice such that $(\mathbf{\Lambda}_G)^{\sharp} \cong (\mathbf{\Lambda}^G)^{\sharp}$. Since the action on $(\mathbf{\Lambda}^G)^{\sharp}$ is trivial, then

also the action on $(\mathbf{\Lambda}_G)^\sharp$ is trivial by Corollary 1.5.2 in [30]. By Lemma 3.7, there exists a primitive embedding $\mathbf{\Lambda}_G \hookrightarrow \mathbf{M}$. Recalling the construction of the gluing subgroup in Subsection 2.2, since $\mathbf{M}$ is unimodular, the gluing subgroup K coincide with $(\mathbf{\Lambda}_G)^\sharp$. Since the action on $(\mathbf{\Lambda}_G)^\sharp$ is trivial, then also the action on the discriminant group of the orthogonal complement $\mathbf{\Lambda}_G^\perp \subset \mathbf{M}$ is trivial due to the anti-isometry between $(\mathbf{\Lambda}_G)^\sharp$ and $(\mathbf{\Lambda}_G^\perp)^\sharp$. In order to define a group $G' \subset \mathrm{O}(\mathbf{M})$ such that G' restricts to G on $\mathbf{\Lambda}_G$ and such that $\mathbf{\Lambda}_G \cong \mathbf{M}_{G'}$, we need an isometry of $\mathbf{\Lambda}_G^\perp \subset \mathbf{M}$ such that the induced action on $(\mathbf{\Lambda}_G^\perp)^\sharp$ is trivial. We show now that this isometry is $\mathrm{id}_{|\mathbf{\Lambda}_G^\perp}$. Indeed, the action of G on $\mathbf{\Lambda}_G$ extends to an action on $\mathbf{M}$ by gluing $G_{|\mathbf{\Lambda}_G}$ and $\mathrm{id}_{|\mathbf{\Lambda}_G^\perp}$ since the action of both on the gluing subgroups is trivial. In this way we get $\mathbf{\Lambda}_G \cong \mathbf{M}_{G'}$ by construction. On the complex space $\mathbf{M} \otimes \mathbb{C}$ we have a weight-two Hodge structure, where $(\mathbf{M} \otimes \mathbb{C})^{2,0} \oplus (\mathbf{M} \otimes \mathbb{C})^{0,2} = \mathbf{M}_{G'} \otimes \mathbb{C}$ is of signature $(2, \mathrm{rk}(S) - 2)$. By the Torelli theorem, there exists a K3 surface X such that $H^{2,0}(X) \oplus H^{0,2}(X) = \mathbf{M}_{G'} \otimes \mathbb{C}$ and the action on the integral cohomology is nonsymplectic, as we want. ∎

Artebani–Sarti–Taki [2] for $p = 5, 7$, and Artebani–Sarti [1] for $p = 3$, find a classification of invariant and coinvariant sublattices with respect to a nonsymplectic group of prime order p on a K3 surface. As a consequence, using Proposition 3.9, we can determine which p-elementary sublattices $\mathbf{\Lambda}_G$ classified in Table 1 are the coinvariant sublattices with respect to an isometry of prime order p of $\mathbf{\Lambda}$. If this is the case, there exists a group $G \subset \mathrm{O}(\mathbf{\Lambda})$ of prime order p such that $(\mathbf{\Lambda}^G, \mathbf{\Lambda}_G)$ are the invariant and the coinvariant sublattices.

3.4. Proof of Theorem 1.1

In the following we prove Theorem 1.1, dividing it into five cases.

Proposition 3.10. *If the order of $G \subset \mathrm{O}(\mathbf{\Lambda})$ is 2 and if $\mathrm{sgn}(\mathbf{\Lambda}_G) = (2, \mathrm{rk}(\mathbf{\Lambda}_G) - 2)$, then there are fourteen possible pairs of invariant and coinvariant lattices $(\mathbf{\Lambda}^G, \mathbf{\Lambda}_G)$ with respect to the action of G on $\mathbf{\Lambda}$.*

Proof. Since $|G| = 2$, then $\mathbf{\Lambda}^G$ and $\mathbf{\Lambda}_G$ are 2-elementary lattices. It holds that $\mathrm{rk}(\mathbf{\Lambda}^G) = 10 - \mathrm{rk}(\mathbf{\Lambda}_G)$, hence $a \leq 10 - \mathrm{rk}(\mathbf{\Lambda}_G)$. We use Lemma 2.6 to bound the number a that occurs there. For each possible value of a we apply Theorem 3.6.2 and Corollary 1.13.5 in [30] and we obtain the classification in Table 1. ∎

Proposition 3.11. *If the order of $G \subset \mathrm{O}(\mathbf{\Lambda})$ is 2 and if $\mathrm{sgn}(\mathbf{\Lambda}_G) = (3, \mathrm{rk}(\mathbf{\Lambda}_G) - 3)$, then there are thirteen possible pairs of invariant and coinvariant lattices $(\mathbf{\Lambda}^G, \mathbf{\Lambda}_G)$ with respect to the action of G on $\mathbf{\Lambda}$.*

Proof. Recall that $\mathrm{rk}(\mathbf{\Lambda}^G) = 10 - \mathrm{rk}(\mathbf{\Lambda}_G)$, hence $a \leq 10 - \mathrm{rk}(\mathbf{\Lambda}_G)$. The result is a direct application of Lemma 2.6, and Theorem 3.6.2 and Corollary 1.13.5 in [30]. The classification is summarized in Table 1. ∎

Proposition 3.12. *If the order of $G \subset \mathrm{O}(\mathbf{\Lambda})$ is 3 and if $\mathrm{sgn}(\mathbf{\Lambda}_G) = (2, \mathrm{rk}(\mathbf{\Lambda}_G) - 2)$, then there are five possible pairs of invariant and coinvariant lattices $(\mathbf{\Lambda}^G, \mathbf{\Lambda}_G)$ with respect to the action of G on $\mathbf{\Lambda}$.*

No.	$\lvert G\rvert$	$\mathbf{\Lambda}_G$	$\mathbf{\Lambda}^G$	$\operatorname{sgn}(\mathbf{\Lambda}_G)$	a	δ
1	2	$\mathbf{U}^{\oplus 2}\oplus[-2]^{\oplus 3}$	$[2]^{\oplus 3}$	$(2,5)$	3	1
2	2	$\mathbf{U}\oplus[-2]^{\oplus 3}\oplus[2]$	$[2]^{\oplus 3}\oplus[-2]$	$(2,4)$	4	1
3	2	$\mathbf{U}^{\oplus 2}\oplus[-2]^{\oplus 2}$	$\mathbf{U}\oplus[2]^{\oplus 2}$	$(2,4)$	2	1
4	2	$[2]^{\oplus 2}\oplus[-2]^{\oplus 3}$	$[-2]^{\oplus 2}\oplus[2]^{\oplus 3}$	$(2,3)$	5	1
5	2	$\mathbf{U}\oplus[-2]^{\oplus 2}\oplus[2]$	$\mathbf{U}\oplus[2]^{\oplus 2}\oplus[-2]$	$(2,3)$	3	1
6	2	$\mathbf{U}^{\oplus 2}\oplus[-2]$	$\mathbf{U}^{\oplus 2}\oplus[2]$	$(2,3)$	1	1
7	2	$[2]^{\oplus 2}\oplus[-2]^{\oplus 2}$	$\mathbf{U}\oplus[2]^{\oplus 2}\oplus[-2]^{\oplus 2}$	$(2,2)$	4	1
8	2	$\mathbf{U}(2)^{\oplus 2}$	$\mathbf{U}\oplus\mathbf{U}(2)^{\oplus 2}$	$(2,2)$	4	0
9	2	$\mathbf{U}\oplus[2]\oplus[-2]$	$\mathbf{U}^{\oplus 2}\oplus[2]\oplus[-2]$	$(2,2)$	2	1
10	2	$\mathbf{U}\oplus\mathbf{U}(2)$	$\mathbf{U}^{\oplus 2}\oplus\mathbf{U}(2)$	$(2,2)$	2	0
11	2	$\mathbf{U}^{\oplus 2}$	$\mathbf{U}^{\oplus 3}$	$(2,2)$	0	0
12	2	$[2]^{\oplus 2}\oplus[-2]$	$\mathbf{U}^{\oplus 2}\oplus[-2]^{\oplus 2}\oplus[2]$	$(2,1)$	3	1
13	2	$\mathbf{U}\oplus[2]$	$\mathbf{U}^{\oplus 3}\oplus[-2]$	$(2,1)$	1	1
14	2	$[2]^{\oplus 2}$	$\mathbf{U}^{\oplus 3}\oplus[-2]^{\oplus 2}$	$(2,0)$	2	1
1	2	$\mathbf{U}^{\oplus 3}\oplus[-2]^{\oplus 2}$	$[2]^{\oplus 2}$	$(3,5)$	2	1
2	2	$\mathbf{U}^{\oplus 2}\oplus[2]\oplus[-2]^{\oplus 2}$	$[2]^{\oplus 2}\oplus[-2]$	$(3,4)$	3	1
3	2	$\mathbf{U}\oplus[2]^{\oplus 2}\oplus[-2]^{\oplus 2}$	$[2]^{\oplus 2}\oplus[-2]^{\oplus 2}$	$(3,3)$	4	1
4	2	$\mathbf{U}\oplus\mathbf{U}(2)^{\oplus 2}$	$\mathbf{U}(2)^{\oplus 2}$	$(3,3)$	4	0
5	2	$\mathbf{U}^{\oplus 2}\oplus[2]\oplus[-2]$	$\mathbf{U}\oplus[2]\oplus[-2]$	$(3,3)$	2	1
6	2	$\mathbf{U}^{\oplus 2}\oplus\mathbf{U}(2)$	$\mathbf{U}\oplus\mathbf{U}(2)$	$(3,3)$	2	0
7	2	$\mathbf{U}^{\oplus 3}$	$\mathbf{U}^{\oplus 2}$	$(3,3)$	0	0
8	2	$[-2]^{\oplus 2}\oplus[2]^{\oplus 3}$	$[2]^{\oplus 2}\oplus[-2]^{\oplus 3}$	$(3,2)$	5	1
9	2	$\mathbf{U}\oplus[-2]\oplus[2]^{\oplus 2}$	$\mathbf{U}\oplus[2]\oplus[-2]^{\oplus 2}$	$(3,2)$	3	1
10	2	$\mathbf{U}^{\oplus 2}\oplus[2]$	$\mathbf{U}^{\oplus 2}\oplus[-2]$	$(3,2)$	1	1
11	2	$\mathbf{U}(2)\oplus[2]^{\oplus 2}$	$\mathbf{U}\oplus\mathbf{U}(2)\oplus[-2]^{\oplus 2}$	$(3,1)$	4	1
12	2	$\mathbf{U}\oplus[2]^{\oplus 2}$	$\mathbf{U}^{\oplus 2}\oplus[-2]^{\oplus 2}$	$(3,1)$	2	1
13	2	$[2]^{\oplus 3}$	$\mathbf{U}^{\oplus 2}\oplus[-2]^{\oplus 3}$	$(3,0)$	3	1
1	3	$\mathbf{U}^{\oplus 2}\oplus\mathbf{A}_2(-1)$	$\mathbf{U}\oplus\mathbf{A}_2$	$(2,4)$	1	–
2	3	$\mathbf{U}\oplus\mathbf{U}(3)\oplus\mathbf{A}_2(-1)$	$\mathbf{U}(3)\oplus\mathbf{A}_2$	$(2,4)$	3	–
3	3	$\mathbf{U}^{\oplus 2}$	$\mathbf{U}^{\oplus 3}$	$(2,2)$	0	–
4	3	$\mathbf{U}\oplus\mathbf{U}(3)$	$\mathbf{U}^{\oplus 2}\oplus\mathbf{U}(3)$	$(2,2)$	2	–
5	3	$\mathbf{A}_2$	$\mathbf{U}^{\oplus 3}\oplus\mathbf{A}_2(-1)$	$(2,0)$	1	–
1	5	$\mathbf{U}\oplus\mathbf{H}_5$	$\mathbf{U}^{\oplus 2}\oplus\mathbf{H}_5$	$(2,2)$	1	–
1	7	$\mathbf{U}^{\oplus 2}\oplus\mathbf{K}_7$	$\mathbf{U}\oplus\mathbf{K}_7(-1)$	$(2,4)$	1	–

Table 1. Pairs $(\mathbf{\Lambda}^G,\mathbf{\Lambda}_G)$ for $G\subset \mathrm{O}(\mathbf{\Lambda})$ of prime order $p=2$ and $\operatorname{sgn}(\mathbf{\Lambda}_G)=(2,\operatorname{rk}(\mathbf{\Lambda}_G)-2)$, or $p=2$ and $\operatorname{sgn}(\mathbf{\Lambda}_G)=(3,\operatorname{rk}(\mathbf{\Lambda}_G)-3)$, or $p\in\{3,5,7\}$ and $\operatorname{sgn}(\mathbf{\Lambda}_G)=(2,\operatorname{rk}(\mathbf{\Lambda}_G)-2)$.

Proof. The result is a direct application of Lemma 2.6, the theorem in §1 of [36], Theorem 1.13.3 and Corollary 1.13.5 in [30], and Proposition 2.7. The coinvariant sublattices that we find for $p=3$ are $\mathbf{\Lambda}_G=\mathbf{U}^{\oplus 2}\oplus\mathbf{A}_2(-1)$, $\mathbf{\Lambda}_G=\mathbf{U}\oplus\mathbf{A}_2(-1)\oplus\mathbf{U}(3)$, $\mathbf{\Lambda}_G=\mathbf{U}^{\oplus 2}$, $\mathbf{\Lambda}_G=\mathbf{U}\oplus\mathbf{U}(3)$ and $\mathbf{\Lambda}_G=\mathbf{A}_2$ and due to the classification of Artebani–Sarti [1], they are coinvariant sublattices with respect to an isometry of order 3 on $\mathbf{\Lambda}$. The classification is summarized in Table 1. ∎

Proposition 3.13. *If the order of $G \subset \mathrm{O}(\Lambda)$ is 5 and if $\mathrm{sgn}(\Lambda_G) = (2, \mathrm{rk}(\Lambda_G) - 2)$, then there is one possible pair of invariant and coinvariant lattices (Λ^G, Λ_G) with respect to the action of G on Λ.*

Proof. Recall that $\mathrm{rk}(\Lambda_G)$ has to be a multiple of 4 and, due to the assumptions on the signature, $\mathrm{rk}(\Lambda_G) \leq 7$. The result is a direct application of Lemma 2.6, the theorem in §1 of [36], Corollary 1.13.5 in [30] and Proposition 2.7. In this case $\Lambda_G = \mathbf{U} \oplus \mathbf{H}_5$ and $\Lambda^G = \mathbf{U}^{\oplus 2} \oplus \mathbf{H}_5$. By Proposition 3.9, we check that this pair corresponds to invariant and coinvariant lattices (Λ^G, Λ_G) with respect to the action of a group $G \subset \mathrm{O}(\Lambda)$ of order 5. The classification is summarized in Table 1. ∎

Proposition 3.14. *If the order of $G \subset \mathrm{O}(\Lambda)$ is 7 and if $\mathrm{sgn}(\Lambda_G) = (2, \mathrm{rk}(\Lambda_G) - 2)$, then there is one possible pair of invariant and coinvariant lattices (Λ^G, Λ_G) with respect to the action of G on Λ.*

Proof. Recall that $\mathrm{rk}(\Lambda_G)$ has to be a multiple of 6 and, due to the assumptions on the signature, $\mathrm{rk}(\Lambda_G) \leq 7$. The result is a direct application of Lemma 2.6, the theorem in §1 of [36], Corollary 1.13.5 in [30] and Proposition 2.7. We obtain a unique coinvariant lattice $\Lambda_G = \mathbf{U}^{\oplus 2} \oplus \mathbf{K}_7$ and the orthogonal complement is $\Lambda^G = \mathbf{U} \oplus \mathbf{K}_7$. By Proposition 3.9, we check that this pair corresponds to the action of a group $G \subset \mathrm{O}(\Lambda)$ of order 7. The classification is summarized in Table 1. ∎

In Table 1, in the column δ we indicate whether the 2-elementary quadratic form of the discriminant group of the lattice is integer valued, $\delta = 0$, or not, $\delta = 1$ (cf. Definition 2.1). Moreover, a is the length of the discriminant group of Λ_G and of Λ^G since Λ is unimodular.

4. Invariant and coinvariant lattices of L

To classify effective nonsymplectic groups of isometries $G \subset \mathrm{O}(\mathbf{L})$ means to classify invariant and coinvariant sublattices $(\mathbf{L}^G, \mathbf{L}_G)$ of $\mathbf{L}$. As we have already explained in Section 1 there are two levels of classification and to classify group actions is finer than classifying pairs $(\mathbf{L}^G, \mathbf{L}_G)$. The classification in (1) is equivalent to counting the different primitive embeddings of $\mathbf{L}_G$ in $\mathbf{L}$. In Subsections 4.1 and 4.2 we classify pairs $(\mathbf{L}^G, \mathbf{L}_G)$ for effective nonsymplectic groups $G \subset \mathrm{O}(\mathbf{L})$ of order 2 and in Subsection 4.3 we classify pairs $(\mathbf{L}^G, \mathbf{L}_G)$ for effective nonsymplectic groups $G \subset \mathrm{O}(\mathbf{L})$ of order 3, 5, 7. In Subsection 4.4 we prove Theorem 1.2.

4.1. Order 2, trivial action on the discriminant group

Lemma 4.1. *If L is a lattice and $G \subset \mathrm{O}(L)$ is a group of order 2 generated by φ, then $L_\varphi = L^{-\varphi}$.*

Proof. We have $L_\varphi = \mathrm{Ker}(\varphi + \mathrm{id}) = \mathrm{Ker}(-\varphi - \mathrm{id}) = \mathrm{Ker}((-\varphi) - \mathrm{id}) = L^{-\varphi}$. ∎

Lemma 4.2. *If $G \subset \mathrm{O}(\mathbf{L})$ is a group such that $|G| = 2$ and $|G^\sharp| = 1$, then $\mathbf{L}_G$ and $\mathbf{L}^G$ are 2-elementary lattices.*

Proof. The lattice $\mathbf{L}_G$ is 2-elementary by Lemmas 3.1 and 2.5 using the primitive embedding $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$. If φ is a generator of G, then $\varphi' = \varphi \oplus -\mathrm{id}_{\mathbf{L}^\perp}$ is an extension of φ to $\mathbf{\Lambda}$ (cf. Corollary 1.5.2 in [30]). The isometry $-\varphi' = -\varphi \oplus \mathrm{id}_{\mathbf{L}^\perp}$ is an isometry of order 2 of $\mathbf{\Lambda}$ hence by Lemma 2.5 $\mathbf{\Lambda}_{-\varphi'}$ is a 2-elementary lattice. Moreover by construction $\mathbf{\Lambda}_{-\varphi'} \cong \mathbf{L}_{-\varphi}$ and by Lemma 4.1 $\mathbf{L}_{-\varphi} \cong \mathbf{L}^{\varphi}$ so $\mathbf{L}^G \cong \mathbf{L}^{\varphi}$ is 2-elementary. ∎

Proposition 4.3. *Let $G \subset \mathrm{O}(\mathbf{L})$ be a subgroup such that $|G| = 2$ and $|G^\sharp| = 1$. Let $H \subset \mathbf{L}^\sharp$ be the embedding subgroup of a primitive embedding $\mathbf{L}_G \hookrightarrow \mathbf{L}$.*

(a) *If $H = \{\mathrm{id}\}$, then $l((\mathbf{L}^G)^\sharp) = l((\mathbf{L}_G)^\sharp) + 2$.*

(b) *If $H = \mathbb{Z}/2\mathbb{Z}$, then $l((\mathbf{L}^G)^\sharp) = l((\mathbf{L}_G)^\sharp)$.*

(c) *If $H = (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$, then $l((\mathbf{L}^G)^\sharp) = l((\mathbf{L}_G)^\sharp) - 2$.*

Proof. Recall that an embedding $\mathbf{L}_G \hookrightarrow \mathbf{L}$ is given by an isometry between a subgroup of $(\mathbf{L}_G)^\sharp$ and a subgroup of $\mathbf{L}^\sharp$. We know that $\mathbf{L}^\sharp \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ hence there are three choices for the the embedding subgroup $H \subset \mathbf{L}^\sharp$. By Lemma 4.2, the lattices $\mathbf{L}$, $\mathbf{L}_G$ and $\mathbf{L}^G$ are 2-elementary, so the formula

$$|\det(\mathbf{L}^G)| = |\det(L)| \cdot |\det(\mathbf{L}_G)|/|H|^2$$

gives

$$l((\mathbf{L}^G)^\sharp) = l((\mathbf{L}_G)^\sharp) + 2 - 2n,$$

where $H \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus n}$. Replacing n with 0, 1 and 2, we obtain the statements (a), (b) and (c), respectively. ∎

Remark 4.4. By Theorem 3.6.2 in [30], if $\delta = 0$, then $\mathrm{sgn}(\mathbf{L}^G) = (1, 1)$ or $\mathrm{sgn}(\mathbf{L}^G) = (1, 5)$.

Here we show how to use Proposition 4.3 to compute the only possible primitive embeddings of the coinvariant sublattices $\mathbf{L}_G = \mathbf{\Lambda}_G$ in $\mathbf{L}$ in order to obtain the invariant sublattices $\mathbf{L}^G$. We do it in one case, the same strategy is applied for all the others. The first case of Table 1 is $\mathbf{\Lambda}_G = \mathbf{L}_G = \mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 3}$. Using Proposition 4.3 we know that if the embedding subgroup H is equal to the identity then $l((\mathbf{L}^G)^\sharp)$ is equal to $l((\mathbf{L}_G)^\sharp) + 2$, hence the lattice $\mathbf{L}^G$ has rank 1 and $l((\mathbf{L}^G)^\sharp) = 3 + 2 = 5$, hence this case cannot happen by Theorem 1.10.1(2) in [30]. If the embedding subgroup H is equal to $\mathbb{Z}/2\mathbb{Z}$ then $l((\mathbf{L}^G)^\sharp)$ is equal to$l((\mathbf{L}_G)^\sharp)$, hence the lattice $\mathbf{L}^G$ has rank 1 and $l((\mathbf{L}^G)^\sharp)$ is equal to 3, hence neither this case can happen. As last choice we have the embedding subgroup H equal to$(\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$ and the following relation holds: $l((\mathbf{L}^G)^\sharp) = l((\mathbf{L}_G)^\sharp) - 2$. In this case the lattice $\mathbf{L}^G$ has rank 1 and $l((\mathbf{L}^G)^\sharp)$ is equal to 1. This case can happen and computing the embedding we get $\mathbf{L}^G = [2]$. Starting from the classification in Table 1 we classify invariant and coinvariant sublattices of G in $\mathbf{L}$ in Table 5.

4.2. Order 2, nontrivial action on the discriminant group

Consider an embedding $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$ and a group of isometries $G \subset \mathrm{O}(\mathbf{\Lambda})$ such that $|G| = 2$ and $|G^\sharp| = 2$. By Lemma 2.5 we computed the thirteen possible isometry classes of $\mathbf{\Lambda}_G$ in Table 1.

Let r_1 and r_2 be two orthogonal generators of $\mathbf{R}$. If $|G^\sharp| = 2$ then by Proposition 3.6, $\mathbf{R}_G \cong [4]$ is generated by the $(r_1 - r_2)$ and $\mathbf{R}^G \cong [4]$ is generated by the $(r_1 + r_2)$. By Proposition 1.15.1 in [30], we compute the possible primitive embeddings $\mathbf{R}_G \cong [4] \hookrightarrow \mathbf{\Lambda}_G$ in order to find $S \cong (\mathbf{R}_G)^\perp \subset \mathbf{\Lambda}_G$. The results of this computation are the possible isometry classes of S summarized in Table 2.

No.	$\mathbf{\Lambda}_G$	$S = [4]^{\perp_{\Lambda_G}}$
1	$\mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 2}$	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4]$
2	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{U} \oplus [2] \oplus \mathbf{A}_3(-1)$
3	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]^{\oplus 2}$	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2} \oplus [-4]$
4	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$[2]^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4]$
5	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]^{\oplus 2}$	$\mathbf{U} \oplus [-2]^{\oplus 2} \oplus [4]$
6	$\mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	$\mathbf{U}(2)^{\oplus 2} \oplus [-4]$
7	$\mathbf{U} \oplus \mathbf{U}(2)^{\oplus 2}$	$\mathbf{U} \oplus \mathbf{U}(2) \oplus [-4]$
8	$\mathbf{U}^{\oplus 2} \oplus [2] \oplus [-2]$	$\mathbf{U} \oplus [2] \oplus [-2] \oplus [-4]$
9	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2)$	$\mathbf{U} \oplus \mathbf{U}(2) \oplus [-4]$
10	$\mathbf{U}^{\oplus 2} \oplus \mathbf{U}(2)$	$\mathbf{U}^{\oplus 2} \oplus [-4]$
11	$\mathbf{U}^{\oplus 3}$	$\mathbf{U}^{\oplus 2} \oplus [-4]$
12	$[2]^{\oplus 3} \oplus [-2]^{\oplus 2}$	$[2] \oplus [-2]^{\oplus 2} \oplus [4]$
13	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]$	$[2]^{\oplus 2} \oplus [-2] \oplus [-4]$
14	$\mathbf{U} \oplus [2]^{\oplus 2} \oplus [-2]$	$\mathbf{U} \oplus [-2] \oplus [4]$
15	$\mathbf{U}^{\oplus 2} \oplus [2]$	$\mathbf{U} \oplus [2] \oplus [-4]$
16	$\mathbf{U}(2) \oplus [2]^{\oplus 2}$	$\mathbf{U}(2) \oplus [4]$
17	$\mathbf{U}(2) \oplus [2]^{\oplus 2}$	$[2]^{\oplus 2} \oplus [-4]$
18	$\mathbf{U} \oplus [2]^{\oplus 2}$	$[2]^{\oplus 2} \oplus [-4]$
19	$\mathbf{U} \oplus [2]^{\oplus 2}$	$\mathbf{U} \oplus [4]$
20	$[2]^{\oplus 3}$	$[2] \oplus [4]$

Table 2. Orthogonal complements of $[4] \hookrightarrow \mathbf{\Lambda}_G$.

Then for each S we obtain $T = S^\perp$ by computing the primitive embeddings $S \hookrightarrow \mathbf{L}$ by Proposition 1.15.1 in [30]. All the pairs (T, S) are summarized in Table 3. Only when $S \cong \mathbf{U}(2)^{\oplus 2} \oplus [-4]$ and when $S \cong \mathbf{U} \oplus [2] \oplus \mathbf{A}_3(-1)$ we do not find any primitive embedding $S \hookrightarrow \mathbf{L}$.

By Lemma 3.2 in [19], we know that if $|G| = 2$ and $|G^\sharp| = 2$ then $|\det(\mathbf{L}_G)| = |\det(\mathbf{L}^G)|$. This result excludes the candidate cases 1, 2, 3, 5, 11, 13, 17, 19, 21, 23, 25, 27 of Table 3.

Lemma 4.5. *In the case $|G| = 2$ and $|G^\sharp| = 2$, if $|\det(\mathbf{L}_G)| = |\det(\mathbf{L}^G)|$, the gluing subgroup of $\mathbf{L}_G \hookrightarrow \mathbf{L}$ contains no elements of order* 4.

Proof. Let H be the gluing subgroup and suppose that it contains an element of order 4. If H' is the image of H in $(\mathbf{L}^G)^\sharp$, then $H \oplus H'$ contains an element of order 4 which is also the unique element of order 4 in $(\mathbf{L}_G)^\sharp \oplus (\mathbf{L}^G)^\sharp$. Hence $H^\perp \oplus (H')^\perp \subset (\mathbf{L}_G)^\sharp \oplus (\mathbf{L}^G)^\sharp$ does not contain elements of order 4 and, in particular, it contains only elements of order two. Since $\mathbf{L}^\sharp$ is generated by elements in $H^\perp \oplus (H')^\perp$ that are not in $H \oplus H'$, G acts trivially on these elements and this is a contradiction. ∎

We compute the gluing subgroups for the possible S and T in Table 3 and Lemma 4.5 excludes the cases 4, 6, 8, 9, 14, 15, 16, 18, 22. In case 25 we have $(S, T) \cong ([2]^{\oplus 2} \oplus [-4], [4] \oplus \mathbf{D}_4(-1))$. These two pairs of lattices do not admit a gluing subgroup in $\mathbf{L}$ since the discriminant form of $S^\sharp$ and the discriminant form of $T(-1)^\sharp$ do not admit any isometric subgroup in both cases. In these two cases the lattices (T, S) can not be invariant and coinvariant sublattices with respect to an action of a group $G \subset \mathrm{O}(\mathbf{L})$ such that $|G| = 2$ and $|G^\sharp| = 2$. For the cases left i.e. 7, 10, 12, 20, 24, 26 we exhibit an isometry that generates a group $G \subset \mathrm{O}(\mathbf{L})$ such that $|G| = 2$, $|G^\sharp| = 2$ and it holds that $\mathbf{L}_G = S$ and $\mathbf{L}^G = T$, i.e. (T, S) are invariant and coinvariant sublattices with respect to the action of G on $\mathbf{L}$. We denote these cases by ♣ in Table 3. For these cases we give in Table 4 an example of $G \subset \mathrm{O}(\mathbf{L})$ written with respect to the standard basis of $\mathbf{L} \cong \mathbf{U}^{\oplus 3} \oplus [-2]^{\oplus 2}$ with $|G| = 2$, $|G^\sharp| = 2$ and $(\mathbf{L}_G, \mathbf{L}^G)$ as invariant and coinvariant sublattices (T, S).

4.3. Order 3, 5 and 7

If $G \subset \mathrm{O}(\mathbf{L})$ and $|G| = p$, where $p \geq 3$, we have $|G^\sharp| = 1$, so by Proposition 3.5 we have $\mathbf{L}_G \cong \mathbf{\Lambda}_G$ and $\mathbf{L}^G \cong \mathbf{R}^\perp \subset \mathbf{\Lambda}^G$.

Proposition 4.6. *If S is a p-elementary lattice with $p \in \{3, 5, 7\}$ and if there exists a primitive embedding $S \hookrightarrow \mathbf{L}$, then the embedding is unique up to isometry of $\mathbf{L}$.*

Proof. If $S \hookrightarrow \mathbf{L}$ is a primitive embedding then the embedding subgroup H is such that $H \subset S^\sharp \cong (\mathbb{Z}/p\mathbb{Z})^{\oplus a}$ and $H \subset \mathbf{L}^\sharp \cong (\mathbb{Z}/2\mathbb{Z})^{\oplus 2}$. Since $p \geq 3$ then $H = \{\mathrm{id}\}$ hence the isometry $\gamma = \mathrm{id}$ which means that $\Gamma^\perp/\Gamma \cong (S^\perp)^\sharp \cong S^\sharp \oplus \mathbf{L}^\sharp$ and $q_{S^\perp} \cong q_{\mathbf{L}} - q_S$. By [30, Proposition 1.15.1] we conclude. ∎

We know that $\mathbf{L} \hookrightarrow \mathbf{\Lambda}$ is an embedding such that if $G \subset \mathrm{O}(\mathbf{L})$ and $|G| = p$ with $p \geq 3$ then $|G^\sharp| = 1$. Due to this fact it is possible to extend the action on $\mathbf{\Lambda}$ trivially on $\mathbf{L}_G^\perp \cong \mathbf{L}^G$ which means that $\mathbf{L}_G \cong \mathbf{\Lambda}_G$ is a p-elementary lattice. We compute an orthogonal complement $\mathbf{L}_G^\perp$ of a primitive embedding $\mathbf{L}_G \hookrightarrow \mathbf{L}$ and by Proposition 4.6 all the other primitive embeddings are equivalent with respect to the level (1) of the classification described in Subsection 1.2.

Remark 4.7. In cases $|G| = 3$ and $|G| = 7$ we can use Theorem 2.9 in [3] to show that there is a unique embedding $\mathbf{L}_G \hookrightarrow \mathbf{L}$ up to isometry of $\mathbf{L}$. In fact, we have $\mathrm{rk}(\mathbf{L}_G) \leq 2$ (or $\mathrm{rk}(\mathbf{L}^G) \leq 2$) and $\mathbf{U}^{\oplus 3} \subset \mathbf{L}$. We obtain the level (1) of the classification explained in Subsection 1.2.

If $|G| = 3$, the only possible coinvariant sublattices that we find in Table 1 are $\mathbf{L}_G \cong \mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$, $\mathbf{L}_G \cong \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus \mathbf{U}(3)$, $\mathbf{L}_G \cong \mathbf{U}^{\oplus 2}$, $\mathbf{L}_G \cong \mathbf{U} \oplus \mathbf{U}(3)$ and $\mathbf{L}_G \cong \mathbf{A}_2$. By Proposition 4.6, we know that the primitive embedding $\mathbf{L}_G \hookrightarrow \mathbf{L}$, if there exists, is

No.	S	T	♣
1	$\mathbf{U}^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4]$	$[4]$	–
2	$\mathbf{U} \oplus [2] \oplus [-2]^{\oplus 2} \oplus [-4]$	$[4] \oplus [-2]$	–
3	$[2]^{\oplus 2} \oplus [-2]^{\oplus 2} \oplus [-4]$	$[-2]^{\oplus 2} \oplus [4]$	–
4	$\mathbf{U} \oplus [-2]^{\oplus 2} \oplus [4]$	$[2] \oplus [-2] \oplus [-4]$	–
5	$\mathbf{U} \oplus [-2]^{\oplus 2} \oplus [4]$	$\mathbf{U} \oplus [-4]$	–
6	$\mathbf{U} \oplus [-2]^{\oplus 2} \oplus [4]$	$\mathbf{U}(2) \oplus [-4]$	–
7	$\mathbf{U} \oplus \mathbf{U}(2) \oplus [-4]$	$\mathbf{U}(2) \oplus [-4]$	♣
8	$\mathbf{U} \oplus \mathbf{U}(2) \oplus [-4]$	$[-2]^{\oplus 2} \oplus [4]$	–
9	$\mathbf{U} \oplus [2] \oplus [-2] \oplus [-4]$	$[-2]^{\oplus 2} \oplus [4]$	–
10	$\mathbf{U} \oplus [2] \oplus [-2] \oplus [-4]$	$[2] \oplus [-2] \oplus [-4]$	♣
11	$\mathbf{U}^{\oplus 2} \oplus [-4]$	$[-2]^{\oplus 2} \oplus [4]$	–
12	$\mathbf{U}^{\oplus 2} \oplus [-4]$	$\mathbf{U} \oplus [-4]$	♣
13	$[2] \oplus [-2]^{\oplus 2} \oplus [4]$	$\mathbf{U} \oplus [-2] \oplus [-4]$	–
14	$[2] \oplus [-2]^{\oplus 2} \oplus [4]$	$[2] \oplus [-2]^{\oplus 2} \oplus [-4]$	–
15	$[2]^{\oplus 2} \oplus [-2] \oplus [-4]$	$[-2]^{\oplus 3} \oplus [4]$	–
16	$[2]^{\oplus 2} \oplus [-2] \oplus [-4]$	$\mathbf{U} \oplus [-2] \oplus [-4]$	–
17	$\mathbf{U} \oplus [-2] \oplus [4]$	$[2] \oplus [-2]^{\oplus 2} \oplus [-4]$	–
18	$\mathbf{U} \oplus [-2] \oplus [4]$	$\mathbf{U} \oplus [-2] \oplus [-4]$	–
19	$\mathbf{U} \oplus [2] \oplus [-4]$	$[-2]^{\oplus 3} \oplus [4]$	–
20	$\mathbf{U} \oplus [2] \oplus [-4]$	$\mathbf{U} \oplus [-2] \oplus [-4]$	♣
21	$\mathbf{U}(2) \oplus [4]$	$\mathbf{U}(2) \oplus [-2]^{\oplus 2} \oplus [-4]$	–
22	$\mathbf{U}(2) \oplus [4]$	$\mathbf{U} \oplus [-2]^{\oplus 2} \oplus [-4]$	–
23	$[2]^{\oplus 2} \oplus [-4]$	$[-2]^{\oplus 4} \oplus [4]$	–
24	$[2]^{\oplus 2} \oplus [-4]$	$\mathbf{U} \oplus [-2]^{\oplus 2} \oplus [-4]$	♣
25	$\mathbf{U} \oplus [4]$	$\mathbf{U} \oplus [-2]^{\oplus 2} \oplus [-4]$	–
26	$\mathbf{U} \oplus [4]$	$\mathbf{U} \oplus \mathbf{A}_3(-1)$	♣
27	$[2] \oplus [4]$	$\mathbf{U} \oplus [-2]^{\oplus 3} \oplus [-4]$	–

Table 3. Orthogonal complements of $S \hookrightarrow \mathbf{L}$.

unique up to isometry of $\mathbf{L}$. The embedding is given by choosing $H = \{\mathrm{id}\}$ as embedding subgroup. If $\mathbf{L}_G \cong \mathbf{U}^{\oplus 2} \oplus \mathbf{A}_2(-1)$ then $\mathbf{L}^G \cong [-2] \oplus [6]$. If $\mathbf{L}_G \cong \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus \mathbf{U}(3)$ then there are no primitive embeddings of this lattice in $\mathbf{L}$; in fact if such an embedding exists, then the length of the discriminant group of the orthogonal complement is greater than the rank. If $\mathbf{L}_G \cong \mathbf{U}^{\oplus 2}$ then $\mathbf{L}^G \cong \mathbf{U} \oplus [-2]^{\oplus 2}$. If $\mathbf{L}_G \cong \mathbf{U} \oplus \mathbf{U}(3)$ then $\mathbf{L}^G \cong \mathbf{U}(3) \oplus [-2]^{\oplus 2}$ and if $\mathbf{L}_G \cong \mathbf{A}_2$ then $\mathbf{L}^G \cong \mathbf{U} \oplus \mathbf{A}_2(-1) \oplus [-2]^{\oplus 2}$.

If $|G| = 5$, the only possible coinvariant sublattice that we find in Table 1 is $\mathbf{L}_G \cong \mathbf{U} \oplus \mathbf{H}_5$. By Proposition 4.6, we know that the primitive embedding $\mathbf{L}_G \hookrightarrow \mathbf{L}$ is unique up to isometry of $\mathbf{L}$. The embedding is given by choosing the trivial embedding subgroup $H = \{\mathrm{id}\}$ and we obtain $\mathbf{L}^G \cong [-2] \oplus [-10] \oplus \mathbf{U}$.

If $|G| = 7$, the only possible coinvariant sublattice that we find in Table 1 is $\mathbf{L}_G \cong \mathbf{U} \oplus \mathbf{K}_7$. By Proposition 4.6, we know that the primitive embedding $\mathbf{L}_G \hookrightarrow \mathbf{L}$ is unique up to isometry of $\mathbf{L}$. The embedding is given by choosing the trivial embedding subgroup $H = \{\mathrm{id}\}$ and we obtain $\mathbf{L}^G \cong [-2] \oplus [14]$.

No.	$\mathbf{L}_G$	$\mathbf{L}^G$	example
1	$\mathbf{U} \oplus \mathbf{U}(2) \oplus [-4]$	$\mathbf{U}(2) \oplus [-4]$	$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
2	$\mathbf{U} \oplus [2] \oplus [-2] \oplus [-4]$	$[2] \oplus [-2] \oplus [-4]$	$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
3	$\mathbf{U}^{\oplus 2} \oplus [-4]$	$\mathbf{U} \oplus [-4]$	$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
4	$\mathbf{U} \oplus [2] \oplus [-4]$	$\mathbf{U} \oplus [-2] \oplus [-4]$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
5	$[2]^{\oplus 2} \oplus [-4]$	$\mathbf{U} \oplus [-2]^{\oplus 2} \oplus [-4]$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$
6	$\mathbf{U} \oplus [4]$	$\mathbf{U} \oplus \mathbf{A}_3(-1)$	$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 1 \\ 0 & 0 & 0 & 0 & 2 & 2 & 1 & 2 \end{pmatrix}$

Table 4. Invariant and coinvariant sublattices of nonsymplectic groups $G \subset \mathrm{O}(\mathbf{L})$ of order 2 and $|G^\sharp| = 2$ on manifolds of OG6 type.

4.4. Proof of Theorem 1.2

The computations in Sections 4.1, 4.2, and 4.3 are the proof of Theorem 1.2. In fact, the pairs $(\mathbf{L}^G, \mathbf{L}_G)$ are invariant and coinvariant sublattices with respect to effective nonsymplectic groups $G \subset \mathrm{O}(\mathbf{L})$ of prime order on a manifold X of OG6 type. Then there exist $G' \subset \mathrm{Aut}(X)$ such that $G = \eta_*(G')$ where η_* is the representation map recalled in equation (1.1). In this way, nonsymplectic groups $G \subset \mathrm{Aut}(X)$ of prime order on manifolds X of OG6 type are completely classified. ∎

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No.	$\lvert G\rvert$	$\lvert G^\sharp\rvert$	$\mathbf{L}_G$	$\mathbf{L}^G$
1	2	1	$\mathbf{U}^{\oplus 2}\oplus[-2]^{\oplus 3}$	$[2]$
2	2	1	$\mathbf{U}\oplus[2]\oplus[-2]^{\oplus 3}$	$[2]\oplus[-2]$
3	2	1	$\mathbf{U}^{\oplus 2}\oplus[-2]^{\oplus 2}$	$\mathbf{U}$
4	2	1	$\mathbf{U}^{\oplus 2}\oplus[-2]^{\oplus 2}$	$[2]\oplus[-2]$
5	2	1	$\mathbf{U}^{\oplus 2}\oplus[-2]^{\oplus 2}$	$\mathbf{U}(2)$
6	2	1	$[2]^{\oplus 2}\oplus[-2]^{\oplus 3}$	$[2]\oplus[-2]^{\oplus 2}$
7	2	1	$\mathbf{U}\oplus[-2]^{\oplus 2}\oplus[2]$	$\mathbf{U}\oplus[-2]$
8	2	1	$\mathbf{U}\oplus[-2]^{\oplus 2}\oplus[2]$	$[2]\oplus[-2]^{\oplus 2}$
9	2	1	$\mathbf{U}^{\oplus 2}\oplus[-2]$	$[-2]^{\oplus 2}\oplus[2]$
10	2	1	$\mathbf{U}^{\oplus 2}\oplus[-2]$	$\mathbf{U}\oplus[-2]$
11	2	1	$[2]^{\oplus 2}\oplus[-2]^{\oplus 2}$	$\mathbf{U}\oplus[-2]^{\oplus 2}$
12	2	1	$[2]^{\oplus 2}\oplus[-2]^{\oplus 2}$	$[-2]^{\oplus 3}\oplus[2]$
13	2	1	$\mathbf{U}(2)^{\oplus 2}$	$\mathbf{U}(2)\oplus[-2]^{\oplus 2}$
14	2	1	$\mathbf{U}\oplus[2]\oplus[-2]$	$[2]\oplus[-2]^{\oplus 3}$
15	2	1	$\mathbf{U}\oplus[2]\oplus[-2]$	$\mathbf{U}\oplus[-2]^{\oplus 2}$
16	2	1	$\mathbf{U}\oplus\mathbf{U}(2)$	$\mathbf{U}(2)\oplus[-2]^{\oplus 2}$
17	2	1	$\mathbf{U}\oplus\mathbf{U}(2)$	$\mathbf{U}\oplus[-2]^{\oplus 2}$
18	2	1	$\mathbf{U}^{\oplus 2}$	$\mathbf{U}\oplus[-2]^{\oplus 2}$
19	2	1	$[2]^{\oplus 2}\oplus[-2]$	$[-2]^{\oplus 4}\oplus[2]$
20	2	1	$[2]^{\oplus 2}\oplus[-2]$	$\mathbf{U}\oplus[-2]^{\oplus 3}$
21	2	1	$\mathbf{U}\oplus[2]$	$\mathbf{U}\oplus[-2]^{\oplus 3}$
22	2	1	$[2]^{\oplus 2}$	$\mathbf{U}\oplus[-2]^{\oplus 4}$
23	2	1	$[2]^{\oplus 2}$	$\mathbf{U}\oplus\mathbf{D}_4(-1)$
24	2	1	$[2]^{\oplus 2}$	$\mathbf{U}(2)\oplus\mathbf{D}_4(-1)$
1	2	2	$\mathbf{U}\oplus\mathbf{U}(2)\oplus[-4]$	$\mathbf{U}(2)\oplus[-4]$
2	2	2	$\mathbf{U}\oplus[2]\oplus[-2]\oplus[-4]$	$[2]\oplus[-2]\oplus[-4]$
3	2	2	$\mathbf{U}^{\oplus 2}\oplus[-4]$	$\mathbf{U}\oplus[-4]$
4	2	2	$\mathbf{U}\oplus[2]\oplus[-4]$	$\mathbf{U}\oplus[-2]\oplus[-4]$
5	2	2	$[2]^{\oplus 2}\oplus[-4]$	$\mathbf{U}\oplus[-2]^{\oplus 2}\oplus[-4]$
6	2	2	$\mathbf{U}\oplus[4]$	$\mathbf{U}\oplus\mathbf{A}_3(-1)$
1	3	1	$\mathbf{U}^{\oplus 2}\oplus\mathbf{A}_2(-1)$	$[-2]\oplus[6]$
2	3	1	$\mathbf{U}^{\oplus 2}$	$\mathbf{U}\oplus[-2]^{\oplus 2}$
3	3	1	$\mathbf{U}\oplus\mathbf{U}(3)$	$\mathbf{U}(3)\oplus[-2]^{\oplus 2}$
4	3	1	$\mathbf{A}_2$	$\mathbf{U}\oplus\mathbf{A}_2(-1)\oplus[-2]^{\oplus 2}$
1	5	1	$\mathbf{U}\oplus\mathbf{H}_5$	$[-2]\oplus[-10]\oplus\mathbf{U}$
1	7	1	$\mathbf{U}^{\oplus 2}\oplus\mathbf{K}_7$	$[-2]\oplus[14]$

Table 5. Invariant and coinvariant sublattices of effective nonsymplectic groups $G\subset \mathrm{O}(\mathbf{L})$ on manifolds of OG6 type.

References

[1] Artebani, M. and Sarti, A.: Non-symplectic automorphisms of order 3 on K3 surfaces. *Math. Ann.* **342** (2008), no. 4, 903–921.

[2] Artebani, M., Sarti, A. and Taki, S.: K3 surfaces with non-symplectic automorphisms of prime order. *Math. Z.* **268** (2011), no. 1-2, 507–533.

[3] Barth, W. P., Hulek, K., Peters, C. A. M. and Van de Ven, A.: *Compact complex surfaces*. 2nd ed. Ergebnisse der Mathematik und ihrer Grenzgebiete 4, Springer-Verlag, Berlin, 2004.

[4] Beauville, A.: Some remarks on Kähler manifolds with $c_1 = 0$. In *Classification of algebraic and analytic manifolds (Katata, 1982)*, pp. 1–26. Progr. Math. 39, Birkhäuser Boston, Boston, MA, 1983.

[5] Beauville, A.: Variétés Kähleriennes dont la première classe de Chern est nulle. *J. Differential Geom.* **18** (1983), no. 4, 755–782 (1984).

[6] Beauville, A.: Antisymplectic involutions of holomorphic symplectic manifolds. *J. Topol.* **4** (2011), no. 2, 300–304.

[7] Boissière, S., Camere, C., Mongardi, G. and Sarti, A.: Isometries of ideal lattices and hyperkähler manifolds. *Int. Math. Res. Not. IMRN* (2016), no. 4, 963–977.

[8] Boissière, S., Camere, C. and Sarti, A.: Classification of automorphisms on a deformation family of hyper-Kähler four-folds by p-elementary lattices. *Kyoto J. Math.* **56** (2016), no. 3, 465–499.

[9] Boissière, S., Nieper-Wiß kirchen, M. and Tari, K.: Some algebraic and geometric properties of hyper-Kähler fourfold symmetries. In *Geometry at the frontier – symmetries and moduli spaces of algebraic varieties*, pp. 57–70. Contemp. Math. 766, Amer. Math. Soc., Providence, RI, 2021.

[10] Boissière, S., Nieper-Wißkirchen, M. and Sarti, A.: Smith theory and irreducible holomorphic symplectic manifolds. *J. Topol.* **6** (2013), no. 2, 361–390.

[11] Brandhorst, S. and Cattaneo, A.: Prime order isometries of unimodular lattices and automorphisms of ihs manifolds. Preprint 2020, arXiv: 1912.07119v3.

[12] Camere, C.: Symplectic involutions of holomorphic symplectic four-folds. *Bull. Lond. Math. Soc.* **44** (2012), no. 4, 687–702.

[13] Camere, C. and Cattaneo, A.: Non-symplectic automorphisms of odd prime order on manifolds of $\mathrm{K3}^{[n]}$-type. *Manuscripta Math.* **163** (2020), no. 3-4, 299–342.

[14] Camere, C., Cattaneo, A. and Cattaneo, A.: Non-symplectic involutions on manifolds of $\mathrm{K3}^{[n]}$-type. *Nagoya Math. J.* **243** (2021), 278–302.

[15] Camere, C., Kapustka, G., Kapustka, M. and Mongardi, G.: Verra four-folds, twisted sheaves, and the last involution. *Int. Math. Res. Not. IMRN* (2019), no. 21, 6661–6710.

[16] Conway, J. H. and Sloane, N. J. A.: *Sphere packings, lattices and groups*. Third edition. Grundlehren der mathematischen Wissenschaften 290, Springer-Verlag, New York, 1999.

[17] Fujiki, A.: On primitively symplectic compact Kähler V-manifolds of dimension four. In *Classification of algebraic and analytic manifolds (Katata, 1982)*, pp. 71–250. Progr. Math. 39, Birkhäuser, Boston, MA, 1983.

[18] Grossi, A.: Induced birational transformations on O'Grady's sixfolds. *J. London Math. Soc.* **105** (2022), no. 1, 665–690.

[19] Grossi, A., Onorati, C. and Veniani, D. C.: Symplectic birational transformations of finite order on O'Grady's sixfolds. To appear in *Kyoto J. Math.*

[20] Huybrechts, D.: A global Torelli theorem for hyperkähler manifolds (after M. Verbitsky). In *Séminaire Bourbaki Vol. 2010/2011, Exposés 1027–1042*, pp. 375–403. Asterique 348, Soc. Math. France, 2012.

[21] Huybrechts, D.: *Lectures on K3 surfaces*. Cambridge Studies in Advanced Mathematics 158, Cambridge University Press, Cambridge, 2016.

[22] Joumaah, M.: Non-symplectic involutions of irreducible symplectic manifolds of $K3^{[n]}$-type. *Math. Z.* **283** (2016), no. 3-4, 761–790.

[23] Markman, E.: A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, pp. 257–322. Springer Proc. Math. 8, Springer, Heidelberg, 2011.

[24] Miranda, R. and Morrison, D. R.: *Embeddings of integral quadratic forms*. Preliminary draft, 2009. Electronically available at: web.math.ucsb.edu/~drm/manuscripts/eiqf.pdf.

[25] Mongardi, G.: Symplectic involutions on deformations of $K3^{[2]}$. *Cent. Eur. J. Math.* **10** (2012), no. 4, 1472–1485.

[26] Mongardi, G. and Rapagnetta, A.: Monodromy and birational geometry of O'Grady's sixfolds. *J. Math. Pures Appl. (9)* **146** (2021), 31–68.

[27] Mongardi, G., Tari, K. and Wandel, M.: Prime order automorphisms of generalised Kummer fourfolds. *Manuscripta Math.* **155** (2018), no. 3-4, 449–469.

[28] Mukai, S.: Symplectic structure of the moduli space of sheaves on an abelian or K3 surface. *Invent. Math.* **77** (1984), no. 1, 101–116.

[29] Nikulin, V. V.: Finite groups of automorphisms of Kählerian surfaces of type K3. *Uspehi Mat. Nauk* **31** (1976), no. 2 (188), 223–224.

[30] Nikulin, V. V.: Integer symmetric bilinear forms and some of their geometric applications. *Izv. Akad. Nauk SSSR Ser. Mat.* **43** (1979), no. 1, 111–177.

[31] O'Grady, K. G.: Desingularized moduli spaces of sheaves on a K3. *J. Reine Angew. Math.* **512** (1999), 49–117.

[32] O'Grady, K. G.: A new six-dimensional irreducible symplectic variety. *J. Algebraic Geom.* **12** (2003), no. 3, 435–505.

[33] Ohashi, H. and Wandel, M.: Non-natural non-symplectic involutions on symplectic manifolds of $K3^{[2]}$-type. Preprint 2013, arXiv: 1305.6353v2.

[34] Onorati, C.: On the monodromy group of desingularised moduli spaces of sheaves on K3 surfaces. *J. Algebraic Geom.* **31** (2022), 425–465.

[35] Rapagnetta, A.: Topological invariants of O'Grady's six dimensional irreducible symplectic variety. *Math. Z.* **256** (2007), no. 1, 1–34.

[36] Rudakov, A. N. and Shafarevich, I. R.: Surfaces of type K3 over fields of finite characteristic. In *Current problems in mathematics, Vol. 18*, pp. 115–207. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981.

[37] Verbitsky, M.: Mapping class group and a global Torelli theorem for hyperkähler manifolds. *Duke Math. J.* **162** (2013), no. 15, 2929–2986.

[38] Verbitsky, M.: Errata for "Mapping class group and a global Torelli theorem for hyperkähler manifolds" by Misha Verbitsky. *Duke Math. J.* **169** (2020), no. 5, 1037–1038.

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